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PERFECT AND QUASIPERFECT DOMINATION IN TREES

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A $k-$quasiperfect dominating set of a connected graph G is a vertex subset S such that every vertex not in S is adjacent to at least one and at most k vertices in S. The cardinality of a minimum k-quasiperfect dominating set in G is denoted by $\gamma_{1k}(G)$. These graph parameters were first introduced by Chellali et al. (2013) as a generalization of both the perfect domination number $\gamma_{11}(G)$ and the domination number $\gamma(G)$. The study of the so-called quasiperfect domination chain $\gamma_{11}(G) \geq \gamma_{12}(G) \geq \cdots \geq \gamma_{1\Delta}(G) = \gamma(G)$ enable us to analyze how far minimum dominating sets are from being perfect. In this paper, we provide, for any tree T and any positive integer k, a tight upper bound of $\gamma_{1k}(T)$. We also prove that there are trees satisfying all possible equalities and inequalities in this chain. Finally a linear algorithm for computing $\gamma_{1k}(T)$ in any tree T is presented.

1. INTRODUCTION

All the graphs considered are finite, undirected, simple and connected. For undefined basic concepts we refer the reader to introductory graph theoretical literature as [**6**]. Recall that a *tree* is a connected acyclic graph. A *leaf* is a vertex of degree 1 and vertices of degree at least 2 are called *interior* vertices. A *support vertex* is a vertex having at least one leaf in its neighborhood and a *strong support vertex* is a support vertex adjacent to at least two leaves.

Given a graph G, a subset S of its vertices is a *dominating set* of G if every vertex not in S is adjacent to at least one vertex in S. The *domination number*

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$\gamma(G)$ is the minimum cardinality of a dominating set of G, and a dominating set of such cardinality is called a *γ-set* [**13**].

Different variants of domination have been considered as models of interactions among the nodes in a network. For instance consider a network in which a small set S of main processors manages all the system's resources. It may be desirable that, any other processor should be in contact (neighbor) with at least one of the main nodes in order to keep it active. At the same time, it could not be convenient to have a processor directly connected with too many main processors due to security reasons, since a failure in such processor may affect an important subset of S and therefore the integrity of the network. In this situation S must be a dominant set into the network but with the additional condition of bound the maximum number of connections (neighbors) that other processors can have in S.

An extreme way of domination occurs when every vertex not in S is adjacent to exactly one vertex in S. In that case, S is called a *perfect dominating set* [**8**]. The *pefect domination number* $\gamma_{11}(G)$ is the minimum cardinality of a perfect dominating set of G, and a dominating set of cardinality $\gamma_{11}(G)$ is called a *γ_{11}-set*. This concept has been studied in [**9, 10, 14, 15**].

In a perfect dominating set what it is gained from the point of view of perfection it is lost in size, comparing it to a minimum dominating set. Between both notions, there is a graduation of definitions given by the so-called k-quasiperfect domination. A *k-quasiperfect dominating set* [**7, 18**] is a dominating set S such that every vertex not in S is adjacent to at least one and at most k vertices of S. As above, the *k-quasiperfect domination number* $\gamma_{1k}(G)$ is the minimum cardinality of a k-quasiperfect dominating set of G and a *γ_{1k}-set* is a k-quasiperfect dominating set of cardinality $\gamma_{1k}(G)$. These parameters have recently been studied in [**2, 16**].

Given a graph G of order n and maximum degree Δ, Δ-quasiperfect dominating sets are precisely dominating sets. Thus, there exists a chain of quasiperfect domination parameters going from perfect domination to domination, that we will call the *QP-chain* of G:

$$n \geq \gamma_{11}(G) \geq \gamma_{12}(G) \geq \ldots \geq \gamma_{1\Delta}(G) = \gamma(G)$$

We began the study of the QP-chain in [**3**] and now our attention is focused on the behavior of these parameters in the particular case of trees. This paper is organized as follows. In the next Section basic and known results about quasiperfect parameters are revisited. In Section 3 we obtain a general upper bound for the quasiperfect domination numbers in terms of the domination number and we prove that it is tight. Section 4 is devoted to study the QP-chain, introducing a realization-type theorem for it. Finally, in Section 5 we provide an algorithm to compute the k-quasiperfect domination number of a tree in linear time.

2. BASIC AND GENERAL RESULTS

In this Section, we revise some known results concerning both domination and perfect domination that constitute the building blocks for the rest of results of the paper.

Let's begin recalling the family of graphs that reach the known upper bound of the domination number in terms of the order: corona graphs. The *corona* of a graph G, denoted by $\mathrm{cor}(G)$, is the graph obtained by attaching a leaf to each vertex of G.

Theorem 1. *[**12, 17**] If a graph G has order n and no isolated vertices, then $\gamma(G) \le n/2$. Moreover, for a graph G with even order n and no isolated vertices, then $\gamma(G) = n/2$ if and only if the connected components of G are either the cycle C_4 or the corona $cor(H)$ for any connected graph H.*

Graphs of odd order n and maximum domination number $\gamma(G) = \lfloor n/2 \rfloor$ are also completely characterized in [**1**], as a list of six graph classes.

On the other hand, it is clear that for every graph G of order $n \ge 3$ with n_1 vertices of degree 1, $\gamma_{11}(G) \le n - n_1$, since the set of all vertices that are no leaves is a perfect dominating set. This property leads to the following observations for trees.

Remark 1. *If T is a tree of order at least 3, then there exists a γ-set containing no leaves, since the set obtained by removing a leaf and adding its support vertex, if necessary, is also a dominating set.*

Remark 2. *Any γ-set of a tree contains all its strong support vertices. Assume, on the contrary, that v is a strong support vertex not of a γ-set S. Then, S must contain at least two leaves, x and y, adjacent to v, implying that the set $(S \setminus \{x, y\}) \cup \{v\}$ is a dominating set with less vertices than S, a contradiction.*

Similar results are known for the perfect domination number of trees.

Proposition 1. *[**4**] Let T be a tree of order $n \ge 3$. Then,*

1. *every γ_{11}-set of T contains all its strong support vertices.*
2. *$\gamma_{11}(T) \le n/2$.*
3. *$\gamma_{11}(T) = n/2$ if and only if $T = cor(T')$, for some tree T'.*

The following corollary is a consequence of the preceding results.

