

Recovering the conductances on grids

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ABSTRACT. In this work, we present an overview of the work developed by the authors in the context of inverse problems on finite networks. This study performs an extension of the pioneer studies by E.B. Curtis and J.A. Morrow, and sets the theoretical basis for solving inverse problems on networks. We present just a glance of what we call overdetermined partial boundary value problems, in which any data are not prescribed on a part of the boundary, whereas in another part of the boundary both the values of the function and of its normal derivative are given. The resolvent kernels associated with these problems are described and they are the fundamental tool to perform an algorithm for the recovery of the conductance of a 3-dimensional grid. We strongly believe that the columns of the partial overdetermined Poisson kernel are the discrete counterpart of the so-called CGO solutions (complex geometrical optic solutions) that, in their turn, are the key to solve inverse continuous problems on planar domains. Finally, we display the steps needed to recover the conductances in a 3-dimensional grid.

1. Introduction

The first applications of inverse boundary value problems are found in geophysical electrical prospecting and electrical impedance tomography. The objective in physical electrical prospecting is to deduce internal terrain properties from surface electrical measurements, which is of great interest in the engineering field. Electrical impedance tomography is a medical imaging technique where the aim is to obtain visual information of the body densities from some electrodes placed on the skin of the patient.

The corresponding mathematical problem is whether it is possible to determine the conductivity of a body from boundary measurements and global equilibrium conditions. That is, an *inverse boundary value problem* consists in the recovery of the internal structure or conductivity information of a body using only external data and general conditions on the body. In general, inverse problems are exponentially ill-posed, since they are highly sensitive to changes in the boundary data. For this problem is poorly arranged, at times the target is only a partial reconstruction of

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the conductivity data or the addition of morphological conditions so as to perform a full internal recovery. However, in this work we deal with a situation where the recovery of the conductance is feasible: grid networks.

De Verdière *et al.* [D2] and Curtis *et al.* [C2] studied the inverse problem for circular planar electrical networks when the Dirichlet-to-Neumann map associated with the combinatorial Laplacian is supposed to be known. Specifically, they considered networks embedded in a disk without crossings and with boundary vertices located on the boundary of the disk and proved that the edge conductances of a critical circular planar electrical network can be recovered uniquely from the response matrix; that is, the matrix associated with the Dirichlet-to-Neumann map. Moreover, they proved that the response matrices realizable by circular planar networks are all singular diagonally dominant M -matrices having all circular minors nonnegative.

Some of the present authors have developed an adequate framework to the study of discrete inverse problem through a series of three papers [A1, A2, A3]. These works represent an extension of the above mentioned works and define a global setting for the study of discrete inverse problems. In [A2], we established the theoretical foundations for the study of overdetermined partial boundary value problems associated with a Schrödinger operator on a network, which constitute the appropriate framework to the recovery of the conductances of a network. In it we gave a necessary and sufficient condition for the existence and uniqueness of solution as well as a recovery algorithm for the conductances of spider networks when then data is the Dirichlet-to-Robin map associated with a Schrödinger operator. The data for an inverse problem on a network is the Dirichlet-to-Robin map since it contains the boundary information. Therefore, we raised in [A1] the analysis of the properties of Dirichlet-to-Robin map, proving in particular the alternating property for general network. The consideration of Schrödinger operators has allowed us to consider as response matrices a wide class of matrices, not necessarily singular nor weakly diagonally dominant. Therefore, our results represent a generalization of those obtained in [C2, C3]. Once we have characterized those matrices that are the response matrices of certain networks, we raise the problem of constructing an algorithm to recover the conductances. With this end, we provided a new necessary and sufficient condition for the existence and uniqueness of solution of overdetermined partial boundary value problems and described its resolvent kernels; named *overdetermined partial Green, Poisson and Robin kernels*, see [A3].

In the present study we want to reflect the power of the mentioned works in an unified approach. For this, we summarize the main result of the mentioned papers and we present some novel result in the setting of the recovery of the conductances. The first one establishes a boundary spike formula that generalizes the one obtained in [C2], since we consider arbitrary networks and we work with positive semi-definite Schrödinger operators. Then, we present a new algorithm for the recovery of the conductances in a 3-dimensional grid with arbitrary conductances when the data is the Dirichlet-to-Robin map. This algorithm makes use of the mentioned boundary spike formula and of the characteristics relative to the zero zone of the overdetermined partial Poisson kernel. A draw of the algorithm for the case of 2-dimensional grids was given in [A4]. In an unpublished paper by Oberlin [O], we can also find a sketch of an algorithm for 2-dimensional grids where the hypothesis of diagonal dominance of the response matrix is required.

2. Preliminaires

Let $\Gamma = (V, c)$ be a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V . If E denotes the set of edges of the network, each $(x, y) \in E$ has been assigned a *conductance* $c(x, y)$, where $c : V \times V \rightarrow [0, +\infty)$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. Then, $x, y \in V$ are *adjacent*, $x \sim y$, iff $c(x, y) > 0$. We denote by $N_r(x)$, the set of vertices of $x \in V$ that are at distance less or equal to r .

The set of real value functions on a subset $F \subseteq V$, denoted by $\mathcal{C}(F)$, and the set of non-negative functions on F , $\mathcal{C}^+(F)$, are naturally identified with $\mathbb{R}^{|F|}$ and the nonnegative cone of $\mathbb{R}^{|F|}$, respectively. Moreover, if F is a non empty subset of V , its characteristic function is denoted by $\mathbf{1}_F$. When $F = \{x\}$, its characteristic function is denoted by ε_x . If $u \in \mathcal{C}(V)$, we define the *support* of u as $\text{supp}(u) = \{x \in V : u(x) \neq 0\}$. Clearly, $\mathcal{C}(F)$ is identified with the set of functions in $\mathcal{C}(V)$ that vanish outside F .

If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F . The vertices of $\delta(F)$ are called *boundary vertices* and when a boundary vertex $x \in \delta(F)$ has a unique neighbour in F we call the edge joining them a *boundary spike*. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is. Of course networks do not have boundaries by themselves, but starting from a network we can define a *network with boundary* as $\Gamma = (\bar{F}, c_F)$ where F is a proper subset and $c_F = c \cdot \mathbf{1}_{(\bar{F} \times \bar{F}) \setminus (\delta(F) \times \delta(F))}$. From now on we will work with networks with boundary. Moreover, for the sake of simplicity we denote $c = c_F$. In addition, we assume that $|F| = n$ and $|\delta(F)| = m$.

On $\mathcal{C}(\bar{F})$ we consider the standard inner product defined as $\langle u, v \rangle = \sum_{x \in \bar{F}} u(x)v(x)$.

Any function $\omega \in \mathcal{C}^+(\bar{F})$ such that $\text{supp}(\omega) = \bar{F}$ and $\sum_{x \in \bar{F}} \omega^2(x) = 1$ is called *weight* on \bar{F} . The set of weights is denoted by $\Omega(\bar{F})$. We call (*generalized*) *degree* of $x \in V$, the value $\kappa(x) = \sum_{y \in V} c(x, y)$.

Clearly, functions and operators can be identified with vectors and matrices, after giving a label on the vertex set. Along the paper we use the convention that operators and their associated matrices, and functions and their associated vectors, are denoted with the same letter, operators in calligraphic font and matrices and vectors in sans serif font. In general, given a matrix M and A, B sets of vertices, $M(A; B)$ denote the matrix obtained from M with rows indexed by the vertices of A and columns indexed by the vertices of B . Also, given a vector v and a set of vertices A , $v(A)$ denotes the entries of v indexed by the vertices of A .

