

Bartering integer commodities with exogenous prices

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Research Report UPC-DEIO DR 2013-07

November, 2013

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Abstract

The analysis of markets with indivisible goods and fixed exogenous prices has played an important role in economic models, especially in relation to wage rigidity and unemployment. This paper provides a novel mathematical programming based approach to study pure exchange economies where discrete amounts of commodities are exchanged at fixed prices. Barter processes, consisting in sequences of elementary reallocations of couple of commodities among couples of agents, are formalized as local searches converging to equilibrium allocations. A direct application of the analyzed processes in the context of computational economics is provided, along with a Java implementation of the approaches described in this paper: <http://www-eio.upc.edu/~nasini/SER/launch.html>.

Key words: Microeconomic Theory, Combinatorial optimization, Multiobjective optimization, Multiagent systems

1. Introduction

The bargaining problem concerns the allocation of a fixed quantity among a set of self-interested agents. The characterizing element of a bargaining problem is that many allocations might be simultaneously suitable for all the agents.

Definition 1. *Let $\mathcal{V} \subset R^n$ be the space of allocations of an n agents bargaining problem. Points in \mathcal{V} can be compared by saying that $v^* \in V$ strictly dominates $v \in V$ if each component of v^* is not less than the corresponding component of v and at least one component is strictly greater, that is, $v_i \leq v_i^*$ for each i and $v_i < v_i^*$ for some i . This is written as $v \prec v^*$. Then, the Pareto frontier is the set of points of \mathcal{V} that are not strictly dominated by others.*

Since the very beginning of the Economic Theory [13, 10], the bargaining problem has generally be adopted as the basic mathematical framework for the study of markets of excludable and rivalrous goods and a long-standing line of research focused on axiomatic approaches for the determination of a uniquely allocation, satisfying agent's interests (for details, see Nash [15] and Rubinstein [19]).

More recently, an increasing attention has been devoted to the cases where the quantity to be allocated is not infinitesimally divisible. The technical difficulties associated to those markets have been pointed out since Shapley and Shubik [22], who characterized the equilibria of markets where each agent can consume at most one indivisible good. After them,

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many authors have been studying markets with indivisible goods (see for example, Kaneko [14], Quinzii [18], Scarf [21], and the most recent literature like Danilov et al. [8], Caplin and Leahy [6]). The main focus was to address the question of existence of market clearing prices in the cases of not infinitesimally divisible allocations.

Another subclass of the family of bargaining problems is associated to markets with fixed prices (for details, see Dreze [9] and Auman and Dreze [3]), which have played an important role in maroeconomic models, especially on those models related to wage rigidities and unemployment. Dreze [9] described price rigidity as inequality constraints on individual prices.

Efficient algorithms to find non-dominated Pareto allocations of bargaining problems associated to markets with not infinitesimally divisible goods and fixed exogenous prices have been recently studied by Vazirani et al. [24] and by Ozlen, Azizoglu and Burton [17]. Our goal is to provide novel mathematical-programming based approaches to analyze barter processes, which are commonly used in everyday life by economic agents to solve bargaining problems associated to n -consumer- m -commodity markets of not infinitesimally divisible goods and fixed exogenous prices. These processes are based on *elementary reallocations* (ER) of two commodities among two agents, sequentially selected from the $m(m-1)n(n-1)/4$ possible combinations. Under fixed prices, markets do not clear and the imbalance between supply and demand is resolved by some kind of quantity rationing [9]. In our analysis this quantity rationing is implicit in the process and not explicitly taken into account.

Based on this multi-agent approach, many economical systems might be simulated [26], as we will see in the computational application illustrated in section 5.

Section 2 illustrates the fundamental properties of the allocation space. Section 3 provides a general mathematical programming formulation and derives an analytical expression for the Pareto frontier of the *elementary reallocation problem* (ERP). It will be shown that the *sequence of elementary reallocations* (SER) (the chain of ERP performed by agents along the interaction process) follows the algorithmic steps of a local search in the integer allocation space with exogenous prices. Section 4 introduces the case of network structures restricting agents interactions to be performed only among adjacent agents. In section 5 the performance of these barter processes is compared with the one of a global optimization algorithm (branch and cut).

2. The integer allocation space with exogenous prices

The key characteristic of an economy is: a collection \mathcal{A} of n agents, a collection \mathcal{C} of m types of commodities, a commodity space X (usually represented by the nonnegative orthant in \mathbb{R}^m), the initial endowments $e_j^i \in X$ for $i \in \mathcal{A}$, $j \in \mathcal{C}$ (representing a budget of initial amount of commodities owned by each agent), a preference relation \preceq_i on X for each agent $i \in \mathcal{A}$. Arrow and Debreu [2] showed that if the set $\{(x, y) \in X \times X : x \preceq_i y\}$ is closed relative to $X \times X$ the preference relation can be represented by a real-valued function $u^i : X \mapsto \mathbb{R}$, such that, for each a and b belonging to X , $u^i(a) \leq u^i(b)$ if and only if $a \preceq b$.

When agents attempt to simultaneously maximize their respective utilities, conditioned to balance constraints, the resulting problems are $\max u^i(\mathbf{x})$ s.to $\sum_{i \in \mathcal{A}} x_j^i = \sum_{i \in \mathcal{A}} e_j^i$ for $j \in \mathcal{C}$, where $x_j^i \in X$, is the amount of commodity j demanded by agent i (from now on the superindex shall denote the agent and the subindex shall denote the commodity).

Arrow and Debreu [2] showed that under certain economic conditions (convex preferences, perfect competition and demand independence) there must be a vector of prices $\hat{P} = (\hat{p}_1, \hat{p}_2, \hat{p}_3, \dots, \hat{p}_m)^T$, such that aggregate supplies will equal aggregate demands for every commodity in the economy.

As studied by Drze (1975) and by Auman and Drze (1986), when prices are regarded as fixed, markets do not clear and the imbalance between supply and demand is resolved by some kind of quantity rationing. The system of linear constraints associated a n -consumer- m -commodity market with fixed prices exhibits a block angular structure with rank $m+n-1$:

$$\begin{bmatrix} p_1 p_2 \dots p_m & & & & \\ & p_1 p_2 \dots p_m & & & \\ & & \ddots & & \\ & & & p_1 p_2 \dots p_m & \\ I & & I & \dots & I \end{bmatrix} \mathbf{x} = \begin{bmatrix} p_1 e_1^1 + \dots + p_m e_m^1 \\ p_1 e_1^2 + \dots + p_m e_m^2 \\ \vdots \\ p_1 e_1^n + \dots + p_m e_m^n \\ \mathbf{e}^1 + \dots + \mathbf{e}^n \end{bmatrix}, \quad (1)$$

where p_1, \dots, p_m are relative prices between commodities, $\mathbf{e}^i = (e_1^i, \dots, e_m^i)^T$, and $\mathbf{x} = (x_1^1, \dots, x_m^1, \dots, x_1^n, \dots, x_m^n)^T$. The constraints matrix of (1) could also be written as $\begin{pmatrix} I \otimes P \\ \mathbf{1} \otimes I \end{pmatrix}$,

where $P = (p_1, p_2, p_3, \dots, p_m)$ and \otimes is the Kronecker product between two matrices. Note that the linking constraints (i.e., the conservation of commodities $(\mathbf{1} \otimes I)\mathbf{x} = \mathbf{e}^1 + \dots + \mathbf{e}^n$) are implied by the balance equations of a network flow among the agents. This fact will be analyzed in Section 5, where we introduced costs associated to the flow.

All the feasible allocations lay in a $(m+n-1)$ dimensional hyperplane defined by the prices (always containing at least one solution, which is represented by the vector of initial endowments \mathbf{e}), and restricted to the fact that agents are rational: $u^i(\mathbf{x}) \geq u^i(\mathbf{e})$, for $i \in \mathcal{A}$.

