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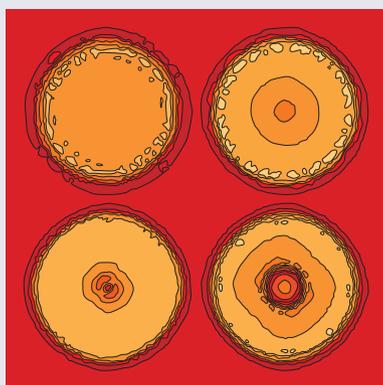
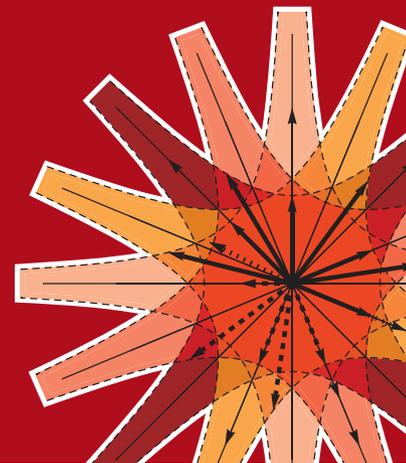
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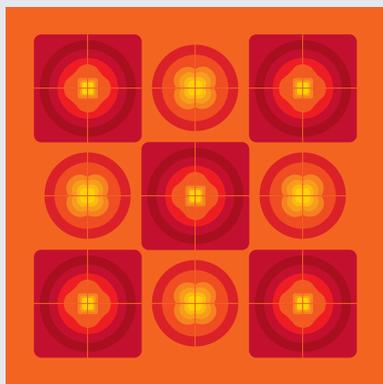
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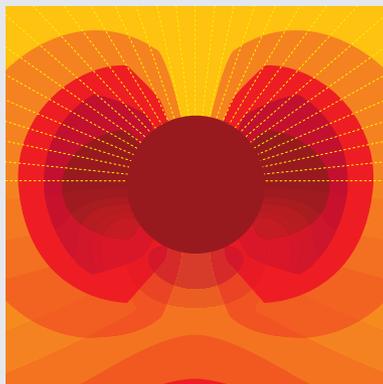
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How can holonomy corrections be introduced in $f(R)$ gravity?

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Abstract – We study the introduction of holonomy corrections in $f(R)$ gravity. We will show that there are infinitely many ways, as many as canonical transformations, to introduce this kind of corrections, depending on the canonical variables (two coordinates and its conjugate momenta) used to obtain the Hamiltonian. In each case, these corrections lead, at effective level, to different modified holonomy-corrected Friedmann equations in $f(R)$ gravity, which are in practice analytically unworkable, *i.e.* only numerical analysis can be used to understand its dynamics. In fact, there are infinitely many quantum theories, as many as canonical transformations, leading to different quantum cosmologies. Then, the problem is to know which of these quantum theories apply correctly to our Universe. We do not have an answer to this problem, and we could only give the following argument in favour of one preferred set of variables: the one that conformally maps $f(R)$ to Einstein gravity, because for these variables the Hamiltonian of the system has the same form as in standard Loop Quantum Cosmology, which provides a map between the dynamics obtained in holonomy-corrected $f(R)$ gravity and the one of standard Loop Quantum Cosmology.

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Introduction. – It is well known that, in general, $f(R)$ gravity (see [1] for a general introduction to $f(R)$ theories) contains singularities, for example: all solutions of R^2 gravity in flat Friedmann-Lemaître-Robertson-Walker (FLRW) are singular at early times (see for instance [2]). A way to avoid these singularities is to introduce holonomy corrections as in Loop Quantum Cosmology (see [3] for recent reviews on Loop Quantum Cosmology), because this kind of corrections provides a big bounce that avoids singularities like the Big Bang and Big Rip (see for example [4]). Moreover, due to the big bounce, holonomy-corrected $f(R)$ belongs into the set of bouncing scenarios that could be a viable alternative to the inflationary paradigm (see [5] for a review of bouncing cosmologies).

