Bounce Loop Quantum Cosmology Corrected Gauss-Bonnet Gravity

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Abstract

We develop an effective Gauss-Bonnet extension of Loop Quantum Cosmology, by introducing holonomy corrections in modified $F(\mathcal{G})$ theories of gravity. Within the context of our formalism, we provide a perturbative expansion in the critical density, a parameter characteristic of Loop Quantum Gravity theories, and we result in having leading order corrections to the classical $F(\mathcal{G})$ theories of gravity. After extensively discussing the formalism, we present a reconstruction method that makes possible to find the Loop Quantum Cosmology corrected $F(\mathcal{G})$ theory that can realize various cosmological scenarios. Specifically, we studied exponential and power-law bouncing cosmologies, emphasizing on the behavior near the bouncing point and in some cases, the behavior for all the values of the cosmic time is obtained. We exemplify our theoretical constructions by using bouncing cosmologies, and we investigate which Loop Quantum Cosmology corrected Gauss-Bonnet modified gravities can successfully realize such cosmologies.

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1 Introduction

The Big Bang era is one of the less understood periods of the evolution of our Universe, and the physics behind this era is still inconceivable. The standard approach that provides a classical physical description is based on the assumption that our Universe is described by a Friedman-Lemaitre-Robertson-Walker (FLRW) metric, which captures most of the present date characteristics of our Universe, i.e. its space uniformity, large scale homogeneity etc. However, the classical cosmological approach leads inevitably to an initial singularity, which is a rather "embarrassing" feature of the classical description, because due to this singularity, the closed time-like geodesics which pass from this singularity, have a finite proper length, but no end points to normal space away from the singularity. In addition to this, there has been conjectured for some years [1], that naked singularities should be well hidden behind horizons, the so-called cosmic censorship conjecture. However, no formal proof exists up to date for this cosmic censorship hypothesis, nevertheless it is considered a physically appealing hypothesis. If a cosmological initial singularity exists, then it belongs to a period of time that quantum physics, or more appropriately, quantum cosmology governs the physical phenomena.

One of the most appealing quantum cosmological theories is holonomy corrected Loop Quantum Cosmology (LQC) [2], which is promising from various points of view. For an important stream of papers and reviews on this fastly developing research topic, consult [2], and references therein. In the context of LQC, many theoretical inconsistencies or ellipses of the classical cosmological theories, could consistently be explained. Having such a promising theoretical framework at hand, it is compelling to investigate theoretical modifications, along the research lines of modified gravity. In this context, $F(R)$ extensions of LQC have recently been developed in [3, 4], where, following the idea of [5], holonomy corrections are introduced in Einstein frame (EF), because in that frame the gravitational part of the Hamiltonian is linear in the scalar curvature and thus the procedure to introduce these effects is the same as in General Relativity. These corrections have been applied to R^2 gravity [6], obtaining a bouncing model free of singularities. On the other hand, inspired in the introduction of holomy effects in the theory of cosmological perturbations [40], an effective way to introduce these corrections in $F(R)$ gravity was developed in [7], where the EF formulation is not necessary to be used. The main characteristic of it is that it could be applied to other theories, where there is not a conformal equivalence with EF, and therefore allowing to introduce these corrections to Gauss-Bonnet gravity in an explicit way. Gauss-Bonnet gravity is characterized by the Gauss-Bonnet invariant curvature which we denote $\mathcal{G}$ in this paper.

Adopting the research lines of [7], we perform an effective Gauss-Bonnet extension of LQC. Once we have obtained this new theory, we generalize the reconstruction method obtained in [8], and we reconstruct some holonomy corrected LQC-$F(\mathcal{G})$ gravity theories. We shall focus our reconstruction procedure to realizing mainly bouncing cosmologies. The reason for that specialization is that bouncing cosmologies have the very appealing feature of the absence of an initial singularity [9, 10, 11, 12, 13, 14, 15, 16, 17, 18], plus accelerating expansion can consistently be described in the context of these theories [20, 21, 22]. These kind of cosmologies are very interesting owing to the fact that nowadays they represent the most promising alternative theory to the inflationary paradigm, because if at very early times the Universe is nearly matter dominated in the contracting phase, the power-spectrum of the modes that leave the Hubble radius during this regime will be almost flat [23]. Moreover, is has been shown in [24] that bouncing cosmologies provide theoretical values of the cosmological parameters, such as the spectral index and its running, that fit well with recent Planck data [25].