Proposition 1 below shows that an asymptotic approximation of an upper bound of the number of nonnegative solutions of (1) is $\mathcal{O}(\frac{n^{(mb)}}{b^m})$, where b is the average amount of each commodity, i.e., $b = \frac{\sum_{j=1}^m (\sum_{h=1}^n e_j^h)}{m}$.

Proposition 1. *Let Λ be the set of nonnegative solutions of (1), i.e., the allocation space of a problem of bargaining integer amounts of m commodities among n agents with fixed prices. If the allocation space satisfies the mild conditions $b_j = \sum_{h=1}^n e_j^h \geq n$ and $b_j \in \mathcal{O}(n)$, $j = 1, \dots, m$ (where b_j is the overall amount of commodity j in the system), then $|\Lambda| \in \mathcal{O}(\frac{n^{(mb)}}{b^m})$.*

Proof. The set of nonnegative solutions of (1) is a subset of the union of bounded sets, as $\Lambda \subset \bigcup_{j=1}^m \{(x_j^1 \dots x_j^n) \in \mathbb{R}^n : x_j^1 + \dots + x_j^n = e_j^1 + \dots + e_j^n; x_j^1 \dots x_j^n \geq 0\}$. Therefore, Λ is a finite set, as it is the intersection between \mathbb{Z} and a bounded subset of \mathbb{R}^{mn} . Let Λ' be the set of nonnegative solutions of (1), without considering the price constraints, i.e., the n diagonal blocks $p_1 x_1^h + p_2 x_2^h + \dots + p_m x_m^h = p_1 e_1^h + p_2 e_2^h + \dots + p_m e_m^h$, for $h = 1, \dots, n$. We know that $|\Lambda'| \geq |\Lambda|$. However, $|\Lambda'|$ can be easily calculated, as the number of solutions of m independent Diophantine equations with unitary coefficients. The number of nonnegative integer solutions of any equation of the form $\sum_{h=1}^n x_j^h = b_j$, $j = 1, \dots, m$, might be seen as the number of distributions of b_j balls among m boxes: $\frac{(n+b_j-1)!}{(n-1)! b_j!}$. Since we have m independent Diophantine equations of this form, then the number of possible solutions for all of them is $\prod_{j=1}^m \frac{(n+b_j-1)!}{(n-1)! b_j!}$. Thus, we know that $|\Lambda| \leq \prod_{j=1}^m \frac{(n+b_j-1)(n+b_j-2)\dots n}{b_j!} \leq \prod_{j=1}^m \frac{(n+b_j-1)^{b_j}}{b_j!} \leq \frac{\prod_{j=1}^m (n+b_j-1)^{b_j}}{b^m}$, where the last inequality holds because $b_j \geq n \geq 2$. Since $b_j \in \mathcal{O}(n)$ we have that $\frac{\prod_{j=1}^m (n+b_j-1)^{b_j}}{b^m} \in \mathcal{O}(\frac{n^{(mb)}}{b^m})$. Hence, $|\Lambda| \in \mathcal{O}(\frac{n^{(mb)}}{b^m})$. \square

In the next section we define a barter process of integer quantities of m commodities among n agents as a local search in the allocation space Λ (obtained as a sequence of

elementary reallocations) and show that the Pareto frontier of the ERP might be analytically obtained without the use of any iterative procedure.

3. The sequence of elementary reallocations

As previously seen, the linear system characterizing the space of possible allocations is (1). Here the conservation of commodity (i.e., the overall amount of commodity of each type must be preserved) is generalized to include arbitrary weights in the last m rows of (1). Based on this observation consider the following multi-objective integer non-linear optimization problem (MINOP)

$$\max [u^i(\mathbf{x}), i = 1, \dots, n] \quad (2a)$$

s. to

$$\begin{bmatrix} P & & & & \\ & P & & & \\ & & \ddots & & \\ & & & P & \\ d^1 I & d^2 I & \dots & d^n I & \end{bmatrix} \mathbf{x} = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \\ \mathbf{b}^0 \end{bmatrix} \quad (2b)$$

$$\begin{aligned} u^i(\mathbf{x}) &\geq u^i(\mathbf{e}) \quad i = 1, \dots, n \\ \mathbf{x} &\in \mathbb{Z}^{mn} \geq 0, \end{aligned} \quad (2c)$$

where $u^i : \mathbb{R}^{mn} \rightarrow \mathbb{R}$, $P \in \mathbb{Q}^{1 \times m}$, $d^i \in \mathbb{Q}$, $b^i \in \mathbb{Q}$, $i = 1, \dots, n$, and $\mathbf{b}^0 \in \mathbb{Q}^m$. The conditions $u^i(\mathbf{x}) \geq u^i(\mathbf{e})$, $i = 1, \dots, n$, guarantee that no agent gets worse under a feasible reallocation, which is known in general bargaining literature as the *disagreement point*. The constraint matrix has a primal block-angular structure with n identical diagonal blocks involving m decision variables. Problem (1) is a particular case of (2) for $d_i = 1, i = 1, \dots, n$.

From a multi-objective optimization point of view, a suitable technique to generate the Pareto frontier of (2) is the ε -constraint method, as proposed by Haimes, Lasdon and Wismer [12]. Recently, Ozlen and Azizoglu [16] developed a general approach to generate all nondominated objective vectors, by recursively identifying upper bounds on individual objectives using problems with fewer objectives.

3.1. The elementary reallocation problem

In everyday life, barter processes among people tends to achieve the Pareto frontier of problem (2) by a sequence of reallocations. We consider a process based on a sequence of two-commodity-two-agent reallocations, denoted as SER. Any step of this sequence requires the solution of a MINOP involving 4 variables and 4 constraints of problem (2).

Let \mathbf{e} be a feasible solution of (2b) and (2c) and suppose we want to produce a feasible change of 4 variables, such that 2 of them belong to the i th and j th position of the diagonal block h and the other belong to the i th and j th position of the diagonal block k .

It can be easily shown that a feasibility condition of any affine change of these 4 variables $e_i^h + \Delta_i^h, e_j^k + \Delta_j^k, e_i^h + \Delta_i^h, e_j^k + \Delta_j^k$ is that $\Delta_i^h, \Delta_j^k, \Delta_i^h, \Delta_j^k$ must be an integer solution of the following system of equations

$$\begin{bmatrix} p_i & p_j & 0 & 0 \\ 0 & 0 & p_i & p_j \\ d^h & 0 & d^k & 0 \\ 0 & d^h & 0 & d^k \end{bmatrix} \begin{bmatrix} \Delta_i^h \\ \Delta_j^k \\ \Delta_i^h \\ \Delta_j^k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (3)$$

The solution set are the integer points in the null space of the matrix of system (3), which will be named A . A is a two-agent-two-commodity constraint matrix, and its rank is three (just note that the first column is a linear combination of the other three using coefficients $\alpha_2 = \frac{p_i}{p_j}$, $\alpha_3 = \frac{d^h}{d^k}$ and $\alpha_4 = -\frac{p_i d^h}{p_j d^k}$). Therefore the null space has dimension one, and its integer solutions are found on the line

$$\begin{bmatrix} \Delta_i^h \\ \Delta_j^h \\ \Delta_i^k \\ \Delta_j^k \end{bmatrix} = q \begin{bmatrix} p_j d^k \\ -p_i d^k \\ -p_j d^h \\ p_i d^h \end{bmatrix}, \quad (4)$$

for some $q = \alpha F(p_i, p_j, d^k, d^h)$, where $\alpha \in \mathbb{Z}$ and $F : \mathbb{Q}^4 \rightarrow \mathbb{Q}$ provides a factor which transforms the null space direction in the nonzero integer null space direction of smallest norm. We note that this factor can be computed as $F(p_i, p_j, d^k, d^h) = G(p_j d^k, p_i d^k, p_j d^h, p_i d^h)$, where

$$G(v_i = \frac{r_i}{q_i}, i = 1, \dots, l) = \frac{\text{lcm}(q_i, i = 1, \dots, l)}{\text{gcd}(\text{lcm}(q_i, i = 1, \dots, l) \cdot v_i, i = 1, \dots, l)}, \quad (5)$$

r_i and q_i being the numerator and denominator of v_i ($q_i = 1$ if v_i is integer), and lcm and gcd being, respectively, the least common multiple and greatest common divisor functions.

