Shifting linear quadratic control of constrained continuous-time descriptor LPV systems ⋆

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Abstract: Recently, the concept of *shifting linear quadratic control* (SLQC), where some varying parameters are introduced and used to schedule the weighting matrices of a quadratic cost function, has been introduced. This paper further explores this concept by considering the presence of constraints in the system to be controlled. In particular, two types of constraints are considered: a) algebraic constraints between the variables of the system; and b) constraints on the allowed values for the input and the state variables. The proposed solution, investigated under the descriptor linear parameter varying (D-LPV) framework, requires solving a set of linear matrix inequalities (LMIs), a problem for which efficient solvers are available nowadays. A numerical example illustrates the application of the proposed theory.

Keywords: Linear parameter-varying (LPV) systems, descriptor systems, optimal control.

1. INTRODUCTION

Linear quadratic control (LQC) (Anderson and Moore, 1989) has played an important role in the last decades, concerning the development of the modern optimal control theory (Li and Wayne Schmidt, 1997, Liu et al., 2014). Since the pioneer work by Kalman (Kalman, 1960), many extensions of LQC have been proposed (Jacobson, 1977), e.g. risk-sensitive optimal control (Whittle, 1990) and multiple LQC (Li, 1993). The study of LQC is still a hot topic of research, as demonstrated by the amount of works that have appeared recently, showing further developments of this theory (Zhang et al., 2015, Videcoq et al., 2015, Modares and Lewis, 2014, Song and Yan, 2014, Alt and Seydenschwanz, 2014).

In the last decades, the need of stability and performance requirements for nonlinear systems in a wide set of operating conditions pushed towards a rapid adoption of gain scheduled systems (Rugh and Shamma, 2000). Due to the important role of gain scheduling in many applications, the necessity for systematic analysis and design tools for gain-scheduled controllers arose, leading to the linear parameter varying (LPV) paradigm as one of the most successful approaches.

LPV systems were introduced by Shamma (1988) to distinguish such systems from linear time invariant (LTI) and linear time varying (LTV) ones (Shamma, 2012). More specifically, LPV systems are a particular class of LTV systems, where the time-varying elements depend on measurable parameters that can vary over time (White et al., 2013). The LPV framework has proved to be suitable for controlling nonlinear systems by embedding the nonlinearities in the varying parameters, that will depend on some endogenous signals, e.g. states, inputs or outputs. In this case, the system is referred to as quasi-LPV, to make a further distinction with respect to pure LPV systems, where the varying parameters only depend on exogenous signals (Marcos and Balas, 2004). The LPV paradigm has been successfully applied to many applications (Hoffmann and Werner, 2015), e.g. mobile robots (Rotondo et al., 2015c) and aeromechanical systems (Rotondo et al., 2013). In recent times, the research interest has been attracted by the more general class of descriptor LPV (D-LPV) systems, where algebraic constraints between the physical variables of the system are included (López-Estrada et al., 2013, Rodrigues et al., 2014).

In a recent work, an LQC design procedure for LPV systems has been proposed (Rotondo et al., 2015b). The solution is based on the results obtained by Ostertag (2011), and is expressed in terms of linear matrix inequalities (LMIs), a formulation that allows solving complicated control problems very efficiently, and with a remarkable degree of simplicity. Furthermore, in Rotondo et al. (2015b), the *shifting LQC (SLQC)*, where some varying parameters are introduced and used to schedule both the controller and the weighting matrices, following the idea presented in Rotondo et al. (2015a), has been introduced.

However, the work in Rotondo et al. (2015b) has not considered the presence of constraints in the system to be controlled. In fact, in many applications, the boundedness of the control actions and the presence of limits on the state values, needed to keep the controlled plant within *safe* operating points, lead to developing control techniques able to handle such constraints (Sussmann et al., 1994, Gilbert and Kolmanovsky, 1999, Bemporad and Mosca, 1998). Moreover, when there are algebraic constraints between the physical variables of the system, a descriptor formulation should be used.

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The goal of the present paper is to address the problem of SLQC in presence of these constraints, following the results presented by Kothare et al. (1996) and Pluymers et al. (2006).

The structure of the paper is as follows. In Section 2, the SLQC problem is introduced. Section 3 shows how the quadratic Lyapunov framework can be used to derive a set of parametrized LMIs that solve the SLQC problem. These parametrized LMIs correspond to an infinite number of constraints, and cannot be used from a practical point of view. Hence, in Section 4, they are reduced to a finite number of LMIs using a polytopic simplification. The application of the proposed theory to a numerical example is described in Section 5. Finally, Section 6 outlines the conclusions.