The paper is organized as follows: In section 2, using the method of Lagrange multipliers, we construct the classical dynamical equations in Gauss-Bonnet gravity, for a flat FLRW geometry. In section 3, we review the way to introduce holonomy corrections in standard Einstein-Hilbert general relativity, for the particular case of the flat FLRW geometry. In section 4, we extend LQC to Gauss-Bonnet gravity obtaining a holonomy corrected Friedmann equation, which contains all the dynamical information of the system. Section 5 is devoted to explain the reconstruction method in holonomy corrected Gauss-Bonnet gravity. Sections 6 and 7 contain examples of reconstruction, for power law, bouncing exponential models and generic bouncing models respectively. The conclusions follow at the end of the paper.

2 Gauss-Bonnet $F(\mathcal{G})$ gravity "a la Ostrogradsky"

When one considers the flat FLRW geometry, the Lagrangian of Gauss-Bonnet $F(\mathcal{G})$ gravity in the vacuum is given by (in units $\hbar = c = 8\pi G = 1$),

$$\mathcal{L}(V, \dot{V}, \ddot{V}) = \frac{1}{2}V(R + F(\mathcal{G})), \tag{1}$$

where $V = a^3$ is the volume and $\mathcal{G}$ denotes the Gauss-Bonnet curvature, which can be expressed in terms of the volume and its higher derivatives,

$$\mathcal{G} = 24H^2(\dot{H} + H^2) = \frac{8\dot{V}^2}{3V^2}\left(\frac{\ddot{V}}{3V} - \frac{2\dot{V}^2}{9V^2}\right) \tag{2}$$

Ostrogradsky's idea to obtain the Hamiltonian from a Lagrangian containing higher order derivatives, is epitomized in the introduction of a Lagrange multiplier term in the Lagrangian, which we denote μ, in the following way [26, 7]:

$$\mathcal{L}_1(V, \dot{V}, \ddot{V}, \mathcal{G}) = \frac{1}{2}V(R + F(\mathcal{G})) + \mu\left(\frac{8\dot{V}^2}{3V^2}\left(\frac{\ddot{V}}{3V} - \frac{2\dot{V}^2}{9V^2}\right) - \mathcal{G}\right). \tag{3}$$

Maximizing with respect to the Gauss-Bonnet invariant $\mathcal{G}$ yields $\mu = \frac{1}{2}VF'(\mathcal{G})$. In order for the second derivative of V to be removed, we subtract from the Lagrangian the following total derivative term,

$$\frac{d}{dt}\left(\frac{4\dot{V}^3}{27V^2}F'(\mathcal{G}) + \dot{V}\right), \tag{4}$$

which does not change the dynamics of the system, and by replacing the Lagrange multiplier μ by its value $\frac{1}{2}VF'(\mathcal{G})$, we finally obtain,

$$\widetilde{\mathcal{L}}(V, \dot{V}, \mathcal{G}, \dot{\mathcal{G}}) = -\frac{\dot{V}^2}{3V} + \frac{1}{2}VF(\mathcal{G}) - \frac{1}{2}VF'(\mathcal{G})\mathcal{G} - \frac{4\dot{V}^3}{27V^2}F''(\mathcal{G})\dot{\mathcal{G}}. \tag{5}$$

As it is obvious by Eq. (5), we can see that the Lagrangian (5) depends on the variables $(V, \mathcal{G})$ and their first derivatives. We can therefore obtain the corresponding canonically conjugate momenta, which are,

$$p_V \equiv \frac{\partial \widetilde{\mathcal{L}}(V, \dot{V}, \mathcal{G}, \dot{\mathcal{G}})}{\partial \dot{V}} = -\frac{2\dot{V}}{3V} - \frac{4\dot{V}^2}{9V^2} F''(\mathcal{G})\dot{\mathcal{G}},$$

$$p_{\mathcal{G}} \equiv \frac{\partial \widetilde{\mathcal{L}}(V, \dot{V}, \mathcal{G}, \dot{\mathcal{G}})}{\partial \dot{\mathcal{G}}} = -\frac{4\dot{V}^3}{27V^2} F''(\mathcal{G}), \tag{6}$$

and consequently the classical gravitational part of the Hamiltonian becomes,

$$\mathcal{H}_{grav}(V, \mathcal{G}, p_V, p_{\mathcal{G}}) \equiv \dot{V} p_V + \dot{\mathcal{G}} p_{\mathcal{G}} - \widetilde{\mathcal{L}}(V, \dot{V}, \mathcal{G}, \dot{\mathcal{G}})$$

$$= \frac{3}{V}\left(\frac{V^2 p_{\mathcal{G}}}{4F''(\mathcal{G})}\right)^{2/3} - 3p_V\left(\frac{V^2 p_{\mathcal{G}}}{4F''(\mathcal{G})}\right)^{1/3} + \frac{V}{2}\left(\mathcal{G}F'(\mathcal{G}) - F(\mathcal{G})\right). \tag{7}$$