Hence, given a feasible point \mathbf{e} , one can choose 4 variables, such that 2 of them belong to the i th and j th position of a diagonal block h and the others belong to the i th and j th position of a diagonal block k , in $m(m-1)n(n-1)/4$ ways. Each of them constitutes an ERP, whose Pareto frontier is in $\mathbf{e} + \text{null}(A)$. The SER is a local search, which repeatedly explores a neighborhood and chooses both a locally improving direction among the $m(m-1)n(n-1)/4$ possible ERPs and a feasible step length $q = \alpha F(p_i, p_j, d^k, d^h)$, $\alpha \in \mathbb{Z}$. For problems of the form of (2) the SER might be written as follows:

$$\mathbf{x}^{t+1} = \mathbf{x}^t + \alpha F(p_i, p_j, d^k, d^h) \begin{bmatrix} \vdots \\ p_j d^k \\ \vdots \\ -p_i d^k \\ \vdots \\ -p_j d^h \\ \vdots \\ p_i d^h \\ \vdots \end{bmatrix} \begin{bmatrix} \vdots \\ h, i \\ \vdots \\ h, j \\ \vdots \\ k, i \\ \vdots \\ k, j \\ \vdots \end{bmatrix} = \mathbf{x}^t + \alpha F(p_i, p_j, d^k, d^h) \Delta_{ij}^{kh}, \quad (6)$$

t being the iteration counter. In shorter notation, we write (6) as $\mathbf{x}^{t+1} = \mathbf{x}^t + \alpha S_{ij}^{kh}$, where

$$S_{ij}^{kh} = F(p_i, p_j, d^k, d^h) \Delta_{ij}^{kh} \quad (7)$$

is a direction of integer components. Since the nonnegativity of \mathbf{x} have to be kept along the iterations, then we have that

$$-\frac{\max \left\{ x_i^h / (p_j d^k), x_j^k / (p_i d^h) \right\}}{F(p_i, p_j, d^k, d^h)} \leq \alpha \leq \frac{\min \left\{ x_j^h / (p_i d^k), x_i^k / (p_j d^h) \right\}}{F(p_i, p_j, d^k, d^h)}, \quad (8)$$

or, equivalently,

$$-\max \left\{ x_i^h / (p_j d^k), x_j^k / (p_i d^h) \right\} \leq q \leq \min \left\{ x_j^h / (p_i d^k), x_i^k / (p_j d^h) \right\}. \quad (9)$$

(The step length is forced to be nonnegative when the direction is both feasible and a descent direction; in our case the direction is only known to be feasible, and then negative step lengths are also considered.)

An important property of an elementary reallocation is that under the assumptions that $\frac{\partial u^k(\mathbf{x})}{\partial x_i^k} : \mathbb{R}^{mn} \rightarrow \mathbb{R}$ is (i) non increasing, (ii) nonnegative and (iii) $\frac{\partial u^k(\mathbf{x})}{\partial x_i^j} = 0$ for $j \neq k$ (i.e., u^k only depends on \mathbf{x}^k), which are quite reasonable requirements for consumer utilities, then $u^k(\mathbf{x} + \alpha S_{ij}^{kh})$ is a unimodal function with respect to α , as shown by the next proposition.

Proposition 2. *Under the definition of u^k and S_{ij}^{kh} , for every feasible point $\mathbf{x} \in R^{mn}$, $u^k(\mathbf{x} + \alpha S_{ij}^{kh})$ is a unimodal function with respect to α in the interval defined by (8).*

Proof. Let us define $g(\alpha) = u^k(\mathbf{x} + \alpha S_{ij}^{kh})$, differentiable with respect to α . It will be shown that for all α in the interval (8), and $0 < \tau \in \mathbb{R}$, $g'(\alpha) < 0$ implies $g'(\alpha + \tau) < 0$, which is a sufficient condition for the unimodality of $g(\alpha)$. By the chain rule, and using (6) and (7), the derivative of $g(\alpha)$ can be written as

$$\begin{aligned} g'(\alpha) &= \nabla_{\mathbf{x}} u^k(\mathbf{x} + \alpha S_{ij}^{kh}) S_{ij}^{kh} \\ &= F(p_i, p_j, d^k, d^h) \left(\frac{\partial u^k(\mathbf{x} + \alpha S_{ij}^{kh})}{\partial x_i^k} (-p_j d^h) + \frac{\partial u^k(\mathbf{x} + \alpha S_{ij}^{kh})}{\partial x_j^k} p_i d^h \right). \end{aligned} \quad (10)$$

If $g'(\alpha) < 0$ then, from (10) and since $F(p_i, p_j, d^k, d^h) > 0$, we have that

$$\frac{\partial u^k(\mathbf{x} + \alpha S_{ij}^{kh})}{\partial x_i^k} p_j d^h > \frac{\partial u^k(\mathbf{x} + \alpha S_{ij}^{kh})}{\partial x_j^k} p_i d^h. \quad (11)$$

Since from (6) the component (k, i) of S_{ij}^{kh} is $F(p_i, p_j, d^k, d^h)(-p_j d^h) < 0$, and $\frac{\partial u^k(\mathbf{x})}{\partial x_i^k}$ is non increasing, we have that for $\tau > 0$

$$\frac{\partial u^k(\mathbf{x} + (\alpha + \tau) S_{ij}^{kh})}{\partial x_i^k} \geq \frac{\partial u^k(\mathbf{x} + \alpha S_{ij}^{kh})}{\partial x_i^k}. \quad (12)$$

Similarly, since the component (k, j) of S_{ij}^{kh} is $F(p_i, p_j, d^k, d^h)(p_i d^h) > 0$, we have

$$\frac{\partial u^k(\mathbf{x} + \alpha S_{ij}^{kh})}{\partial x_j^k} \geq \frac{\partial u^k(\mathbf{x} + (\alpha + \tau) S_{ij}^{kh})}{\partial x_j^k}. \quad (13)$$

Multiplying both sides of (12) and (13) by, respectively, $p_j d^h$ and $p_i d^h$, and connecting the resulting inequalities with (11) we have that

$$\frac{\partial u^k(\mathbf{x} + (\alpha + \tau) S_{ij}^{kh})}{\partial x_i^k} p_j d^h > \frac{\partial u^k(\mathbf{x} + (\alpha + \tau) S_{ij}^{kh})}{\partial x_j^k} p_i d^h,$$

which proves that $g'(\alpha + \tau) < 0$. \square

Using Proposition 2 and the characterization of the space of integer solutions of (3), we are able to derive a closed expression of the Pareto frontier of the ERP, based on the behavior of $u(\mathbf{x} + \alpha S_{ij}^{kh})$ (see Corollary 1 below), as it is shown in this example:

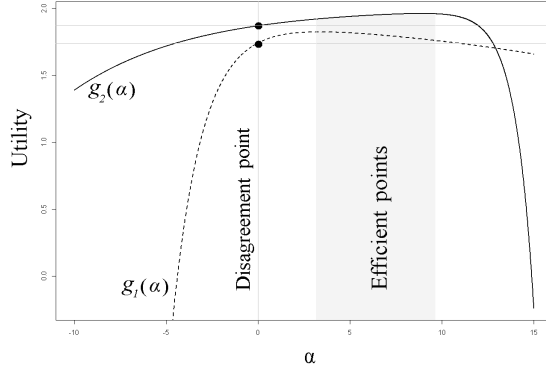


Figure 1: Plots of $g^1(\alpha)$ and $g^2(\alpha)$, and interval of α associated to the Pareto frontier. The disagreement point corresponds to $g^1(0)$ and $g^2(0)$, the utilities in the current iterate.