Notation: Throughout the paper, if a matrix $M \in \mathbb{R}^{n\times n}$ is symmetric, then $M \in \mathbb{S}^{n\times n}$. The symbols I and O are used to denote the identity matrix and the zero matrix of appropriate dimensions, respectively. A matrix $M \in \mathbb{S}^{n\times n}$ is said *positive definite* ($M \succ O$) if all its eigenvalues are positive, and *negative definite* ($M \prec O$) if all its eigenvalues are negative. For brevity, symmetric elements in a matrix are denoted by $*$ and $M+M^T$ will be indicated as $He\{M\}$. Also, the notation $M_{[i,:]}$ is used to denote the i-th row of a matrix M.

2. PROBLEM FORMULATION

Let us consider the following continuous-time D-LPV system:

$$E\dot{x}(t) = A(\theta(t))x(t) + B(\theta(t))u(t) \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the input vector, and $A(\theta(t)) \in \mathbb{R}^{n_x\times n_x}$ and $B(\theta(t)) \in \mathbb{R}^{n_x\times n_u}$ are parameter-varying matrices, scheduled by the value of the varying parameter vector $\theta \in \Theta \subset \mathbb{R}^{n_\theta}$, and $E \in \mathbb{R}^{n_x\times n_x}$ is a singular matrix, i.e. $r = rank(E) < n_x$. Without loss of generality, it is assumed that the D-LPV descriptor system (1) is given in a normalized singular value decomposition (SVD) form (Verghese et al., 1981, Rehm and Allgower, 2002), i.e.:

$$E = \begin{pmatrix} I_r & O \\ O & O \end{pmatrix} \tag{2}$$

$$A(\theta(t)) = \begin{pmatrix} A_{11}(\theta(t)) & A_{12}(\theta(t)) \\ A_{21}(\theta(t)) & A_{22}(\theta(t)) \end{pmatrix} \tag{3}$$

$$B(\theta(t)) = \begin{pmatrix} B_1(\theta(t)) \\ B_2(\theta(t)) \end{pmatrix} \tag{4}$$

Also, let us consider that (1) is subject to state and input constraints, as follows:

$$x \in \mathscr{X} \triangleq \{x \in \mathbb{R}^{n_x} : |\Lambda_x x| \leq 1_v\} \tag{5}$$

$$u \in \mathscr{U} \triangleq \{u \in \mathbb{R}^{n_u} : |\Lambda_u u| \leq 1_v\} \tag{6}$$

where Λ_x and Λ_u are matrices that weigh the constraints and 1_v denotes the column vector whose components are all equal to 1. Then, provided that the initial condition $x(0) = x_0$ has been specified, it is wished to design a state-feedback control law of the form:

$$u(t) = K(\theta(t), p(t))x(t) \tag{7}$$

where $p(t) \in \Pi \subset \mathbb{R}^{n_p}$ is a vector of scheduling parameters used to achieve the shifting linear quadratic control (SLQC), i.e. such that for $p(t) = p$ it minimizes:

$$J(p) = \int_0^\infty \left(x(t)^T Q(p)x(t) + u(t)^T R(p)u(t)\right) dt \tag{8}$$

where $Q(p) = H(p)^T H(p) \succ O$ and $R(p) \succ O \ \forall p \in \Pi$, with $H(p) \in \mathbb{R}^{n_x\times n_x}$, denote the state and input cost weighting matrices, respectively. The minimization of $J(p)$ is performed such that the controller does not violate the constraints (5)-(6) over the possible trajectories of $x(t)$ starting from x_0.

Remark 1. Despite in this paper the problem of controller design using SLQC is considered for the case of D-LPV systems, the proposed method is useful for LTI systems too. In this case, the controller is scheduled by means of the vector of parameters $p(t)$, such that, even though the plant to be controlled is D-LTI, the overall system is D-LPV and the mathematical reasoning developed hereafter can be applied. The reason to do so is that in this way the performance of the closed-loop system can be varied in time according to some criterium, e.g. energetic or economic costs.

3. DESIGN CONDITIONS

A practical approach for minimizing $J(p)$ is to search a control law (7) which guarantees that the criterion J is below some number γ, such that the minimization of J becomes the minimization of γ (Duc, 2002).