The *combinatorial Laplacian* of Γ is the linear operator $\mathcal{L} : \mathcal{C}(\bar{F}) \rightarrow \mathcal{C}(\bar{F})$ that assigns to each $u \in \mathcal{C}(\bar{F})$ the function defined for all $x \in \bar{F}$ as

$$\mathcal{L}(u)(x) = \sum_{y \in \bar{F}} c(x, y) (u(x) - u(y)).$$

Given $q \in \mathcal{C}(\bar{F})$, the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathcal{L}_q : \mathcal{C}(\bar{F}) \rightarrow \mathcal{C}(\bar{F})$ that assigns to each $u \in \mathcal{C}(\bar{F})$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$. It is well-known that any Schrödinger operator is self-adjoint. Moreover, if L denotes the matrix associated with \mathcal{L}_q , then L is an irreducible and symmetric M -matrix.

We define the *normal derivative* of $u \in \mathcal{C}(\bar{F})$ on F as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathbf{n}_F} \right) (x) = \sum_{y \in F} c(x, y) (u(x) - u(y)), \quad \text{for any } x \in \delta(F).$$

For any weight $\sigma \in \Omega(\bar{F})$, the so-called *potential associated with σ* is the function in $\mathcal{C}(\bar{F})$ defined as $q_\sigma = -\sigma^{-1} \mathcal{L}(\sigma)$ on F , $q_\sigma = -\sigma^{-1} \frac{\partial \sigma}{\partial \mathbf{n}_F}$ on $\delta(F)$. It is worth to note that the definition of q_σ is a discrete analogous of the Liouville transform, see [S]. In [A1], we proved that the Energy is positive semi-definite on $\mathcal{C}(\bar{F})$ if there exist $\lambda \geq 0$ and $\sigma \in \Omega(\bar{F})$ such that $q = q_\sigma + \lambda \chi_{\delta(F)}$. In this case, it is positive definite iff $\lambda > 0$. So, through this paper, we will suppose that the above condition $q = q_\sigma + \lambda \chi_{\delta(F)}$ holds with $\sigma \in \Omega(\bar{F})$ and $\lambda \geq 0$. Therefore, for any $f \in \mathcal{C}(F)$ and $g \in \mathcal{C}(\delta(F))$ the following Dirichlet problem

$$\mathcal{L}_q(u) = f \quad \text{on } F \quad \text{and} \quad u = g \quad \text{on } \delta(F),$$

has a unique solution. The existence and uniqueness of solution implies that the operator \mathcal{L}_q is invertible on F and its inverse is called the *Green operator* for F and it is denoted by \mathcal{G}_q . Observe that \mathcal{G}_q is a self-adjoint operator whose associated matrix will be denoted by \mathbf{G} .

When $f = 0$, we denote the unique solution of the corresponding Dirichlet problem as u_g . Then, the map $\Lambda_q: \mathcal{C}(\delta(F)) \rightarrow \mathcal{C}(\delta(F))$ that assigns to any function $g \in \mathcal{C}(\delta(F))$ the function $\Lambda_q(g) = \frac{\partial u_g}{\partial \mathbf{n}_F} + qg$ is called *Dirichlet-to-Robin map*.

In [A1], the authors proved that the Dirichlet-to-Robin map is a self-adjoint, positive semi-definite operator. Moreover, λ is the lowest eigenvalue of Λ_q and its associated eigenfunctions are multiple of σ . In addition, the matrix, Λ , associated with Λ_q is an irreducible and symmetric M -matrix, usually called the *response matrix* of the network and it is the Schur complement of $\mathbf{L}(F; F)$ in \mathbf{L} (see [H]); that is,

$$\Lambda = \mathbf{L} / \mathbf{L}(F; F) = \mathbf{D}(\delta(F); \delta(F)) - \mathbf{C}(\delta(F); F) \cdot \mathbf{G} \cdot \mathbf{C}(\delta(F); F)^\top,$$

where \mathbf{D} is the diagonal matrix whose diagonal entries are given by $\kappa + q$ and $\mathbf{C} = (c(x, y))_{x, y \in \bar{F}}$. This map is the response matrix of a Schrödinger type matrix, and therefore it can be assumed to be known, since it provides boundary reactions to boundary actions, the type of data that we can measure.

Moreover, we proved that the edge conductances of a critical circular planar electrical network can be recovered from the Dirichlet-to-Robin map associated with a positive semi-definite Schrödinger operator. In addition, we proved that the response matrices realizable by circular planar networks are all M -matrices having all circular minors nonnegative. Therefore, these results are a generalization of those proved in [D2, C2].

Unlike the planar network case, very few results have been obtained for general networks. In [L], the authors carried out a study of the inverse Dirichlet-to-Neumann problem for networks embedded in a cylinder. For the class of purely cylindrical networks, they obtained a characterization of the response matrices and proved that the network can be recovered from the Dirichlet-to-Neumann map.

3. Overdetermined Partial boundary value problems

In this section we summarize some of the results obtained in [A2, A3] that allows us to perform the algorithm for the recovery of the conductances in a grid. We fix a proper and connected subset $F \subset V$ and $A, B \subset \delta(F)$ non-empty subsets such that $A \cap B = \emptyset$. Moreover we denote by R the set $R = \delta(F) \setminus (A \cup B)$, so $\delta(F) = A \cup B \cup R$ is a partition of $\delta(F)$. We remark that R can be an empty set. We consider a new type of non self-adjoint boundary value problems in which the values of the functions and their normal derivatives are known at the same part of the boundary, which represents an overdetermined problem, and there exists another part of the boundary where no data is known.

For any $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$, the *overdetermined partial Dirichlet-Neumann boundary value problem on F with data f, g, h* consists in finding a function $u \in \mathcal{C}(\bar{F})$ such that

$$(3.1) \quad \mathcal{L}_q(u) = f \text{ on } F, \quad \frac{\partial u}{\partial \mathbf{n}_F} = h \text{ on } A \quad \text{and} \quad u = g \text{ on } A \cup R.$$

In [A2] the authors proved the existence and uniqueness of solution of this problem for any data $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$, $h \in \mathcal{C}(A)$ iff $|A| = |B|$ and $\Lambda(A; B)$ is invertible, or equivalently iff $|A| = |B|$ and $L(A \cup F; F \cup B)$ is invertible, see [A3].

From now on we will suppose that $L(A \cup F; F \cup B)$ is invertible. In this case, the following result holds.

PROPOSITION 3.1. [A3, Corollary 2.10] *If $u \in \mathcal{C}(\bar{F})$ is the unique solution of the overdetermined partial boundary value problem (3.1), then*

$$\begin{aligned} u(B) &= \Lambda(A; B)^{-1} \cdot \left(C(A; F) \cdot G \cdot f - \Lambda(A; A \cup R) \cdot g + h \right), \\ u(F) &= G \cdot \left(f + C(F; B) \cdot u_B + C(F; A \cup R) \cdot g \right) \end{aligned}$$

and, clearly, $u(A \cup R) = g$.

