Analysis of the Strategy "Hiring above the α-quantile"*

Ahmed Helmi† Conrado Martínez† Alois Panholzer‡

Abstract

We study here the strategy hiring above the α-quantile of the hired staff. This strategy was introduced first by Archibald and Martínez in 2009. We show here more results like the lower and upper bounds for many interesting quantities for the general case, i.e., $0 < \alpha < 1$. For the main parameter, number of hired candidates, we were able to obtain the exact and limiting distributions for $\alpha = \frac{1}{d}$, $d \in \mathbb{N}$.

1 Introduction

The hiring problem is a recent research problem, which has been introduced and studied first by Broder et al. [2] in 2008. It belongs to the category of on-line decision making under uncertainty. In such kind of research, the input is a sequence of instances and a decision must be taken for each instance depending on the instances seen so far while no information on the future is available. The hiring problem can be considered as a natural extension of the well-known secretary problem [4], where the employer is now looking for many candidates rather than only one (as it is the case for the secretary problem). Here the goal is to design some hiring strategy to meet the demands of the employer, which essentially are to obtain a good quality staff at a reasonable hiring rate.

Archibald and Martínez [1] have reformulated the problem for a discrete model that considers the relative ranks amongst candidates as it is the case in the secretary problem. We give an overview of this model in the next lines.

Random permutation model

Here, the interviewed candidate is given a *rank* which is relative to the ranks of the previous candidates. The ranking scheme considers the *larger* rank as *better* than *lower* one, that is the contrary to ranking schemes in secretary problems and percentile rules [6]. But of course both ways are totally equivalent and lead to the random permutation model for the sequence of candidates. Thus, for a sequence $S = s_1, \ldots, s_i, \ldots$ of candidates, s_i denotes the rank of the ith candidate among all interviewd ones, where the best candidate seen so far among the n gets a rank n, while the worst

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†Dept. Llenguatges i Sistemes Informàtics, Universitat Politècnica de Catalunya, E-08034 Barcelona, Spain. `{ahelmi,conrado}-at-lsi.upc.edu`

‡Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, 1040 Wien, Austria. `Alois.Panholzer-at-tuwien.ac.at`

one gets rank 1. Then the n ranks of the n candidates form finally a permutation $\sigma^{(n)} = \{1, \ldots, n\}$. Then any permutation σ representing the scores is equally likely. In this context, the *hiring set* of a permutation σ is the set of indices that would be hired by applying a specific strategy to the permutation σ.

Given a permutation $\sigma^{(n-1)}$ of length $n-1$ and a value (relative rank) j, $1 \le j \le n$, $\sigma^{(n-1)} \circ j$ denotes the resulting permutation of size n after relabelling $j, j+1, \ldots, n$ and appending j to the end. For example, if we have this sequence of relative ranks: $1, 1, 3, 2, 2$ then the corresponding permutations are $\sigma^{(1)} = 1$, $\sigma^{(2)} = \sigma^{(1)} \circ 1 = 21$, $\sigma^{(3)} = \sigma^{(2)} \circ 3 = 213$, $\sigma^{(4)} = \sigma^{(3)} \circ 2 = 3142$ and $\sigma^{(5)} = \sigma^{(4)} \circ 2 = 41532$. The notation $\mathcal{H}(\sigma)$ will denote the set of indices of the hired candidates or the *hiring set* of the permutation σ. This hiring set has some parameters to be studied *w.r.t.* a given hiring strategy such as its size $h(\sigma)$, the gap of last hired candidate $g(\sigma)$, the index of last hired candidate $L(\sigma)$ and other useful parameters as we will see later. The letters $h_n, g_n, L_n, \ldots$ will denote the corresponding random variables of these parameters. The gap of last hired candidate is defined as $1 - r(\sigma)/|\sigma|$ where $r(\sigma)$ is the score of last hired candidate.
The behaviour of *hiring above the α-quantile* can be explained according to the following definition, as given in [1]:

Definition 1 *The strategy "hiring above the α-quantile" hires the first candidate in the sequence, then any further candidate is hired if and only if her relative rank is larger than the α-quantile, $0 < \alpha < 1$, of the already hired staff. The α-quantile of a sequence $x_1 < x_2 < \cdots < x_k$ of k elements is the element x_j with $j = \lceil \alpha k \rceil$.*

We discuss in Section 2 using the framework in [1] to analyze hiring above the α-quantile. This approach can give us what we call "lower" and "upper" bounds for the studied parameters, that indicates at least the order of growth of those parameters, for general α, with $0 < \alpha < 1$, but the desired results for the parameters are missing due to coping with the ceiling function. We give first a summary of the main results for the bounds of three parameters: the *size of hiring set,* h_n that is the number of hired candidates, the *gap of last hired candidate,* g_n, and the *number of replacements,* f_n. f_n is the quantity of interest in "hiring with replacement", that is a technique that enables one discarded candidate by the basis strategy to be hired anew instead of the worst hired one (see [5] for more discussion). The introduced theorems quantify precisely the order of growth of the expectation of the mentioned parameters, while similar results for other parameters are quite involved.
Moreover, we show that the framework of Archibald and Martínez can be used to analyze other probabilistic selection rules rather than hiring strategies in a systematic analytic way. One example is the seating plan $(\frac{1}{2}, 1)$ of the Chinese Restaurant Process (CRP) [7], which is exactly the upper bound of "hiring above the median", then we can obtain similar results for classes of seating plans like $(\alpha, 1)$ and $(\alpha, 0)$, $0 < \alpha < 1$. After that we extend our recursive approach used to analyze hiring above the median, to obtain explicit results for the main quantity, *size of hiring set* in case of $\alpha = \frac{1}{d}$, $d \in \mathbb{N}$.

2 Lower and upper bounds

We follow here the framework introduced in [1] which uses directly the generating functions to perform the analysis. The first step is always to characterize the quantity $X(\sigma)$ of the strategy under study. $X(\sigma)$ specifies how many candidates can be hired in the next step right after σ,

which is a unique value for each unique hiring strategy. For hiring above the α-quantile, $X(\sigma) = \lceil(1-\alpha)(h(\sigma)+1)\rceil$, with $0 < \alpha < 1$. For simplicity, we write $X(\sigma) = \lceil a(h(\sigma)+1)\rceil$, where $a = 1-\alpha$. Rather than dealing with the ceilings, we shall consider the *lower* and *upper* bounds $X_\ell(\sigma) = ah(\sigma)+a$ and $X_u(\sigma) = ah(\sigma)+1$, when $|\sigma| > 0$, and for the empty permutation, we set $X_\ell(\epsilon) = X_u(\epsilon) = 1$.
The lower and upper bounds X_ℓ and X_u will yield lower and upper bounds for the *expected values* of several parameters, i.e., $\mathbb{E}\{h_n^{(\alpha,\alpha)}\} \leq \mathbb{E}\{h_n\} \leq \mathbb{E}\{h_n^{(\alpha,1)}\}$. We make use of the following proposition to establish such relationships for bounds of the considered parameters,

Proposition 1 *Let A and B be two pragmatic hiring strategies such that, for all σ, $X_A(\sigma) \leq X_B(\sigma)$. Since both strategies are pragmatic, that means that if strategy A hires a candidate with score j, then the candidate will be hired also by strategy B. Furthermore,*

(i) $h_n^{(A)} \leq_{st} h_n^{(B)}$.