The first attempt to introduce holonomy corrections in $f(R)$ gravity has been recently performed in [6,7]. The basic idea in those works is the conformal equivalence between $f(R)$ and Einstein gravity, more precisely, the equivalence between the dynamical equations of $f(R)$ gravity in the vacuum, obtained using as variables the coefficients of the metric, and the dynamical equations in general relativity when matter is represented by a scalar field, provided that the coefficients of a conformal metric were used as dynamical variables.

In cosmology, where the universe is approximately described by a flat FLRW geometry, the introduction of holonomies in $f(R)$ gravity is greatly simplified because the only relevant geometrical variables are the Hubble parameter and its first and second derivatives. In this context, we study the different ways to extend LQC to $f(R)$ theories, *i.e.*, the ways to introduce holonomy corrections in $f(R)$ gravity, and, as a consequence, the deduction of the effective equations provided by these extensions.

These extensions will depend on the variables (two coordinates and two momenta) used to obtain the Hamiltonian of the system, and since there are infinitely many canonical transformations, this means that there are infinite many ways to introduce holonomies in $f(R)$ gravity, leading to different quantum theories, which give rise to different effective holonomy-corrected Friedmann equations in $f(R)$ gravity. These modified Friedmann equations will depend, in a very complicated way, on the Hubble parameter H and its first and second derivatives, meaning that the natural phase space containing the orbits of the system is the plane (H, \dot{H}) . These equations are more complicated than the classical Friedmann equation for $f(R)$ gravity, which is already analytically unworkable, meaning that when one deals with holonomy corrections, in practice, only

numerical methods can be used to understand the dynamics of the system.

Finally, it is important to stress that we do not have any criterium to decide which canonical transformation is the correct one in the sense that its corresponding quantum theory could depict correctly our Universe. We can only argue that the variables that conformally map $f(R)$ with Einstein gravity could be preferred because they provide a clear physical interpretation of the dynamical equations: the same as in Loop Quantum Cosmology, which allows us to understand the dynamics in these coordinates and, with the help of this conformal map, to have a clear representation of the phase portrait in the plane (H, \dot{H}) .

The letter is organized as follows: In the second section, we use the well-known Ostrogradsky construction to obtain the Hamiltonian of the system that leads to the modified Friedmann equation in $f(R)$ gravity. In the following section we introduce the canonical transformation that conformally maps $f(R)$ gravity which Einstein gravity, and we remark that since there are infinitely many canonical transformations, there will be infinitely many ways to formulate $f(R)$ gravity. The fourth section is devoted to reviewing the way in which holonomy corrections are introduced in standard Loop Quantum Cosmology. In the fifth section we provide a recipe to introduce holonomy corrections in $f(R)$ gravity. We show that different canonical transformations lead to different holonomy-corrected Friedmann equations in $f(R)$ gravity, and as an example we discuss the dynamics of the system corresponding to the canonical transformation that conformally maps $f(R)$ gravity into Einstein gravity.

$f(R)$ gravity “à la Ostrogradsky”. – When one considers the flat FLRW geometry, the Lagrangian of $f(R)$ gravity in the vacuum is given by

$$\mathcal{L}(V, \dot{V}, \ddot{V}) = \frac{1}{2} V f(R), \quad (1)$$

where $V = a^3$ is the volume and $R = \frac{2\ddot{V}}{V} - \frac{2\dot{V}^2}{3V^2}$ is the scalar curvature in terms of the volume and its derivatives.