We call *overdetermined partial Green operator*, $\tilde{\mathcal{G}}_q : \mathcal{C}(F) \longrightarrow \mathcal{C}(F \cup B)$, the operator that assigns to any $f \in \mathcal{C}(F)$ the unique solution of problem

$$(3.2) \quad \mathcal{L}_q(u) = f \text{ on } F, \quad \frac{\partial u}{\partial \mathbf{n}_F} = 0 \text{ on } A \quad \text{and} \quad u = 0 \text{ on } A \cup R.$$

We call *overdetermined partial Poisson operator*, $\tilde{\mathcal{P}}_q : \mathcal{C}(A \cup R) \longrightarrow \mathcal{C}(\bar{F})$, the operator that assigns to any $g \in \mathcal{C}(A \cup R)$ the unique solution of problem

$$(3.3) \quad \mathcal{L}_q(u) = 0 \text{ on } F, \quad \frac{\partial u}{\partial \mathbf{n}_F} = 0 \text{ on } A \quad \text{and} \quad u = g \text{ on } A \cup R.$$

The matrix associated with the overdetermined partial Poisson operator plays an important role in the following section, it will be denoted by $\tilde{\mathcal{P}}$. In fact, the functions $\tilde{\mathcal{P}}_q(\varepsilon_x) = \tilde{\mathcal{P}}(\cdot, x)$ for each $x \in A \cup R$ are the key tool for the recovery algorithm and hence they can be seen as the discrete counterpart of the so-called *complex geometrical optic solutions*, see [U]. In [A3], we have obtained explicit expressions for the overdetermined partial resolvent kernels for a generalized cylinder.

On the other hand, a consequence of the invertibility of $L(A \cup F; F \cup B)$ is the following extension of the boundary spike formula obtained in [C2] for circular planar networks and for the combinatorial Laplacian. This so-called extension covers all networks, not only circular planar ones, and all Schrödinger operators, not

only Laplacian operators. Moreover, this formula is a powerful tool for recovery algorithms since when a certain diagonal value of the overdetermined partial Green matrix $\tilde{\mathbf{G}}$ is null it allows to recover the conductance of a certain boundary spike.

PROPOSITION 3.2 (Boundary spike formula). *If $x \in R$ has a unique neighbour $y \in F$, then*

$$\Lambda(x; x) - \Lambda(x; B) \cdot \Lambda(A; B)^{-1} \cdot \Lambda(A; x) = \lambda + \frac{\omega(y)}{\omega(x)} c(x, y) - \tilde{\mathbf{G}}(y, y) c(x, y)^2.$$

Proof. As the matrices $\mathbf{L}(F; F)$ and $\mathbf{L}(A \cup F; F \cup B)$ are invertible, applying the properties of the Schur complement we get that

$$\begin{aligned} & \mathbf{L}(x; x) - \mathbf{L}(x; F \cup B) \cdot \mathbf{L}(A \cup F; F \cup B)^{-1} \cdot \mathbf{L}(A \cup F; x) \\ &= \mathbf{L}(\{x\} \cup A \cup F; \{x\} \cup F \cup B) \Big/_{\mathbf{L}(A \cup F; F \cup B)} = \mathbf{L}(\{x\} \cup A \cup F; \{x\} \cup F \cup B) \Big/_{\mathbf{L}(F; F)} \Big/_{\mathbf{L}(A \cup F; F \cup B) \Big/_{\mathbf{L}(F; F)}}. \end{aligned}$$

The last equality can be written as

$$\begin{aligned} & \mathbf{L}(x; x) - \mathbf{L}(x; F \cup B) \cdot \mathbf{L}(A \cup F; F \cup B)^{-1} \cdot \mathbf{L}(A \cup F; x) \\ (3.4) \quad &= \Lambda(\{x\} \cup A; \{x\} \cup B) \Big/_{\Lambda(A; B)} \\ &= \Lambda(x; x) - \Lambda(x; B) \cdot \Lambda(A; B)^{-1} \cdot \Lambda(A; x). \end{aligned}$$

On the other hand, it is clear that

$$(3.5) \quad \mathbf{L}(x, x) = \mathcal{L}_q(\varepsilon_x)(x) = c(x, y) - \frac{1}{\omega(x)} \frac{\partial \omega}{\partial \mathbf{n}_F}(x) + \lambda = \frac{\omega(y)}{\omega(x)} c(x, y) + \lambda$$

and

$$\begin{aligned} & \mathbf{L}(x; B \cup F) \cdot \mathbf{L}(A \cup F; B \cup F)^{-1} \cdot \mathbf{L}(A \cup F; x) \\ (3.6) \quad &= \mathbf{C}(x; y) \cdot \left[\mathbf{L}(A \cup F; B \cup F) \right]^{-1}(y; y) \cdot \mathbf{C}(y; x) \\ &= c(x, y)^2 \left[\mathbf{L}(A \cup F; B \cup F) \right]^{-1}(y; y) \end{aligned}$$

because the unique neighbour of x is y . The result follows when we properly join Equations (3.4), (3.5), (3.6) and using the equality

$$\left[\mathbf{L}(A \cup F; B \cup F) \right]^{-1}(y; y) = \tilde{\mathbf{G}}(y, y)$$

proved in [A3, Proposition 2.12]. \square

4. Recovering the conductance on grids

Our purpose is to use the results of the previous section in order to recover all the conductances on 3-dimensional grids. To the best of our knowledge, this is the first work that shows a recovery algorithm for non-planar networks. First, we define this family of networks for $n = 3$ and then we prove all the steps for the recovery algorithm. The 2-dimensional case was presented in the IX JMDA conference held in Tarragona, Spain, and whose extended abstract, without proofs, can be found in [A4]. A similar algorithm for spider networks was presented in [A1].

A 3-dimensional grid is the discretization of any cuboid in \mathbb{R}^3 . Let us take three integers $\ell_i \in \mathbb{N}$, $i = 1, 2, 3$. We define the *three dimensional grid with boundary* as the network $\Gamma = (V, c)$ with vertex set

$$V = \{x_{ijk} : i = 0, \dots, \ell_1 + 1, j = 0, \dots, \ell_2 + 1, k = 0, \dots, \ell_3 + 1\}$$

and conductivity function c given for all $i = 1, \dots, \ell_1$, $j = 1, \dots, \ell_2$ and $k = 1, \dots, \ell_3$, by

$$c(x_{ijk}, x_{rst}) > 0 \text{ when } \begin{cases} r = i \pm 1, & s = j \text{ and } t = k, \\ r = i, & s = j \pm 1 \text{ and } t = k, \\ r = i, & s = j \text{ and } t = k \pm 1, \end{cases}$$

$c(x_{ij0}, x_{ij1}) > 0$, $c(x_{ij\ell_3}, x_{ij\ell_3+1}) > 0$, $c(x_{0jk}, x_{1jk}) > 0$, $c(x_{\ell_1jk}, x_{\ell_1+1jk}) > 0$, $c(x_{i0k}, x_{i1k}) > 0$ and $c(x_{i\ell_2k}, x_{i\ell_2+1k}) > 0$ and $c(x, y) = 0$ otherwise.

We define for $j = 0, \dots, \ell_2 + 1$, the following sets of vertices

$$A_j = \{x_{ijk} : i = 1, \dots, \ell_1, k = 1, \dots, \ell_3\},$$

and for $j = 1, \dots, \ell_2$, the sets

$$R_j = \{x_{ijk} : i = 0, \ell_1 + 1, k = 1, \dots, \ell_3\} \cup \{x_{ijk} : i = 1, \dots, \ell_1, k = 0, \ell_3 + 1\}.$$

If we consider the set $F = \bigcup_{j=1}^{\ell_2} A_j$, then $\delta(F) = A \cup R \cup B$, where

$$A = A_0, \quad R = \bigcup_{i=1}^{\ell_2} R_i \quad \text{and} \quad B = A_{\ell_2+1}.$$

Figure 1 displays a three dimensional grid with $\ell_1 = \ell_3 = 2$ and $\ell_2 = 3$, and the boundary sets A , B and R_1, \dots, R_{ℓ_2} .