(ii) $g_n^{(A)} \leq_{st} g_n^{(B)}$.

(iii) $f_n^{(A)} \geq_{st} f_n^{(B)}$.

Where for any two positive random variables Y and Z, $Y \leq_{st} Z$ (reads: "Y is stochastically smaller than or equal to Z") means that $\mathbb{P}\{Y > t\} \leq \mathbb{P}\{Z > t\}$, for all $t \geq 0$. Moreover, $Y \leq_{st} Z$ implies $\mathbb{E}\{Y\} \leq \mathbb{E}\{Z\}$.

Proof

(i) It follows directly, since $X_A(\sigma) \leq X_B(\sigma)$.

(ii) It holds from [1] that $\mathbb{E}\{g_n\} = \frac{1}{2n}\big(\mathbb{E}\{X_n\} - 1\big)$. Then, as $X(\sigma)$ increases the gap increases and vice-versa. discard a candidate that B would hire.

(iii) Because strategy A discards at least as many candidates as strategy B. Then more candidates will induce a replacement for A than for B.

■

It's important to clarify that both selection rules defined by X_ℓ and X_u are "pragmatic" but do not correspond to actual hiring strategies. Pragmaticity conditions (see [1] for the definition of pragmatic hiring strategies) hold here but since we are dealing with rank-based strategies, then the function $X(\sigma)$ should give an *integer* value that is the number of choices to hire the next candidate. When $X(\sigma)$ is not always integer-valued then we can't always specify the *threshold candidate* during hiring; in other terms the threshold candidate doesn't always exist in the hiring set.
For example, consider hiring above the median [1, 5] which is a special case when $\alpha = 1/2$, then $X_{med} = \lceil \frac{1}{2}(k+1)\rceil$, where k is the number of hired candidates so far, and its lower and upper bounds with $X_\ell = \frac{1}{2}k + \frac{1}{2}$ and $X_u = \frac{1}{2}k + 1$ respectively. Then for odd $k = 2t-1$, we have $X_{med} = t$, $X_\ell = t$ and $X_u = t + \frac{1}{2}$, while for even $k = 2t-2$, we have $X_{med} = t$, $X_\ell = t - \frac{1}{2}$ and $X_u = t$.
We conclude that X_ℓ and X_u define two probabilistic selection rules in terms of the *probabilities of selection* (similar to the seating plan $(\alpha, 1)$ of the CRP [7]).

2.1 Main results for bounds on hiring above the α-quantile

Theorem 1 *For hiring above the α-quantile, with $0 < \alpha < 1$, let h_n denote the size of hiring set, then*

$$\mathbb{E}\{h_n^{(\alpha,\alpha)}\} \leq \mathbb{E}\{h_n\} \leq \mathbb{E}\{h_n^{(\alpha,1)}\}$$

where asymptotically as $n \to \infty$,

$$\mathbb{E}\{h_n^{(\alpha,\alpha)}\} = \frac{2}{2-\alpha} \cdot \frac{n^{1-\alpha}}{\Gamma(2-\alpha)} \cdot \left(1 + O\Big(\frac{1}{n}\Big)\right),$$
$$\mathbb{E}\{h_n^{(\alpha,1)}\} = \frac{1}{1-\alpha} \cdot \frac{n^{1-\alpha}}{\Gamma(2-\alpha)} \cdot \left(1 + O\Big(\frac{1}{n}\Big)\right).$$

Theorem 2 *For hiring above the α-quantile, with $0 < \alpha < 1$, let g_n denote the gap of last hired candidate, then*

$$\mathbb{E}\{g_n^{(\alpha,\alpha)}\} \leq \mathbb{E}\{g_n\} \leq \mathbb{E}\{g_n^{(\alpha,1)}\}$$

where asymptotically as $n \to \infty$,

$$\mathbb{E}\{g_n^{(\alpha,\alpha)}\} = \frac{1-\alpha}{2-\alpha} \cdot \frac{n^{-\alpha}}{\Gamma(2-\alpha)} + O\Big(\frac{1}{n}\Big),$$
$$\mathbb{E}\{g_n^{(\alpha,1)}\} = \frac{1}{2} \cdot \frac{n^{-\alpha}}{\Gamma(2-\alpha)} + O\Big(\frac{1}{n}\Big).$$

Theorem 3 *For hiring above the α-quantile, with $0 < \alpha < 1$, let f_n denote the number of replacements, then*

$$\mathbb{E}\{f_n^{(\alpha,\alpha)}\} \leq \mathbb{E}\{f_n\} \leq \mathbb{E}\{f_n^{(\alpha,1)}\}$$

where asymptotically as $n \to \infty$,

$$\mathbb{E}\{f_n^{(\alpha,\alpha)}\} = \frac{2\alpha}{(1-\alpha)(2-\alpha)} \cdot \frac{n^{1-\alpha}}{\Gamma(2-\alpha)} - \ln n + O(1),$$
$$\mathbb{E}\{f_n^{(\alpha,1)}\} = \frac{\alpha}{(1-\alpha)^2} \cdot \frac{n^{1-\alpha}}{\Gamma(2-\alpha)} - \frac{1}{1-\alpha} \ln n + O(1).$$

Theorem 4 *Let $h_n^{(\frac{1}{2},\frac{1}{2})}$ and $h_n^{(\frac{1}{2},1)}$, denote the size of hiring set for the selection rules that are defined by $X_\ell(\sigma) = \frac{1}{2}h(\sigma) + \frac{1}{2}$ and $X_u(\sigma) = \frac{1}{2}h(\sigma) + 1$ respectively, then both rules are "bounding" hiring above the median. The explicit distributions of $h_n^{(\frac{1}{2},\frac{1}{2})}$ and $h_n^{(\frac{1}{2},1)}$ are given as follows:*

$$\mathbb{P}\{h_n^{(\frac{1}{2},\frac{1}{2})} = k\} = \frac{k}{n} 2^{k+1-2n} \sum_{j=0}^{n-k} 2^j \binom{2n-k-j-1}{n-1},$$
$$\mathbb{P}\{h_n^{(\frac{1}{2},1)} = k\} = (k+1) 2^{k-2n} \frac{k}{n} \binom{2n-k-1}{n-1}.$$