Corollary 1. *If T is a tree of order $n \ge 3$, then the following conditions are equivalent:*

1. *$\gamma(T) = n/2$.*
2. *$\gamma_{11}(T) = n/2$.*
3. *$T = cor(T')$, for some tree T'.*

This corollary shows that the QP-chain adopts its shortest form in graphs which are the corona of a tree. For instance, the comb graph $\mathrm{cor}(P_m)$, which is, for every integer $m \geq 3$, the corona of the path P_m, satisfies that $n/2 = m = \gamma_{11}(\mathrm{cor}(P_m)) = \gamma(\mathrm{cor}(P_m))$. There are other simple tree families having a constant QP-chain. For instance, if P_n and $K_{1,n-1}$ denote the path and the star of order n, respectively, then

$$\gamma_{11}(P_n) = \gamma_{12}(P_n) = \gamma(P_n) = \lceil n/3 \rceil,$$
$$\gamma_{11}(K_{1,n-1}) = \gamma_{12}(K_{1,n-1}) = \cdots = \gamma_{1,n-1}(K_{1,n-1}) = \gamma(K_{1,n-1}) = 1.$$

Finally, recall that a *caterpillar* is a tree that has a dominating path. This special class of trees has a particular behavior regarding the QP-chain.

Proposition 2. *[7] If T is a caterpillar, then $\gamma(T) = \gamma_{12}(T)$.*

3. BOUNDS FOR QUASIPERFECT DOMINATION IN TREES

Although the QP-chain provides natural bounds for the quasiperfect domination numbers, it is not a surprise that for the case of trees it is possible to get better bounds in terms of the domination number. This section is devoted to obtain them and to prove that they are tight, characterizing the family of trees that attain it.

3.1 General upper bound

The QP-chain shows that the domination number $\gamma(T)$ of a tree T is the natural lower bound, for every positive integer k, of the quasiperfect domination number $\gamma_{1k}(T)$. Furthermore, this bound is reached, as commented in the previous section, for instance, when T is either a path or a star. Our main result in this subsection provides an upper bound of the quasiperfect domination numbers of a tree in terms of its domination number.

For a tree T and a subset $S \subseteq V(T)$, we denote by $T[S]$ the subgraph of T induced by the vertices of S.

Lemma 1. *Let T be a tree and let S be a dominating set of T. Then, every vertex not in S has at most one neighbor at each connected component of the subgraph $T[S]$.*

Proof. If a vertex not in S has two neighbors in a connected component of $T[S]$, then T has a cycle, a contradiction. □

As a consequence, the following result is obtained:

Corollary 2. *Let T be a tree and S a dominating set of T such that the subgraph $T[S]$ has at most k connected components. Then S is a k-quasiperfect dominating set.*

Theorem 2. *For every tree T and for every integer $k \geq 1$,*

$$\gamma_{1k}(T) \leq \gamma(T) + \left\lceil \frac{\gamma(T)}{k} \right\rceil - 1 \tag{1}$$

and this bound is tight.

Proof. Let S be a γ-set of T. If S is a k-quasiperfect dominating set, then inequality (1) trivially holds.

Suppose, on the contrary, that S is not a k-quasiperfect dominating set. We intend to construct a k-quasiperfect dominating set S^* containing S and satisfying the desired inequality. Let r be the number of connected components of the subgraph $T[S]$. Then, $\gamma(T) \geq r$ and, by Corollary 2, $r > k$.

Consider a vertex $x_0 \in V(T) \setminus S$ with at least $k+1$ neighbors in S and let $S_1 = S \cup \{x_0\}$. By Lemma 1, all the neighbors of x_0 in S lie in different connected components of $T[S]$. Therefore S_1 is a dominating set inducing a subgraph $T[S_1]$ with at most $r-k$ connected components. If S_1 is a k-quasiperfect dominating set, let $S^* = S_1$. Otherwise, consider a vertex $x_1 \in V(T) \setminus S_1$ having at least $k+1$ neighbors in S_1 and let $S_2 = S_1 \cup \{x_1\}$. Again all the neighbors of x_1 in S_1 lie in different connected components of $T[S_1]$, and thus S_2 is a dominating set inducing a subgraph $T[S_2]$ with at most $(r-k)-k = r-2k$ connected components. If S_2 is a k-quasiperfect dominating set, let $S^* = S_2$.

Observe that this proceeding will end since $T[S_i]$ has at most $r - ik$ connected components, and this number sequence is strictly decreasing. In the worst case, you should consider $j = \lceil \frac{r-k}{k} \rceil$ with S_j having at most $r - jk$ connected components because in this case $r - jk \leq k$ and S_j must be a k-quasiperfect dominating set. So $|S^*| \leq |S| + j = \gamma(T) + \lceil \frac{r-k}{k} \rceil$ and

$$\gamma_{1k}(T) \leq |S^*| \leq |S| + j = \gamma(T) + \left\lceil \frac{r-k}{k} \right\rceil \leq \gamma(T) + \left\lceil \frac{\gamma(T)-k}{k} \right\rceil = \gamma(T) + \left\lceil \frac{\gamma(T)}{k} \right\rceil - 1.$$

We finally show the tightness of the bound. Notice that if $k \geq \gamma(T)$, then $\gamma_{1k}(T) = \gamma(T)$ and $\left\lceil \frac{\gamma(T)}{k} \right\rceil = 1$, so in this case $\gamma_{1k}(T) = \gamma(T) + \left\lceil \frac{\gamma(T)}{k} \right\rceil - 1$.

Next, suppose that $\gamma(T) = a$, $a \geq 2$ and $k < a$. Consider the graph in Figure 1, where $a = q \cdot k + r$, $q \geq 1, 1 \leq r \leq k$. It is clear that the set of squared vertices is a γ-set, so $\gamma(T) = a$, and that the set of black vertices is a γ_{1k}-set, so

$$\gamma(T) + \left\lceil \frac{\gamma(T)}{k} \right\rceil - 1 = a + \left\lceil \frac{qk+r}{k} \right\rceil - 1 = a + q + 1 - 1 = qk + r + q = \gamma_{1k}(T). \quad \square$$

3.2 Trees satisfying $\gamma_{11}(T) = 2\gamma(T) - 1$

In the particular case of the perfect domination number, the upper bound shown in Theorem 2 is the following

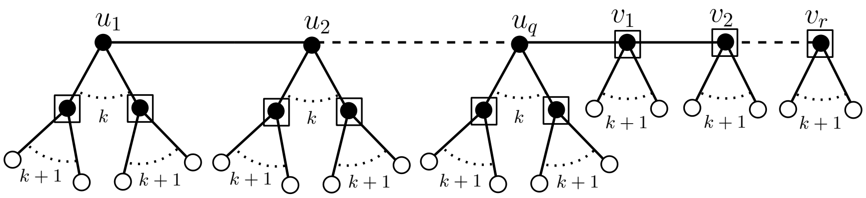

Figure 1: Squared vertices are a γ-set and black vertices are a γ_{1k}-set

$$\gamma_{11}(T) \leq \gamma(T) + \left\lceil \frac{\gamma(T)}{1} \right\rceil - 1 = 2\gamma(T) - 1$$

Notice that this bound is far from being true for general graphs and, as a matter of fact, the difference between both parameters can be as large as desired. For instance, the graph shown in Figure 2 satisfies $\gamma(G) = 2$ and $\gamma_{11}(G) = |V(G)| > 2\gamma(G) - 1$.