The Hamiltonian constraint $\mathcal{H}_{grav}(V, \mathcal{G}, p_V, p_{\mathcal{G}}) = 0$ leads to the well-known modified Friedmann equation in the vacuum for non-LQC corrected $F(\mathcal{G})$ gravity, which is, [27]

$$6H^2 + 24H^3\dot{\mathcal{G}}F''(\mathcal{G}) - \mathcal{G}F'(\mathcal{G}) + F(\mathcal{G}) = 0. \tag{8}$$

Note that in ordinary Einstein-Hilbert gravity, where $F(\mathcal{G}) = 0$, the canonically conjugate momenta are equal to,

$$p_V = -\frac{2\dot{V}}{3V} = -2H, \quad \frac{p_{\mathcal{G}}}{F''(\mathcal{G})} = \frac{1}{2}p_V^3 V \tag{9}$$

and therefore, the classical gravitational part of the Hamiltonian takes the following simplified form,

$$\mathcal{H}_{grav}(V, p_V) = -\frac{3}{4}p_V^2 V = -3H^2 V. \tag{10}$$

Remark 2.1. *Note that, the Hamilton equations $\dot{V} = \frac{\partial \mathcal{H}_{grav}}{\partial p_V}$ and $\dot{\mathcal{G}} = \frac{\partial \mathcal{H}_{grav}}{\partial p_{\mathcal{G}}}$ are simple identities. The equation $\dot{p_{\mathcal{G}}} = -\frac{\partial \mathcal{H}_{grav}}{\partial \mathcal{G}}$ is equivalent to the modified Friedmann equation (8), and the equation $\dot{p_V} = -\frac{\partial \mathcal{H}_{grav}}{\partial V}$ which corresponds to the modified Raychauduri equation in $F(\mathcal{G})$ gravity, could be obtained taking the derivative with respect to the cosmic time of the modified Friedmann equation. As a consequence, in the vacuum the dynamics in $F(\mathcal{G})$ gravity is only modelled by the modified Friedmann equation (8), and when one considers matter one has to include the conservation equation $\dot{\rho} = -3H(\rho + P)$, being P the pressure.*

Before ending this section, we need to stress an important issue: Since in principle there are infinitely many canonical transformations, this means that the modified $F(\mathcal{G})$ gravity could be formulated using infinitely many sets of variables (two coordinates and their corresponding canonically conjugate momenta). Note that some of these sets of variables will be meaningless physically speaking, because they are built using a combination of both coordinates and momenta, giving new quantities with a very difficult physical interpretation. Moreover, since the introduction of holonomy effects critically depend on the set of variables used, in effect there are infinitely many ways to introduce holonomy effects in modified $F(\mathcal{G})$ gravity. Consequently, there will be infinitely many different effective holonomy corrected Friedmann equations in $F(\mathcal{G})$ gravity.

3 Introduction of holonomy corrections

In this section and in order for the article to be maintained self-complete, we shall describe the technique of introducing holonomy corrections in standard Einstein-Hilbert gravity. Assuming a flat FLRW geometry, first of all one can consider the variable $\beta = -\frac{\gamma}{2}p_V = \gamma H$ ([28]), where γ is the Barbero-Immirzi parameter. In terms of the parameter β, the Hamiltonian (10) becomes $\mathcal{H}_{grav}(V,\beta) = -\frac{3\beta^2}{\gamma^2}V$. However, in LQC, due to the discrete nature of space, the quantum operator $\hat{\beta}$ is not well defined (see for instance [29] or [30] for a status report on this issue). Then, in order to build the quantum theory, the gravitational part of the Hamiltonian must be redefined. To be precise, we will consider holonomies of the form $h_j(\lambda) \equiv e^{-i\frac{\lambda\beta}{2}\sigma_j}$, where σ_j denote the Pauli matrices and λ is the square root of the minimum eigenvalue of the area operator in loop quantum gravity. Since β^2 does not have a well-defined quantum operator, in order for a consistent quantum Hamiltonian operator to be constructed, an almost periodic function that approaches β^2 for small values of β is needed. This can be done using the general formulae of loop quantum gravity to obtain the holonomy corrected Hamiltonian, which is equal to,

$$\mathcal{H}_{hol,grav}(V,\beta) \equiv -\frac{2V}{\gamma^3\lambda^3}\sum_{i,j,k}\varepsilon^{ijk}Tr\left[h_i(\lambda)h_j(\lambda)h_i^{-1}(\lambda)\right.$$
$$\left.\times h_j^{-1}(\lambda)h_k(\lambda)\{h_k^{-1}(\lambda),V\}\right] . \tag{11}$$

The Hamiltonian (11) captures the underlying loop quantum dynamics, and for a detailed account on this issue see for instance [31]).