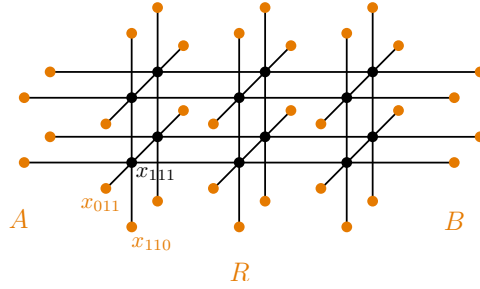


FIGURE 1. An example of 3 dimensional grid with $\ell_1 = \ell_3 = 2$ and $\ell_2 = 3$.

For fixed $\sigma \in \Omega(\bar{F})$ and $\lambda \geq 0$, such that $q = q_\sigma + \lambda \chi_{\delta(F)}$, it is satisfied that

$$\mathbf{L}(A \cup F; F \cup B) = \begin{pmatrix} -C_{01} & 0 & 0 & \cdots & 0 & 0 & 0 \\ Q_1 & -C_{12} & 0 & \cdots & 0 & 0 & 0 \\ -C_{12}^\top & Q_2 & -C_{23} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & -C_{\ell_1-2\ell_1-1} & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & Q_{\ell-1} & -C_{\ell_1-1\ell_1} & 0 \\ 0 & \cdots & \cdots & \cdots & -C_{\ell_1-1\ell_1}^\top & Q_{\ell_1} & -C_{\ell_1\ell_1+1} \end{pmatrix}$$

where $C_{ii+1} = C(A_i; A_{i+1})$ for any $i = 0, \dots, \ell_1$ and $Q_i = D(A_i; A_i) - C(A_i; A_i)$. We recall that D is the diagonal matrix whose diagonal entries are given by $\kappa + q$.

PROPOSITION 4.1. *For any $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$, the overdetermined partial Dirichlet–Neumann boundary value problem on F with data f, g, h on a 3–dimensional grid*

$$\mathcal{L}_q(u) = f \text{ on } F, \quad \frac{\partial u}{\partial \mathbf{n}_F} = h \text{ on } A \quad \text{and} \quad u = g \text{ on } A \cup R,$$

has a unique solution.

Proof. Note that the matrix $L(A \cup F; F \cup B)$ is invertible, since C_{ii+1} is a diagonal matrix whose diagonal entries are positive. Then, the result follows by applying Theorem 2.1 in [A3]. \square

In the following results we analyze some properties of the overdetermined partial Green and Poisson kernels for a grid that will be useful for the recovery algorithm. Notice that a pole $y \in R \cup A$, is of the form $y = x_{ijk}$ for $i = 1, \dots, \ell_1$, $j = 1, \dots, \ell_2$ and $k = 0, \ell_3 + 1$ when $y \in R_j$ or $y = x_{i0k}$ for $i = 1, \dots, \ell_1$, $k = 1, \dots, \ell_3$ when $y \in A$. So, from now on we denote by $u_{ijk} = \tilde{P}(\cdot, x_{ijk})$.

In order to prove the properties of the overdetermined partial resolvent kernels, we will need the following result that was proved in [A3].

LEMMA 4.1. [A3, Corollary 2.11] *It is satisfied that*

$$\tilde{G}(A_1; F) = 0, \quad \tilde{P}(A_1; R) = 0 \quad \text{and} \quad \tilde{P}(A_1; A) = 1.$$

In the following result we analyze the zero–set of the function u_{ijk} as well as the sign pattern along certain paths starting from $A \cup R$. This behavior is in accordance with the alternating property proved in [A1].

PROPOSITION 4.2. *The following properties hold:*

- (i) *If $k = 0, \ell_3 + 1$, then for any $i = 1, \dots, \ell_1$, $j = 1, \dots, \ell_2$,*

$$u_{ijk}(x) = 0 \quad \text{if} \quad x \in \bigcup_{s=0}^j A_s \cup \bigcup_{r=j+1}^{\ell_2+1} (A_r \setminus N_{r-j}(x_{irk})).$$

Moreover, for any $h = 0, \dots, \ell_2 - j$, it is satisfied that

$$\begin{aligned} & c(x_{ij+h|k-h-1|}, x_{ij+h+1|k-h-1|})u_{ijk}(x_{ij+h+1|k-h-1|}) = \\ & - c(x_{ij+h|k-h-1|}, x_{ij+h|k-h|})u_{ijk}(x_{ij+h|k-h|}), \end{aligned}$$

which implies $(-1)^h u_{ijk}(x_{ij+h|k-h|}) > 0$.

- (ii) *If $j = 0$, then for $i = 1, \dots, \ell_1$, $k = 1, \dots, \ell_3$*

$$u_{i0k}(x) = 0 \quad \text{if} \quad x \in \bigcup_{r=1}^{\ell_2+1} (A_r \setminus N_{r-1}(x_{irk}))$$

and $u_{i0k}(x_{i1k}) = 1$. Moreover, for any $h = 1, \dots, \ell_3 - 1$, it is satisfied

$$c(x_{ihh+1}, x_{ih+1h+1})u_{i01}(x_{ih+1h+1}) = -c(x_{ihh+1}, x_{ihh})u_{i01}(x_{ihh}),$$

which implies $(-1)^h u_{i01}(x_{ih+1h+1}) > 0$.

Proof. (i) Since $x_{ijk} \in R_j$, then

$$\mathcal{L}_q(u_{ijk}) = 0 \text{ on } F, \quad u_{ijk} = \varepsilon_{x_{ijk}} \text{ on } A \cup R \quad \text{and} \quad \frac{\partial u_{ijk}}{\partial \mathbf{n}_F} = 0 \text{ on } A.$$

The proof is by induction on s . The case $s = 0$, is trivial from the definition of u_{ijk} and the case $s = 1$, follows from Lemma 4.1. Suppose now that the result is true for any $s \leq j - 1$. Let $x_{ms+1t} \in A_{s+1}$ and then, by hypothesis of induction,

$$0 = \mathcal{L}_q(u_{ijk})(x_{mst}) = -c(x_{mst}, x_{ms+1t})u_{ijk}(x_{ms+1t}),$$

and hence $u_{ijk} = 0$ on A_{s+1} .

Suppose now that $s = j + 1$, then for any $x_{mj+1t} \in A_s \setminus N_1(x_{ij+1k})$,

$$0 = \mathcal{L}_q(u_{ijk})(x_{mjt}) = -c(x_{mjt}, x_{mj+1t})u_{ijk}(x_{mj+1t}),$$

and hence $u_{ijk}(x_{mj+1t}) = 0$. Suppose now that the result is true for any $r < \ell_2 - j + 1$ and let $r + 1$, then for any $x_{mj+r+1t} \in A_{j+r+1} \setminus N_{r+1}(x_{ij+r+1k})$

$$0 = \mathcal{L}_q(u_{ijk})(x_{mj+rt}) = -c(x_{mj+rt}, x_{mj+r+1t})u_{ijk}(x_{mj+r+1t})$$

and hence $u_{ijk}(x_{mj+r+1t}) = 0$.