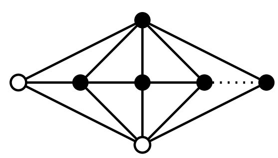

Figure 2: The pair of white vertices form a γ-set meanwhile $\gamma_{11}(G) = |V(G)|$.

Let T be a tree satisfying $\gamma_{11}(T) = 2\gamma(T) - 1$. Then, for any γ-set S of T, the associated perfect dominating set S^* constructed in Theorem 2 satisfies $|S^*| = 2\gamma(T) - 1$, so it is also a γ_{11}-set. However, some trees contain γ_{11}-sets which can not be obtained from this construction. For instance, the tree shown in Figure 3 has a γ_{11}-set which does not contain any γ-set.

Our next goal is to characterize the family of trees achieving this bound. To this end, we review the construction of the perfect dominating set associated with a γ-set given in Theorem 2. Let S be a γ-set of a tree T which is not a γ_{11}-set. Notice that since S is not a perfect dominating set, there exists at least one vertex $x \notin S$ that is not a leaf. Denote by $C_1, \ldots, C_k$, $k \geq 1$, the connected components of the graph $T - (S \cup L')$, where L' is the set of leaves of T not in S such that for some $r \in \{1, \ldots, k\}$, at least one vertex in each $C_1, \ldots, C_r$ has two or more neighbors in S and vertices in $C_{r+1}, \ldots, C_k$ (if $r < k$) have exactly one neighbor in S. In Proposition below, we follow this notation and a precise description of the perfect dominating set S^* associated with S is provided.

Proposition 3. *Let S be a γ-set of a tree T which is not a perfect dominating set. Then, $S^* = S \cup (\bigcup_{i=1}^{r} V(C_i))$ has the following properties.*

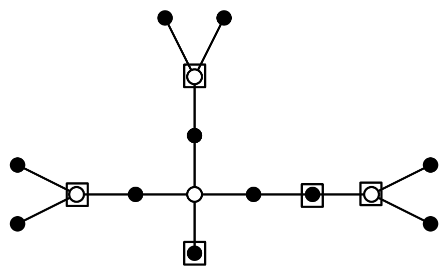

Figure 3: Squared vertices form the unique γ_{11}-set and they do not contain the unique γ-set consisting on white vertices.

1. S^* is a perfect dominating set of T.

2. S^* has at most $2\gamma(T)-1$ vertices.

3. If S' is a perfect dominating set of T containing S, then $S^* \subseteq S'$.

Proof. 1. Let $u \in V(T) \setminus S^*$. If u is a leaf, then it has just one neighbor in S^*. Suppose now that $u \notin S^* \cup L'$. Then, there exists $i \in \{r+1, \ldots k\}$ such that $u \in V(C_i)$ and it has just one neighbor in S. Using that the connected components are pairwise disjoint, it is clear that u has exactly one neighbor in S^*, as desired.

2. Consider the tree $T-L'$. By construction, $V(T-L') = S \cup V(C_1) \cup \cdots \cup V(C_k)$, where $S, V(C_1), \ldots, V(C_k)$ are pairwise disjoint sets. Therefore,

$$|E(T-L')| = |V(T-L')| - 1 = |S| + \sum_{i=1}^{k} |V(C_i)| - 1. \tag{2}$$

Now, observe that the edges of $T-L'$ connect two vertices of one of the connected components C_i, or two vertices of the γ-set S, or a vertex of S with a vertex of some C_i. For any pair of subsets of vertices A, B, let us denote $E(A:B)$ the set of edges with an endpoint in A and the other one in B. With this notation, we have that:

$$E(T-L') = \Big(\bigcup_{i=1}^{k} E(C_i:S)\Big) \cup \Big(\bigcup_{i=1}^{k} E(C_i:C_i)\Big) \cup E(S:S).$$

Moreover, the $2k+1$ subsets involved in this union are pairwise disjoint.

By hypotheses, $|E(C_i:S)| = |V(C_i)| + \delta_i$, where $\delta_i \geq 1$ for all $i \in \{1, \ldots, r\}$, and $|E(C_i:S)| = |V(C_i)|$ for all $i \in \{r+1, \ldots, k\}$. On the other hand,

$|E(C_i : C_i)| = |V(C_i)| - 1$ for all $i \in \{1, \ldots, k\}$, using that each C_i is a tree. From these observations we obtain

(3)

$$\begin{aligned}
|E(T - L')| &= \sum_{i=1}^{k} |E(C_i : S)| + \sum_{i=1}^{k} |E(C_i : C_i)| + |E(S : S)| \\
&= \sum_{i=1}^{k} (|V(C_i)| - 1) + \sum_{i=1}^{r} |E(C_i : S)| + \sum_{i=r+1}^{k} |E(C_i : S)| + |E(S : S)| \\
&= \sum_{i=1}^{k} |V(C_i)| - k + \sum_{i=1}^{r} (|V(C_i)| + \delta_i) + \sum_{i=r+1}^{k} |V(C_i)| + |E(S : S)| \\
&= \sum_{i=1}^{k} |V(C_i)| - k + \sum_{i=1}^{r} |V(C_i)| + \sum_{i=1}^{r} \delta_i + \sum_{i=r+1}^{k} |V(C_i)| + |E(S : S)| \\
&= \sum_{i=1}^{k} |V(C_i)| - k + |S^*| - |S| + \sum_{i=1}^{r} \delta_i + \sum_{i=r+1}^{k} |V(C_i)| + |E(S : S)|
\end{aligned}$$