A simple calculation [32, 33, 34] shows that the Hamiltonian of Eq. (11) acquires the simple form

$$\mathcal{H}_{hol,grav}(V,\beta) = -3\frac{\sin^2(\lambda\beta)}{\lambda^2\gamma^2}V, \tag{12}$$

which indicates that effectively, holonomy effects can be introduced by explicitly performing the replacement $\beta \to \frac{\sin(\lambda\beta)}{\lambda}$ or equivalently $p_V \to -\frac{2\sin(\lambda\beta)}{\lambda\gamma}$.

In order to obtain the holonomy corrected Friedmann equation, in principle it is necessary to use the full Hamiltonian, i.e., the Hamiltonian that contains both, gravitational and matter parts,

$$\mathcal{H}_{hol}(V,\beta) = -3\frac{\sin^2(\lambda\beta)}{\lambda^2\gamma^2}V + \rho V, \tag{13}$$

Then, by combining the Hamilton equation,

$$\dot{V} = -\frac{\gamma}{2}\frac{\partial\mathcal{H}_{hol}(V,\beta)}{\partial\beta} = -3V\frac{\sin(2\lambda\beta)}{2\lambda\gamma}, \tag{14}$$

with the Hamiltonian constraint $\mathcal{H}_{hol}(V,\beta) = 0$, we obtain the well-known modified Friedmann equation [35],

$$H^2 = \frac{\rho}{3}\left(1 - \frac{\rho}{\rho_c}\right), \tag{15}$$

where $\rho_c \equiv \frac{3}{\lambda^2\gamma^2}$ is the so-called *critical density*. Practically this density is what measures the *strength* of the loop quantum effects in cosmology, so if it is infinite, and for finite energy densities, the holonomy corrected Friedmann equation will be reduced to its well known form in standard cosmology.

4 Holonomy corrected $F(\mathcal{G})$ gravity

Recently, holonomy correction are introduced in $F(R)$ gravity using the property that $F(R)$ gravity in the Jordan Frame (JF) is equivalent to General Relativity in the Einstein Frame (EF). Then, introducing, as in standard LQC, holonomy corrections in EF and comming back to the JF one obtains what is named as Loop Quantum $F(R)$ gravity [3, 4].

Unfortunately, Gauss-Bonnet gravity is not equivalent to General Relativity in any frame. Then, to introduce holonomy corrections, other strategy must be used. What we will set up is an effective LQC theory for $F(\mathcal{G})$ gravity based in this key point: As we have seen in previous Section, stardard holohomy corrected LQC, could essentially be obtained from the replacement $\beta \to \frac{\sin(\lambda\beta)}{\lambda}$ or equivalently $p_V \to -\frac{2\sin(\lambda\beta)}{\lambda\gamma}$. Then, when there is is not an established way to perform a kinematical loop quantization of the phase space, as when one deals with cosmological perturbations. In that case, to incorporate loop corrections an effective Hamiltonian is built adopting as a prescription the replacement of the Ashtekar connection by a suitable function of that connection [40]. Then, with the same spirit, we will introduce holonomy corrections in general $F(\mathcal{G})$ using the following recipe:

To introduce the holonomy correction in general $F(\mathcal{G})$ we will adopt the following recipe: In analogy with the standard Einstein-Hilbert gravity case, where $F(\mathcal{G}) = 0$, we will replace the momentum that in Einstein gravity case corresponds to $-2\frac{\beta}{\gamma}$ by $-\frac{2\sin(\lambda\beta)}{\lambda\gamma}$. For example, if the variables $(V, \mathcal{G}, p_V, p_\mathcal{G})$ are used, we make the replacement $p_V \to -\frac{2\sin(\lambda\beta)}{\lambda\gamma}$ in the Hamiltonian of Eq. (7). It is conceivable that this way of introducing holonomy corrections, critically depends on the set of variables used to formulate the $F(\mathcal{G})$ theory, which means that the use of other canonically conjugate variables will lead to different corrections. In the case that matter fluids are taken into account, considering the Hamiltonian (7) and its corresponding matter part, namely $\mathcal{H}_{matter} = \rho V$, upon making the replacement, $p_V \to -\frac{2\sin(\lambda\beta)}{\lambda\gamma}$, we finally obtain the following Hamiltonian,