Let us introduce the Lyapunov function (Chadli et al., 2014):

$$V(x(t)) = x(t)^T E^T P x(t) \tag{9}$$

with:

$$E^T P = PE \succeq O \tag{10}$$

and such that:

$$V(x_0) = x_0^T E^T P x_0 < \gamma \tag{11}$$

$$\dot{V}(x(t)) + x(t)^T Q(p)x(t) + u(t)^T R(p)u(t) < 0 \tag{12}$$

It is quite straightforward to check that P should have the following form in order to accomplish (10):

$$P = \begin{pmatrix} P_1 & O \\ O & P_2 \end{pmatrix}, \quad P_1 \succ O \tag{13}$$

By integrating (12) from $t = 0$ to $t = \infty$, one obtains:

$$-V(x_0) + J(p) < 0 \tag{14}$$

that, due to (11), is equivalent to:

$$J(p) < V(x_0) < \gamma \tag{15}$$

that clearly demonstrates that the minimization of γ would minimize $J(p)$ as well.

Combining the inequality (12), taking into account the control law (7), and:

$$\dot{V}(x(t)) = x(t)^T A_{cl}(\theta,p)^T P x(t) + x(t) P A_{cl}(\theta,p) x(t) \tag{16}$$

with $A_{cl}(\theta,p) = A(\theta) + B(\theta)K(\theta,p)$, leads to the parameter-dependent LMI:

$$\begin{aligned}(A(\theta)+B(\theta)K(\theta,p))^T P + P(A(\theta)+B(\theta)K(\theta,p)) & \quad \forall\theta\in\Theta \\ +Q(p) + K(\theta,p)^T R(p)K(\theta,p) \prec O & \quad \forall p \in \Pi\end{aligned} \tag{17}$$

that, through the change of variables $Z = \gamma P^{-1}$ and $\Gamma(\theta,p) = K(\theta,p)Z$, the pre- and post-multiplication by Z, and the division by γ, as long as $P_2 \succ O$, becomes [1] :

$$\begin{aligned}&(A(\theta)Z + B(\theta)\Gamma(\theta,p))^T + A(\theta)Z + B(\theta)\Gamma(\theta,p) \\ &+\frac{1}{\gamma}\left(ZQ(p)Z + \Gamma(\theta,p)^T R(p)\Gamma(\theta,p)\right) \prec O\end{aligned} \tag{18}$$

Taking into account that:

$$ZQ(p)Z + \Gamma(\theta,p)^T R(p)\Gamma(\theta,p) = \begin{pmatrix} Z \\ \Gamma(\theta,p) \end{pmatrix}^T \begin{pmatrix} Q(p) & O \\ O & R(p) \end{pmatrix} \begin{pmatrix} Z \\ \Gamma(\theta,p) \end{pmatrix} \tag{19}$$

[1] In the following, the fact that the matrix inequalities should hold $\forall\theta\in\Theta$ and $\forall p \in \Pi$ is omitted for sake of space.

and applying Schur complements (Schur, 1917), recalling that $Q(p) = H(p)^T H(p)$, the following is obtained:

$$\begin{pmatrix} -He\{A(\theta)Z + B(\theta)\Gamma(\theta,p)\} & * & * \\ H(p)Z & \gamma I & * \\ R(p)^{1/2}\Gamma(\theta,p) & O & \gamma I \end{pmatrix} \succ O \tag{20}$$

such that, pre- and post-multiplying by $diag\left(I, I, R(p)^{-1/2}\right)$, becomes:

$$\begin{pmatrix} -He\{A(\theta)Z + B(\theta)\Gamma(\theta,p)\} & * & * \\ H(p)Z & \gamma I & * \\ \Gamma(\theta,p) & O & \gamma R(p)^{-1} \end{pmatrix} \succ O \tag{21}$$

The application of Schur complements to (11), taking into account (2) and (13), leads to the LMI form:

$$\begin{pmatrix} \gamma & x_{0,r}^T \\ x_{0,r} & \gamma^{-1} Z_1 \end{pmatrix} \succ O \tag{22}$$

where $Z_1 = \gamma P_1^{-1}$, and $x_{r,0} \in \mathbb{R}^r$ is the subset of x_0 that corresponds to the matrix I_r in (2).