Ostrogradsky’s idea to obtain the Hamiltonian from a Lagrangian containing higher order derivatives is to introduce a Lagrange multiplier, namely μ , in the Lagrangian as follows [8]:

$$\mathcal{L}_1(V, \dot{V}, \ddot{V}, R) = \frac{1}{2} V f(R) + \mu \left(\frac{2\ddot{V}}{V} - \frac{2\dot{V}^2}{3V^2} - R \right). \quad (2)$$

Extremization with respect to R gives $\mu = \frac{1}{2} V f_R(R)$, where we have introduced the notation $f_R(R) \equiv \partial_R f(R)$. In order to remove the second derivative of V , we subtract to the Lagrangian the following total derivative $\frac{d}{dt} \left(f_R(R) \dot{V} \right)$, which does not change the dynamics of the system, and we replace the Lagrange multiplier μ by its

value $\frac{1}{2} V f_R(R)$, obtaining

$$\begin{aligned} \tilde{\mathcal{L}}(V, \dot{V}, R, \dot{R}) &= \frac{1}{2} V f(R) \\ &- \frac{1}{2} V f_R(R) \left(\frac{2\dot{V}^2}{3V^2} + R \right) - f_{RR}(R) \dot{R} \dot{V}. \end{aligned} \quad (3)$$

We can see that the Lagrangian (3) depends on the variables (V, R) and its first derivatives. The corresponding canonically conjugate momenta are

$$\begin{aligned} p_V &\equiv \frac{\partial \tilde{\mathcal{L}}(V, \dot{V}, R, \dot{R})}{\partial \dot{V}} = -\frac{2\dot{V}}{3V} f_R - f_{RR} \dot{R}, \\ p_R &\equiv \frac{\partial \tilde{\mathcal{L}}(V, \dot{V}, R, \dot{R})}{\partial \dot{R}} = -f_{RR} \dot{V}, \end{aligned} \quad (4)$$

and the classical Hamiltonian of the system becomes

$$\begin{aligned} \mathcal{H}_c(V, R, p_V, p_R) &\equiv \dot{V} p_V + \dot{R} p_R - \tilde{\mathcal{L}}(V, \dot{V}, R, \dot{R}) \\ &= -\frac{p_V p_R}{f_{RR}} + \frac{1}{3V} \frac{f_R}{f_{RR}^2} p_R^2 + \frac{V}{2} (R f_R - f). \end{aligned} \quad (5)$$

The Hamiltonian constrain $\mathcal{H}_c(V, R, p_V, p_R) = 0$ leads to the well-known modified Friedmann equation in $f(R)$ gravity

$$6f_{RR} \dot{R} H + 6H^2 f_R - (R f_R - f) = 0. \quad (6)$$

Note that in Einstein gravity $f(R) = R$, one has $\frac{p_R}{f_{RR}} = -\dot{V}$ and $p_V = -\frac{2\dot{V}}{3V} = -2H$ and, thus, the classical Hamiltonian reduces to

$$\mathcal{H}_c(V, p_V) = -\frac{3}{4} p_V^2 V = -3H^2 V. \quad (7)$$

Canonical transformations. – It is well known that $f(R)$ gravity is conformally equivalent to Einstein gravity (see for instance [8]), which means that there exists a canonical transformation \mathcal{T} of the form

$$\mathcal{T} : (V, R, p_V, p_R) \mapsto (\bar{V}, \bar{\phi}, p_{\bar{V}}, p_{\bar{\phi}}), \quad (8)$$

defined by

$$\mathcal{T}(V, R, p_V, p_R) = \left(f_R^{3/2} V, \sqrt{\frac{3}{2}} \ln f_R, \frac{p_V}{f_R^{3/2}}, \sqrt{\frac{3}{2}} \left(\frac{2}{3} \frac{p_R f_R}{V f_{RR}} - p_V \right) \right). \quad (9)$$

Remark: The canonical transformation \mathcal{T} can be obtained as follows: The Lagrangian (1) is equivalent to

$$\bar{\mathcal{L}} = -\frac{(\bar{V}')^2}{3\bar{V}} + \left(\frac{(\bar{\phi}')^2}{2} + W(\bar{\phi}) \right) \bar{V}, \quad (10)$$

with $W(\bar{\phi}) = \frac{R f_R - f}{2 f_R^2}$, and $' \equiv \frac{d}{d\bar{t}}$ where we have introduced the conformal time $d\bar{t} = e^{\frac{\bar{\phi}}{\sqrt{6}}} dt$. And, finally, from this Lagrangian one obtains the conjugated momenta

$$p_{\bar{V}} \equiv \frac{\partial \bar{\mathcal{L}}}{\partial \bar{V}'}, \quad p_{\bar{\phi}} \equiv \frac{\partial \bar{\mathcal{L}}}{\partial \bar{\phi}'}. \quad (11)$$