Moreover, for any $h \in \{0, \dots, \ell_2 - j\}$, it holds that

$$\begin{aligned} u_{ijk}(x_{ij+h|k-h-1|}) &= u_{ijk}(x_{ij+h-1|k-h-1|}) = u_{ijk}(x_{i-1j+h|k-h-1|}) \\ &= u_{ijk}(x_{i+1j+h|k-h-1|}) = u_{ijk}(x_{ij+h|k-h-2|}) = 0, \end{aligned}$$

and hence

$$\begin{aligned} &c(x_{ij+h|k-h-1|}, x_{ij+h+1|k-h-1|})u_{ijk}(x_{ij+h+1|k-h-1|}) = \\ &-c(x_{ij+h|k-h-1|}, x_{ij+h|k-h|})u_{ijk}(x_{ij+h|k-h|}) \end{aligned}$$

since $\mathcal{L}_q(u_{ijk})(x_{ij+h|k-h-1|}) = 0$.

As $u_{ijk}(x_{ijk}) = 1$, the above identity implies that $(-1)^h u_{ijk}(x_{ij+h|k-h|}) > 0$.

(ii) Suppose now that $j = 0$. Then,

$$\mathcal{L}_q(u_{i0k}) = 0 \text{ on } F, \quad u_{i0k} = \varepsilon_{x_{i0k}} \text{ on } A \cup R \quad \text{and} \quad \frac{\partial u_{i0k}}{\partial \mathbf{n}_F} = 0 \text{ on } A.$$

The proof is by induction on r . Suppose that $r = 1$, then for any $x_{m1t} \in A_1 \setminus \{x_{i1k}\}$

$$0 = \frac{\partial u_{i0k}}{\partial \mathbf{n}_F}(x_{m0t}) = -c(x_{m0t}, x_{m1t})u_{i0k}(x_{m1t}).$$

Therefore, $u_{i0k} = 0$ on $A_1 \setminus \{x_{i1k}\}$. Moreover,

$$0 = \frac{\partial u_{i0k}}{\partial \mathbf{n}_F}(x_{i0k}) = c(x_{i0k}, x_{i1k})(1 - u_{i0k}(x_{i1k}))$$

and hence $u_{i0k}(x_{i1k}) = 1$. The rest of the proof is analogue to the case $x_{ijk} \in R_j$.

Moreover, for any $h = 1, \dots, \ell_3 - 1$, it holds that

$$u_{i01}(x_{ihh+1}) = u_{i01}(x_{ihh+2}) = u_{i01}(x_{ih-1h+1}) = u_{i01}(x_{i+1hh+1}) = u_{i01}(x_{i-1hh+1}) = 0,$$

and hence

$$c(x_{ihh+1}, x_{ih+1h+1})u_{i01}(x_{ih+1h+1}) = -c(x_{ihh+1}, x_{ihh})u_{i01}(x_{ihh})$$

since $\mathcal{L}_q(u_{i01})(x_{ihh+1}) = 0$.

As $u_{i01}(x_{i11}) = 1$, the above identity implies that $(-1)^h u_{i01}(x_{ih+1h+1}) > 0$. \square

PROPOSITION 4.3. *For a 3-dimensional grid it is satisfied that*

$$\tilde{\mathcal{G}}(A_i, A_j) = 0, \text{ for all } j = 1, \dots, \ell_2 \text{ and } 1 \leq i \leq j.$$

Proof. The proof is analogous to the one of Proposition 4.2 but considering the overdetermined partial boundary value problem

$$\mathcal{L}_q(v) = \varepsilon_{x_{ijk}} \text{ on } F, \quad v = 0 \text{ on } A \cup R \quad \text{and} \quad \frac{\partial v}{\partial \mathbf{n}_F} = 0 \text{ on } A$$

instead; that is, $v = \tilde{\mathcal{G}}_q(\varepsilon_{x_{ijk}})$ on \bar{F} . \square

4.1. Recovery algorithm. Without loss of generality we assume that $\ell_3 \leq \ell_2$ and we consider the *partial layers* of vertices

$$\begin{aligned} D_0 &= \{x_{ij0} \in \bar{F} : i = 1, \dots, \ell_1, j = 1, \dots, \ell_2\}, \\ D_k &= \{x_{ijk} \in \bar{F} : i = 1, \dots, \ell_1, j = k, \dots, \ell_2\}, \text{ for } k = 1, \dots, \ell_3, \\ D_{\ell_3+1} &= \{x_{ij\ell_3+1} \in \bar{F} : i = 1, \dots, \ell_1, j = \ell_3 + 1, \dots, \ell_2\}. \end{aligned}$$

In particular, $D_{\ell_3+1}, D_0 \subset R$.

The recovery of conductances on a 3-dimensional grid from its response matrix is an iterative process, for we are not able to give explicit formulae for all the conductances at the same time but we can give a recovery algorithm instead. Hence, we describe the algorithm in steps, each of them requiring the information obtained in the last one.

To start with, let Λ be an irreducible and symmetric M -matrix of order $2(\ell_1\ell_2 + \ell_1\ell_3 + \ell_2\ell_3)$ satisfying that is a response matrix and which is suppose to be known. Let $\lambda \geq 0$ be the lowest eigenvalue of Λ and $\sigma \in \Omega(\delta(F))$ the eigenvector associated with λ . In addition, we choose $\omega \in \Omega(\bar{F})$ such that $\omega = k\sigma$ on $\delta(F)$ with $0 < k < 1$.

Step 1. Let us fix the indices $\mathbf{r} \in \{1, \dots, \ell_1\}$ and $\mathbf{s} \in \{1, \dots, \ell_2\}$ and consider $u_{\mathbf{r}\mathbf{s}0}$. We already know that $u_{\mathbf{r}\mathbf{s}0} = 0$ on $A \cup (R \setminus \{x_{\mathbf{r}\mathbf{s}0}\})$ and $u_{\mathbf{r}\mathbf{s}0}(x_{\mathbf{r}\mathbf{s}0}) = 1$. Moreover, the values of $u_{\mathbf{r}\mathbf{s}0}$ on B are given by

$$u_{\mathbf{r}\mathbf{s}0}(B) = -\Lambda(A; B)^{-1} \cdot \Lambda(A; x_{\mathbf{r}\mathbf{s}0})$$

because of Corollary 3.1.

Notice that this means that all the values of $u_{\mathbf{r}\mathbf{s}0}$ on B are known, for the Dirichlet-to-Robin map is known. In consequence, $u_{\mathbf{r}\mathbf{s}0}$ is known on all the boundary $\delta(F)$. In Figure 2(b) we show all the information obtained at the end of this step.

If $\ell_2 = \ell_3$, let us now also fix the indices $\mathbf{p} \in \{1, \dots, \ell_1\}$ and $\mathbf{t} \in \{1, \dots, \ell_3\}$ and consider $u_{\mathbf{p}0\mathbf{t}}$. We already know that $u_{\mathbf{p}0\mathbf{t}} = 0$ on $R \cup (A \setminus \{x_{\mathbf{p}0\mathbf{t}}\})$ and $u_{\mathbf{p}0\mathbf{t}}(x_{\mathbf{p}0\mathbf{t}}) = 1$. Then, the values of $u_{\mathbf{p}0\mathbf{t}}$ on B are given by

$$u_{\mathbf{p}0\mathbf{t}}(B) = -\Lambda(A; B)^{-1} \cdot \Lambda(A; x_{\mathbf{p}0\mathbf{t}})$$

because of Corollary 3.1.