In order to take into account the presence of the constraints (5)-(6), let us remind that, given a column vector $x \in \mathbb{R}^{n_x}$ and a row vector $a \in \mathbb{R}^{1 \times n_x}$, it is true for the scalar ax that if:

$$\begin{pmatrix} P^{-1} & P^{-1}a^T \\ aP^{-1} & \frac{1}{\gamma} \end{pmatrix} \succ O \tag{23}$$

holds, then $|ax| \leq 1 \; \forall x \in \left\{x : x^T P x \leq \gamma\right\}$ (Nguyen and Jabbari, 2000). Notice that (23) is equivalent to:

$$\begin{pmatrix} Z & Za^T \\ aZ & 1 \end{pmatrix} \succ O \tag{24}$$

and the constraint (5) can be expressed element-wise with $a = \Lambda_{x[l,:]}$, such that (24) becomes:

$$\begin{pmatrix} Z & * \\ \Lambda_{x[l,:]}Z & 1 \end{pmatrix} \succ O \qquad l = 1, \ldots, n_x \tag{25}$$

Similarly, taking into account that $u = K(\theta, p)\, x$, the constraint (6) can be expressed element-wise with $a = \Lambda_{u[l,:]} K(\theta, p)$, such that (24) becomes:

$$\begin{pmatrix} Z & * \\ \Lambda_{u[l,:]}\Gamma(\theta,p) & 1 \end{pmatrix} \succ O \qquad l = 1, \ldots, n_u \tag{26}$$

Remark 2. In the case in which the quadratic cost function (8) contains only the weighting state term $x(t)^T Q(p) x(t)$, the LMI condition (21) simplifies to:

$$\begin{pmatrix} -He\{A(\theta)Z + B(\theta)\Gamma(\theta,p)\} & ZH(p)^T \\ H(p)Z & \gamma I \end{pmatrix} \succ O \tag{27}$$

Similarly, in the case in which the quadratic cost function (8) contains only the weighting input term $u(t)^T R(p) u(t)$, the LMI condition (21) simplifies to:

$$\begin{pmatrix} -He\{A(\theta)Z + B(\theta)\Gamma(\theta,p)\} & \Gamma(\theta,p)^T \\ \Gamma(\theta,p) & \gamma R(p)^{-1} \end{pmatrix} \succ O \tag{28}$$

4. POLYTOPIC SIMPLIFICATION

The conditions provided in Section 3 for the design of the shifting linear quadratic controller, i.e. (21)-(22) and (25)-(26), cannot be used from a practical point of view, because (21) and (26) correspond to an infinite number of constraints.

Under the assumption that the matrix B is constant, that the matrix $A(\theta(t))$ is polytopic, i.e. it satisfies the following property:

$$A(\theta(t)) = \sum_{i=1}^{N} \alpha_i(\theta(t)) A_i \qquad \forall \theta \in \Theta \tag{29}$$

with:

$$\begin{cases} \sum_{i=1}^{N} \alpha_i(\theta(t)) = 1 \\ \alpha_i(\theta(t)) \geq 0 \quad \forall i = 1, \ldots, N \end{cases} \qquad \forall \theta \in \Theta \tag{30}$$

and that:

$$\begin{pmatrix} H(p) \\ R(p)^{-1} \end{pmatrix} = \sum_{j=1}^{P} \pi_j(p) \begin{pmatrix} H_j \\ \check{R}_j \end{pmatrix} \qquad \forall p \in \Pi \tag{31}$$

with:

$$\begin{cases} \sum_{j=1}^{P} \pi_j(p) = 1 \\ \pi_j(p) \geq 0 \quad \forall j = 1, \ldots, P \end{cases} \qquad \forall p \in \Pi \tag{32}$$

it is possible to choose the control law (7) to be polytopic as well:

$$u(t) = \sum_{i=1}^{N} \alpha_i(\theta(t)) \sum_{j=1}^{P} \pi_j(p(t)) K_{ij} x(t) \tag{33}$$

in order to reduce (21) and (26) to a finite number of LMIs, as follows:

$$\begin{pmatrix} -He\left\{A_i Z + B\Gamma_{ij}\right\} & ZH_j^T & \Gamma_{ij}^T \\ H_j Z & \gamma I & O \\ \Gamma_{ij} & O & \gamma \check{R}_j \end{pmatrix} \succ O \qquad \begin{matrix} \forall i = 1, \ldots, N \\ \forall j = 1, \ldots, P \end{matrix} \tag{34}$$

$$\begin{pmatrix} Z & * \\ \Lambda_{u[l,:]}\Gamma_{ij} & 1 \end{pmatrix} \succ O \qquad \begin{matrix} \forall i = 1, \ldots, N \\ \forall j = 1, \ldots, P \\ \forall l = 1, \ldots, n_u \end{matrix} \tag{35}$$

This is possible thanks to the basic property of matrices that any linear combination with non-negative coefficients, of which at least one different from zero, of positive (negative) definite matrices is positive (negative) definite as well (Horn and Johnson, 1990).