From Equations 2 and 3 and using that $|V(C_i)| \geq 2$ for all $i \in \{r+1, \ldots, k\}$, because as otherwise the unique vertex in C_i would be a leaf, we obtain

$$\begin{aligned}
2|S| - 1 = & |S^*| + \sum_{i=1}^{r} \delta_i + \sum_{i=r+1}^{k} |V(C_i)| - k + |E(S : S)| \\
& \geq |S^*| + r + 2(k - r) - k \\
& = |S^*| + k - r \\
& \geq |S^*|.
\end{aligned}$$

3. Let S' be a perfect dominating set of T containing S and suppose, on the contrary, that $V(C_i) \setminus S' \neq \emptyset$ for some $i \in \{1, \ldots, r\}$. Let $v \in V(C_i) \setminus S'$ and let $u_i \in V(C_i)$ be a vertex with at least two neighbors in S. It is clear that $u_i \in S'$. Consider a $u_i - v$ path P in C_i and let w be the first vertex of the path not in S'. Then, w has at least one neighbor in $S \subseteq S'$ and a neighbor in $S' \cap V(P) \subseteq S' \setminus S$, contradicting the fact that S' is a perfect dominating set.

□

Next, we present some properties involving γ-sets and its associated perfect dominating sets, when the upper bound is reached. For a vertex set C, we denote by $N(C)$ the set of all neighbors of the vertices of C. We also denote by L, the set of leaves of T.

Lemma 2. *Let T be a tree such that $\gamma_{11}(T) = 2\gamma(T) - 1$. Let S be a γ-set of T and let L' be the set of leaves not in S. Then,* *1. S is an independent set and every connected component C_i of $T-(S\cup L')$ satisfies $|N(V(C_i))\cap S| = |V(C_i)|+1$.*

2. $S^* = V(T)\setminus L'$. Moreover, if S does not contain leaves, then $S^* = V(T)\setminus L$.

Proof. 1. If $\gamma_{11}(T) = 2\gamma(T)-1$, then S^* is a γ_{11}-set and $|S^*| = 2|S|-1$. From Equation 2, we deduce that $|E(S:S)| = 0$, $r = k$ and $\delta_i = 1$, for every $i\in\{1,\dots,r\}$. Therefore, S is an independent set and for any $i\in\{1,\dots,k\}$, $|E(C_i:S)| = |V(C_i)|+\delta_i = |V(C_i)|+1$. Since two different vertices of the same connected component C_i have no common neighbor in S, we obtain that $|N(V(C_i))\cap S| = |E(C_i:S)| = |V(C_i)|+1$.

2. It is a direct consequence of both the construction of S^* and the preceding item.

□

Remark 3. *Condition 1 in the Lemma 2 means that there exists exactly one vertex in each connected component C_i with exactly two neighbors in S and the rest of vertices of C_i have an unique neighbor in S.*

We need also some properties of the set of support vertices on trees reaching the upper bound.

Lemma 3. *Let T be a tree such that $\gamma_{11}(T) = 2\gamma(T)-1$. Then,*

1. the set of support vertices of G is a dominating set.

2. Every support vertex of G is a strong support vertex. Moreover, the set of strong support vertices is the unique γ-set of T.

Proof. 1. Let S be a γ-set of T containing all support vertices and assume, on the contrary, that there exists $v\in S$ such that v is not a support vertex. By hypothesis, and according to Lemma 2, the perfect dominating set associated with S is $S^* = V(T)\setminus L$, with $|S^*| = 2\gamma(T)-1$, so it is also a γ_{11}-set. We are going to construct a smaller perfect dominating set of T, leading a contradiction.

Denote by $N(v) = \{u_1,\dots,u_s\}$, $s\geq 2$, the set of neighbors of v. Observe that S is an independent set, so $N(v)\cap S = \emptyset$. Denote by D_i the connected component of $T-S$ containing u_i, $i\in\{1,\dots,s\}$. Firstly, suppose that each u_i has exactly two neighbors in S (see Figure 4(a)). Notice that one of them is the vertex v and that D_i contains no leaves of T. We define $R = (S^*\setminus\{v\})\setminus\big(\bigcup_{i=2}^{s} D_i\big)$ (see Figure 4(b)), so $V(T)\setminus R = L\cup\big(\bigcup_{i=2}^{s} D_i\big)\cup\{v\}$. Note that any leaf has one neighbor in R, also the unique neighbor of v in R is u_1 and any vertex in D_i, $i\in\{2,\dots,s\}$ is dominated by exactly one vertex in R. So R is a perfect dominating set of T, with smaller cardinality than S^*, a contradiction.

Hence, assume that there exists an integer $t \in \{1, \dots, s\}$, in such a way that that vertices $u_1, \dots, u_t$ have exactly one neighbor in S, that must be thus v, and that vertices $u_{t+1}, \dots, u_s$ have two neighbors in S (see Figure 5(a)). Using that u_i is not a leaf and condition 1 in Lemma 2, we denote by D_i^*, $i \in \{1, \dots, t\}$ the connected component of $D_i - \{u_i\}$ containing the unique vertex of D_i with two neighbors in S and let $\widehat{D_i} = D_i - D_i^*$. Then, $R = (S^* \setminus \{v\}) \setminus \big((\bigcup_{i=2}^t \widehat{D_i}) \bigcup (\bigcup_{j=t+1}^s D_j)\big)$ (see Figure 5(b)) is a perfect dominating set of T, with smaller cardinality than S^*, again a contradiction.

2. Let S be the γ-set of T consisting on all support vertices and assume, on the contrary, that there exists $v \in S$ which is not a strong support vertex. Again the associated perfect dominating set satisfies $S^* = V(T) \setminus L$, with $|S^*| = 2\gamma(T) - 1$.

   Denote by $N(v) = \{u_1, \dots, u_s\}$, $s \geq 2$, the set of neighbors of v, where u_1 is the unique neighbor that is a leaf, and by D_i the connected component of $T - S$ containing u_i, $i \in \{2, \dots, s\}$. We repeat the construction above, so firstly suppose that each $u_i, i \in \{2, \dots, s\}$ has exactly two neighbors in S. Then, $R = \big((S^* \cup \{u_1\}) \setminus \{v\}\big) \setminus \bigcup_{i=2}^s D_i$ is a perfect dominating set of T, with smaller cardinality than S^*, a contradiction.