Asymptotically as $n \to \infty$*:*
The normalized r.v. $\frac{h_n^{(\frac{1}{2},\frac{1}{2})}}{\sqrt{n}} \xrightarrow{(d)} Y$*, where* Y *has the probability density function:*

$$f(y) = \frac{y}{\sqrt{\pi}} \int_{t=y}^{\infty} e^{-\frac{t^2}{4}} dt,$$

similarly, $\frac{h_n^{(\frac{1}{2},1)}}{\sqrt{2n}} \xrightarrow{(d)} Z$*, where* Z *follows a Chi distribution with* $k = 3$*, namely Maxwell-Boltzmann distribution with parameter* $\sqrt{2n}$*.*

2.2 Analysis

2.2.1 Size of hiring set

We have to apply the following theorem that describes the PDE of the size of hiring set,

Theorem 5 (Archibald and Martínez, 2009) *Let* $H(z,u)$ *be the generating function*

$$H(z,u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)},$$

where $h(\sigma)$ *is the size of hiring set in* σ *and* $\mathcal{P}$ *is the class of all permutations.*
Then

$$(1-z)\frac{\partial}{\partial z} H(z,u) - H(z,u) = (u-1) \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)}.$$

Also we need to define $X(\sigma)$ which is $ah(\sigma) + b$ as mentioned above. Remember that we use $a = 1 - \alpha$ for simplicity. Thus, the PDE of $H(z,u)$ takes the following form

$$(1-z)\frac{\partial H_{a,b}}{\partial z} - au(u-1)\frac{\partial H_{a,b}}{\partial u} - (1 + b(u-1))H_{a,b}(z,u) = (u-1)(1-b). \tag{1}$$

It is difficult to obtain a closed form of $H(z,u)$ from the last equation, so we will go directly to the next step by differentiating $w.r.t.$ u and setting $u = 1$. Then

$$(1-z)\frac{\partial}{\partial z} h_{a,b}(z) - (1+a)h_{a,b}(z) - \frac{b}{1-z} = 1 - b.$$

The solution turns out to be

$$h_{a,b}(z) = \frac{-1}{(1-z)^{1+a}} \left(\frac{(1-z)^a (az(b-1) + a + b)}{a(1+a)} + C \right),$$

with the initial condition $h(0) = 0$; we get thus $C = -\frac{b+a}{a(1+a)}$. From singularity analysis [3], we have then

$$[z^n]h_{a,b}(z) = \frac{n^a}{\Gamma(1+a)} \cdot \frac{b+a}{a(1+a)} \cdot \left(1 + O\Big(\frac{1}{n}\Big)\right), \tag{2}$$

where $\Gamma(\cdot)$ is the Gamma function. Then replacing b by a in (2) we have a lower bound

$$[z^n]h_{a,a}(z) = \frac{2}{a+1} \cdot \frac{n^a}{\Gamma(1+a)} \cdot \left(1 + O\Big(\frac{1}{n}\Big)\right). \tag{3}$$

Replacing b by 1 in (2), we have the upper bound

$$[z^n]h_{a,1}(z) = \frac{1}{a} \cdot \frac{n^a}{\Gamma(1+a)} \cdot \left(1 + O\Big(\frac{1}{n}\Big)\right). \tag{4}$$

We get the same result as [1], where for hiring above the α-quantile, we have $\mathbb{E}\{h_n\} = \Theta(n^{1-\alpha})$. We replace a by $1-\alpha$ in (3) and (4) to obtain the results for $\mathbb{E}\{h_n^{(\alpha,\alpha)}\}$ and $\mathbb{E}\{h_n^{(\alpha,1)}\}$, respectively, in Theorem 1.

2.2.2 Gap of last hired candidate

Let us consider

$$X(z) = \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!}.$$

Then for $X(\sigma) = ah(\sigma) + b$, we have

$$\begin{aligned}
\sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} &= \sum_{\sigma \in \mathcal{P}} (ah(\sigma) + b) \frac{z^{|\sigma|}}{|\sigma|!} \\
&= a \cdot \sum_{\sigma \in \mathcal{P}} h(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} + b \cdot \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} \\
&= a \cdot h_{a,b}(z) + b.
\end{aligned}$$

We have the value of $[z^n]h_{a,b}(z)$ from (2) hence,

$$\frac{1}{2}([z^n]X_{a,b}(z) - 1) = \frac{b+a}{2(1+a)} \cdot \frac{n^{a-1}}{\Gamma(1+a)} + O\Big(\frac{1}{n}\Big) \ . \tag{5}$$

Then following [1], we can obtain the lower bound for the gap as

$$\frac{a}{1+a} \cdot \frac{n^{a-1}}{\Gamma(1+a)} + O\Big(\frac{1}{n}\Big), \tag{6}$$

while the upper bound is

$$\frac{1}{2} \cdot \frac{n^{a-1}}{\Gamma(1+a)} + O\Big(\frac{1}{n}\Big). \tag{7}$$

In general, as given before [1], for hiring above the α-quantile, we get $\mathbb{E}\{g_n\} = \Theta(n^{-\alpha})$. Observe that for any $\alpha < 1$, $g_n \to 0$ as $n \to \infty$. We substitute $a = 1-\alpha$ in (6) and (7) to get the results stated in Theorem 2.

2.2.3 Number of replacements

We begin with the following trivariate generating function

$$\mathsf{F}(z,u,v) = \sum_{\sigma\in\mathbb{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{f(\sigma)} v^{h(\sigma)}, \tag{8}$$

where $f(\sigma)$ is the number of replacements made to process the permutation σ. Again, we use the catalytic variable v to be able to proceed in the analysis of strategies that have $X(\sigma) = f(h(\sigma))$.

In hiring with replacements, of the $|\sigma|+1$ possible rankings coming after σ, $|\sigma|+1-h(\sigma)$ will be discarded, $X(\sigma)$ will be hired without replacement and $h(\sigma)-X(\sigma)$ will be hired with replacement. So that the recurrence of $f(\sigma)$ will take the following form:

$$f(\sigma\circ j) = \begin{cases} f(\sigma), & \text{if } 1 \le j \le |\sigma|+1-h(\sigma) \text{ (j is discarded)},\\[2ex] f(\sigma)+1, & \text{if } |\sigma|+2-h(\sigma) \le j \le |\sigma|+1-X(\sigma) \text{ (replaces worst)},\\[2ex] f(\sigma), & \text{if } |\sigma|+2-X(\sigma) \le j \le |\sigma|+1 \text{ (j is hired)}. \end{cases}$$

This yields the next theorem, whose proof we omit as it closely follows that one of Theorem 5.