$$\mathcal{H}_{hol}(V, \mathcal{G}, p_V, p_\mathcal{G}) = \frac{3}{V}\left(\frac{V^2 p_\mathcal{G}}{4F''(\mathcal{G})}\right)^{2/3}$$
$$+\frac{6\sin(\lambda\beta)}{\lambda\gamma}\left(\frac{V^2 p_\mathcal{G}}{4F''(\mathcal{G})}\right)^{1/3} + \frac{V}{2}\left(\mathcal{G}F'(\mathcal{G}) - F(\mathcal{G})\right) + \rho V. \tag{16}$$

The corresponding Hamilton equations are equal to,

$$\dot{V} = -\frac{\gamma}{2}\frac{\partial \mathcal{H}_{hol}}{\partial \beta}; \quad \dot{\mathcal{G}} = \frac{\partial \mathcal{H}_{hol}}{\partial p_\mathcal{G}}, \tag{17}$$

which together with the Hamiltonian constraint $\mathcal{H}_{hol}(V, \mathcal{G}, p_V, p_\mathcal{G}) = 0$, lead to the following equations:

$$
\begin{aligned}
H &= -\cos(\lambda\beta)\widetilde{p}_{\mathcal{G}}^{\frac{1}{3}} \\
\dot{\mathcal{G}} &= \frac{1}{2F''(\mathcal{G})}\left(\widetilde{p}_{\mathcal{G}}^{-\frac{1}{3}} + \widetilde{p}_{\mathcal{G}}^{-\frac{2}{3}}\frac{\sin(\lambda\beta)}{\lambda\gamma}\right) \\
3\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}} + \widetilde{p}_{\mathcal{G}}^{\frac{1}{3}}\frac{6\sin(\lambda\beta)}{\lambda\gamma} &+ \frac{1}{2}\left(\mathcal{G}F'(\mathcal{G}) - F(\mathcal{G})\right) + \rho = 0,
\end{aligned}
\tag{18}
$$

where for notational simplicity, we introduced the variable $\widetilde{p}_{\mathcal{G}} = \frac{p_{\mathcal{G}}}{4VF''(\mathcal{G})}$.

Before dealing with these equations, we will check that in standard Einstein-Hilbert gravity where $F(\mathcal{G}) = 0$, we can obtain the holonomy corrected Friedmann equation of Eq. (15). In Einstein-Hilbert gravity, the equations (18) take the following form,

$$
\begin{aligned}
H &= -\cos(\lambda\beta)\widetilde{p}_{\mathcal{G}}^{\frac{1}{3}} \\
\widetilde{p}_{\mathcal{G}}^{-\frac{1}{3}} + \widetilde{p}_{\mathcal{G}}^{-\frac{2}{3}}\frac{\sin(\lambda\beta)}{\lambda\gamma} &= 0 \\
3\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}} + \widetilde{p}_{\mathcal{G}}^{\frac{1}{3}}\frac{6\sin(\lambda\beta)}{\lambda\gamma} &+ \rho = 0.
\end{aligned}
\tag{19}
$$

The second and third equations lead to the relation $\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}} = \frac{\rho}{3}$, and the first and second equation can be written as follows,

$$
\begin{aligned}
\sin^2(\lambda\beta) &= 1 - \frac{H^2}{\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}}} = 1 - \frac{3H^2}{\rho} \\
\sin^2(\lambda\beta) &= \frac{3}{\rho_c}\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}} = \frac{\rho}{\rho_c},
\end{aligned}
\tag{20}
$$

which eventually lead to the Friedmann equation (15) in LQC. Coming back to the general equation (18), these three equations have to be used to obtain a general relation of the form $G(H, \mathcal{G}, \dot{\mathcal{G}}, \rho) = 0$ which will correspond to the modified Friedmann equation in Gauss-Bonnet $F(\mathcal{G})$ gravity. This equation in conjunction with the energy-momentum conservation equation $\dot{\rho} = -3H(\rho + P)$ and the equation of state $P = P(\rho)$, are the equations that will depict the dynamics of the system.