Summarizing, in order to solve the SLQC problem for constrained LPV systems, the following optimization problem should be solved:

$$\min_{\gamma, Z, \Gamma_{11}, \ldots, \Gamma_{ij}, \ldots, \Gamma_{NP}} \gamma \tag{36}$$

subject to (22), (25), (34) and (35). Once a solution γ^*, Z^*, $\Gamma_{11}^*, \ldots, \Gamma_{ij}^*, \ldots, \Gamma_{NP}^*$ has been found, the controller gains are easily obtained as:

$$K_{ij} = \Gamma_{ij}^* (Z^*)^{-1} \tag{37}$$

Remark 3. The polytopic versions of (27) and (28) are:

$$\begin{pmatrix} -He\left\{A_i Z + B\Gamma_{ij}\right\} & ZH_j^T \\ H_j Z & \gamma I \end{pmatrix} \succ O \qquad \begin{matrix} \forall i = 1, \ldots, N \\ \forall j = 1, \ldots, P \end{matrix} \tag{38}$$

and:

$$\begin{pmatrix} -He\left\{A_i Z + B\Gamma_{ij}\right\} & \Gamma_{ij}^T \\ \Gamma_{ij} & \gamma \check{R}_j \end{pmatrix} \succ O \qquad \begin{matrix} \forall i = 1, \ldots, N \\ \forall j = 1, \ldots, P \end{matrix} \tag{39}$$

Remark 4. Notice that the assumption of a constant B is not restrictive, since in the case of a varying $B(\theta(t))$, a prefiltering of the input $u(t)$ would lead to obtain a new system with a constant input matrix $\tilde{B}$ (Apkarian et al., 1995). More specifically, for the system (1), let us define a new control input $\tilde{u}(t)$ such that:

$$\dot{x}_u(t) = A_u(\theta(t))\, x_u(t) + B_u \tilde{u}(t) \tag{40}$$

$$u(t) = C_u(\theta(t))\, x_u(t) \tag{41}$$

with $A_u(\theta(t))$ stable. Then, the resulting LPV system would be:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{x}_u(t) \end{pmatrix} = \tilde{A}(\theta(t)) \begin{pmatrix} x(t) \\ x_u(t) \end{pmatrix} + \tilde{B}\tilde{u}(t) \tag{42}$$

with:

$$\tilde{A}(\theta(t)) = \begin{pmatrix} A(\theta(t)) & B(\theta(t))C_u \\ O & A_u(\theta(t)) \end{pmatrix} \qquad \tilde{B} = \begin{pmatrix} O \\ B_u \end{pmatrix}$$

5. EXAMPLE

Consider a continuous-time D-LPV system as in (1)-(4), with matrices defined as follows:

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$A(\theta(t)) = \begin{pmatrix} -\theta(t) & 2 & 1 \\ 2 & 1 & -\theta(t) \\ 1 & 1 & 1 \end{pmatrix}$$

with $\theta(t) \in [1,2]$, and subject to the constraints (5)-(6) with:

$$\Lambda_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Lambda_u = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The controller gains $K(\theta(t), p(t))$ in (7) are designed such that, starting from $x_1(0) = 0.5$ and $x_2(0) = 0.5$, they minimize the quadratic criterion (8) in the following three cases:

- Case A, $K_A(\theta(t))$

$$J_A \quad \Rightarrow \quad Q_A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- Case B, $K_B(\theta(t))$

$$J_B \quad \Rightarrow \quad Q_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 100 & 0 \\ 0 & 0 & 100 \end{pmatrix} \quad R_B = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{10} \end{pmatrix}$$