One can easily check that this transformation is canonical introducing the Poisson bracket

$$\{M, N\} \equiv (\partial_{p_V} M \partial_V N - \partial_V M \partial_{p_V} N) + (\partial_{p_R} M \partial_R N - \partial_R M \partial_{p_R} N), \quad (12)$$

a simple calculation yields

$$\{\bar{V}, p_{\bar{V}}\} = \{\bar{\phi}, p_{\bar{\phi}}\} = -1, \\ \{\bar{V}, \bar{\phi}\} = \{\bar{V}, p_{\bar{\phi}}\} = \{p_{\bar{V}}, \bar{\phi}\} = \{p_{\bar{V}}, p_{\bar{\phi}}\} = 0. \quad (13)$$

After Legendre's transformation the corresponding Hamiltonian in new variables is given by

$$\bar{\mathcal{H}}_c(\bar{V}, \bar{\phi}, p_{\bar{V}}, p_{\bar{\phi}}) = \left(-\frac{3}{4} p_{\bar{V}}^2 \bar{V} + \frac{p_{\bar{\phi}}^2}{2\bar{V}} + W(\bar{\phi}) \bar{V} \right) e^{\frac{\bar{\phi}}{\sqrt{6}}}, \quad (14)$$

where $W(\bar{\phi}) = \frac{Rf_R - f}{2f_R^2}$.

The Hamilton equations are $\dot{\mathcal{A}} = \{\bar{\mathcal{H}}_c, \mathcal{A}\}$, where $\mathcal{A} = \bar{V}, \bar{\phi}, p_{\bar{V}}$ and $p_{\bar{\phi}}$. Using the conformal time \bar{t} introduced above, the Hamilton equations become $\frac{d\mathcal{A}}{d\bar{t}} \equiv \mathcal{A}' = \{\tilde{\mathcal{H}}_c, \mathcal{A}\}$, where the Hamiltonian $\tilde{\mathcal{H}}_c$ has the more familiar form

$$\tilde{\mathcal{H}}_c(\bar{V}, \bar{\phi}, p_{\bar{V}}, p_{\bar{\phi}}) = -\frac{3}{4} p_{\bar{V}}^2 \bar{V} + \frac{p_{\bar{\phi}}^2}{2\bar{V}} + W(\bar{\phi}) \bar{V}. \quad (15)$$

In fact, this Hamiltonian corresponds to a dynamical system given by a scalar field $\bar{\phi}$ under the action of the potential $W(\bar{\phi})$ in Einstein Cosmology (EC). Introducing the energy density

$$\bar{\rho} = \frac{p_{\bar{\phi}}^2}{2\bar{V}^2} + W(\bar{\phi}), \quad (16)$$

the Hamiltonian can be written as

$$\tilde{\mathcal{H}}_c(\bar{V}, \bar{\phi}, p_{\bar{V}}, p_{\bar{\phi}}) = -\frac{3}{4} p_{\bar{V}}^2 \bar{V} + \bar{\rho} \bar{V}. \quad (17)$$

Hamiltonian (17) shows the canonical equivalence between $f(R)$ and Einstein gravity. Moreover, due to the conformal change of variable $\bar{V} = f_R^{3/2} V$ (the first component on the right-hand side of (9)), one can argue that $f(R)$ and Einstein gravity are conformally equivalent.

Note also that when $f(R) = R$, $p_{\bar{V}}$ reduces to $-2H$, and Hamiltonian (17) reduces to (7).

