ADAPTIVE SPECTRUM ESTIMATION WITH LINEAR CONSTRAINTS

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A general constrained adaptive method is developed to be applied to the spectral estimation problem. The method presented can be used in a wide range of situations, this is, we can get different estimators with it. The algorithm is formulated in a variational approach context, and the non linear system obtained is solved with a constrained adaptive method applied to a digitized version of the spectrum. The set of constraints is considered to be a set of known correlation values, and they can be located in non consecutive lags. A generalization of the method is done, so it can be used in a multidimensional framework. As an example, a bidimensional maximum entropy spectrum is presented.

1 Introduction

The problem of estimating the power spectral density of a random process, from a finite number of samples of a single realization, has been widely studied. Working in an unidimensional framework, there are a lot of different methods developed, obtaining both, parametric models and non parametric ones. Some of these methods have been generalized to be applied in a bidimensional (or multidimensional) framework, but unfortunately this generalization is not always possible. Another limitation that usually appears when using most of the estimators available, is that although they work efficiently when they are applied to particular situations, they become nearly useless in other cases. A well known example is the maximum entropy estimator; in one dimension and using consecutive correlation lags, Levinson's algorithm solves the problem optimally, but it cannot be used if the correlation lags are non consecutive or if we are in a two dimensional environment. In this work a general iterative method to solve the spectrum estimation problem is presented. The problem is first formulated in a variational approach framework, and a gradient adaptive method with linear constraints is used. The method can be applied to multidimensional spectral estimation and it is not imposed that the correlation lags be consecutive.

2 The variational approach in spectral estimation

One of the most interesting frameworks to study the estimation problem is the so called variational approach [1]. An objective function is to be optimized subject to a set of constraints. The optimization is found by means of the Lagrange multipliers λ_n. In this work we are considering a general function $F(S(f))$ of the spectra $S(f)$ to be optimized

$$F(S(f)) = \int \phi(S(f)) df \qquad (1)$$

the set of constraints D is a set of Q correlation lags. It must include the origin correlation point, but no other restrictions are imposed.

$$r(n) = \int S(f) e^{j2\pi n f} df \qquad n \in D \qquad (2)$$

The Lagrangian

$$L(S(f)) = \phi(S(f)) - \sum_{n \in D} \lambda_n S(f) e^{-j2\pi n f} \qquad (3)$$

is derived with respect to $S(f)$ and equaling the derivative to zero a non linear system is obtained. The Lagrange multipliers λ_n are

$$\lambda_n = \int \frac{\partial \phi(S(f))}{\partial S} e^{j2\pi n f} df \qquad n \in D \qquad (4)$$

and they are in fact the spectrum parametres.
Different estimators can be achieved choosing $\phi(S)$. For example if $\phi(S) = Ln(S)$ a maximum entropy spectrum should be obtained, or if $\phi(S) = (\frac{S}{P} - Ln\frac{S}{P} - 1)$

the spectrum that should be achieved is a minimum cross entropy one, where $P(f)$ is the previous spectrum knowaledge we have. On the other hand, if other sets of constrains were considered, the estimators achieved would also be different, but they are not studied here.

3 Adaptive gradient methods with constraints

To solve the previous problem, an adaptive gradient method [2] with costraints is proposed to be applied to a digitalized version of the spectrum to be estimated. Two different algorithms are discussed. The first one is based on Frost's method [3], and the second one on the General Sidelobe Canceller (GSC) proposed by Griffiths [4].

Considering that the N points of the spectrum to be estimated form the vector $\underline{S} = (S(0), .., S(k), .., S(n-1))^T$, and the Q correlation values that form the set of constraints are $\underline{r} = (r(0), .., r(i), .., r(M-1))^T$ where $i \in D$, a matrix $\underline{M}$ (QxN) can be formed and will be the relationship between $\underline{S}$ and $\underline{r}$. Thus, it can be written

$$\underline{r} = \frac{1}{N}\underline{M}\underline{S} \qquad (5)$$

where

$$M_{ik} = e^{j\frac{2\pi}{N}ik} \qquad i \in D, \quad k = 0, .., N-1 \qquad (6)$$

Equation 5 is in fact the constraint equation.
The objective function is then approximated by

$$F(\underline{S}) = \sum_{k=0}^{N-1} \phi(\underline{S}(k)) \qquad (7)$$

and so, the Lagrangian comes to be

$$L(\underline{S}) = \sum_{k=0}^{N-1} \phi(\underline{S}(k)) - \underline{\lambda}^T(\frac{1}{N}\underline{M}\underline{S} - \underline{r}) \qquad (8)$$

where $\underline{\lambda} = (\lambda_0, ..., \lambda_{M-1})^T$ is the vector formed by the Q Lagrange multipliers. The Lagrangian gradient with respect to $\underline{S}$ is

$$\nabla L = \underline{\phi\prime} - \frac{1}{N}\underline{M}^H\lambda \qquad (9)$$

where

$$\underline{\phi\prime} = (\frac{\partial\phi}{\partial S_0}, ..., \frac{\partial\phi}{\partial S_{N-1}})^H \qquad (10)$$

Generalizing the results presented by the authors in [5] in a maximum entropy context, the iterative algorithm comes to be

$$\underline{S}_{i+1} = (\underline{I}_N - \frac{1}{N}\underline{M}^H\underline{M})(\underline{S}_i + \mu\underline{\phi}_i\prime) + \underline{S}_0 \qquad (11)$$

where $\underline{S}_0 = \underline{M}^H\underline{r}$, and is called the minimum norm spectrum. $\underline{I}_N$ is the NxN identity matrix.