- Case C, $K_C(\theta(t), p(t))$

$$J_C(p) \quad \Rightarrow \quad \begin{cases} Q_C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+9p)^2 & 0 \\ 0 & 0 & (1+9p)^2 \end{pmatrix} \\ R_C = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1+9p} \end{pmatrix} \end{cases}$$

  with $p \in [0,1]$. Notice that $Q(p)$ and $R(p)$ correspond to:

$$H_C(p) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+9p & 0 \\ 0 & 0 & 1+9p \end{pmatrix} \quad R_C(p)^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1+9p \end{pmatrix}$$

  which can be easily expressed as (31), with the relevant feature that the values of $H_C(p)$ and $R_C(p)^{-1}$ in the case $p = 0$ ($p = 1$) correspond to H_A and R_A^{-1} (H_B and R_B^{-1}), where H_A (H_B) is selected to satisfy $Q_A = H_A^T H_A$ ($Q_B = H_B^T H_B$).

The procedure described in Section 4 provides the controller gains with the following bounds on the quadratic criterion: $\gamma_A = 1.27$, $\gamma_B = 2.35$ and $\gamma_C = 2.35$. It should be remarked that the conditions presented in Sections 3-4 are not strictly LMIs due to the term $\gamma^{-1}Z_1$ in (22). However, (22) can be brought to LMI form by performing the minimization of γ through a line search procedure. Then, a solution to the LMIs can be found efficiently using available software. In this paper, the YALMIP

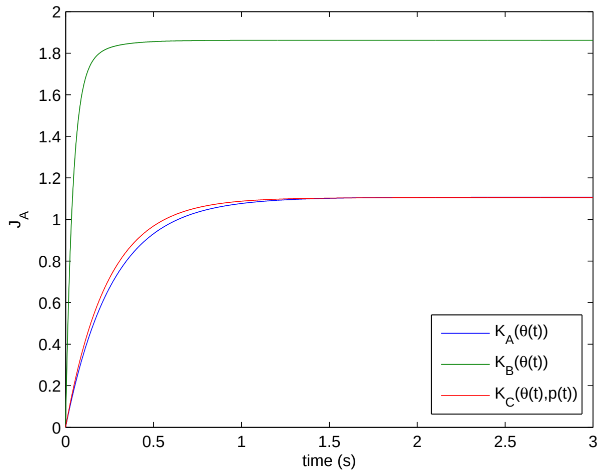

Fig. 1. Criterion J_A with the designed controllers.

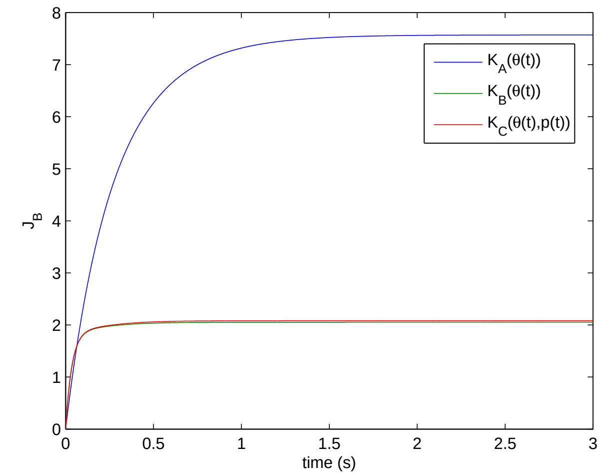

Fig. 2. Criterion J_B with the designed controllers.

toolbox (Löfberg, 2004) with SeDuMi solver (Sturm, 1999) has been used.

Figs. 1-3 show the evolutions of the performance indexes J_A, J_B and $J_C(p(t))$, obtained with $\theta(t) = 1.5 + 0.5\sin(t)$ and $p(t) = \max(1-t, 0)$ using the designed controllers. As expected, when considering J_A, $K_A(\theta(t))$ provides the best performance $J_A < \gamma_A = 1.27$. However, even though the upper bound for the performance of $K_C(\theta(t), p(t))$ is $\gamma_C = 2.35$, the loss of performance due to considering a wider range of possible quadratic cost weighting matrices using the SLQC approach is negligible. Similar considerations hold for J_B, for which looking at Fig. 2, it can be seen that the loss of performance using $K_C(\theta(t), p(t))$ with respect to the best performance obtained by $K_B(\theta(t))$ is negligible as well.

However, the strong appeal of the proposed approach becomes evident when considering the quadratic criterion $J_C(p(t))$, for which the weighting matrices vary in time, as shown in Fig. 3, where the controller $K_C(\theta(t), p(t))$ clearly outperforms $K_A(\theta(t))$ and $K_B(\theta(t))$.