Example 1. Consider the following ERP with initial endowments [40, 188, 142, 66].

$$\max [2 - e^{-0.051x_1^1} - e^{-0.011x_2^1}, 2 - e^{-0.1x_1^2} - e^{-0.031x_2^2}]$$

s. to

$$\begin{aligned} 5x_1^1 + 10x_2^1 &= 2080 \\ 5x_1^2 + 10x_2^2 &= 1370 \end{aligned}$$

$$\begin{aligned} 5x_1^1 + 6x_2^1 &= 1052 \\ 5x_2^1 + 6x_2^2 &= 1336 \end{aligned} \tag{14}$$

$$\begin{aligned} 2 - e^{-0.051x_1^1} - e^{-0.011x_2^1} &\geq 1.68 \\ 2 - e^{-0.1x_1^2} - e^{-0.031x_2^2} &\geq 1.50 \end{aligned}$$

$$x_j^i \geq 0 \in Z \quad i = 1, 2; \quad j = 1, 2;$$

The utility functions $g^1(\alpha) = u^1(\mathbf{x} + \alpha S_{12}^{12})$ and $g^2(\alpha) = u^2(\mathbf{x} + \alpha S_{12}^{12})$ are

$$g^1(\alpha) = u^1(\mathbf{x} + \alpha S_{12}^{12}) = u^1 \left(\begin{bmatrix} 40 \\ 188 \\ 142 \\ 66 \end{bmatrix} + \alpha \begin{bmatrix} 12 \\ -6 \\ -10 \\ 5 \end{bmatrix} \right) = 2 - e^{-0.051(40+12\alpha)} - e^{-0.011(188-6\alpha)}$$

$$g^2(\alpha) = u^2(\mathbf{x} + \alpha S_{12}^{12}) = u^2 \left(\begin{bmatrix} 40 \\ 188 \\ 142 \\ 66 \end{bmatrix} + \alpha \begin{bmatrix} 12 \\ -6 \\ -10 \\ 5 \end{bmatrix} \right) = 2 - e^{-0.1(142-10\alpha)} - e^{-0.031(66+5\alpha)},$$

which are plotted in Figure 1. The continuous optimal step lengths for the two respective agents are $\operatorname{argmax} g^1(\alpha) = 3.33$ and $\operatorname{argmax} g^2(\alpha) = 8.94$. Due to the unimodality of $u^k(\mathbf{x} + \alpha S_{ij}^{hk})$, all efficient solutions of (14) are given by integer step lengths $\alpha \in [3.33, 8.94]$ (see Fig. 1), i.e., for $\alpha \in \{4, 5, 6, 7, 8\}$ we have

$$\begin{aligned} g^1(4) &= 1.82412 & g^1(5) &= 1.81803 & g^1(6) &= 1.80882 & g^1(7) &= 1.79752 & g^1(8) &= 1.78465, \\ g^2(4) &= 1.93043 & g^2(5) &= 1.94035 & g^2(6) &= 1.94873 & g^2(7) &= 1.95558 & g^2(8) &= 1.96057. \end{aligned}$$

Due to the unimodality of both utility functions with respect to α , no efficient solution exists for an α outside the segment [3.33, 8.94].

The above example illustrates a case where the segment between $\operatorname{argmax} u^h(\mathbf{x} + \alpha S_{ij}^{kh})$ and $\operatorname{argmax} u^k(\mathbf{x} + S_{ij}^{kh})$ contains five integer points, associated with the feasible step lengths.

The following statements give a constructive characterization of the Pareto frontier of an ERP for the case of concave utility function and linear utility functions respectively.

Corollary 1. *Let Γ be the set of integer points in the interval $[a^{\text{down}}, a^{\text{up}}]$, where $a^{\text{down}} = \min\{\text{argmax}_{\alpha} u^k(\mathbf{x} + \alpha S_{ij}^{kh}), \text{argmax}_{\alpha} u^h(\mathbf{x} + \alpha S_{ij}^{kh})\}$ and $a^{\text{up}} = \max\{\text{argmax}_{\alpha} u^k(\mathbf{x} + \alpha S_{ij}^{kh}), \text{argmax}_{\alpha} u^h(\mathbf{x} + \alpha S_{ij}^{kh})\}$, and let $[\alpha^{\text{down}}, \alpha^{\text{up}}]$ be the interval of feasible step lengths defined in (8). Then, due to Proposition 2, the set \mathcal{V}^* of Pareto efficient solutions of an ERP can be obtained as follows:*

- i. $\mathcal{V}^* = \{[u^h(\mathbf{x} + \alpha S_{ij}^{kh}), u^k(\mathbf{x} + \alpha S_{ij}^{kh})] : \alpha \in \Gamma\}$ if Γ is not empty and does not contain the zero.
- ii. If Γ is empty and there exists an integer point between 0 and a^{down} but no integer point between a^{up} and α^{up} then \mathcal{V}^* contains the unique point given by $[u^h(\mathbf{x} + \alpha S_{ij}^{kh}), u^k(\mathbf{x} + \alpha S_{ij}^{kh})]$ such that α is the greatest integer between 0 and a^{down} .
- iii. If Γ is empty and there exists an integer point between a^{up} and α^{up} but no integer point between 0 and a^{down} then \mathcal{V}^* contains either the unique point given by $[u^h(\mathbf{x} + \alpha S_{ij}^{kh}), u^k(\mathbf{x} + \alpha S_{ij}^{kh})]$ such that α is the smallest integer between a^{up} and α^{up} , or $\alpha = 0$, or both of them if they do not dominate each other. (In this case the three possibilities must be checked, since if for only one of the utilities —let it be h , for instance— $u^h(\mathbf{x}) > u^h(\mathbf{x} + \bar{\alpha} S_{ij}^{kh})$, $\bar{\alpha}$ being the smallest integer between a^{up} and α^{up} , then both values 0 and $\bar{\alpha}$ are Pareto efficient.)
- iv. If Γ is empty and there are integer points both between a^{up} and α^{up} and between 0 and a^{down} then \mathcal{V}^* contains the points given by $[u^h(\mathbf{x} + \alpha S_{ij}^{kh}), u^k(\mathbf{x} + \alpha S_{ij}^{kh})]$ such that α is either the smallest integer between a^{up} and α^{up} , or the greatest integer between 0 and a^{down} , or both points if they do not dominate each other.
- v. In the case that Γ contains the zero, then no point dominates the initial endowment \mathbf{x} , so that the only point in the Pareto frontier is \mathbf{x} .

Corollary 2. *Consider the case of an economy where agents have linear utility functions with gradients $\mathbf{c}^1, \dots, \mathbf{c}^n$ and let again Γ be the set of integer points in the interval $[a^{\text{down}}, a^{\text{up}}]$, where $a^{\text{down}} = \min\{\text{argmax}_{\alpha} \alpha \mathbf{c}^k S_{ij}^{kh}, \text{argmax}_{\alpha} \alpha \mathbf{c}^h S_{ij}^{kh}\}$ and $a^{\text{up}} = \max\{\text{argmax}_{\alpha} \alpha \mathbf{c}^k S_{ij}^{kh}, \text{argmax}_{\alpha} \alpha \mathbf{c}^h S_{ij}^{kh}\}$, and let $[\alpha^{\text{down}}, \alpha^{\text{up}}]$ be the interval of feasible step lengths defined in (8). It might be easily seen that either $\Gamma = \mathbb{Q}$ or $\Gamma = \emptyset$. The set $\Gamma = \mathbb{Q}$ in the case $(c_i^h p_j d^k - c_j^h p_i d^k)$ and $(c_j^k p_i d^h - c_i^k p_j d^h)$ have opposite signs, whereas $\Gamma = \emptyset$ if $(c_i^h p_j d^k - c_j^h p_i d^k)$ and $(c_j^k p_i d^h - c_i^k p_j d^h)$ have the same sign. Then, due to Proposition 2, the set \mathcal{V}^* of Pareto efficient solutions of an ERP may contain at most one point:*