Theorem 6 *Let* $\mathsf{F}(z,u,v)$ *be the generating function defined in* (8)*. Let* $X(\sigma)$ *denote the number of ranks* j*,* $1 \le j \le |\sigma|+1$*, such that a candidate with score* j *will be hired without replacing anyone, if interviewed right after* σ*.*

Then

$$(1-z)\frac{\partial}{\partial z}\mathsf{F}(z,u,v) - \mathsf{F}(z,u,v) = v(u-1)\frac{\partial}{\partial v}\mathsf{F}(z,u,v) + (v-u)\sum_{\sigma\in\mathbb{P}} X(\sigma)\frac{z^{|\sigma|}}{|\sigma|!}u^{f(\sigma)}v^{h(\sigma)}.$$

In order to compute the expected number of replacements for a random permutation of size n, we can differentiate $\mathsf{F}(z,u,v)$ *w.r.t.* u and set $u=1$. Then the differential equation given in Theorem 6 transforms into

$$\begin{aligned}(1-z)\frac{\partial}{\partial z}\mathsf{f}(z,v) - \mathsf{f}(z,v) &= v\frac{\partial}{\partial v}\mathsf{F}(z,1,v) + (v-1)\sum_{\sigma\in\mathbb{P}} X(\sigma)f(\sigma)\frac{z^{|\sigma|}}{|\sigma|!}v^{h(\sigma)}\\ &\quad - \sum_{\sigma\in\mathbb{P}} X(\sigma)\frac{z^{|\sigma|}}{|\sigma|!}v^{h(\sigma)},\end{aligned} \tag{9}$$

with

$$\mathsf{f}(z,v) = \left.\frac{\partial}{\partial u}\mathsf{F}(z,u,v)\right|_{u=1}.$$

Now, $\mathsf{F}(z,1,v) = H(z,v)$, and we have to set $v=1$ in (9) to get rid of the catalytic variable v and thus obtain the generating function $f(z)$ for the expected values f_n:

$$(1-z)\frac{d}{dz}f(z) - f(z) = h(z) - \sum_{\sigma\in\mathbb{P}} X(\sigma)\frac{z^{|\sigma|}}{|\sigma|!}, \tag{10}$$

where

$$h(z) = \frac{\partial}{\partial v} H(z,v)\bigg|_{v=1}$$

is the generating function for $\mathbb{E}\{h_n\}$, and the initial condition is $f(0) = 0$.
Now, we set $X(\sigma) = a \cdot h(\sigma) + b$ in (10), thus we have

$$(1-z)\frac{d}{dz} f_{a,b}(z) - f_{a,b}(z) = (1-a)h_{a,b}(z) - \frac{bz}{1-z},$$

where

$$h_{a,b}(z) = \frac{-1}{(1-z)^{a+1}}\left(\frac{(1-z)^a(az(b-1)+a+b)}{a(a+1)} - \frac{b+a}{a(1+a)}\right),$$

as explained above in this subsection. The solution for $f_{a,b}(z)$ is

$$f_{a,b}(z) = \frac{(a+b)(1-a)}{a^2(1+a)} \cdot \frac{1}{(1-z)^{1+a}} - \frac{b}{a} \cdot \frac{1}{1-z} \ln\left(\frac{1}{1-z}\right) + \frac{C + z(2b+a-1)}{1-z}, \qquad (11)$$

where C is a constant, that can be computed using the initial condition $f(0) = 0$. However, the value of C is irrelevant, as the last term in (11) can be ignored as it is not dominant. Using again singularity analysis to obtain the asymptotic of the nth coefficient of both the lower bound ($b = a$) and upper bound ($b = 1$), we get

$$[z^n]f_{a,a}(z) = \frac{2(1-a)}{a(1+a)} \cdot \frac{n^a}{\Gamma(1+a)} - \ln n + O(1), \qquad (12)$$

$$[z^n]f_{a,1}(z) = \frac{(1-a)}{a^2} \cdot \frac{n^a}{\Gamma(1+a)} - \frac{1}{a}\ln n + O(1). \qquad (13)$$

Thus, for hiring above the α-quantile, $\mathbb{E}\{f_n\} = \Theta(n^{1-\alpha})$. In particular, $\mathbb{E}\{f_n\} = \Theta\big(\mathbb{E}\{h_n\}\big)$. As usual, we replace a by $1-\alpha$ to get the results in Theorem 3.

2.2.4 Bounds on hiring above the median

As we mentioned before, hiring above the median is a special case of hiring above the α-quantile when $\alpha = 1/2$. So that we set $a = 1/2$ in the general case equations to obtain the bounds of the parameters of this strategy.
Substituting $a = 1/2$ in (3) and (4), we have the following bounds

$$\mathbb{E}\{h_n^{(\frac{1}{2},\frac{1}{2})}\} = \frac{8\sqrt{n}}{3\sqrt{\pi}}\left(1 + O\left(\frac{1}{n}\right)\right), \qquad (14)$$

$$\mathbb{E}\{h_n^{(\frac{1}{2},1)}\} = \frac{4\sqrt{n}}{\sqrt{\pi}}\left(1 + O\left(\frac{1}{n}\right)\right). \qquad (15)$$

This implies that, for hiring above the median, $\mathbb{E}\{h_n\} = \Theta(\sqrt{n})$ as already given in [1, 5].
The solution of the PDE for $H_{a,b}(z)$ is difficult in general, but not to when $a = \frac{1}{2}$ and $b \in \{\frac{1}{2}, 1\}$, which give us lower and upper bounds on the probability distribution of h_n. So, substituting $a = 1/2$ and $b = 1/2$ in (1) and using the initial condition $H(0,u) = 1$, we find the solution as

$$H_{\frac{1}{2},\frac{1}{2}}(z,u) = \frac{1-u^2}{(1-u)^2} - \frac{2u}{(1-u)(1-u+u\sqrt{1-z})}$$

$$+\frac{2u}{(1-u)^2}\left(\frac{1}{2}\ln\Big(\frac{1}{1-z}\Big)-\ln\Big(\frac{1}{1-u+u\sqrt{1-z}}\Big)\right).$$

For the upper bound, we set $a=1/2$ and $b=1$, then

$$H_{\frac{1}{2},1}(z,u)=\frac{1}{(1-u+u\sqrt{1-z})^2}.$$

Since we have these closed forms, then we can obtain the following useful information. First we can give the factorial moments of each r.v. there:

$$\begin{aligned}
\mathbb{E}\{h_n^{(\frac{1}{2},\frac{1}{2})\underline{r}}\} &= \Theta(n^{r/2}),\\
\mathbb{E}\{h_n^{(\frac{1}{2},1)\underline{r}}\} &= [z^n]\frac{(r+1)!\Big(1-\sqrt{1-z}\Big)^r}{(1-z)^{r/2+1}}\\
&= (-1)^r(r+1)!\sum_{k=0}^{r}(-1)^k\binom{r}{k}\binom{n+k/2}{n}\\
&= \Theta(n^{r/2})
\end{aligned}$$

A more precise asymptotic estimation of $\mathbb{E}\{h_n^{(\frac{1}{2},1)\underline{r}}\}$ is given later in (16).