The combination of the first and third equations of (18) leads to the relation,

$$
12\rho_c(\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}} - H^2) = \left(\frac{1}{2}\left(\mathcal{G}F'(\mathcal{G}) - F(\mathcal{G})\right) + \rho + 3\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}}\right)^2, \tag{21}
$$

which allows us to isolate $\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}}$ as a function of H, $\mathcal{G}$ and ρ, giving as a result

$$
\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}} = \frac{4\rho_c - A(\mathcal{G})}{6}\left(1 - \sqrt{1 - \frac{A^2(\mathcal{G}) + 46\rho_c H^2}{(4\rho_c - A(\mathcal{G}))^2}}\right), \tag{22}
$$

where the notation $A(\mathcal{G}) \equiv \mathcal{G}F'(\mathcal{G}) - F(\mathcal{G}) + 2\rho$ has been introduced.

On the other hand, the combination of the three equations in (18) leads to the dynamical equation that corresponds to the modified Friedmann equation in LQC-$F(\mathcal{G})$ gravity,

$$4\widetilde{p}_{\mathcal{G}}^{\frac{4}{3}}(F''(\mathcal{G}))^2\dot{\mathcal{G}}^2 = \frac{\rho_c}{3}\left(1 - \frac{H^2}{\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}}}\right) - \frac{1}{6}(\mathcal{G}F'(\mathcal{G}) - F(\mathcal{G})) - \frac{\rho}{3}. \tag{23}$$

To sum up, the dynamical holonomy corrected equations in Gauss-Bonnet gravity are given by the modified Friedmann equation in LQC-$F(\mathcal{G})$ gravity and the conservation equation

$$\begin{aligned} 4\widetilde{p}_{\mathcal{G}}^{\frac{4}{3}}(F''(\mathcal{G}))^2\dot{\mathcal{G}}^2 = \frac{\rho_c}{3}\left(1 - \frac{H^2}{\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}}}\right) - \frac{1}{6}(\mathcal{G}F'(\mathcal{G}) - F(\mathcal{G})) - \frac{\rho}{3} \\ \dot{\rho} = -3H(\rho + P), \end{aligned} \tag{24}$$

where $\widetilde{p}_{\mathcal{G}}^{\frac{2}{3}}$ is given by (22) and ρ_c is the critical density introduced at the end of the previous section.

Here, as in standard $F(\mathcal{G})$ gravity, only two equations are needed to define the dynamics. All the Hamiltonian equations obtained from (16) are equivalent to (24).

Note finally that, in order to solve Eq. (24), for a barotropic fluid an equation of state of the form $P = P(\rho)$ is needed. In contrast, when one deals with a canonical scalar field φ, one has $\rho = \frac{1}{2}\dot{\varphi}^2 + V(\varphi)$ and $P = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$.

5 Analysis of the theory $F(\mathcal{G})$ with LQC

In this way we shall investigate in which way we can have some LQC corrections in modified $F(\mathcal{G})$ gravity, at least the leading order LQC corrections. These corrections would materialize the first deviations from the classical non-LQC $F(\mathcal{G})$ gravity. In order to find these, we rewrite Eq. (24) in the following form,

$$24(F''(\mathcal{G})\dot{\mathcal{G}})^2\bar{p}_{\mathcal{G}}^2(\mathcal{G}) - 2\rho_c\left(1 - \frac{H^2}{\bar{p}_{\mathcal{G}}(\mathcal{G})}\right) + A(\mathcal{G}) = 0, \tag{25}$$

where the parameter $\bar{p}_{\mathcal{G}}(\mathcal{G})$ stands for,

$$\bar{p}_{\mathcal{G}}(\mathcal{G}) \equiv \frac{1}{6}\left(4\rho_c - A(\mathcal{G}) - 2\sqrt{2\rho_c}\sqrt{2\rho_c - A(\mathcal{G}) - 6H^2}\right).$$

Note that when $\rho_c \to \infty$, the quantity $\bar{p}_{\mathcal{G}}(\mathcal{G})$ can be represented as a series

$$\bar{p}_{\mathcal{G}}(\mathcal{G}) = H^2 + \frac{1}{48}\left(A(\mathcal{G}) + 6H^2\right)^2\frac{1}{\rho_c} + \frac{1}{192}\left(A(\mathcal{G}) + 6H^2\right)^3\frac{1}{\rho_c^2} + o\left(\frac{1}{\rho_c^3}\right).$$