Notice that this means that all the values of $u_{\mathbf{p}0\mathbf{t}}$ on B are known, for the Dirichlet-to-Robin map is known. In consequence, $u_{\mathbf{p}0\mathbf{t}}$ is known on all the boundary $\delta(F)$.

Step 2. From Proposition 4.3, $\tilde{G}(y; y) = 0$ for any $y \in F$ and hence applying the boundary spike formula given in Proposition 3.2 we can recover the conductance of the boundary edges.

COROLLARY 4.1. *The conductances of the edges joining the vertices of D_0 with the vertices of D_1 are given by*

$$c(x_{ij0}, x_{ij1}) = \frac{\omega(x_{ij0})}{\omega(x_{ij1})} (\Lambda(x_{ij0}; x_{ij0}) - \Lambda(x_{ij0}; B) \cdot \Lambda(A; B)^{-1} \cdot \Lambda(A; x_{ij0}) - \lambda)$$

for all $i = 1, \dots, \ell_1$ and $j = 1, \dots, \ell_2$.

By rotating the grid conveniently; that is, by considering another labeling of the vertices of the grid that fit the description at the beginning of this section, we obtain the conductances of all the boundary edges of the grid. In Figure 2(c) we show all the information obtained at the end of this step.

Step 3. In this step we recover all the values of $u_{\mathfrak{r}0}$ and of $u_{\mathfrak{p}0t}$ on D_1 .

LEMMA 4.2. *The values of $u_{\mathfrak{r}0}$ and of $u_{\mathfrak{p}0t}$ on D_1 are given for all $i = 1, \dots, \ell_1$ and $j = 1, \dots, \ell_2$ by*

$$u_{\mathfrak{p}0t}(x_{ij1}) = \frac{1}{c(x_{ij0}, x_{ij1})} \left(-\Lambda(x_{ij0}; x_{\mathfrak{p}0t}) + \Lambda(x_{ij0}; B) \cdot \Lambda(A; B)^{-1} \cdot \Lambda(A; x_{\mathfrak{p}0t}) \right)$$

whereas when $i \neq \mathfrak{r}$ or $j \neq \mathfrak{s}$

$$u_{\mathfrak{r}0}(x_{ij1}) = \frac{1}{c(x_{ij0}, x_{ij1})} \left(-\Lambda(x_{ij0}; x_{\mathfrak{r}0}) + \Lambda(x_{ij0}; B) \cdot \Lambda(A; B)^{-1} \cdot \Lambda(A; x_{\mathfrak{r}0}) \right)$$

and $u_{\mathfrak{r}0}(x_{\mathfrak{r}1}) = 0$.

Proof. Consider the function $u_{\mathfrak{r}0}$. Observe that $u_{\mathfrak{r}0}(x_{\mathfrak{r}1}) = 0$ from Proposition 4.2. Suppose now that $i \neq \mathfrak{r}$ or $j \neq \mathfrak{s}$ then, we can rewrite Problem (3.3) as the Dirichlet problem

$$\mathcal{L}_q(u_{\mathfrak{r}0}) = 0 \quad \text{on } F \quad \text{and} \quad u_{\mathfrak{r}0} = \varepsilon_{x_{\mathfrak{r}0}} + u_{\mathfrak{r}0|_B} \quad \text{on } \delta(F)$$

with the additional condition $\frac{\partial u_{\mathfrak{r}0}}{\partial \mathbf{n}_F} = 0$ on A . Therefore, by the definition of the Dirichlet-to-Robin map, for all $x_{ij0} \in \delta(F)$ it is satisfied that

$$\begin{aligned} \Lambda(x_{ij0}; x_{\mathfrak{r}0}) + \Lambda(x_{ij0}; B) \cdot \mathbf{u}_{\mathfrak{r}0}(B) &= \Lambda_q \left(\varepsilon_{x_{\mathfrak{r}0}} + u_{\mathfrak{r}0|_B} \right) (x_{ij0}) \\ &= \frac{\partial u_{\mathfrak{r}0}}{\partial \mathbf{n}_F} (x_{ij0}) + q(x_{ij0}) u_{\mathfrak{r}0}(x_{ij0}) \\ &= \left(\lambda + \frac{\omega(x_{ij1})}{\omega(x_{ij0})} c(x_{ij0}, x_{ij1}) \right) u_{\mathfrak{r}0}(x_{ij0}) - c(x_{ij0}, x_{ij1}) u_{\mathfrak{r}0}(x_{ij1}) \\ &= -c(x_{ij0}, x_{ij1}) u_{\mathfrak{r}0}(x_{ij1}). \end{aligned}$$

Observe that all the terms of this equality, except the value $u_{\mathfrak{r}0}(x_{ij1})$, are already known. Therefore, we get the result. We proceed analogously to obtain the values of $u_{\mathfrak{p}0t}$. \square

In Figure 2(d) we show all the data gathered from the 3-dimensional grid at the end of this step.

Step 4. Here we find the conductances of all the edges with both ends in D_1 and such that the second indices of their ends are different. However, we state a more general result which is a direct consequence of Proposition 4.2 (i) when $h = 0$.

PROPOSITION 4.4. *Let $h \in \{0, \dots, \ell_3 - 1\}$. For every $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2$, let us suppose that we know the values of $u_{\tau\mathfrak{s}0}$ on D_{h+1} and D_h . Also, we suppose that the conductances of all the edges joining vertices from D_{h+1} and D_h are known. Then, for every $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - h - 1$ the conductances $c(x_{\tau\mathfrak{s}+hh+1}, x_{\tau\mathfrak{s}+h+1h+1})$ are given by*

$$c(x_{\tau\mathfrak{s}+hh+1}, x_{\tau\mathfrak{s}+h+1h+1}) = -\frac{u_{\tau\mathfrak{s}0}(x_{\tau\mathfrak{s}+hh})}{u_{\tau\mathfrak{s}0}(x_{\tau\mathfrak{s}+h+1h+1})}c(x_{\tau\mathfrak{s}+hh+1}, x_{\tau\mathfrak{s}+hh}).$$

When $h = 0$, Proposition 4.4 shows that $c(x_{\tau\mathfrak{s}1}, x_{\tau\mathfrak{s}+11})$ is known for all $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - 1$. See Figure 2(e) in order to see all the known information at the end of this step.

Step 5. In this step we give the conductances of all the edges with both ends in D_1 that are still unknown. Furthermore, we state a more general result.