Now, extracting the coefficients of $[u^k z^n]H_{\frac{1}{2},1}(z,u)$ gives us the probability mass functions as follows

$$\begin{aligned}
\mathbb{P}\{h_n^{(\frac{1}{2},\frac{1}{2})}=k\} &= \frac{k}{n}2^{k+1-2n}\sum_{j=0}^{n-k}2^j\binom{2n-k-j-1}{n-1},\\
\mathbb{P}\{h_n^{(\frac{1}{2},1)}=k\} &= (k+1)2^{k-2n}\frac{k}{n}\binom{2n-k-1}{n-1}.
\end{aligned}$$

We can also obtain the limiting distribution in both cases. In case of lower bound, first we use the absolute approximation [3]

$$\binom{2n-k-1}{n-1}\sim 2^{2n-k}\frac{n}{2n-k}\frac{e^{-k^2/4n}}{\sqrt{\pi n}}.$$

Then

$$\begin{aligned}
\mathbb{P}\{h_n^{(\frac{1}{2},\frac{1}{2})}=k\} &\sim \frac{k}{\sqrt{\pi}n^{3/2}}\sum_{j=0}^{n-k}e^{-\frac{(k+j)^2}{4n}}\\
&\sim \frac{k}{\sqrt{\pi}n}\left(\int_{t=\frac{k}{\sqrt{n}}}^{\sqrt{n}}e^{-\frac{t^2}{4}}\,dt+\Theta(\frac{1}{\sqrt{n}})\right)\\
&\sim \frac{k}{\sqrt{\pi}n}\int_{t=\frac{k}{\sqrt{n}}}^{\infty}e^{-\frac{t^4}{4}}\,dt, \quad \text{as } n\to\infty.
\end{aligned}$$

So that the normalized r.v. $\frac{h_n^{(\frac{1}{2},\frac{1}{2})}}{\sqrt{n}}\xrightarrow{(d)}Y$, where Y has the probability density function:

$$f(y)=\frac{y}{\sqrt{\pi}}\int_{t=y}^{\infty}e^{-\frac{t^2}{4}}\,dt$$

Moreover, we can compute the moments as follows:

$$\begin{aligned}\mathbb{E}\{Y^r\} &= \int_0^\infty y^r \cdot f(y)dy \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{y^{r+2}}{r+2} e^{-\frac{y^2}{4}} dy \\ &= \frac{2^{r+2}}{\sqrt{\pi}(r+2)} \cdot \Gamma(\frac{r+3}{2}).\end{aligned}$$

For the upper bound, we start with following exact closed form of the moment generating function

$$\begin{aligned}\mathbb{E}\{h_n^{(\frac{1}{2},1)\underline{r}}\} &= [z^n] \frac{(r+1)!\Big(1-\sqrt{1-z}\Big)^r}{(1-z)^{r/2+1}} \\ &= [z^n] \frac{(r+1)!}{1-z} \Big(\frac{1}{\sqrt{1-z}} - 1\Big)^r.\end{aligned}$$

Then using the singularity analysis, one can write

$$\begin{aligned}\mathbb{E}\{h_n^{(\frac{1}{2},1)\underline{r}}\} &\sim [z^n] \frac{(r+1)!}{(1-z)^{1+r/2}} \\ &= (r+1)! \binom{n+r/2}{n} \\ &= (r+1)! \frac{(n+\frac{r}{2})(n+\frac{r}{2}-1)(n+\frac{r}{2}-2)\dots(n+1)}{(\frac{r}{2})!} \\ &= n^{r/2} \frac{(r+1)\cdot r!}{\Gamma(\frac{r}{2}+1)} \cdot \Big(1+\Theta(n^{r/2-1})\Big). \end{aligned} \tag{16}$$

Now we can use the following property of Gamma function:

$$\Gamma(\frac{r}{2}+\frac{1}{2}) = \frac{\sqrt{\pi}\ 2^{-r}\ r!}{\Gamma(\frac{r}{2}+1)},$$

thus

$$\mathbb{E}\{h_n^{(\frac{1}{2},1)\underline{r}}\} \sim n^{r/2}\ 2^r\ \frac{(r+1)\Gamma(\frac{r}{2}+\frac{1}{2})}{\sqrt{\pi}}.$$

If we consider this normalized r.v. $h_n^{'} = \frac{h_n^{(\frac{1}{2},1)}}{\sqrt{2n}}$, then

$$\mathbb{E}\{h_n^{'\underline{r}}\} \sim 2^{r/2}\ \frac{(r+1)\Gamma(\frac{r}{2}+\frac{1}{2})}{\sqrt{\pi}}, \tag{17}$$

since the moment generating function of the Chi distribution with $k=3$ is given as

$$M_r = \frac{2^{r/2}\ \Gamma(\frac{r+3}{2})}{\Gamma(3/2)} = \frac{2^{r/2}\ (\frac{r+1}{2})\ \Gamma(\frac{r+1}{2})}{\frac{1}{2}\ \Gamma(1/2)} = \frac{2^{r/2}\ (r+1)\ \Gamma(\frac{r+1}{2})}{\sqrt{\pi}}. \tag{18}$$

Using the method of moments, from equations(17) and (18) we can proof that asymptotically as $n \to \infty$, $\frac{h_n^{(\frac{1}{2},1)}}{\sqrt{2n}} \xrightarrow{(d)} Z$, where Z follows a Chi distribution with $k=3$, namely Maxwell-Boltzmann distribution with parameter $\sqrt{2n}$.

3 Hiring above the $\frac{1}{d}$-quantile

Our approach to the study of hiring above the median (see [5]) enables us to understand well the hiring process under such strategy, but we are not able to find a suitable combinatorial explanation for the probability distribution of h_n. This reflects that we have to investigate other special cases of α in order to generalize our results if possible. For example, hiring above the $\frac{1}{3}$-quantile: in this case we have three different states for the automaton according to the number of hired candidates k; where k is congruent to 0,1, or 2 modolu 3. We can write the recurrences of the quantities $a^{[1]}_{n,\ell}$, $a^{[2]}_{n,\ell}$ and $a^{[3]}_{n,\ell}$, which are the probabilities that after receiving n candidates, the threshold candidate has the ℓ-th largest score in the hiring set and k is 1(mod 3), 2(mod 3) and 0(mod 3) respectively. We give the main result for *hiring above the* $\frac{1}{d}$*-quantile*, then we discuss the analysis of the special case $\alpha = \frac{1}{3}$, followed by the proof of our theorem.

