Rings of Weak Dimension One and Syzygetic Ideals

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Abstract

We prove that rings of weak dimension one are the rings with all (three-generated) ideals syzygetic. This leads to a characterization of these rings in terms of the André-Quillen homology.

Let I be an ideal of a commutative ring A. There is a canonical morphism of graded A-algebras $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ from the symmetric algebra of I onto its Rees algebra. The ideal I is said to be of linear type if α is an isomorphism. If $\alpha_2 : \mathbf{S}_2(I) \to I^2$ is an isomorphism, I is said to be syzygetic.

In [**C**] (Theorem 4), Costa showed that a domain A is Prüfer if and only if I is of linear type for every two-generated ideal I of A and I is syzygetic for every three-generated ideal I of A. In this note we show that the preliminary hypothesis that A is a domain can be removed by changing the Prüfer condition to the condition $wd(A) \leq 1$, weak dimension of A one or less. Moreover, the condition that every two-generated ideal of A be of linear type is not necessary. Concretely,

Theorem 1 *Let A be a commutative ring. The following conditions are equivalent:*

i) $wd(A) \leq 1$.

ii) *Every ideal of A is of linear type.*

iii) *Every ideal of A is syzygetic.*

iv) *Every three-generated ideal of A is syzygetic.*

Recall that $wd(A)$ is the supremum of the flat dimensions of all A-modules. Von Neumann regular rings are those of weak dimension zero. Semihereditary rings (i.e. rings with all its finitely generated ideals projective) have weak dimension one or less. In fact, A is a semihereditary ring if and only if $wd(A) \leq 1$ and A is coherent. A semihereditary domain is called a Prüfer ring. For a domain A, to be Prüfer is equivalent to $wd(A) \leq 1$ (see [**B**]

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or [**R**]). In particular, if A is a domain, Theorem 1 characterizes Prüfer rings as domains with every three-generated ideal being syzygetic.

Before proving Theorem 1 we need the following two lemmas.

Lemma 2 *Let $(A, \mathfrak{m})$ be a local ring. Supose that there exist two nonzero elements $a, b \in A$ with $ab = 0$. If (a) is syzygetic, then (a, b) it is not.*

Proof. Let $I = (a, b) \subset \mathfrak{m}$ be the ideal of A generated by the zero divisors a, b. If $x \in (a) \cap (b)$, $x = ac = bd$, and multiplying by a, $c \in (0 : a^2)$. Since (a) is syzygetic, $c \in (0 : a)$ (see [**V**], page 31) and $x = ca = 0$. Therefore, $(a) \cap (b) = 0$.

Let $0 \to Z_1 \to A^2 \xrightarrow{f} I \to 0$ be the free presentation of I defined by $f((1,0)) = a$, $f((0,1)) = b$. Consider $0 \to N \to A[X,Y] \xrightarrow{s} \mathbf{R}(I) \to 0$, the induced free presentation of $\mathbf{R}(I) = \oplus_{q \geq 0} I^q t^q$, defined by $s(X) = at$, $s(Y) = bt$.

If $I = (a, b)$ were syzygetic, the quadratic relation XY on a, b could be written in terms of linear relations on a, b ([**V**], page 29), i.e.

$$XY = (a_1 X + b_1 Y)(c_1 X + d_1 Y) + \cdots + (a_r X + b_r Y)(c_r X + d_r Y) \tag{1}$$

with $c_i X + d_i Y \in N_1 = Z_1$. In particular, $c_i a = -d_i b \in (a) \cap (b) = 0$. Therefore, $c_i a = d_i b = 0$ and as $a, b \neq 0$, then c_i, d_i would be zero divisors, in particular, elements of $\mathfrak{m}$. Comparing the coefficients of XY in both members of (1) we would get the contradiction $1 = \sum_{i=1}^{r} a_i d_i + b_i c_i \in \mathfrak{m}$. ∎

Lemma 3 *Let $(A, \mathfrak{m}, k)$ be a local ring. Let I be a non principal finitely generated ideal of A. If I is syzygetic, then I^2 it is not.*