PROPOSITION 4.5. *Let $h \in \{0, \dots, \ell_3 - 1\}$. For every $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2$, let us suppose that we know the values of $u_{\tau\mathfrak{s}0}$ and of $u_{\tau01}$ on D_{h+1} and D_h . Also, let us assume that we know the conductances of all the edges joining vertices from D_{h+1} and D_h , and the ones of the edges with both ends in D_{h+1} and such that the ends have different second component. Then, for every $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - h - 1$ the conductances $c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau\mathfrak{s}+h+1h+1})$ are given by*

$$\begin{aligned} c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau\mathfrak{s}+h+1h+1}) &= -c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau+1\mathfrak{s}+h+2h+1})\frac{u_{\tau\mathfrak{s}0}(x_{\tau+1\mathfrak{s}+h+2h+1})}{u_{\tau\mathfrak{s}0}(x_{\tau\mathfrak{s}+h+1h+1})} \\ &\quad -c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau+1\mathfrak{s}+h+1h})\frac{u_{\tau\mathfrak{s}0}(x_{\tau+1\mathfrak{s}+h+1h})}{u_{\tau\mathfrak{s}0}(x_{\tau\mathfrak{s}+h+1h+1})}. \end{aligned}$$

Moreover, the conductances $c(x_{\tau+1h+1h+1}, x_{\tau h+1h+1})$ are given by

$$\begin{aligned} c(x_{\tau+1h+1h+1}, x_{\tau h+1h+1}) &= -c(x_{\tau+1h+1h+1}, x_{\tau+1h+2h+1})\frac{u_{\tau01}(x_{\tau+1h+2h+1})}{u_{\tau01}(x_{\tau h+1h+1})} \\ &\quad -c(x_{\tau+1h+1h+1}, x_{\tau+1h+1h})\frac{u_{\tau01}(x_{\tau+1h+1h})}{u_{\tau01}(x_{\tau h+1h+1})}. \end{aligned}$$

Proof. We fix the indices $h \in \{0, \dots, \ell_3 - 1\}$, $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - h - 1$. Then, using Proposition 4.2 (i),

$$\begin{aligned} 0 &= \mathcal{L}_q(u_{\tau\mathfrak{s}0})(x_{\tau+1\mathfrak{s}+h+1h+1}) \\ &= -c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau\mathfrak{s}+h+1h+1})u_{\tau\mathfrak{s}0}(x_{\tau\mathfrak{s}+h+1h+1}) \\ &\quad - c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau+1\mathfrak{s}+h+2h+1})u_{\tau\mathfrak{s}0}(x_{\tau+1\mathfrak{s}+h+2h+1}) \\ &\quad - c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau+1\mathfrak{s}+h+1h})u_{\tau\mathfrak{s}0}(x_{\tau+1\mathfrak{s}+h+1h}). \end{aligned}$$

The value $c(x_{\tau+1\mathfrak{s}+h+1h+1}, x_{\tau\mathfrak{s}+h+1h+1})$ is the unique unknown term of this equality and by Proposition 4.2(i) we know that $u_{\tau\mathfrak{s}0}(x_{\tau\mathfrak{s}+h+1h+1}) \neq 0$.

Moreover, we fix the indices $h \in \{0, \dots, \ell_3 - 1\}$, $\tau = 1, \dots, \ell_1$. Then, using Proposition 4.2 (ii),

$$\begin{aligned}
0 &= \mathcal{L}_q(u_{\tau 01})(x_{\tau+1h+1h+1}) \\
&= -c(x_{\tau+1h+1h+1}, x_{\tau h+1h+1})u_{\tau 01}(x_{\tau h+1h+1}) \\
&\quad - c(x_{\tau+1h+1h+1}, x_{\tau+1h+2h+1})u_{\tau 01}(x_{\tau+1h+2h+1}) \\
&\quad - c(x_{\tau+1h+1h+1}, x_{\tau+1h+1h})u_{\tau 01}(x_{\tau+1h+1h}).
\end{aligned}$$

The value $c(x_{\tau+1h+1h+1}, x_{\tau h+1h+1})$ is the unique unknown term of this equality and by Proposition 4.2(ii) we know that $u_{\tau 01}(x_{\tau h+1h+1}) \neq 0$. \square

When $h = 0$, Proposition 4.5 shows that $c(x_{\tau s+11}, x_{\tau+1s+11})$ is known for all $\tau = 1, \dots, \ell_1$ and $s = 1, \dots, \ell_2 - 1$ and the same happens with $c(x_{\tau+111}, x_{\tau 11})$ for all $\tau = 1, \dots, \ell_1$. See Figure 2(f) in order to see all the information gathered at the end of this step.

Step 6. Let us define the linear operator $\wp: \mathcal{C}(\bar{F}) \rightarrow \mathcal{C}(F)$ given by the values

$$\begin{aligned}
\wp(v)(x_{ijk}) &= c(x_{ijk}, x_{ijk-1})v(x_{ijk-1}) + c(x_{ijk}, x_{i-1jk})v(x_{i-1jk}) + c(x_{ijk}, x_{i+1jk})v(x_{i+1jk}) \\
&\quad + c(x_{ijk}, x_{ij-1k})v(x_{ij-1k}) + c(x_{ijk}, x_{ij+1k})v(x_{ij+1k})
\end{aligned}$$

for all $v \in \mathcal{C}(\bar{F})$ and $x_{ijk} \in F$. This operator will be useful in this and also in the following steps because of its relation with the Schrödinger operator for any $x_{ijk} \in F$

$$\begin{aligned}
(4.1) \quad \mathcal{L}_q(v)(x_{ijk}) &= \frac{v(x_{ijk})}{\omega(x_{ijk})} \wp(\omega)(x_{ijk}) - \wp(v)(x_{ijk}) \\
&\quad - c(x_{ijk}, x_{ijk+1})v(x_{ijk+1}) + \frac{\omega(x_{ijk+1})}{\omega(x_{ijk})} c(x_{ijk}, x_{ijk+1})v(x_{ijk}).
\end{aligned}$$

In this step we give the conductances of all the edges joining the vertices from D_1 and D_2 . Furthermore, we state a more general result.

PROPOSITION 4.6. *Let $h \in \{0, \dots, \ell_3 - 1\}$. For every $\tau = 1, \dots, \ell_1$ and $s = 1, \dots, \ell_2$, let us suppose that we know the values of $u_{\tau s 0}$ on D_{h+1} and D_h . Also, let us suppose that we know the conductances of all the edges joining vertices from D_{h+1} and D_h , and the ones of the edges with both ends in D_{h+1} . Then, for every $\tau = 1, \dots, \ell_1$ and $s = 1, \dots, \ell_2 - h - 1$ the conductances $c(x_{\tau s+h+1h+1}, x_{\tau s+h+1h+2})$ are given by*

$$c(x_{\tau s+h+1h+1}, x_{\tau s+h+1h+2}) = \frac{\wp(u_{\tau s 0})(x_{\tau s+h+1h+1})\omega(x_{\tau s+h+1h+1})}{u_{\tau s 0}(x_{\tau s+h+1h+1})\omega(x_{\tau s+h+1h+2})} - \frac{\wp(\omega)(x_{\tau s+h+1h+1})}{\omega(x_{\tau s+h+1h+2})}.$$

Proof. We fix the indices $h \in \{0, \dots, \ell_3 - 1\}$, $\tau = 1, \dots, \ell_1$ and $s = 1, \dots, \ell_2 - 1$. Observe that $\wp(\omega)(x_{\tau s+h+1h+1})$ and $\wp(u_{\tau s 0})(x_{\tau s+h+1h+1})$ are already known. Then, from Equation (4.1)

$$\begin{aligned}
0 &= \mathcal{L}_q(u_{\tau s 0})(x_{\tau s+h+1h+1}) \\
&= u_{\tau s 0}(x_{\tau s+h+1h+1})\wp(\omega)(x_{\tau s+h+1h+1}) - \omega(x_{\tau s+h+1h+1})\wp(u_{\tau s 0})(x_{\tau s+h+1h+1}) \\
&\quad + \omega(x_{\tau s+h+1h+2})c(x_{\tau s+h+1h+1}, x_{\tau s+h+1h+2})u_{\tau s 0}(x_{\tau s+h+1h+1})
\end{aligned}$$

and hence $c(x_{\tau s+h+1h+1}, x_{\tau s+h+1h+2})$ is the only unknown term of this equality. \square

In particular, when $h = 0$, Proposition 4.6 shows that $c(x_{\tau\mathfrak{s}+11}, x_{\tau\mathfrak{s}+12})$ is known for all $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - 1$. See Figure 2(g) in order to see all the information obtained at the end of this step.