Finally, to conclude the results analysis, let us look at the states and inputs obtained with the controller $K_C(\theta(t), p(t))$ in

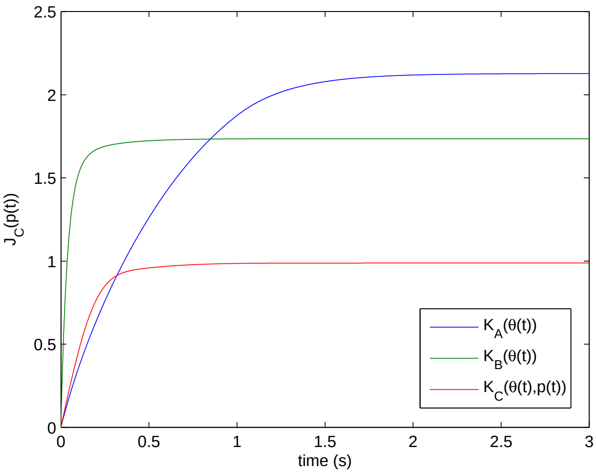

Fig. 3. Criterion $J_C\left(p(t)\right)$ with the designed controllers.

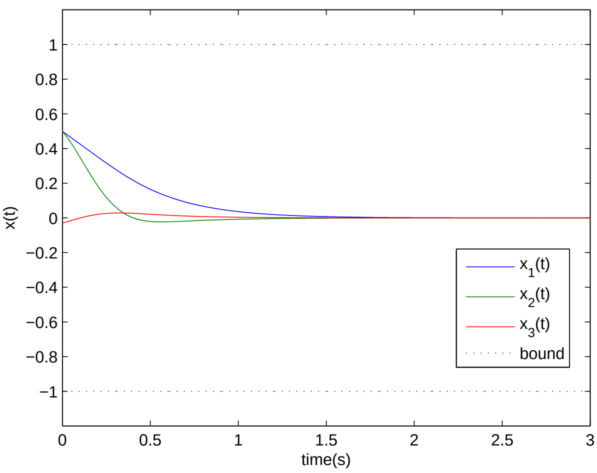

Fig. 4. States obtained with $K_C\left(\theta(t), p(t)\right)$ in case C.

the case of varying weighting matrices (case C). As expected, all the states and inputs are inside the bounds defined by the constraints (5)-(6).

6. CONCLUSIONS

This paper has presented an LMI approach for designing shifting linear quadratic controllers in the case of descriptor LPV systems. The feature of this technique, that distinguishes SLQC from standard LQC is that the weighting matrices of the quadratic performance criterion are assumed to vary in time, and these variations are used to schedule the controller. The descriptor formulation allows to consider the presence of algebraic constraints between the variables of the system. Moreover, constraints on the allowed values for the input and the state variables have been considered too.

The proposed solution requires solving a set of LMIs, a problem for which efficient solvers are available nowadays. The results obtained with a numerical example have shown that for the case of fixed weighting matrices, the controller designed to be optimal for the considered weighting matrices outperforms the other controllers. However, in these cases, the shifting linear quadratic controller performance is only slightly worse than the best obtained performance. The strong appeal of the proposed approach becomes evident when considering a criterion varying in time, in which case the shifting linear quadratic controller clearly proves to be the best one.

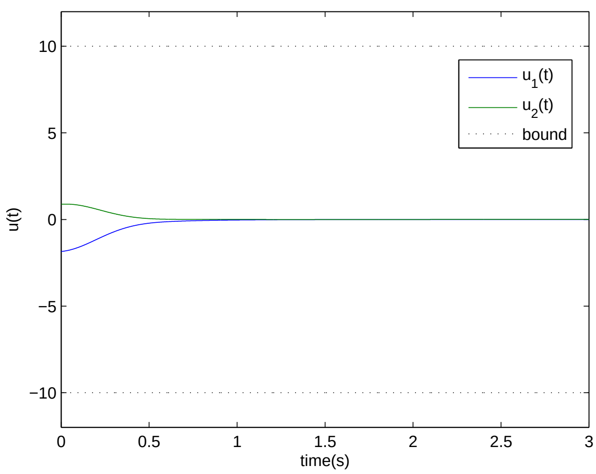

Fig. 5. Inputs obtained with $K_C\left(\theta(t), p(t)\right)$ in case C.

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