To end this section, a final remark is in order: Since there are infinitely many canonical transformations, this means that $f(R)$ gravity could be formulated using infinitely many sets of variables (two coordinates and their corresponding momenta). Note that some of these sets of variables will be meaningless physically speaking, because they are built using a combination of both coordinates and momenta, giving new quantities with a very difficult physical interpretation. Moreover, we will show that the introduction of holonomy effects will depend on the set of variables used, *i.e.*, there will be infinitely many ways to introduce holonomy effects in $f(R)$ gravity, and consequently, there will be infinitely many different effective holonomy-corrected Friedmann equations in $f(R)$ gravity.

Introduction of holonomy corrections. – For the flat FLRW geometry, when one deals with Einstein gravity to introduce holonomies: first of all one can consider the variable $\beta = -\frac{\gamma}{2} p_V = \gamma H$ ([9]) where γ is the Barbero-Immirzi parameter. In term of β the Hamiltonian (7) becomes $\mathcal{H}_c(V, \beta) = -\frac{3\beta^2}{\gamma^2} V$. However, in Loop Quantum Cosmology, due to the discrete nature of space, the quantum operator $\hat{\beta}$ is not well defined (see for instance [10] or [11] for a status report). Then, in order to build the quantum theory, one needs to re-define the gravitational part of the Hamiltonian. To be precise, we will consider the holonomies $h_j(\lambda) \equiv e^{-i\frac{\lambda\beta}{2}\sigma_j}$, where σ_j are the Pauli matrices and λ is the square root of the minimum eigenvalue of the area operator in Loop Quantum Gravity. Since β^2 does not have a well-defined quantum operator, to construct a consistent quantum Hamiltonian operator, one needs an almost periodic function that approaches β^2 for small values of β . This can be done using the general formulae of Loop Quantum Gravity to obtain the holonomy-corrected Hamiltonian

$$\mathcal{H}_{hc}(V, \beta) \equiv -\frac{2V}{\gamma^3 \lambda^3} \sum_{i,j,k} \varepsilon^{ijk} \text{Tr}[h_i(\lambda) h_j(\lambda) h_k^{-1}(\lambda) \times h_j^{-1}(\lambda) h_k(\lambda) \{h_k^{-1}(\lambda), V\}], \quad (18)$$

which captures the underlying loop quantum dynamics (see for instance [12,13]).

A simple calculation proves [14–16] that (18) acquires the simple form

$$\mathcal{H}_{hc}(V, \beta) = -3 \frac{\sin^2(\lambda\beta)}{\lambda^2 \gamma^2} V, \quad (19)$$

which shows that, at effective level, holonomy effects can be introduced performing the replacement $\beta \rightarrow \frac{\sin(\lambda\beta)}{\lambda}$ (equivalently, $p_V \rightarrow -\frac{2\sin(\lambda\beta)}{\lambda\gamma}$).

To obtain the holonomy-corrected Friedmann equation one has to use the full Hamiltonian

$$\mathcal{H}_{full}(V, \beta) = -3 \frac{\sin^2(\lambda\beta)}{\lambda^2 \gamma^2} V + \rho V, \quad (20)$$

to calculate the Hamilton equation

$$\dot{V} = -\frac{\gamma}{2} \frac{\partial \mathcal{H}_{full}(V, \beta)}{\partial \beta} = -3V \frac{\sin(2\lambda\beta)}{2\lambda\gamma}, \quad (21)$$

that, together with the Hamiltonian constrain $\mathcal{H}_{full}(V, \beta) = 0$, leads to the well-known modified Friedmann equation

$$H^2 = \frac{\rho}{3} \left(1 - \frac{\rho}{\rho_c} \right), \quad (22)$$

where $\rho_c \equiv \frac{3}{\lambda^2 \gamma^2}$ is the so-called *critical density*.