3.1 Main result

Theorem 7 *Let h_n denote the number of hired candidates if we apply the strategy hiring above the $\frac{1}{d}$-quantile, $d \in \mathbb{N}$, for n candidates, then the normalized random variable $\frac{h_n}{n^{\frac{d-1}{d}}} \xrightarrow{(d)} X$, where X has the density function:*

$$f(x) = \frac{1}{d^{\frac{1}{d-1}}(\frac{1}{d-1})!} \cdot x^{\frac{1}{d-1}} \cdot \exp\Big(-\frac{(d-1)^{d-1}}{d^d} \cdot x^d\Big), \quad x > 0.$$

3.2 Analysis

In general, the α-quantile of a sequence $r_1 < r_2 < \cdots < r_k$ of k elements is the element r_j with $j = \lceil \alpha k \rceil$, hence it is the $(k - j + 1)$-th largest one. For $\alpha = \frac{1}{3}$ we can track the evolution of the threshold candidate which is the ℓ-th largest one in the hiring set via the following table:

k	1	2	3	4	5	6	7	$\cdots$
ℓ	1	2	3	3	4	5	5	$\cdots$

Notice that when we move from the state $a^{[3]}_{n,\ell}$ to $a^{[1]}_{n,\ell}$, ℓ is still the same while ℓ is incremented when moving from $a^{[1]}_{n,\ell}$ to $a^{[2]}_{n,\ell}$ or from $a^{[2]}_{n,\ell}$ to $a^{[3]}_{n,\ell}$. Thus for $n \geq 2$ and $1 \leq \ell \leq n$:

$$a^{[1]}_{n,\ell} = (1 - \frac{\ell}{n}) \cdot a^{[1]}_{n-1,\ell} + \frac{\ell}{n} \cdot a^{[3]}_{n-1,\ell}$$
$$a^{[2]}_{n,\ell} = (1 - \frac{\ell}{n}) \cdot a^{[2]}_{n-1,\ell} + \frac{\ell - 1}{n} \cdot a^{[1]}_{n-1,\ell-1}$$
$$a^{[3]}_{n,\ell} = (1 - \frac{\ell}{n}) \cdot a^{[3]}_{n-1,\ell} + \frac{\ell - 1}{n} \cdot a^{[2]}_{n-1,\ell-1},$$

with the initial conditions: $a^{[1]}_{1,1} = 1$, $a^{[2]}_{1,1} = a^{[3]}_{1,1} = 0$.
We simplify those recurrences by introducing a suitable normalization:

$$c^{[i]}_{n,\ell} = \frac{n!}{(n-\ell)! \cdot (\ell-1)!} \cdot a^{[i]}_{n,\ell}, \quad i = 1, 2, 3 \tag{19}$$

which leads us to the following relations:

$$\begin{aligned}(n-\ell)\cdot c_{n,\ell}^{[1]} &= (n-\ell)\cdot c_{n-1,\ell}^{[1]} + \ell\cdot c_{n-1,\ell}^{[3]}\\ c_{n,\ell}^{[2]} &= c_{n-1,\ell}^{[2]} + c_{n-1,\ell-1}^{[1]}\\ c_{n,\ell}^{[3]} &= c_{n-1,\ell}^{[3]} + c_{n-1,\ell-1}^{[2]},\end{aligned}$$

Now we introduce the following generating function:

$$C^{[i]}(z,u) := \sum_{n\geq 1}\sum_{1\leq \ell\leq n} c_{n,\ell}^{[i]}\cdot z^{n-\ell}u^{\ell}, \quad i=1,2,3$$

which when applied to the above system of recurrences gives:

$$\begin{aligned}\frac{\partial}{\partial z}\Big((1-z)C^{[1]}(z,u)\Big) &= u\cdot\frac{\partial}{\partial u}C^{[3]}(z,u)\\ C^{[2]}(z,u) &= \frac{u}{1-z}\cdot C^{[1]}(z,u)\\ C^{[3]}(z,u) &= \frac{u}{1-z}\cdot C^{[2]}(z,u),\end{aligned}$$

It is easy to infer that $C^{[3]}(z,u) = \frac{u^2}{(1-z)^2}\cdot C^{[1]}(z,u)$ to obtain the following PDE:

$$(1-z)^2\frac{\partial}{\partial z}\Big((1-z)C^{[1]}(z,u)\Big) = u\cdot\frac{\partial}{\partial u}\Big(u^2C^{[1]}(z,u)\Big),$$

whose explicit solution contains some arbitrary function $F(\cdot)$ as follows:

$$C^{[1]}(z,u) := \frac{1}{(1-z)u^2}\cdot F\left(\frac{u^2-(1-z)^2}{u^2(1-z)^2}\right).$$

But the initial condition $a_{1,1}^{[1]} = 1$ implies that $C^{[1]}(0,u) = \sum_{n\geq 1} c_{n,n}^{[1]}u^n = u$; this shows that:

$$F(x) = \frac{1}{(1-x)^{3/2}}.$$

Then we get:

$$\begin{aligned}C^{[1]}(z,u) &= \frac{1}{(1-z)u^2}\frac{1}{\left(1-\frac{u^2-(1-z)^2}{u^2(1-z)^2}\right)^{3/2}}\\ &= \frac{u}{(1-z)\left(1-\left(\frac{1-(1-z)^2}{(1-z)^2}\right)u^2\right)^{3/2}}.\end{aligned}$$

Now we can extract the coefficients of $c_{n,\ell}^{[1]}$ as follows:

$$\begin{aligned}c_{n,\ell}^{[1]} = [z^{n-\ell}u^{\ell}]C^{[1]}(z,u) &= [z^{n-\ell}u^{\ell-1}]\frac{1}{(1-z)\left(1-\left(\frac{1-(1-z)^2}{(1-z)^2}\right)u^2\right)^{3/2}}\\ &= [z^{n-\ell}(u^2)^{(\ell-1)/2}]\frac{1}{(1-z)\left(1-\left(\frac{1-(1-z)^2}{(1-z)^2}\right)u^2\right)^{3/2}}\end{aligned}$$

$$
\begin{aligned}
&= \binom{\frac{\ell}{2}}{\frac{\ell-1}{2}} \cdot [z^{n-\ell}] \frac{1}{1-z} \left(\frac{1-(1-z)^2}{(1-z)^2} \right)^{(\ell-1)/2} \\
&= \binom{\frac{\ell}{2}}{\frac{\ell-1}{2}} \cdot [z^{n-\frac{3\ell}{2}+\frac{1}{2}}] \frac{(1+(1-z))^{(\ell-1)/2}}{(1-z)^{\ell}} \\
&= \binom{\frac{\ell}{2}}{\frac{\ell-1}{2}} \cdot [z^{n-\frac{3\ell}{2}+\frac{1}{2}}] \sum_{j=0}^{\frac{\ell-1}{2}} \binom{\frac{\ell-1}{2}}{j} \cdot \frac{1}{(1-z)^{\ell-j}} \\
&= \binom{\frac{\ell}{2}}{\frac{\ell-1}{2}} \sum_{j=0}^{\frac{\ell-1}{2}} \binom{\frac{\ell-1}{2}}{j} \cdot \binom{n-\frac{\ell}{2}-j-\frac{1}{2}}{\ell-j-1}.
\end{aligned}
$$

Remember the normalization used in (21), then we have:

$$
a_{n,\ell}^{[1]} = \begin{cases} 0, & \text{if } \ell \text{ is even}, \\ \frac{1}{\ell\binom{n}{\ell}} \cdot \binom{\frac{\ell}{2}}{\frac{\ell-1}{2}} \sum_{j=0}^{\frac{\ell-1}{2}} \binom{\frac{\ell-1}{2}}{j} \cdot \binom{n-\frac{\ell}{2}-j-\frac{1}{2}}{\ell-j-1}, & \text{if } \ell \text{ is odd} \end{cases} \tag{20}
$$

For the other cases of $a_{n,\ell}^{[\cdot]}$, which turn out to be easier, we proceed as above: we obtain the corresponding generating function $C^{[\cdot]}(z,u)$ first, then we extract the coefficients.