- i. if there is at least one non-null integer between $-\max\{x_i^h/(p_j d^k), x_j^k/(p_i d^h)\}/F(p_i, p_j, d^k, d^h)$ and $\min\{x_j^h/(p_i d^k), x_i^k/(p_j d^h)\}/F(p_i, p_j, d^k, d^h)$ and $\Gamma = \emptyset$, then \mathcal{V}^* only contains the unique point corresponding to the allocation $\mathbf{x}^{t+1} = \mathbf{x}^t + \alpha S_{ij}^{kh}$ for a step-length α which is either equal to $-\max\{x_i^h/(p_j d^k), x_j^k/(p_i d^h)\}/F(p_i, p_j, d^k, d^h)$ (if $(c_i^h p_j d^k - c_j^h p_i d^k)$ and $(c_j^k p_i d^h - c_i^k p_j d^h)$ are negative) or for equal to $\min\{x_j^h/(p_i d^k), x_i^k/(p_j d^h)\}/F(p_i, p_j, d^k, d^h)$ (if $(c_i^h p_j d^k - c_j^h p_i d^k)$ and $(c_j^k p_i d^h - c_i^k p_j d^h)$ are positive).
- ii. \mathcal{V}^* only contains the disagreement point in the opposite case.

Having a characterization of the Pareto frontier for any ERP in the sequence allows not just a higher efficiency in simulating the process but also the possibility of measuring the number of non dominated endowments of each of the $m(m-1)n(n-1)/4$ ERPs, which might be used as a measure of uncertainty of the process. Indeed, the uncertainty of a barter process of this type might come from different sides: i) how to choose the couple of agents and commodities in each step? ii) which Pareto efficient solution of each ERP to use to update the endowments of the system? In the next subsection we shall study different criteria for answering the first two questions.

Note that the set of non-dominated solutions of the ERP, obtained by the local search movement (6) might give rise to imbalances between supply and demand, as described by Dreze [9] for the continuous case. To resolve this imbalance Dreze introduce a quantity rationing, which can be also extended to the ERP.

We define a rationing scheme for the ERP as a pair of vectors $l \in \mathbb{Z}^m$, $L \in \mathbb{Z}^m$, with $L \geq 0 \geq l$, such that for a specified vector of relative prices $P \in \mathbb{Z}^m$, the allocation space is redefined by introducing in (1) the additional constraints $l_i \leq \mathbf{x}_i - \mathbf{e}_i \leq L_i$, for $i = 1, \dots, n$, where \mathbf{e}_i and \mathbf{x}_i are n -dimensional vectors associated to the initial endowment and to the allocation of the i^{th} commodities among the n agents respectively; l_i and L_i are the i^{th} components of l and L respectively. Since in each ERP the vector \mathbf{x}^t is the current allocation at the t^{th} interaction, a local rationing scheme might be defined $l_i \leq \mathbf{x}_i^{t+1} - \mathbf{x}_i^t \leq L_i$. Thus, for two given agents h and k and two given commodities i and j we have

$$l_i \leq \alpha F(p_i, p_j, d^k, d^h) \begin{bmatrix} \vdots \\ p_j d^k \\ \vdots \\ -p_j d^h \\ \vdots \end{bmatrix} \leq L_i, \quad l_j \leq \alpha F(p_i, p_j, d^k, d^h) \begin{bmatrix} \vdots \\ -p_i d^k \\ \vdots \\ p_i d^h \\ \vdots \end{bmatrix} \leq L_j, \quad (15)$$

which only requires to introduce two additional inequalities to the range of feasible step-length:

$$-\frac{L_i}{F(p_i, p_j, d^k, d^h) \min \{p_j d^h, p_i d^k\}} \leq \alpha \leq -\frac{l_i}{F(p_i, p_j, d^k, d^h) \max \{p_j d^h, p_i d^k\}}, \quad (16a)$$

$$\frac{l_i}{F(p_i, p_j, d^k, d^h) \min \{p_j d^k, p_i d^h\}} \leq \alpha \leq \frac{L_i}{F(p_i, p_j, d^k, d^h) \max \{p_j d^k, p_i d^h\}}. \quad (16b)$$

An open problem, which is not investigated in this paper, is the formulation of equilibrium conditions for this rationing scheme. One possibility might be the construction of two intervals for l and L which minimize the overall imbalances, under the conditions that (16) is verified in each ERP, as long as l and L are inside the respected intervals. The integrality of the allocation space Λ forbids a straightforward application of the equilibrium criteria proposed by Dreze [9] to the markets we are considering in this work.

3.2. Taking a unique direction of movement

The sequence of elementary reallocations formalized in (3) requires the iterative choice of couples of agents (h, k) and couples of commodities (i, j), i.e., directions of movement among the $m(m-1)n(n-1)/4$ in the neighborhood of the current solution. If we this choice is based on a welfare function (summarizing the utility functions of all the agents), the selection of of couples of agents and couples of commodities can be made mainly in two different ways: first improving and best improving directions of movement.

The best improving direction requires an exhaustive exploration of the neighborhood. Noting that each direction of movement in the current neighborhood constitutes a particular ERP, a welfare criterion might be the uncertainty of each elementary reallocation, measured by the number of points in the Pareto frontier of ERPs, as described in the previous subsection. A usual welfare criterion is a norm of the objective vector (e.g., Euclidean, L_1 or L_∞ norms). Also the average marginal rate of substitution could represent an interesting criterion to select the direction of movement as a high marginal rate of substitution suggests a kind of mismatch between preferences and endowments.

If at iteration t an improving direction exists the respective endowments are updated in accordance with the solution of the selected ERP: for each couple of commodities (i, j) and each couple of agents (h, k) , agent k gives $\alpha F(p_i, p_j, d^k, d^h) p_j d^k$ units of i to agent h and in return he/she gets $\alpha F(p_i, p_j, d^k, d^h) p_i d^k$ units of j , for some $\alpha \in \mathbb{Z}$. At iteration $t + 1$, a second couple of commodities and agents is considered in accordance with the defined criterion. If we use a first improving criterion, the process stops when the endowments keep in *status quo* continuously during $m(m-1)n(n-1)/4$ explorations, i.e., when no improving direction is found in the current neighborhood.

3.3. Observing the paths of all improving directions of movement

When simulating social systems it might be interesting to enumerate all possible stories which are likely to be obtained starting from the known initial point. In this subsection we introduce a method to enumerate possible paths exclusively based on the Pareto efficiency of each elementary reallocation.

The idea is to solve $m(m-1)n(n-1)/4$ ERPs and keep all the efficient solutions generated. If in a given iteration we have r non-dominated solutions, and observe $l_i \leq m(m-1)n(n-1)/4$, for $i = 1, \dots, r$, Pareto improving directions, with $f_{i,j}$ for $j = 1 \dots l_i$ efficient solutions for each of them, we would expect some of the $r + \sum_{i=1}^r \sum_{j=1}^{l_i} f_{i,j}$ solutions to be non-dominated by some others and the incumbent should be updated by adding to the r previous solutions those which are non-dominated and removing those which are dominated by some other. From the point of view of a local search, the incumbent solution of this process is not a unique point in the allocation space but a collection of points which Pareto-dominate the initial endowment and do not dominate each other.

This procedure requires a method to find Pareto-optimal vectors each time $m(m-1)n(n-1)/4$ ERPs are solved. Corley and Moon [7] proposed an algorithm to find the set V^* of Pareto vectors among r given vectors $\mathcal{V} = \{v_1, v_2, \dots, v_r\}$, where $v_i = (v_{i1}, v_{i2}, \dots, v_{in}) \in \mathbb{R}^n$, $i = 1, 2, \dots, r$. Sastry and Mohideen [20] observed that the latter algorithm is incorrect and presented a modified version. In our implementation of the the best-improve barter process, we use the modified Corley and Moon algorithm of [20], shown below.

Step 0. Set $\mathcal{V}^* = \emptyset$.

Step 1. Set $i = 1, j = 2$.