Then, Eq. (25) can be written as follows,

$$
\begin{aligned}
&576H^6(F''(\mathcal{G})\dot{\mathcal{G}})^2 - \left(A(\mathcal{G}) - 6H^2\right)^2 + \\
&\qquad + \frac{\left(A(\mathcal{G}) + 6H^2\right)^2}{48H^2}\left(A^2(\mathcal{G}) - 36H^4 + 1152H^6(F''(\mathcal{G})\dot{\mathcal{G}})^2\right)\frac{1}{\rho_c} + \\
&\qquad\qquad + \frac{\left(A(\mathcal{G}) + 6H^2\right)^3}{2304H^4} \times \\
&\times\left(A^3(\mathcal{G}) - 6A^2(\mathcal{G})H^2 + 432H^6 - 576H^6(A(\mathcal{G} + 30H^2))(F''(\mathcal{G})\dot{\mathcal{G}})^2\right)\frac{1}{\rho_c^2} + \ldots = 0.
\end{aligned}
$$

It is easy to see that in the limit $\rho_c \to \infty$ this equation reduced to the Friedman equation for $F(\mathcal{G})$ theory (see [36]). Therefore, we define the parameter $\varepsilon = 1/\rho_c$, which as $\rho_c \to \infty$, it takes small values, and we seek a solution of this equation by performing a perturbative expansion in terms of this parameter.

In order to provide a consistent solution to the Cauchy problem or a boundary value problem, for the differential equation (25), we shall perform a perturbative expansion in terms of the parameter ε, as $\varepsilon \to 0$. This expansion takes the form,

$$
F(\mathcal{G}) = \sum_{k=0}^{\infty} \varepsilon^k F_k(\mathcal{G}). \tag{26}
$$

Then, the differential equations for each order of the expansion are given below,

$$
24H^3F_0''(\mathcal{G})\dot{\mathcal{G}} - \mathcal{G}F_0'(\mathcal{G}) + F_0(\mathcal{G}) + 6H^2 - 2\rho(t) = 0, \tag{27}
$$

$$
24H^3F_1''(\mathcal{G})\dot{\mathcal{G}} - \mathcal{G}F_1'(\mathcal{G}) + F_1(\mathcal{G}) = -18H^4(1 + 2HF_0''(\mathcal{G})\dot{\mathcal{G}})^2(1 + 6HF_0''(\mathcal{G})\dot{\mathcal{G}}), \tag{28}
$$

$$
\begin{aligned}
&24H^3F_2''(\mathcal{G})\dot{\mathcal{G}} - \mathcal{G}F_2'(\mathcal{G}) + F_2(\mathcal{G}) = \\
&\qquad = -9H^5(3H(1 + 2HF_0''(\mathcal{G})\dot{\mathcal{G}})^3(4 + 39HF_0''(\mathcal{G})\dot{\mathcal{G}}(1 + 2HF_0''(\mathcal{G})\dot{\mathcal{G}})) + \\
&\qquad\qquad + 4\dot{\mathcal{G}}(1 + 2HF_0''(\mathcal{G})\dot{\mathcal{G}})(5 + 18HF_0''(\mathcal{G})\dot{\mathcal{G}})F_1''(\mathcal{G})).
\end{aligned} \tag{29}
$$

where we included only the first three orders of the perturbative expansion of the solution. If it is possible to construct a solution of equation (27) for the function $F_0(\mathcal{G})$, the remaining terms of the expansion $F_k(\mathcal{G})$, $k \geq 1$ are determined by solving linear differential equations. The initial or boundary conditions for the functions $F_k(\mathcal{G})$ can be obtained by taking into account the expansion (26) and the conditions for the equation (25).

Note that from the equations defining the function $F_k(\mathcal{G})$, it follows that the corrections to the $F_0(\mathcal{G})$ are not related to the distribution of matter in the universe. In the following sections, we shall exemplify our results by using illustrative examples, emphasizing in bouncing cosmologies. We shall assume that all matter fluids are absent i.e. $\rho(t) = 0$, and therefore we are interested in vacuum modified LQC-corrected $F(\mathcal{G})$ theories.