   Hence, assume that there exists an integer $t \in \{2, \dots, s\}$, such that vertices $u_2, \dots, u_t$ have exactly one neighbor in S, that must be vertex v, and vertices $u_{t+1}, \dots, u_s$ has two neighbors in S. We use the same notation as above, so the set $R = \big((S^* \cup \{u_1\}) \setminus \{v\}\big) \setminus \big((\bigcup_{i=2}^t \widehat{D_i}) \bigcup (\bigcup_{j=t+1}^s D_j)\big)$ is a perfect dominating set of T, with smaller cardinality than S^*, again a contradiction. □

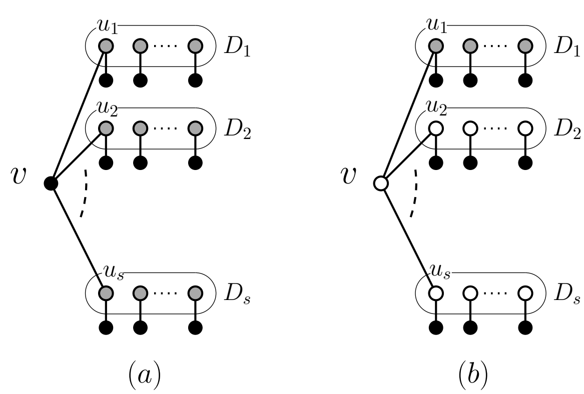

Figure 4: (a) Black vertices are in S. Black and gray vertices are in S^*. (b) There is a perfect dominating set not containing white vertices.

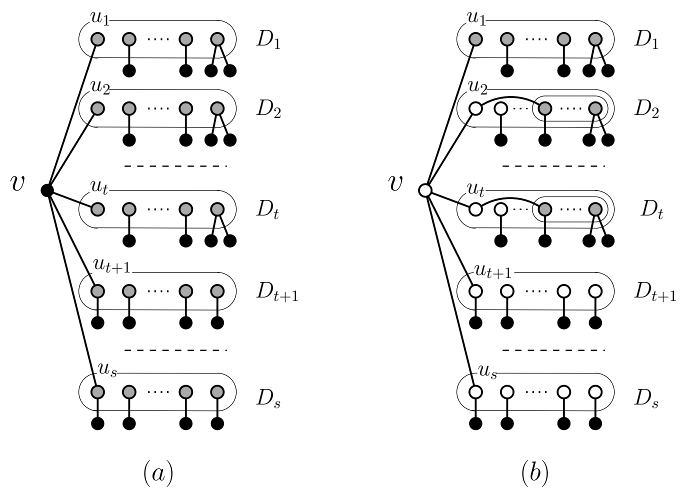

Figure 5: (a) Black vertices are in S. Black and gray vertices are in S^*. (b) There is a perfect dominating set not containing white vertices.

Now, we can characterize trees achieving the upper bound stated in Theorem 2, in the case of perfect domination.

Theorem 3. *Let T be a tree. Then, $\gamma_{11}(T) = 2\gamma(T) - 1$ if and only if the following conditions hold:*

1. *the set S of strong support vertices of T is an independent dominating set,*
2. *any connected component C of $T - (S \cup L)$ satisfies $|N(V(C)) \cap S| = |V(C)| + 1$, that is, every vertex in C has exactly one neighbor in S except one vertex that has two neighbors in S.*

Proof. If $\gamma_{11}(T) = 2\gamma(T) - 1$, then, according to Lemma 2 and Lemma 3, it is clear that T satisfies both conditions.

On the other hand, suppose that T satisfies conditions 1 and 2. Note that the set S of strong support vertices is the unique γ-set of T. Moreover, S and thus also its associated perfect dominating sets S^*, are contained in any perfect dominating set of T, so S^* is the unique γ_{11}-set of T. By hypothesis, S is an independent set, so $E(S : S) = 0$, and also any connected component of $T \setminus (S \cup L)$ has a unique vertex with two neighbors in S, so $r = k$ and $\delta_i = 1$, $i \in \{1, \ldots, r\}$. Finally, using Equation 2 we obtain

$$2|S| - 1 = |S^*| + \sum_{i=1}^{r} \delta_i + \sum_{i=r+1}^{k} |V(C_i)| - k + |E(S : S)| = |S^*| + r - r + 0 = |S^*|$$

and $2\gamma(T) - 1 = \gamma_{11}(T)$, as desired. □

3.3 Realization result

A realization theorem for the short QP-chain $\gamma(T) \le \gamma_{11}(T) \le 2\gamma(T) - 1$ is presented. Note that, for every tree T of order $n \ge 3$, Proposition 1 and Theorem 2 give us two possible situations $\gamma(T) = \gamma_{11}(T) \le n/2$ or $\gamma(T) < \gamma_{11}(T) < n/2$. We show that both of them are feasible and both parameters, γ and γ_{11} can take every possible value in each case.

Proposition 4. *1. Let a, n be integers such that $1 \le a$ and $n \ge 2a$. Then, there exists a caterpillar T of order n such that $\gamma(T) = \gamma_{11}(T) = a$.*

2. Let a, b, n be integers such that $2 \le a < b \le 2a - 1$ and $n > 2b$. Then, there exists a caterpillar T of order n such that $\gamma(T) = a$ and $\gamma_{11}(T) = b$.

Proof. 1. Consider the caterpillar obtained by attaching a leaf to each of the first $a - 1$ vertices of a path of order a and $n - 2a + 1 \ge 1$ leaves to the last vertex of the path (see Figure 6). Then, the set of vertices of the path is both a γ-set and a γ_{11}-set, and $\gamma(T) = \gamma_{11}(T) = a$.

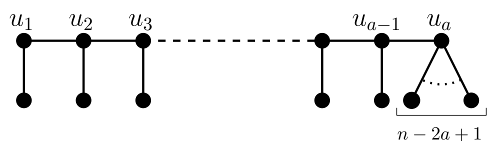

Figure 6: T has order n, $\gamma(T) = \gamma_{11}(T) = a$.

2. Note that $\gamma(T) = 1$ implies $\gamma_{11}(T) = 1$, so if both parameters do not agree, then $\gamma(T) \ge 2$.

Using that $1 \le b - a \le a - 1$, let P be the path of order b with consecutive vertices labeled with

$$u_1, v_1, \ldots, u_{b-a}, v_{b-a}, u_{b-a+1}, u_{b-a+2}, \ldots, u_a$$

and consider the caterpillar obtained by attaching two leaves to each of the vertices $u_1, u_2, \ldots, u_{b-a}$, one leaf to each of the vertices $u_{b-a+2}, u_{b-a+3}, \ldots, u_a$ and $n - 2b + 1$ leaves to vertex u_{b-a+1} (see Figure 7). Since $n - 2b + 1 \ge 2$ we obtain that $\{u_1, u_2, \ldots, u_a\}$ is a γ-set with a vertices and $\{u_1, u_2, \ldots, u_a\} \cup \{v_1, \ldots, v_{b-a}\}$ is a γ_{11}-set with b vertices.