Step 2. If $i = r - 1$, go to Step 6. For $k = 1, 2, \dots, n$, if $v_{jk} \geq v_{ik}$, then go to Step 3; else, if $v_{ik} \geq v_{jk}$, then go to Step 4; otherwise, go to Step 5.

Step 3. Set $i = i + 1, j = i + 1$; go to Step 2.

Step 4. If $j = r$, set $\mathcal{V}^* = \mathcal{V}^* \cup \{v_i\}$ and $v_j = \{\infty, \infty, \dots, \infty\}$ go to Step 3; otherwise, set $v_{jk} = v_{rk}$, where $k = 1, 2, \dots, n$. Set $r = r - 1$ and go to Step 2.

Step 5. If $j = r$, set $\mathcal{V}^* = \mathcal{V}^* \cup \{v_i\}$ go to Step 3; otherwise, set $j = j + 1$ and go to Step 2.

Step 6. For $k = 1, 2, \dots, n$, if $v_{jk} \geq v_{ik}$, then set $\mathcal{V}^* = \mathcal{V}^* \cup \{v_j\}$ and stop; else, if $v_{ik} \geq v_{jk}$, then set $\mathcal{V}^* = \mathcal{V}^* \cup \{v_i\}$ and stop. Otherwise, let $\mathcal{V}^* = \mathcal{V}^* \cup \{v_i, v_j\}$. Return \mathcal{V}^* .

The nice property of the modified Corley and Moon algorithm is that it doesn't necessarily compare each of the $r(r-1)/2$ couples of vectors for each of the n components. This is actually what the algorithm do in the worst case, so that the complexity could be written as $\mathcal{O}(nr^2)$, which is linear with respect of the dimension of the vectors and quadratic with respect to the number of vectors. For the case of linear utilities, the next subsection provides a small numerical example and the pseudo-code of the procedure used to enumerate the paths of all possible stories.

3.4. Linear utilities

In microeconomic theory the utility functions are rarely linear, however the case of linear objectives appears particularly suitable from an optimization point of view and allows a remarkable reduction of operations, as the ERPs cannot have more than one Pareto-efficient solution (see Corollary 1).

Consider a given direction of movement S_{ij}^{kh} . We know that a feasible step length α belongs to the interval defined by (8). Since in the case of one linear objective the gradient is constant, for any direction of movement (i, j, k, h) the best Pareto improvement (if there exists one) must happen in the endpoints of the feasible range of α (let $\alpha^{down}(i, j, k, h)$ and $\alpha^{up}(i, j, k, h)$ denote the left and right endpoints of the feasible range of α , when the direction of movement is (i, j, k, h)). Therefore, the line search reduces to decide either $\alpha^{down}(i, j, k, h)$, $\alpha^{up}(i, j, k, h)$ or none of them. Then for every given point \mathbf{x} , we have a neighborhood of at most $m(m-1)n(n-1)/2$ candidate solutions. The pseudo-code to generate all sequences of elementary reallocations for n linear agents, keeping the Pareto-improvement in each interaction, is shown in Algorithm 1.

Algorithm 1 Generating paths of all improving directions of movement

- 1: Initialize the endowments $E = \langle \mathbf{e}^1, \dots, \mathbf{e}^n \rangle$ and utilities $U = \langle u^1, \dots, u^n \rangle$.
 - 2: Initialize the incumbent allocations $\tilde{E}^t = \{E\}$ and the incumbent utilities $\tilde{U}^t = \{U\}$.
 - 3: **repeat**
 - 4: **for** $\mathbf{x} \in \tilde{E}^t$ **do**
 - 5: Let $\langle S_{\mathbf{x}}, G_{\mathbf{x}} \rangle$ be the set of movements and utilities $\{(\mathbf{x} + \alpha S_{ij}^{kh}, c^T(\mathbf{x} + \alpha S_{ij}^{kh}))\}$ for each couple of commodities and agents (i, j, k, h) and $\alpha \in \{\alpha^{down}(i, j, k, h), \alpha^{up}(i, j, k, h)\}$
 - 6: **end for**
 - 7: Let $\langle S, G \rangle = \bigcup_{\mathbf{x} \in \tilde{E}^t} \langle S_{\mathbf{x}}, G_{\mathbf{x}} \rangle$
 - 8: Let $\langle S, G \rangle = \text{CorleyMoon}(\langle S, G \rangle)$
 - 9: Let $\tilde{E}^{t+1} = \tilde{E}^t \cup S$ and $\tilde{U}^{t+1} = \tilde{U}^t \cup G$
 - 10: **until** $\tilde{E}^t = \tilde{E}^{t-1}$
-

The function $\text{CorleyMoon}()$ applies the modified Corley and Moon algorithm to a set of utility vectors and allocation vectors and returns the Pareto-efficient utility vectors with the associated allocations.

Despite the idea behind the SER is a process among self-interested agents, which are by definition local optimizers, this algorithm could also be applied to any integer linear programming problem of the form of (2) with one linear objective: $u(\mathbf{x}) = c^T \mathbf{x}$. In this case however the branch and cut algorithm is much more efficient even for big instances, as we will show in the next section.

If a first-improve method is applied, an order of commodities and agents is required when exploring the neighborhood and the equilibrium allocation might be highly affected by this order (path-dependence). The pseudocode of algorithm 2 describes the first improve search of the barter algorithm applied to the case of one linear objective function.

Note that if the nonnegativity constraints are not taken into account, problem (2) is unbounded for linear utility functions. This corresponds to the fact that without lower

bounds the linear version of this problem would make people infinitely get into debt. As a consequence, the only possible stopping criterion, when the objective function is linear, is the fulfillment of nonnegativity constraints, i.e. a given point \mathbf{x} is a final endowment (an equilibrium of the barter process) if we have that for any direction of movement and for any given integer α if $c^T(\mathbf{x} + \alpha S_{ij}^{kh}) > c^T \mathbf{x}$ then $\mathbf{x} + \alpha S_{ij}^{kh}$ has some negative component. In some sense the optimality condition is now only based on feasibility.

Algorithm 2 First-improve SER with linear utility function

- 1: Initialize the endowments $E = \langle \mathbf{e}^1, \dots, \mathbf{e}^n \rangle$ and utilities $U = \langle u^1, \dots, u^n \rangle$.
 - 2: Let $t = 0$;
 - 3: Let (i, j, k, h) be the t^{th} direction in the order set of directions;
 - 4: **if** $c^T(x + \alpha^{down}(i, j, k, h)S_{ij}^{kh}) > c^T(x + \alpha^{up}(i, j, k, h)S_{ij}^{kh})$ and $c^T(x + \alpha^{down}(i, j, k, h)S_{ij}^{kh}) > c^T(x)$ **then**
 - 5: Update the incumbent $x = x + \alpha^{down}(i, j, k, h)S_{ij}^{kh}$ and GOTO 3;
 - 6: **else if** $c^T(x + \alpha^{up}(i, j, k, h)S_{ij}^{kh}) > c^T(x + \alpha^{down}(i, j, k, h)S_{ij}^{kh})$ and $c^T(x + \alpha^{up}(i, j, k, h)S_{ij}^{kh}) > c^T(x)$ **then**
 - 7: Update the incumbent $x = x + \alpha^{up}(i, j, k, h)S_{ij}^{kh}$ and GOTO 3;
 - 8: **else**
 - 9: $t = t + 1$;
 - 10: **if** $t < m(m-1)n(n-1)$ **then**
 - 11: GOTO 4;
 - 12: **else**
 - 13: RETURN
 - 14: **end if**
 - 15: **end if**
-

3.5. The final allocation and the convergence of the SER

For the case of continuous commodity space and exogenous prices, Feldman [11] proved that as long as all agents are initially endowed with some positive amount of a commodity, pairwise optimality implies global optimality. Unfortunately, the SER described in this paper does not necessarily lead to Pareto efficient endowments. Let $T_{\mathbf{x}}(\alpha) = \mathbf{x} + \sum_{k \neq h} \sum_{i \neq j} \alpha(i, j, k, h)S_{ij}^{kh}$, representing a simultaneous reallocation of m commodities among n agents, with step length α_{ij}^{kh} for each couple of commodities ij and agents hk , starting from $\mathbf{x} \in \Lambda$. Whereas a SER is required to keep feasibility along the process, a simultaneous reallocation $T_{\mathbf{x}}(\alpha)$ of m commodities among n agents does not consider the particular path and any feasibility condition on the paths leading from \mathbf{x} to $T_{\mathbf{x}}(\alpha)$. Hence, remembering that all SERs described in this section stop when no improving elementary reallocation exists in the current neighborhood, we can conclude that the non existence of a feasible improving ER does not entail the non existence of an improving simultaneous reallocation of m commodities among n agents. In this sense a SER provides a lower bound of any sequence of reallocations of more than two commodities and two agents at a time.