Proof. As I is not principal, $\dim_k(I/\mathfrak{m}I) = n > 1$. By hypothesis, α_2 is an isomorphism and hence, $\alpha_2 \otimes 1_k$ is also an isomorphism. Therefore,

$$\dim_k \left(I^2/\mathfrak{m}I^2\right) = \dim_k \left(\mathbf{S}_2^k(I/\mathfrak{m}I)\right) = \frac{n(n+1)}{2} = p$$

Thus, $\dim_k \left(\mathbf{S}_2^k(I^2/\mathfrak{m}I^2)\right) = \frac{p(p+1)}{2} = \frac{n(n+1)(n^2+n+2)}{8}$. Since α_4 is an epimorphism, $\alpha_4 \otimes 1_k$ is also an epimorphism. So, one has

$$\dim_k \left(I^4/\mathfrak{m}I^4\right) \leq \dim_k \left(\mathbf{S}_4^k(I/\mathfrak{m}I)\right) = \frac{n(n+1)(n^2+5n+6)}{24}$$

Finally, one observes that if $n \neq 0, 1$, then $3(n^2 + n + 2) > (n^2 + 5n + 6)$. In particular, $\mathbf{S}_2^k(I^2/\mathfrak{m}I^2) \not\simeq I^4/\mathfrak{m}I^4$ and $\mathbf{S}_2^A(I^2) \not\simeq I^4$. ∎

Proof of the theorem. Recall $wd(A) \leq 1$ is equivalent to every ideal of A being flat ([**R**]). If I is a flat ideal of A, then I is an ideal of linear type (see Proposition 3 [**MR**] or [**P**]). This proves $i) \Rightarrow ii)$. Let us see $iv) \Rightarrow i)$.

Let I be an ideal of A. To show I is flat, one can suppose I is finitely generated since any ideal is the direct limit of finitely generated ideals and the direct limit of flat modules is again a flat module. Write $I = (x_1, \ldots, x_n)$ and let us see that $I_{\mathfrak{m}}$ is a free $A_{\mathfrak{m}}$-module for every maximal ideal $\mathfrak{m}$ of A. If $I \not\subseteq \mathfrak{m}$, $I_{\mathfrak{m}} = A_{\mathfrak{m}}$. If $I \subseteq \mathfrak{m}$, consider $J = (x_1, x_2) \subseteq I$. By hypothesis $iv)$, J and J^2 are syzygetic ideals of A. Localizing at $\mathfrak{m}$, lemma 3 provides an element $z \in J$ with $J_{\mathfrak{m}} = (\frac{z}{1})$. Therefore $I_{\mathfrak{m}} = (\frac{z}{1}, \frac{x_3}{1}, \ldots, \frac{x_n}{1})$. Now, take $J' = (z, x_3) \subseteq I$. Repeating the process we deduce that $I_{\mathfrak{m}}$ is a principal ideal of $A_{\mathfrak{m}}$. By hypothesis $iv)$ and lemma 2, $A_{\mathfrak{m}}$ is a domain, in particular, $I_{\mathfrak{m}}$ is an ideal generated by a nonzero divisor, i.e. a free $A_{\mathfrak{m}}$-module. ∎

Remark 4 From Lemma 2 and Costa's Theorem 3 of [**C**] one deduces that for a commutative ring A to be locally an integrally closed domain is equivalent to being of linear type for every two-generated ideal of A. Moreover, the same example given by Costa in his paper shows that this last condition is strictly stronger than every two-generated ideal of A being syzygetic.

In terms of the André-Quillen homology (see [**A**]) and as a corollary of Theorem 1, rings of weak dimension one are characterized as follows:

Corollary 5 *Let A be a commutative ring. The following conditions are equivalent:*

i) $wd(A) \leq 1$.

ii) $H_2(A, B, \cdot) = 0$ *for every quotient ring* $B = A/I$ *of* A.

iii) $H_2(A, B, B) = 0$ *for every quotient ring* $B = A/I$ *of* A *by a three-generated ideal* I *of* A.

Proof. It follows from the fact that if I is an ideal of A, $B = A/I$ and $\alpha_2 : \mathbf{S}_2(I) \to I^2$ is the canonical morphism, then $H_2(A, B, B) = \mathrm{Ker}\alpha_2$. Moreover, if I is syzygetic, $H_2(A, B, W) = \mathrm{Tor}_1^B(I/I^2, W)$ for any B-module W (see, for instance, [**BR**]). ∎

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