□

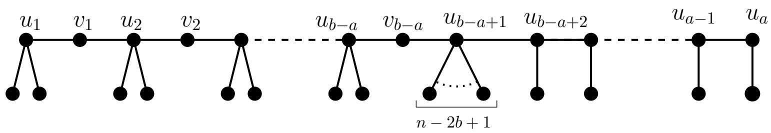

Figure 7: T has order $n > 2b$, $a = \gamma(T) < \gamma_{11}(T) = b \leq 2a - 1$.

4. REALIZATION OF THE QP-CHAIN

In this Section, we present a general realization theorem for the QP-chain. It is interesting to notice that any feasible relationship among the quasiperfect parameters can be achieved with a tree. We begin with some previous technical results.

Lemma 4. *If u is a vertex of a graph G with at least d leaves in its neighborhood, then u is in every h-quasiperfect dominating set, for any $h \in \{1, \ldots, d-1\}$.*

Proof. Let $h \in \{1, \ldots, d-1\}$ be and let S be a h-quasiperfect dominating set of G such that $u \notin S$. Then, every leaf adjacent to u must be in S, so u has at least d neighbors in S, with $d > h$, a contradiction. □

Corollary 3. *If G is a graph with maximum degree Δ and u is a vertex with at least $\Delta - 1$ leaves in its neighborhood, then u is in every γ_{1h}-set, for any $h \in \{1, \ldots, \Delta - 2\}$.*

The following Lemma is trivial.

Lemma 5. *Let T be a tree with maximum degree Δ and s support vertices. Then, $\gamma_{1\Delta}(T) = \gamma(T) \geq s$.*

Let T be a tree with maximum degree $\Delta \geq 3$. The next theorem shows that for each inequality of the QP-chain both possibilities, the equality and the strict inequality, are feasible.

Theorem 4. *For any $\Delta \geq 3$, there exists a tree T with maximum degree Δ satisfying each one of the $2^{\Delta-1}$ possible combinations of the inequalities of the QP-chain*

$$\gamma_{11}(T) \geq \gamma_{12}(T) \geq \gamma_{13}(T) \geq \ldots \geq \gamma_{1(\Delta-1)}(T) \geq \gamma_{1\Delta}(T) = \gamma(T)$$

Proof. Throughout this proof, the symbol $\circledast_i$ denotes, either '=' or '>' in $\gamma_{1i}(T) \geq \gamma_{1(i+1)}(T)$, for every $i \in \{1, \ldots, \Delta - 1\}$.

Case 1. If $\circledast_i$ is '=' for all $i \in \{1, \ldots, \Delta - 2\}$. We distinguish two subcases.

Case 1.1. If $\circledast_{\Delta-1}$ is '='. The star $T = K_{1,\Delta}$ is a tree with maximum degree Δ satisfying:

$$\gamma_{11}(T) = \gamma_{12}(T) = \ldots = \gamma_{1(\Delta-1)}(T) = \gamma_{1\Delta}(T) = \gamma(T) = 1.$$

Case 1.2. If $\circledast_{\Delta-1}$ is '>'. We consider the tree T displayed in Figure 8. It is easily derived from Corollary 3 that $\{x_1,\ldots,x_\Delta\}$ is a γ-set and $\{u,x_1,\ldots,x_\Delta\}$ is a γ_{1i}-set, for any i such that $i<\Delta$. Therefore,

$$\Delta+1=\gamma_{11}(T) = \gamma_{12}(T) = \ldots = \gamma_{1(\Delta-1)}(T) > \gamma_{1\Delta}(T)=\gamma(T)=\Delta.$$

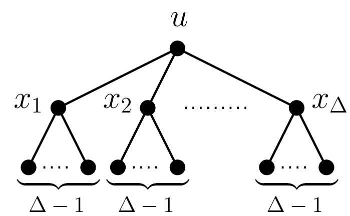

Figure 8: Trees illustrating Case 1.2 of Theorem 4.

Case 2. If $\circledast_i$ is '>' for some $i\in\{1,\ldots,\Delta-2\}$.

If $\Delta=3$, then consider the graphs shown in Figure 9. The tree T on the left side satisfies $6=\gamma_{11}(T) > \gamma_{12}(T) = \gamma_{1,3}(T)=\gamma(T)=4$, since support vertices form a γ-set (and also a γ_{12}-set and a γ_{13}-set), and all vertices but the leaves form a γ_{11}-set. The tree T on the right side satisfies $\gamma_{11}(T)=18 > \gamma_{12}(T)=12 > \gamma_{1,3}(T)=\gamma(T)=11$, since support vertices together with vertex u form a γ-set (and also a γ_{13}-set), support vertices together with vertices u and v form a γ_{12}-set, and all vertices but the leaves form a γ_{11}-set.

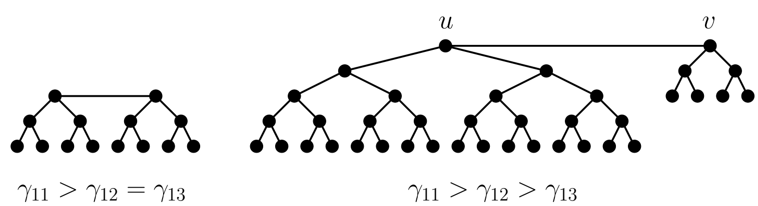

Figure 9: Trees illustrating Case 2 of Theorem 4 when $\Delta=3$.

Suppose now that $\Delta\geq 4$. Let

$$\{i_1,i_2,\ldots,i_k\}=\{j\,:\,\gamma_{1j}(T)>\gamma_{1(j+1)}(T)\,,\,j\leq\Delta-2\},$$

where $k\geq 1$ by hypotheses, and assume that $1\leq i_1<\cdots<i_k\leq\Delta-2$. We distinguish two subcases.

Case 2.1. $\circledast_{\Delta-1}$ is '='.