Axtell [4] studied sequences of k -lateral trades with local Walrasian prices and observed that the convergence to the equilibrium is linear. A similar reasoning could be applied to the SER.

Proposition 3. *The rate of convergence of the SER is linear.*

Proof. Consider the Lyapunov function $U(t) = \sum_{i=1}^n u^i(\mathbf{x}^t)$, associating a real value to each point in the allocation space [23]. As $U(t)$ increases monotonically along the SER (6) and the allocation space is a finite set, then $\lim_{t \rightarrow \infty} U(t) = U^*$.

Since each iteration of the SER makes at least one agent strictly better off without producing any change in the others, then $U(t+1) > U(t)$ and $\frac{|U^* - U(t+1)|}{|U^* - U(t)|} = r < 1$ for all t sufficiently large. \square

4. Bartering on networks

An important extension of the problem of bargaining integer amounts of m commodities among n agents with fixed prices is to define a network structure such that trades among agents are allowed only for some couples of agents who are linked in this network. In this case the conservation of commodities $d^1 \mathbf{x}^1 + d^2 \mathbf{x}^2 + \dots + d^n \mathbf{x}^n = d^1 \mathbf{e}^1 + d^2 \mathbf{e}^2 + \dots + d^n \mathbf{e}^n$ is replaced by balance equations on a network, so that the final allocation of commodity i must verify $A \mathbf{y}_i = D(\mathbf{x}_i - \mathbf{e}_i)$, where \mathbf{y}_i is the flow of commodity i in the system, A is the incidence matrix, and D is a $n \times n$ diagonal matrix containing the weights of the conservation of commodity i , that is $D = \text{diag}(d^1 \dots d^n)$ (for more details on network flows problems see [1]).

It is also possible for the final allocation to have a given maximum capacity, that is, an upper bound of the amount of commodity i that agent h may hold: $x_i^h \leq \bar{x}_i^h$.

The variables of the problem are now x_i^h , which again represent the amount of commodity i hold by agent h , s_i^h which are the slack variables for the upper bounds, and $y_i^{h,k}$ which are the flow of commodity i from agent h to agent k .

The objective functions $\tilde{u}^i(\mathbf{x}, \mathbf{y})$, $i = 1 \dots n$, might depend on both the final allocation \mathbf{x} and the interactions \mathbf{y} , since the network topology could represent a structure of geographical proximity and reachability.

The resulting mathematical programming formulation of the problem of bargaining integer commodities with fixed prices among agents on a network with upper bounds on the final allocations is as follows:

$$\max [\tilde{u}^i(\mathbf{x}, \mathbf{y}), i = 1, \dots, n] \quad (17a)$$

s. to

$$\left[\begin{array}{c|c|c} P & & \\ & \ddots & \\ & & P \\ \hline I & & I \\ & \ddots & \\ & & I \\ \hline & & A \\ \mathbb{D} & & \\ & & \ddots \\ & & A \end{array} \right] \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} b^1 \\ \vdots \\ b^n \\ \bar{x}^1 \\ \vdots \\ \bar{x}^n \\ \mathbf{b}^0 \end{bmatrix} \quad (17b)$$

$$u^i(\mathbf{x}, \mathbf{y}) \geq u^i(\mathbf{e}, \mathbf{0}) \quad i = 1 \dots, n \quad (17c)$$

$$\mathbf{x} \in \mathbb{Z}^{mn} \geq 0, \quad \mathbf{y} \in \mathbb{Z}^{mn(n-1)} \geq 0,$$

where $\tilde{u}^i : \mathbb{R}^{mn} \rightarrow \mathbb{R}$, $P \in \mathbb{Q}^{1 \times m}$, $\mathbb{D} \in \mathbb{Q}^{mn \times mn}$, $b^i \in \mathbb{Q}$, $i = 1, \dots, n$, $A \in \mathbb{Q}^{n \times n(n-1)}$, and $\mathbf{b}^0 \in \mathbb{Q}^{nm}$. Matrix \mathbb{D} is an appropriate permutation of the diagonal matrix made of m copies

of the matrix D with the weights of the conservation of commodity and $\tilde{u}^i(\mathbf{e}, \mathbf{0})$ is the utility function of agent i evaluated in the initial endowments \mathbf{e} with null flow.

Problem (2) had mn variables and $m+n$ constraints, whereas problem (17) has $mn(n+1)$ variables and $n(1+2m)$ constraints. When a SER is applied, the definition of a network structure and the application of upper bounds to the final allocation reduce the number of feasible directions of movement in each iteration and the bound of the interval of feasible step length, as for any incumbent allocation \mathbf{x} , the step length α must be such that $0 \leq \mathbf{x} + \alpha S_{ij}^{kh} \leq \tilde{\mathbf{x}}$.

The effect of network structures on the performance of a barter process has been previously studied by Bell [5] and by Wilhite [25], for the case of endogenous prices and continuous commodity space. In this case the process takes into account how agents update prices each time they perform a bilateral trade. Reasonably, prices should be updated based either on the current state of the only two interacting agents or on the state of the overall population or also on the history of the system, such as previous prices. Bell showed that centralized network structures, such as a stars, exhibit a faster convergence to an equilibrium allocation.

It must be noted that any sequence of bilateral trades intrinsically gives rise to a network structure generated by the set of couples of agents interacting along the process. Such a structure might be statistically analyzed in term of its topological properties, as it is done in the next section with a battery of problems of different sizes.

5. Computational results

We have already seen that a SER can also be applied to any integer linear programming problem of the form (2), where the individual utilities are aggregated in a single welfare function. If this aggregated welfare is defined as a linear function of the endowments of the form $u(\mathbf{x}) = c^T \mathbf{x}$, the comparison of the SERs with the standard branch and cut algorithm is easily carried out. Considering the ERP as the basic operation of a SER and the simplex iteration as the basic operation of the branch and cut algorithm, the comparison between the two methods is numerically shown in Table 1 for three replications of 11 problems with the same number of agents and commodities, which amounts to 33 instances. The branch and cut implementation of the state-of-the-art optimization solver Cplex was used.

In the special case of a unique linear utility function a system of many local optimizers (agents) could be highly inefficient if compared with a global optimizer, who acts for the “goodness” of the system, as in the case of branch and cut. Also the increase of elementary operations of the barter process is much higher than the one of the branch and cut, particularly when the direction of movement is selected in a best-improve way, as it is shown in Tab. 1. Each point is averaged over the three instances for each size (m, n) . The economical interpretation suggests that if the time taken to reach an equilibrium is too long, it is possible that this equilibrium is eventually never achieved since in the meanwhile many perturbing events might happen.