In fact, a geometrical interpretation of Eq. 11 can be easily made. Defining $\underline{P} = \underline{I}_N - \frac{1}{N}\underline{M}^H\underline{M}$ as a projection matrix over the orthogonal unconstrained subspace defined by $\underline{0} = \underline{M}\underline{S}$ and pointing out that Eq. 5 defines the so called constrained hiperplane (where all the estimator candidates lay), the algorithm can be explained as follows. The ith estimation is modified in the objective function gradient direction. The new vector obtained should not lay on the constrained hiperplane, so it must be projected over the orthogonal unconstrained subspace, and adding to that projected vector the minimum norm one, a new spectrum estimation is achieved, which fulfills the set of constraints. The algorithm convergence is controlled by the μ constant, that at the same time bounds the residual error that can be achieved.

The second method presented is based on a decomposition of the vector to be adapted into two orthogonal components. As shown in Eq. 11, the iteration can be expressed as the addition of two vectorial components, one is the minimum norm spectrum, which is time invariant, and the other one, which is orthogonal to it, is in fact which will be adapted along the process. Thus, if we were able to work in the unconstrained subspace, the adaptive constrained algorithm would come to be an unconstrained one. The computational cost is then reduced by two reasons. On one hand the adaptive algorithm is simpler, and on the other hand there are less coeficients to be adapted. First of all a transformation must be made, passing to work in the time domain instead of working in the frequency one. Though, let us consider $\underline{r}$ the constrained vector as before, and $\underline{r}_a$ the adaptive vector. The latest is in fact a correlation extrapolation, and its (N-Q) components are out of the constrained set D. The iteration equation is now an unconstrained one, and is carried on over $\underline{r}_a$ as

$$\underline{r}_{a_{i+1}} = \underline{r}_{a_i} + \mu\frac{1}{N}\underline{A}\underline{S}_i \qquad (12)$$

where

$$A_{ik} = e^{j\frac{2\pi}{N}ik} \qquad i \text{ not in } D, \quad k = 0, .., N-1 \qquad (13)$$

and $\underline{S}_i$ is the ith spectrum estimated.

4 Using the FFT

It has to be pointed out that with the formulation used up to here, the algorithm is solved with a high computational cost. This is mainly due to the fact of the matrix by matrix, matrix by vector multiplication opera-

tions involved in the procedure. In fact, the QxN matrix $\underline{M}$ is partially defining the Inverse Discrete Fourier Transform. There are $N - Q$ rows missing to have the NxN square matrix associated with the IDFT. Adding the needed rows to the matrix, and extendig the Q dimensional vectors in such way that they become N dimensional, introducing no modifications to the formal equations, the FFT algorithm can be ued. This is done in a very simple way. The $\underline{r}$ vector is extended initially with noughts located in the non constrained lags. Besides that, a temporal window $w(n)$ must be used multiplying the inverse Fourier transformoms computed, to ensure that the constraints remain unchanged. This window is equal to one in the constrained lags, and is zero in the unconstrained ones. Thus, with this vector extension Eq. 11 can be rewritten as

$$\underline{S}_{i+1} = \underline{S}_i + \mu\underline{\phi}_i / - FFT\{w(n)FFT^{-1}\{\underline{S}_i + \mu\underline{\phi}_i /\}\} + \underline{S}_0 \tag{14}$$

5 Conclusions

This last equation shows some of the main advantages of the method presented. Among them we mention the following ones. The constrain correlation points are located through the window $w(n)$, and no restriction is imposed to this location, so the correlation lags might be non consecutive. The correlation constraints are always fullfilled. This is due to the last term of Eq. 14, so if the known correlation points were time-varying, an adaptive version of the correlation constrained set shuold be used just modifying this last term in each iteration. A very important advantage that makes the procedure very usefull is that expression 14 is directly generalizable to a multidimensional framework. This is done considering multidimensional Fourier transforms, and extending the vectors and the window to the multidimensional case.
Just to present an example of the results achieved, a maximum entropy spectrum is estimated in a bidimensional case. We consider two sinusoids in white gaussian noise as input signal.

$$x(i,j) = n(i,j) + \sum_{l=1}^{2} \sqrt{2}cos(2\pi(f_1(l)i + f_2(l)j)) \tag{15}$$

The sinusoid frequencies are $(0.1, 0.1)$ and $(0.2, 0.3125)$, and the noise power is $\sigma_n^2 = 1$. The number of points used to compute the estimation is 64x64. The constrained set is a 5x5 square of known correlation points centered in the origin. Figures 1 and 2 show the minimum norm spectrum and the maximum entropy one obtained with the method presented.

References

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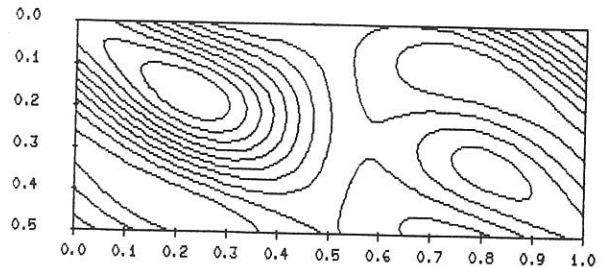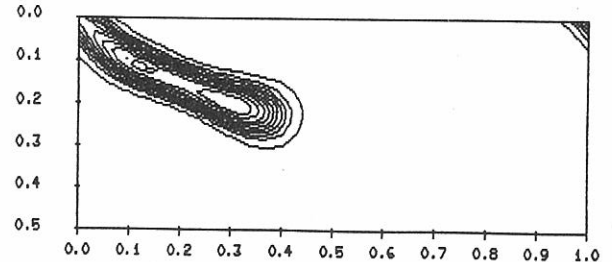
Figure 1: Minimum norm spectrum

Figure 2: Maximum entropy spectrum