Consider a path P of length $k+2$ with consecutive vertices labeled $u_{i_1}, \ldots, u_{i_k}, v, w$. Attach i_j new vertices to u_{i_j} and $\Delta - 1$ leaves to each one of those new vertices. Attach also $\Delta - 2$ leaves to vertex v (see Figure 10, above).

For each vertex x of the path P, let $N'(x)$ be the set of vertices of $N(x)$ not belonging to the path P. Let $A = \cup_{j=1}^{k} N'(u_{i_j})$.

It is clear that $A \cup \{v\}$ is a γ-set of T, and also a $\gamma_{1(\Delta-1)}$-set. Moreover, $A \cup \{v\} \cup \{u_{i_j} : h \leq j \leq k\}$ is a γ_{1i}-set if $i_{h-1} < i \leq i_h$.

Case 2.2. $\circledast_{\Delta-1}$ is '>'.

Consider the tree constructed in Case 2.1. and attach $\Delta - 1$ new vertices to w and $\Delta - 1$ leaves to each one of those new vertices (see Figure 10, below).

With the same notations as in Case 2.1., it is easy to verify that $A \cup \{v\} \cup N'(w)$ is a γ-set of T and $A \cup \{v, w\} \cup N'(w)$ is a $\gamma_{1(\Delta-1)}$-set. Moreover, $A \cup \{v, w\} \cup N'(w) \cup \{u_{i_j} : h \leq j \leq k\}$ is a γ_{1i}-set if $i_{h-1} < i \leq i_h$.

□

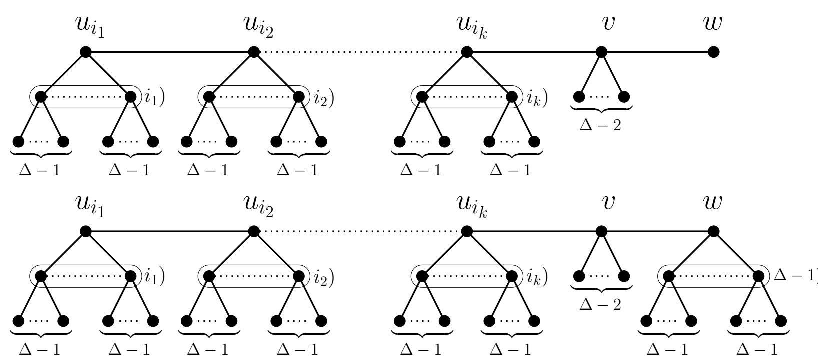

Figure 10: Trees illustrating Case 2.1. (above) and Case 2.2. (below) of Theorem 4.

5. A LINEAR ALGORITHM FOR TREES

The objective of this section is to devise a linear algorithm for computing $\gamma_{1k}(T)$ for a tree T, which answers a question posed in [**7**], where authors show that the decision problem of determining whether a graph has a 2-quasiperfect dominating set of cardinality at most r is NP-complete for bipartite graphs. Moreover, in [**5**] it is shown that the same problem for perfect dominating sets is also NP-complete. We will follow the ideas of dynamic programming which appear in [**11**], where an algorithm to compute the nearly perfect number of a tree in linear time is given.

We will use an operation on rooted trees called *composition*. The composition $(T_1, r_1) \circ (T_2, r_2)$ of two rooted trees is defined as the tree (T, r_1) where $V(T) = V(T_1) \cup V(T_2)$, $E(T) = E(T_1) \cup E(T_2) \cup \{r_1 r_2\}$ and its root is r_1. The class of rooted trees can be constructed by using this operation and K_1 as initial rooted tree with its unique vertex as root.

Let (T, r) be a rooted tree and S a subset of vertices. For a fixed positive integer k, we give the next definitions:

- $S \in A$ if S is a k-quasiperfect dominating set of T, $r \notin S$ and $|N(r) \cap S| = k$.
- $S \in B$ if S is a k-quasiperfect dominating set of T, $r \notin S$ and $|N(r) \cap S| \leq k-1$.
- $S \in C$ if S is a k-quasiperfect dominating set of T and $r \in S$.
- Finally, $S \in D$ if S is a k-quasiperfect dominating set of $T - r$ and $N[r] \cap S = \emptyset$.

Clearly any k-quasiperfect dominating set of T belongs to just one of types A, B or C. The key point in the algorithm is that all the sets of one type can be built in a bottom up form using sets of the above types, which is proved by the next results.

Proposition 5. *Let $(T, r) = (T_1, r) \circ (T_2, r_2)$ be a rooted tree which is the composition of two rooted trees, and let S be a k-quasiperfect dominating set of T. We denote $S_1 = S \cap V(T_1)$ and $S_2 \cap V(T_2)$. Then,*

1. *$S \in A$ if and only if one of the following conditions holds:*
   - *(a) $|N(r) \cap S_1| = k$, $S_1 \in A$ and $S_2 \in A \cup B$,*
   - *(b) $|N(r) \cap S_1| = k - 1$, $S_1 \in B$ and $S_2 \in C$.*
2. *$S \in B$ if and only if one of the following conditions holds:*
   - *(a) $|N(r) \cap S_1| = k - 1$, $S_1 \in B$ and $S_2 \in A \cup B$,*
   - *(b) $|N(r) \cap S_1| \leq k - 2$, $S_1 \in B$ and $S_2 \in A \cup B \cup C$,*
   - *(c) $|N(r) \cap S_1| \leq k - 2$, $S_1 \in D$ and $S_2 \in C$.*
3. *$S \in C$ if and only if $S_1 \in C$ and $S_2 \in B \cup C \cup D$.*

Proof. Since the necessity is clear, we prove only the sufficiency.

1. (a) Assume that $|N(r) \cap S_1| = k$ hence $r_2 \notin S$ and let $S \in A$, hence $r \notin S$. Then, the edge rr_2 joins two vertices not in S and thus S_1 and S_2 are k-quasiperfect dominating sets of T_1 and T_2, respectively. If S_2 is a k-quasiperfect dominating set where $r_2 \notin S_2$, then $S_2 \in A \cup B$. On the other hand, all the neighbors of r in S belong to S_1, hence $|N(r) \cap S_1| = |N(r) \cap S| = k$ and so $S_1 \in A$.