3.2.1 Proof of Theorem 7

The approach used for the analysis still prove useful and as we have seen, the computations for $\alpha = \frac{1}{3}$ to obtain the fundamental quantities $a_{n,\ell}^{[\cdot]}$ were relatively easy and we got the desired results. This encourages us to investigate some generalization in this class of hiring strategies when $\alpha = \frac{1}{d}$; namely *hiring above the* $\frac{1}{d}$*-quantile*, $d \in \mathbb{N}$. We know that for $\alpha = \frac{1}{d}$, we have d recurrences that describe the relationships between the quantities $a_{n,\ell}^{[\cdot]}$, but the trick is always to find some suitable normalization to reduce the resulting system of differential equations into only one PDE in one function, after that we expect that the computations will be a routine task. We start writing the following recurrence relations for $n \geq 2$, $1 \leq \ell \leq n$ and $2 \leq i \leq d$:

$$
\begin{aligned}
a_{n,\ell}^{[1]} &= \frac{\ell}{n} \cdot a_{n-1,\ell}^{[d]} + (1-\frac{\ell}{n}) \cdot a_{n-1,\ell}^{[1]}, \\
a_{n,\ell}^{[i]} &= \frac{\ell-1}{n} \cdot a_{n-1,\ell}^{[i-1]} + (1-\frac{\ell}{n}) \cdot a_{n-1,\ell}^{[i]}.
\end{aligned}
$$

We need again this normalization:

$$
c_{n,\ell}^{[i]} = \frac{n!}{(n-\ell)! \cdot (\ell-1)!} \cdot a_{n,\ell}^{[i]}, \quad 1 \leq i \leq d, \tag{21}
$$

together with the generating function:

$$
C^{[i]}(z,u) := \sum_{n\geq 1} \sum_{1\leq \ell \leq n} c_{n,\ell}^{[i]} \cdot z^{n-\ell} u^{\ell}, \quad 1 \leq i \leq d.
$$

As before we obtain a system of PDEs:

$$
\frac{\partial}{\partial z}\Big((1-z)C^{[1]}(z,u)\Big) = u \cdot \frac{\partial}{\partial u} C^{[d]}(z,u)
$$

$$C^{[i]}(z,u) = \frac{u}{1-z} \cdot C^{[i-1]}(z,u).$$

Thus we have $C^{[d]}(z,u) = \frac{u^{d-1}}{(1-z)^{d-1}} \cdot C^{[1]}(z,u)$, and write the following PDE:

$$(1-z)^{d-1}\frac{\partial}{\partial z}\Big((1-z)C^{[1]}(z,u)\Big) = u \cdot \frac{\partial}{\partial u}\Big(u^{d-1}C^{[1]}(z,u)\Big). \tag{22}$$

A direct solution for this PDE will be too complicated, so we transform it into a simpler form. If we consider the function:

$$C_\ell^{[\cdot]}(z) = [u^\ell]C^{[\cdot]}(z,u) = \sum_{n\geq\ell} c_{n,\ell}^{[\cdot]} z^n,$$

Then treating (22) gives:

$$(1-z)^{d-1}\frac{\partial}{\partial z}\Big((1-z)C_\ell^{[1]}(z)\Big) = \ell \cdot C_{\ell-d+1}^{[1]}(z). \tag{23}$$

We need again to introduce another normalization in order to obtain the last PDE in a useful form. First, we have

$$\hat{C}_\ell^{[\cdot]}(z) = \frac{C_\ell^{[\cdot]}(z)}{\ell!^{(d-1)}},$$

where the notation $n!^{(x)}$ denotes the multifactorial of n defined as follows:

$$n!^{(x)} = \begin{cases} 1, & \text{if } 0 \leq n < x, \\ n \cdot ((n-x)!^{(x)}), & \text{if } n \geq x. \end{cases}$$

or the following alternative definition, which is suitable when $n = 1(\bmod x)$:

$$n!^{(x)} = x^{\frac{n-1}{x}} \frac{\Gamma(\frac{n}{x}+1)}{\Gamma(\frac{1}{x}+1)}. \tag{24}$$

Then equation (23) becomes:

$$(1-z)^{d-1}\frac{\partial}{\partial z}\Big((1-z)\hat{C}_\ell^{[1]}(z)\Big) = \hat{C}_{\ell-d+1}^{[1]}(z),$$

multiplying both sides by u^ℓ and summing over $\ell \geq d-1$ yields:

$$(1-z)^{d-1}\frac{\partial}{\partial z}\Big((1-z)\hat{C}^{[1]}(z,u)\Big) = u^{d-1}\hat{C}^{[1]}(z,u),$$

whose solution is the following:

$$\hat{C}^{[1]}(z,u) = \frac{1}{1-z} \cdot F(u) \cdot \exp\Big(\frac{u^{d-1}}{(d-1)(1-z)^{d-1}}\Big),$$

using the initial condition $\hat{C}^{[1]}(0,u) = u$ gives us

$$F(u) = u \cdot \exp\Big(-\frac{u^{d-1}}{d-1}\Big).$$

Thus we get finally the solution

$$\hat{C}^{[1]}(z,u) = \frac{u}{1-z} \cdot \exp\Big(\frac{u^{d-1}}{d-1}\Big(\frac{1}{(1-z)^{d-1}} - 1\Big)\Big).$$

Now extracting the coefficients is going as follows:

$$\begin{aligned}
\hat{c}^{[1]}_{n,\ell} &= [z^{n-\ell}u^{\ell}]\hat{C}^{[1]}(z,u) \\
&= [z^{n-\ell}(u^{d-1})^{\frac{\ell-1}{d-1}}]\frac{1}{1-z} \cdot \exp\Big(\frac{u^{d-1}}{d-1}\Big(\frac{1}{(1-z)^{d-1}} - 1\Big)\Big) \\
&= [z^{n-\ell}]\frac{1}{1-z} \cdot \frac{1}{(d-1)^{\frac{\ell-1}{d-1}} \cdot (\frac{\ell-1}{d-1})!} \cdot \Big(\frac{1}{(1-z)^{d-1}-1}\Big)^{\frac{\ell-1}{d-1}} \\
&= \frac{1}{(d-1)^{\frac{\ell-1}{d-1}} \cdot (\frac{\ell-1}{d-1})!} \sum_{j=0}^{\frac{\ell-1}{d-1}} \binom{\frac{\ell-1}{d-1}}{j} (-1)^j \binom{n-j(d-1)-1}{\ell-j(d-1)-1}, \qquad \text{for } \ell = 1(\text{mod } (d-1)).
\end{aligned}$$