Step 7. In this step we are able to obtain the unknown values of $u_{\tau\mathfrak{s}0}$ and $u_{\tau01}$ on D_2 for all $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2$. In fact, let us state a more general result.

PROPOSITION 4.7. *Let $h \in \{0, \dots, \ell_3 - 1\}$. For every $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2$, let us suppose that we know the values of $u_{\tau\mathfrak{s}0}$ and of $u_{\tau01}$ on D_{h+1} and D_h . Also, let us suppose that we know the conductances of all the edges joining vertices from D_{h+1} and D_h , from D_{h+2} and D_{h+1} and the ones of the edges with both ends in D_{h+1} . Then, for every $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - h - 2$ the values of $u_{\tau\mathfrak{s}0}$ on D_{h+2} are given by*

$$u_{\tau\mathfrak{s}0}(x_{ijh+2}) = -\frac{\wp(u_{\tau\mathfrak{s}0})(x_{ijh+1})}{c(x_{ijh+1}, x_{ijh+2})} + \frac{\wp(\omega)(x_{ijh+1})}{\omega(x_{ijh+1})c(x_{ijh+1}, x_{ijh+2})}u_{\tau\mathfrak{s}0}(x_{ijh+1}) \\ + \frac{\omega(x_{ijh+2})}{\omega(x_{ijh+1})}u_{\tau\mathfrak{s}0}(x_{ijh+1})$$

for all $i = 1, \dots, \ell_1$ and $j = \mathfrak{s} + h + 2, \dots, \ell_2$. Moreover,

$$u_{\tau01}(x_{ijh+2}) = -\frac{\wp(u_{\tau01})(x_{ijh+1})}{c(x_{ijh+1}, x_{ijh+2})} + \frac{\wp(\omega)(x_{ijh+1})}{\omega(x_{ijh+1})c(x_{ijh+1}, x_{ijh+2})}u_{\tau01}(x_{ijh+1}) \\ + \frac{\omega(x_{ijh+2})}{\omega(x_{ijh+1})}u_{\tau01}(x_{ijh+1})$$

for all $i = 1, \dots, \ell_1$ and $j = h + 2, \dots, \ell_2$.

Proof. Fixed three indices $h \in \{0, \dots, \ell_3 - 1\}$, $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - h - 2$, let $i \in \{1, \dots, \ell_1\}$ and $j \in \{\mathfrak{s} + h + 2, \dots, \ell_2\}$. Observe that $\wp(\omega)(x_{ijh+1})$ and $\wp(u_{\tau\mathfrak{s}0})(x_{ijh+1})$ are already known. Then,

$$0 = \mathcal{L}_q(u_{\tau\mathfrak{s}0})(x_{ijh+1}) = \frac{u_{\tau\mathfrak{s}0}(x_{ijh+1})}{\omega(x_{ijh+1})}\wp(\omega)(x_{ijh+1}) - \wp(u_{\tau\mathfrak{s}0})(x_{ijh+1}) \\ - c(x_{ijh+1}, x_{ijh+2})u_{\tau\mathfrak{s}0}(x_{ijh+2}) + \frac{\omega(x_{ijh+2})}{\omega(x_{ijh+1})}c(x_{ijh+1}, x_{ijh+2})u_{\tau\mathfrak{s}0}(x_{ijh+1})$$

and hence $u_{\tau\mathfrak{s}0}(x_{ijh+2})$ is the unique unknown term of this equality.

The identity for $u_{\tau01}$ can be proved analogously. \square

In particular, when $h = 0$, Proposition 4.7 shows that $u_{\tau\mathfrak{s}0}$ is known on D_2 for all $\tau = 1, \dots, \ell_1$ and $\mathfrak{s} = 1, \dots, \ell_2 - 2$. Figure 2(h) shows the information obtained until this step.

Step 8 and beyond. We keep repeating the same process to obtain more conductances, that is, we keep applying Proposition 4.4 from Step 4, then Proposition 4.5 from Step 5, Proposition 4.6 from Step 6 and then Proposition 4.7 from Step 7 for each $h = 1, \dots, \ell_3 - 1$. We stop when we have obtained all the conductances between and all the vertices in $\bigcup_{k=1}^{\ell_3} D_k$, see Figure 2(j).

The final step left is to consider the pole of the overdetermined partial Poisson Kernel on D_{ℓ_3+1} . By proceeding analogously to the last steps, we obtain the lacking conductances of the 3-dimensional grid, see Figure 2(l).

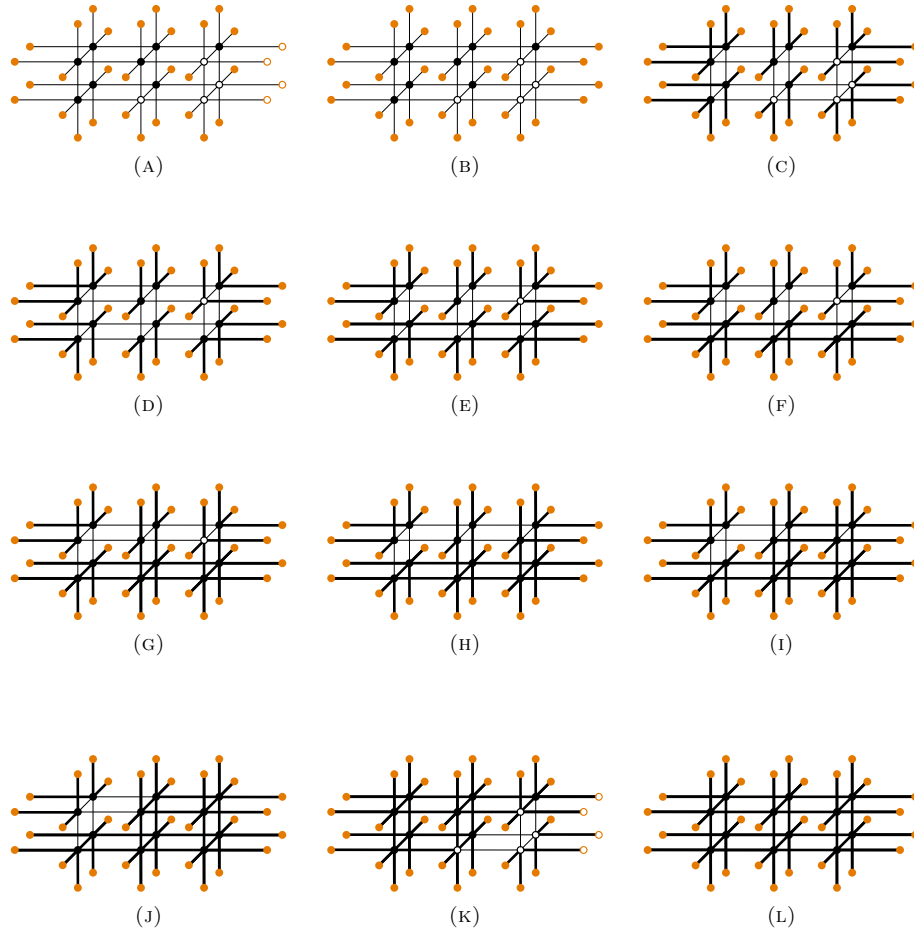


FIGURE 2. The bold items are the ones known at the end of each step for the case $\tau = \varsigma = 1$ and $p = t = 1$.

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