(b) Now, assume that $|N(r) \cap S_1| = k - 1$. Since $S \in A$, the root r has k neighbors in S, so it follows that $r_2 \in S$. Note that all dominations in $V(T_2) \setminus S_2$ are exactly the same as in $V(T_2) \setminus S$, so S_2 is a k-quasiperfect dominating set of T_2 and therefore $S_2 \in C$. On the other hand, although r is not dominated by r_2 in T_1, it has $k-1$ neighbors in S_1, so S_1 is a k-quasiperfect dominating set of T_1 and hence $S_1 \in B$.

2. (a) Suppose that $|N(r) \cap S_1| = k - 1$ and $S \in B$, so the $k-1$ neighbors of r belong to S_1 and thus $r_2 \notin S_2$. Consequently, both S_1 and S_2 are k-quasiperfect dominating set of T_1 and T_2 respectively. Hence, $S_1 \in B$ and $S_2 \in A \cup B$.

   (b) Let $|N(r) \cap S_1| \leq k - 2$ and assume that $1 \leq |N(r) \cap S_1|$. Then, S_1 is a k-quasiperfect dominating set of T_1 and thus $S_1 \in B$. Clearly, S_2 is a k-quasiperfect dominating set of T_2, so $S_2 \in A \cup B \cup C$.

   (c) Now, if $|N(r) \cap S_1| \leq k-2$ and $|N(r) \cap S_1| = 0$, then S_1 is a k-quasiperfect dominating set of $T_1 - r$, so $S_1 \in D$. In this case, $r_2 \in S_2$ and S_2 is a k-quasiperfect dominating set of T_2 with $S_2 \in C$.

3. Let $S \in C$. Any vertex in $V(T_1) \setminus S_1$ is dominated by the same vertices as in S, so S_1 is a k-quasiperfect dominating set of T_1 and $S_1 \in C$. However, r_2 may or may not belong to S. In the former case, we can reason analogously as above and conclude that $S_2 \in C$. In the later case, all the vertices in $V(T_2) \setminus S_2$ except r_2 are dominated by at least one and at most k vertices in S_2, and r_2 by at most $k-1$ vertices. If r_2 is dominated by some vertex in S_2, then S_2 is a k-quasiperfect dominating set in B. Otherwise, S_2 is a k-quasiperfect dominating set of $T_2 - r_2$ in D.

□

Proposition 6. *Let $(T, r) = (T_1, r) \circ (T_2, r_2)$ be a rooted tree which is the composition of two rooted trees, and let S be a subset of its vertices. We denote $S_1 = S \cap V(T_1)$ and $S_2 \cap V(T_2)$. Then, $S \in D$ if and only if $S_1 \in D$ and $S_2 \in A \cup B$.*

Proof. We only prove the sufficiency. Suppose that $S \in D$, i.e., all the vertices in T except r are dominated by at least one and at most k vertices in S. Therefore, S_1 inherits this property for T_1 and $S_1 \in D$. On the other hand, S_2 should be a k-quasiperfect dominating set for T_2. Since r is not dominated in S, the vertex r_2 does not belong to S, hence $S_2 \in A \cup B$. □

In the algorithm, we assume that the vertices of the tree have been numbered from 1 to n such that all vertices have a greater number than its parent. The tree is stored in the array `Parent` in which any vertex `i` points to the location of its parent. At any time of the execution of the second loop, the four variables called `a(i)`, `b(i)`, `c(i)` and `d(i)` store the minimum cardinalities of sets of type A, B, C and D for the trees having `i` as root and previously processed vertices. Any of this variables might be infinite due to either it is not possible to find such sets or there exists a set of different type with the same cardinality. Those variables are initialized with the values corresponding to K_1. It is not difficult to modify the algorithm in order to keep track of the final γ_{1k}-set.

It is necessary to use a fifth variable `z(i)` to decide between the two possible options for the cardinal of type B sets given in Theorem 5. Specifically, `z(i)` is defined as $|N(i) \cap S|$ where $|S|$ has finite cardinality `b(i)`, and to keep internal consistency `z(i)` will be ∞ whenever `b(i)` is infinite. Note that any k-quasiperfect dominating set of a rooted tree T is in $A \cup B \cup C$. Thus, the resulting γ_{1k}-set will have as cardinality the minimum value among `a(1), b(1), c(1)`.

```
Algorithm γ_1k for trees
   Input:the parent array Parent[1...n] for any tree T
   Output:  γ_1k(T)
begin
   for i:=1...n do
      initialize a(i):=∞; b(i):=∞; c(i):=1; d(i):=0; z(i):=∞
   od
   for i:=n...2 do
      j:=Parent[i]; z:=z(j)
      a:=min(a(j)+a(i),a(j)+b(i));
      if a>b(j)+c(i) and z(j)==k-1 then
         a:=b(j)+c(i)
      fi
      b:=min(b(j)+a(i),b(j)+b(i));
      if b>d(j)+c(i) then
         b:=d(j)+c(i);
         z:=1
      fi
      if b>b(j)+c(i) and z(j)≤ k-2 then
         b:=b(j)+c(i);
         z:=z(j)+1
      fi
      if b==∞ then
         z:=∞
      fi
      c:=min(c(j)+b(i),c(j)+c(i),c(j)+d(i));
      d:=min(d(j)+a(i),d(j)+b(i))
      a(j):=a; b(j):=b; c(j):=c; d(j):=d; z(j):=z;
   od
   γ_1k(T):=min(a(1),b(1),c(1));
end.
```

In Figure 11 is shown an example of the output of the algorithm for $k = 3$. The vertices of the tree are labelled as in the initial order. It is also shown the final values of the variables `a(i), b(i), c(i), d(i)` and `z(i)` for the internal vertices.

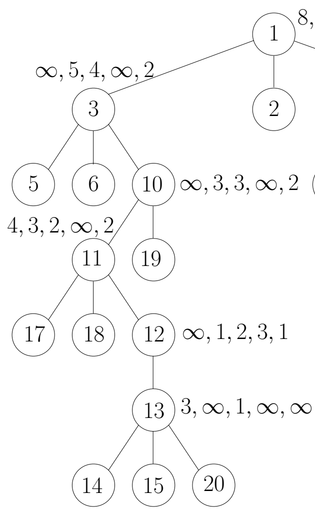

Figure 11: An example of the output of the algorithm γ_{1k} for trees and $k = 3$. For instance, `a(10)=`∞`, b(10)=3, c(10)=3, d(10)=`∞ and `z(10)=2`.

Theorem 5. *For any tree T with n vertices, $\gamma_{1k}(T)$ can be computed in linear time.*

Proof. Clearly, the second loop is iterated n times and the operations within the loop can be computed in constant time. □

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