The result for the quantity which we are interested in follows easily,

$$\begin{aligned}
a^{[1]}_{n,\ell} &= \frac{\ell!^{(d-1)}}{\ell \cdot \binom{n}{\ell}} \cdot \hat{c}^{[1]}_{n,\ell} \\
&= \frac{\ell!^{(d-1)}}{\ell\binom{n}{\ell}(d-1)^{\frac{\ell-1}{d-1}} \cdot (\frac{\ell-1}{d-1})!} \sum_{j=0}^{\frac{\ell-1}{d-1}} \binom{\frac{\ell-1}{d-1}}{j} (-1)^j \binom{n-j(d-1)-1}{\ell-j(d-1)-1}, \qquad \text{for } \ell = 1(\text{mod } (d-1)).
\end{aligned}$$

Limit distribution. First, we have from (24) that

$$\ell!^{(d-1)} = \frac{(d-1)^{\frac{\ell-1}{d-1}} \cdot (\frac{\ell}{d-1})!}{(\frac{1}{d-1})!}, \qquad \text{for } \ell = 1(\text{mod } (d-1)),$$

so that

$$\frac{\ell!^{(d-1)}}{(d-1)^{\frac{\ell-1}{d-1}} \cdot (\frac{\ell-1}{d-1})!} = \frac{(\frac{\ell}{d-1})!}{(\frac{\ell-1}{d-1})! \cdot (\frac{1}{d-1})!} = \frac{(\frac{\ell}{d-1})^{\frac{1}{d-1}}}{(\frac{1}{d-1})!} \cdot \Big(1 + O(\frac{1}{\ell})\Big), \quad \text{as } d \text{ is fixed.}$$

Then we use Stirling's formula as usual to do the asymptotic analysis. We have a sum of terms

$$T_j := \frac{1}{\binom{n-1}{\ell-1}} \cdot \binom{\frac{\ell-1}{d-1}}{j} \binom{n-j(d-1)-1}{\ell-j(d-1)-1}$$

so

$$\begin{aligned}
\log(T_j) =& \log\left(\frac{(\frac{\ell-1}{d-1})!(n-j(d-1)-1)!(\ell-1)!}{(\frac{\ell-1}{d-1}-j)!(\ell-j(d-1)-1)!(n-1)!}\right) \\
&= -j \cdot \log(d-1) + d \cdot j \cdot \log \ell - j(d-1)\log n - \frac{j^2(d-1)}{2(\ell-1)} + \frac{j^2(d-1)^2}{2n} - \frac{j^2(d-1)^2}{2\ell} \\
&\quad + \Theta(\frac{j}{\ell}) + \Theta(\frac{j^3}{\ell^2}) + \Theta(\frac{1}{\ell})
\end{aligned}$$

After that we recover an asymptotic estimate for T_j:

$$T_j = \frac{1}{j!}\left(\frac{\ell^d}{(d-1)n^{d-1}}\right)^j \cdot \exp\Big(-\frac{j^2(d-1)}{2(\ell-1)} + \frac{j^2(d-1)^2}{2n} - \frac{j^2(d-1)^2}{2\ell}\Big) \cdot \Big(1 + \Theta(\frac{j}{\ell}) + \Theta(\frac{j^3}{\ell^2}) + \Theta(\frac{1}{\ell})\Big).$$

Asymptotically as $n \to \infty$, $\frac{\ell^d}{n^{d-1}} = O(1)$ and $\ell \gg d$, thus we have

$$T_j = \frac{(-1)^j}{j!}\left(\frac{\ell^d}{(d-1)n^{d-1}}\right)^j \cdot \Big(1 + \Theta(\frac{j^2}{\ell})\Big),$$

and its summation can be approximated as follows:

$$\sum_{j=0}^{\frac{\ell-1}{d-1}} T_j = \sum_{j=0}^{\infty} T_j - \sum_{j=\frac{\ell-1}{d-1}}^{\infty} T_j.$$

Since

$$\sum_{j=\frac{\ell-1}{d-1}}^{\infty} T_j = O\left(\exp\Big(\frac{-(\ell-1)}{2(d-1)} \cdot \log(\frac{\ell-1}{d-1})\Big)\right),$$

$$\sum_{j=0}^{\infty} T_j \sim \exp\Big(\frac{-\ell^d}{(d-1)n^{d-1}}\Big),$$

we can show the limit distribution for the sequence $a^{[1]}_{n,\ell}$, for $\ell = O(n^{\frac{d-1}{d}})$, is asymptotically

$$a^{[1]}_{n,\ell} \sim \frac{1}{(d-1)^{1/(d-1)}(\frac{1}{d-1})!} \cdot \frac{\ell^{\frac{1}{d-1}}}{n} \cdot \exp\Big(\frac{-\ell^d}{(d-1)n^{d-1}}\Big).$$

Since $a^{[1]}_{n,\ell}$ represents the case when k is $1 \pmod d$ then we can state the results for the size of hiring set as follows:

$$\begin{aligned}\mathbb{P}\{h_n = k\} &= a^{[1]}_{n,k\cdot\frac{d-1}{d}+\frac{1}{d}} \\ &\sim \frac{1}{d^{\frac{1}{d-1}}(\frac{1}{d-1})!} \cdot \frac{k^{\frac{1}{d-1}}}{n} \cdot \exp\Big(-\frac{(d-1)^{d-1}}{d^d} \cdot \frac{k^d}{n^{d-1}}\Big).\end{aligned}$$

If we consider the normalized random variable $\frac{h_n}{n^{\frac{d-1}{d}}}$, then we have

$$\mathbb{P}\left\{\frac{h_n}{n^{\frac{d-1}{d}}} = \frac{k}{n^{\frac{d-1}{d}}}\right\} \sim \frac{1}{n^{\frac{d-1}{d}} d^{\frac{1}{d-1}}(\frac{1}{d-1})!} \cdot \Big(\frac{k}{n^{\frac{d-1}{d}}}\Big)^{\frac{1}{d-1}} \cdot \exp\Big(-\frac{(d-1)^{d-1}}{d^d} \cdot \Big(\frac{k}{n^{\frac{d-1}{d}}}\Big)^d\Big).$$

Thus Theorem 7 follows easily.

4 Conclusions

We were able to obtain some general results for hiring above the α-quantile for $0 < \alpha < 1$. However, those general results offer only lower and upper bounds for the hiring parameters, but they help to obtain at least the order of growth of various parameters. Also the framework given by Archibald and Martínez proves useful to analyze a lot of selection rules in a direct systematic way, just one has to define carefully the crucial quantity $X(\sigma)$ for the studied selection rule. As a future work, we would like to investigate more generalizations of hiring above the α-quantile, i.e., rational α with $\alpha = \frac{p}{q}$ where $gcd(p, q) = 1$.

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