Holonomy-corrected $f(R)$ gravity. – To introduce the holonomy correction in general $f(R)$ we will adopt

the following recipe: In analogy with the linear case $f(R) = R$, we will replace the momentum that in the linear case corresponds to $-2\frac{\beta}{\gamma}$ by $-\frac{2\sin(\lambda\beta)}{\lambda\gamma}$. For example, if the variables (V, R, p_V, p_R) are used, we will perform in Hamiltonian (5) the replacement $p_V \rightarrow -\frac{2\sin(\lambda\beta)}{\lambda\gamma}$, and if we consider the variables $(\bar{V}, \bar{\phi}, p_{\bar{V}}, p_{\bar{\phi}})$, we will replace in the Hamiltonian (17) $p_{\bar{V}}$ by $-\frac{2\sin(\lambda\beta)}{\lambda\gamma}$, because, in both cases, when $f(R) = R$ one has $p_V = p_{\bar{V}} = -2\frac{\beta}{\gamma}$.

It is important to realize that this way to introduce holonomy corrections will depend on the set of variables used to formulate $f(R)$. We can prove it, introducing holonomy corrections in Hamiltonians (5) and (17), and showing that these corrections lead to different differential equations.

- 1) First of all we deal with the Hamiltonian (5). After the replacement one gets

$$\mathcal{H}_{hc} = \frac{2\sin(\lambda\beta)p_R}{\lambda\gamma f_{RR}} + \frac{1}{3V} \frac{f_R}{f_{RR}^2} p_R^2 + \frac{V}{2} (Rf_R - f). \quad (23)$$

The Hamilton equations

$$\begin{aligned} \dot{V} &= -\frac{\gamma}{2} \frac{\partial \mathcal{H}_{hc}}{\partial \beta} = -\frac{\cos(\lambda\beta)p_R}{f_{RR}}, \\ \dot{R} &= \frac{\partial \mathcal{H}_{hc}}{\partial p_R} = \frac{2\sin(\lambda\beta)}{\lambda\gamma f_{RR}} + \frac{2}{3V} \frac{f_R}{f_{RR}^2} p_R, \end{aligned} \quad (24)$$

together with the Hamiltonian constrain $\mathcal{H}_{hc} = 0$, have to be used to obtain a relation of the form $F(H, R, \dot{R}) = 0$ which corresponds to the modified Friedmann equation in $f(R)$ gravity containing holonomy corrections.

To do that, first of all we introduce the notation $p_R = V\bar{p}_R$. Then, eqs. (24) become

$$\sin^2(\lambda\beta) = 1 - \frac{9H^2 f_{RR}^2}{\bar{p}_R^2}, \quad (25)$$

and

$$\sin^2(\lambda\beta) = \frac{3}{4\rho_c} \left(\dot{R}^2 f_{RR}^2 + \frac{2}{3} f_R (Rf_R - f) \right), \quad (26)$$

where to obtain this last equation we have used the Hamiltonian constrain. Equalising both equations one has

$$\bar{p}_R^2 = \frac{9H^2 f_{RR}^2}{1 - \frac{3A}{4\rho_c}}, \quad (27)$$

where $A = f_{RR}^2 \dot{R}^2 + \frac{2}{3} f_R (Rf_R - f)$.

Finally, inserting (25) and (27) in the square of the Hamiltonian constrain one gets

$$\begin{aligned} 9H^2 f_{RR}^2 \dot{R}^2 \left(1 - \frac{3A}{4\rho_c} \right) &= \\ \frac{1}{4} \left[6H^2 f_R - (Rf_R - f) \left(1 - \frac{3A}{4\rho_c} \right) \right]^2. \end{aligned} \quad (28)$$

Remark: It is important to realize that, when $f(R) = R$, eq. (28) leads to eq. (22). To prove that, one only has to introduce an energy density ρ , which can be done replacing $(Rf_R - f)$ by $(Rf_R - f - 2\rho)$ in (28).

- 2) Finally, we deal with the Hamiltonian (17). After the replacement $p_{\bar{V}} \rightarrow -\frac{2\sin(\lambda\beta)}{\lambda\gamma}$ in (17), one obtains

$$\tilde{\mathcal{H}}_{hc} = -3 \frac{\sin^2(\lambda\beta)}{\lambda^2 \gamma^2} \bar{V} + \bar{\rho} \bar{V}. \quad (29)$$

Using the Hamilton equation $\bar{V}' = \{ \tilde{\mathcal{H}}_{hc}, \bar{V} \}$ and the Hamiltonian constrain $\tilde{\mathcal{H}}_{hc} = 0$ one obtains the following holonomy-corrected Friedmann equation

$$\bar{H}^2 = \frac{\bar{\rho}}{3} \left(1 - \frac{\bar{\rho}}{\rho_c} \right), \quad (30)$$

where $\bar{H} = \frac{1}{3} \frac{\bar{V}'}{\bar{V}}$ is a conformal Hubble parameter, which is exactly the same equation as (22) but with the new variables.