As we discussed in Section 4, any SER intrinsically gives rise to a network structure generated by the set of couples of agents interacting along the process. Such a structure might be statistically analyzed in term of its topological properties. We consider two kind of assortativity measure (the preference for an agent to interact with others that are similar or different in some way, often operationalized as a correlation between adjacent node’s properties):

- i. couples of agents with highly different marginal utilities are more often commercial partners (negative Pearson correlation between the Euclidean distance of marginal utilities and the number of interactions of each couple of agents, $cor(\|c^h, c^k\|_2, interactions^{(h,k)})$),

size	initial welfare	first-improve			best-improve		branch and cut	
		neighborhood	ERPs	solution	ERPs	solution	simplex	solution
10	75.134	0.66	267	353.269	91	365.126	87	394.630
10	147.958	0.84	271	763.188	91	767.371	12	769.861
10	1.205.972	0.77	375	3.925.921	74	3.844.165	70	4.060.685
15	297.713	0.70	1.343	1.455.839	215	1.471.387	49	1.488.149
15	326.996	0.71	1.090	2.544.271	237	2.554.755	63	2.614.435
15	625.800	0.71	806	2.640.317	224	2.644.008	76	2.684.016
20	183.573	0.67	2.759	3.432.832	378	3.425.665	110	3.525.421
20	1.064.023	0.81	1.582	4.197.757	361	4.194.187	94	4.331.940
20	201.377	0.78	2.629	1.017.906	351	1.089.860	80	1.180.977
25	228.365	0.89	4.358	2.221.790	648	2.226.152	237	2.271.552
25	687.492	0.65	2.806	3.416.982	572	3.403.937	113	3.462.043
25	323.495	0.61	4.706	2.262.657	666	2.245.817	50	2.474.429
30	973.955	0.79	6.648	5.428.473	975	5.427.207	101	5.377.843
30	1.811.905	0.82	13.126	8.945.605	1.084	8.953.611	127	9.080.651
30	1.302.404	0.85	12.089	7.583.841	957	7.573.400	132	7.605.525
35	653.739	0.87	13.201	3.456.918	1.310	3.458.570	112	3.474.126
35	564.905	0.80	8.772	3.579.713	1.308	3.585.815	77	3.599.639
35	753.056	0.83	14.199	5.132.226	1.290	5.107.933	67	5.333.123
40	482.570	0.87	16.307	2.429.707	1.608	2.428.731	145	2.446.953
40	430.174	0.68	7.885	5.281.060	1.640	5.229.740	90	5.279.631
40	2.795.862	0.79	14.240	19.175.278	1.578	14.503.963	186	19.276.444
45	3.392.010	0.98	62.398	22.681.229	2.300	22.664.443	162	22.728.195
45	842.645	0.92	12.900	6.606.875	2.137	6.642.397	204	6.755.016
45	1.909.859	0.97	48.688	15.979.841	2.173	15.865.744	180	16.071.407
50	839.559	0.93	20.615	4.822.082	2.105	4.859.830	137	4.895.655
50	718.282	0.97	20.744	3.586.560	2.459	3.588.633	160	3.610.194
50	1.570.652	0.99	58.165	18.872.864	2.530	19.018.519	180	19.069.868
55	351.051	0.98	20.344	2.761.203	2.935	2.748.862	1.242	2.799.187
55	413.656	0.96	26.780	4.566.394	2.922	4.569.975	336	4.585.475
55	551.355	0.99	32.053	5.136.295	3.139	5.135.647	253	5.157.444
60	468.575	0.99	27.208	1.941.409	3.568	1.949.786	271	1.995.930
60	501.366	0.99	34.323	5.051.429	3.521	5.051.836	313	5.067.154
60	575.950	0.98	43.227	4.751.072	3.589	4.747.097	273	4.801.179

Table 1: Numerical results of the SER and Branch and Cut for different instances of problem (2). The first column shows the number of agents and commodities of the problem. Columns 'ERPs' provide the number of elementary reallocations and column 'neighborhood' shows the proportion of neighborhood which has been explored. Columns 'solution' give the maximum total utility found. Column 'simplex' gives the number of simplex iterations performed by branch and cut.

Size	First-improve			Best-improve		
	Type i	Type ii	Type iii	Type i	Type ii	Type iii
10	0.40	0.48	0.63	0.70	0.67	0.74
10	0.46	0.66	0.61	0.85	0.63	0.74
10	0.60	0.48	0.75	0.71	0.70	0.75
15	0.49	0.28	0.64	0.74	0.48	0.56
15	0.44	0.52	0.59	0.58	0.44	0.67
15	0.37	0.60	0.54	0.56	0.74	0.66
20	0.22	0.36	0.49	0.39	0.62	0.54
20	0.33	0.53	0.48	0.54	0.48	0.55
20	0.053	0.30	0.47	0.48	0.45	0.42
25	0.37	0.53	0.56	0.55	0.66	0.53
25	0.33	0.63	0.40	0.65	0.56	0.66
25	0.32	0.43	0.56	0.48	0.70	0.49
30	0.10	0.28	0.33	0.42	0.55	0.53
30	0.07	0.30	0.39	0.56	0.62	0.68
30	0.33	0.42	0.59	0.61	0.63	0.65
35	0.27	0.41	0.43	0.44	0.59	0.43
35	0.26	0.33	0.56	0.46	0.55	0.48
35	0.13	0.40	0.47	0.46	0.58	0.53
40	0.20	0.37	0.36	0.44	0.64	0.38
40	0.48	0.48	0.52	0.68	0.52	0.64
40	0.40	0.44	0.59	0.64	0.64	0.60
45	0.10	0.13	0.45	0.62	0.60	0.54
45	0.29	0.45	0.51	0.57	0.59	0.58
45	0.20	0.23	0.52	0.58	0.57	0.68
50	0.17	0.26	0.37	0.35	0.55	0.32
50	0.21	0.28	0.50	0.45	0.62	0.42
50	0.15	0.30	0.42	0.51	0.50	0.65
55	0.14	0.53	0.17	0.39	0.52	0.38
55	0.17	0.33	0.38	0.29	0.53	0.44
55	0.19	0.37	0.38	0.47	0.56	0.43
60	0.35	0.45	0.60	0.54	0.57	0.62
60	0.20	0.30	0.43	0.34	0.50	0.52
60	0.16	0.38	0.29	0.39	0.51	0.48

Table 2: Three types of network assortativity.

- ii. agents who are more sociable (trade more often) interact frequently with agents who are not sociable (negative Pearson correlation between the Euclidean distance of couples of agents with respect to their number of interactions and the number of joint interactions of each couple, $cor(\|degree^h, degree^k\|_2, degree^{(h,k)})$),
- iii. the more two agents are different with respect to their marginal utilities, the more they are different with respect to their number of interactions(negative Pearson correlation between the Euclidean distance of marginal utilities and the number of interactions of each couple of agents, $cor(\|c^h, c^k\|_2, \|degree^h, degree^k\|_2)$)

For the same instances in Tab. 1, Tab. 2 reports the the corresponding assortativities.

This interaction pattern is far from been random and, in the case the barter process is forced to be performed only among agents adjacent in a network, it suggests that highly dissortative structure match pretty well with the best-improve directions of movement, so that none of them is penalized by the network structure.

6. Summary and future directions

We studied the use of barter processes for solving problems of bargaining on a discrete set, representing markets with indivisible goods and fixed exogenous prices. We showed that the allocation space is characterized by a block diagonal system of linear constraints, whose structural properties might be exploited in the construction and analysis of barter processes. Using Proposition 2 and the characterization of the space of integer solutions of the ERP,

we were able to derive a constructive procedure to obtain its Pareto frontier, as shown by Corollary 1 and Corollary 2.

Further research on this topic should include the characterization of the integer points in the null space of a general reallocation problem with fixed prices to obtain a closed form solution of a general problem of reallocating integer amounts of m commodities among n agents with fixed prices.

An open problem, which has not been investigated in this paper, is the formulation of equilibrium conditions for this rationing scheme proposed in Section 3, as suggested by Dreze [9] for the case of continuous allocation space.

In Section 4 we proposed a mathematical programming model for the problem of real-locating integer amounts of m commodities among n agents with fixed prices on a sparse network structure with nodal capacities. Further research on this issue should include a mathematical properties of a SER in dealing with markets with sparsely connected agents, as formulated in (17).

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