Using the variables (V, \dot{V}, R, \dot{R}) this equation becomes (note that $H = \frac{1}{3} \frac{\dot{V}}{V}$)

$$6f_{RR} \dot{R} H + 6H^2 f_R - (Rf_R - f) = -\frac{9A^2}{8f_R^4 \rho_c}, \quad (31)$$

which is completely different from (28), because it contains a linear term on \dot{R} . In the same way, using other different variables we will obtain different effective holonomy-corrected Friedmann equations in $f(R)$ gravity.

What is important in all of these modified Friedmann equations obtained using different variables, is that all of them are autonomous second order differential equations in H , that is, they contain H, \dot{H} and \ddot{H} , and do not contain explicitly the time. This means that the dynamical system is contained in the phase space (H, \dot{H}) . The problem with this kind of equations is that, in general, they are analytically unworkable and only numerical computations can be performed.

However, eq. (31) has an advantage with respect to all the other formulations of holonomy-corrected $f(R)$ gravity: The relation

$$\begin{aligned} H &= e^{\frac{\bar{\phi}}{\sqrt{6}}} \left(\bar{H} - \frac{\bar{\phi}'}{\sqrt{6}} \right), \\ \dot{H} &= \frac{\bar{\phi}' e^{\sqrt{\frac{2}{3}} \bar{\phi}}}{\sqrt{6}} \left(\bar{H} - \frac{\bar{\phi}'}{\sqrt{6}} \right) + e^{\sqrt{\frac{2}{3}} \bar{\phi}} \left(\bar{H}' - \frac{\bar{\phi}''}{\sqrt{6}} \right), \end{aligned} \quad (32)$$

provides a map between the planes (H, \dot{H}) and $(\bar{\phi}, \bar{\phi}')$, where, as we will show, the dynamics of the system is easier to understand. Effectively, the energy density $\bar{\rho}$ has the following form in terms of $(\bar{\phi}, \bar{\phi}')$:

$$\bar{\rho} = \frac{(\bar{\phi}')^2}{2} + W(\bar{\phi}). \quad (33)$$

Then, the holonomy-corrected Friedmann equation (30) relates \bar{H} with $(\bar{\phi}, \bar{\phi}')$, and the dependence of \bar{H}' and $\bar{\phi}''$ with respect to $(\bar{\phi}, \bar{\phi}')$ is obtained through the Raychaudhuri

$$\bar{H}' = -\frac{1}{2} \left(1 - \frac{2\bar{\rho}}{\rho_c} \right) (\bar{\phi}')^2, \quad (34)$$

and conservation equations

$$\bar{\rho}' = -3\bar{H}(\bar{\phi}')^2 \iff \bar{\phi}'' + 3\bar{H}\bar{\phi}' + \partial_{\bar{\phi}}W(\bar{\phi}) = 0. \quad (35)$$

From these equations, the second term in (32) becomes

$$\begin{aligned} \dot{H} = & \frac{\bar{\phi}' e^{\sqrt{\frac{2}{3}}\bar{\phi}}}{\sqrt{6}} \left(\bar{H} - \frac{\bar{\phi}'}{\sqrt{6}} \right) \\ & + e^{\sqrt{\frac{2}{3}}\bar{\phi}} \left(-\frac{1}{2} \left(1 - \frac{(\bar{\phi}')^2 + 2W(\bar{\phi})}{\rho_c} \right) (\bar{\phi}')^2 \right. \\ & \left. + \frac{3\bar{H}\bar{\phi}' + \partial_{\bar{\phi}}W(\bar{\phi})}{\sqrt{6}} \right), \end{aligned} \quad (36)$$

which only depends on the variables $(\bar{\phi}, \bar{\phi}')$.

Consequently, the dynamics in phase space (H, \dot{H}) is obtained from the one in the plane $(\bar{\phi}, \bar{\phi}')$. Working with these variables eq. (30) shows that the universe moves along an ellipse in the plane $(\bar{H}, \bar{\rho})$, like in standard Loop Quantum Cosmology. Moreover the conservation equation (35) shows that the movement is clockwise from the contracting ($\bar{H} < 0$) to the expanding ($\bar{H} > 0$) phase in these variables, bouncing at $\bar{\rho} = \rho_c$.

The dynamics in the phase space $(\bar{\phi}, \bar{\phi}')$ is obtained from the conservation equation (35) which in the contracting phase ($\bar{H} < 0$) is given by

$$\begin{aligned} \bar{\phi}'' - 3\sqrt{\frac{(\bar{\phi}')^2 + W(\bar{\phi})}{3}} \left(1 - \frac{(\bar{\phi}')^2 + W(\bar{\phi})}{\rho_c} \right) \bar{\phi}' \\ + \partial_{\bar{\phi}}W(\bar{\phi}) = 0, \end{aligned} \quad (37)$$

and in the expanding one ($\bar{H} > 0$) by

$$\begin{aligned} \bar{\phi}'' + 3\sqrt{\frac{(\bar{\phi}')^2 + W(\bar{\phi})}{3}} \left(1 - \frac{(\bar{\phi}')^2 + W(\bar{\phi})}{\rho_c} \right) \bar{\phi}' \\ + \partial_{\bar{\phi}}W(\bar{\phi}) = 0. \end{aligned} \quad (38)$$

Strictly speaking eqs. (37) and (38) depict two autonomous dynamical systems in the phase space $(\bar{\phi}, \bar{\phi}')$, meaning that orbits in the contracting phase ($\bar{H} < 0$) intersect with the ones in the expanding phase ($\bar{H} > 0$). The numerical integration of this system, and thus, the numerical phase portrait could be done as follows: Given an initial condition $(\bar{\phi}_0, \bar{\phi}'_0)$, one integrates forward in time eq. (37) for this initial condition. Then, the orbit could move to a critical point of the system $(\bar{\phi}_c, 0)$ where $\bar{\phi}_c$ is a solution of the equation $\partial_{\bar{\phi}}W(\bar{\phi}) = 0$, or it hits tangentially the curve $\bar{\rho} = \rho_c$ at some point $(\bar{\phi}_1, \bar{\phi}'_1)$. In the latter case one has to continue the orbit integrating forward in

time (38) for this new initial condition. In this way one obtains the phase portrait in the plane $(\bar{\phi}, \bar{\phi}')$, and from the map (32) one finally obtains the phase portrait in the plane (H, \dot{H}) . In fact, a realistic application of this method has recently been performed in [2] in order to obtain the phase portrait for $R + \alpha R^2$ Loop Quantum Cosmology.

Conclusions. – In this letter we have introduced a recipe to introduce holonomy corrections in $f(R)$ gravity. We have also showed that, depending on the canonical variables used to build the Hamiltonian, the holonomy corrections lead to different quantum theories, and consequently, to different modified Friedmann equations. This dependence is problematic, because only one of these Friedmann equations depicts correctly our Universe. Unfortunately, we do not have any objective criterion to decide which of these variables has to be used to obtain the fundamental quantum theory. Perhaps working in Einstein frame is the best choice, because in this case the Hamiltonian (17) has the same form as the Hamiltonian in Einstein gravity, what means that from the holonomy-corrected Hamiltonian (18) (where V has to be replaced by \bar{V}) the replacement $p_{\bar{V}}$ by $-\frac{2\sin(\lambda\beta)}{\lambda\gamma}$, like in standard Loop Quantum Cosmology, is completely justified. Anyway, this is a problem that deserves future investigation.

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