6 Power-law Cosmology from LQC-corrected $F(\mathcal{G})$ Gravity

We start of our analysis, by studying a power-law cosmology in the context of LQC-corrected $F(\mathcal{G})$ gravity. Consider an evolution of the Universe for which the scale factor is of the following form,

$$a(t) = \alpha t^{2n}, \quad \alpha \in \mathbb{R}, \quad n \in \mathbb{N}.$$

For this model the Hubble rate reads,

$$H(t) = \frac{2n}{t}, \quad \mathcal{G}(t) = \frac{192n^3(2n-1)}{t^4}.$$

Whence, it follows immediately that,

$$H^2 = \frac{\mathcal{G}^{1/2}n^{1/2}}{2\sqrt{3}\sqrt{2n-1}}, \quad H\dot{\mathcal{G}} = -\frac{\mathcal{G}^{3/2}}{\sqrt{3}\sqrt{n}\sqrt{2n-1}}.$$

We assume that the initial conditions for the Cauchy problem of the differential equation (25), are,

$$F(\mathcal{G}_0) = \gamma_1, \quad F'(\mathcal{G}_0) = \gamma_2,$$

where γ_1, γ_2 are constants, $0 < \mathcal{G}_0 < +\infty$. Then, for the system of differential equations for the functions $F_k(\mathcal{G})$, we obtain the following initial conditions:

$$F_0(\mathcal{G}_0) = \gamma_1, \quad F_0'(\mathcal{G}_0) = \gamma_2,$$
$$F_k(\mathcal{G}_0) = 0, \quad F_k'(\mathcal{G}_0) = 0, \quad k = 1, 2, \ldots. \tag{30}$$

The general solution of the differential equation for the zero order unperturbed function $F_0(\mathcal{G})$, namely Eq. (27), is of the following form:

$$F_0(\mathcal{G}) = c_1\mathcal{G} + c_2\mathcal{G}^{(1-2n)/4} - \frac{2\sqrt{3}\sqrt{\mathcal{G}}\sqrt{n(2n-1)}}{2n+1},$$

where c_1 and c_2 are arbitrary integration constants.

In order to further simplify our analysis, we can choose γ_1 and γ_2 in such a way, so that $c_1 = c_2 = 0$, without great loss of generality. In effect, $F_0(\mathcal{G})$ reads,

$$F_0(\mathcal{G}) = -\frac{2\sqrt{3}\sqrt{\mathcal{G}}\sqrt{n(2n-1)}}{2n+1}. \tag{31}$$

Having $F_0(\mathcal{G})$ at hand, we can easily proceed in finding the first order correction of the perturbative expansion, namely the function $F_1(\mathcal{G})$, which is of first order in the ε expansion. Considering Eqs. (30) and (31), the Cauchy problem for (28) is the following:

$$\frac{4\mathcal{G}^2}{1-2n}F_1''(\mathcal{G}) - \mathcal{G}F_1'(\mathcal{G}) + F_1(\mathcal{G}) = \frac{12\mathcal{G}n^3(n-1)}{(1-2n)(1+2n)^3}, \quad F_1(\mathcal{G}_0) = 0, \quad F_1'(\mathcal{G}_0) = 0.$$

Therefore, one analytic solution of the Cauchy problem is:

$$F_1(\mathcal{G}) = c_3\mathcal{G} + c_4\mathcal{G}^{(1-2n)/4} + \frac{12(1-n)n^3}{(1+2n)^3(3+2n)^2}\left(4\mathcal{G} - (3+2n)\mathcal{G}\ln\mathcal{G}\right),$$

$$c_3 = \frac{12(1-n)n^3 \ln \mathcal{G}_0}{(1+2n)^3(3+2n)}, \quad c_4 = -\frac{48(1-n)n^3 \mathcal{G}_0^{(3+2n)/4}}{(1+2n)^3(3+2n)^2}.$$

Note that when $n = 1$, the Cauchy problem has the trivial solution, $F_1(\mathcal{G}) = 0$. Iteratively, we can calculate the second order correction in the ε-expansion. This can be easily done, since we have the explicit form of the function $F_1(\mathcal{G})$ at hand. Combining Eqs. (30) and (31) and also by taking into account the explicit form of the function $F_1(\mathcal{G})$, the Cauchy problem for the differential equation (29), takes the form :

$$\frac{4\mathcal{G}^2}{1-2n} F_2''(\mathcal{G}) - \mathcal{G} F_2'(\mathcal{G}) + F_2(\mathcal{G}) =$$
$$= c_3 \frac{\sqrt{3} n^{3/2}(2n+3)(5n-2)}{4\sqrt{2n-1}(2n+1)^2} \mathcal{G}^{(3-2n)/4} - \frac{3\sqrt{3} n^{9/2}(32n^3 - 78n^2 + 51n - 20)}{(2n-1)^{3/2}(2n+1)^5(2n+3)} \mathcal{G}^{3/2},$$
$$F_2(\mathcal{G}_0) = 0, \quad F_2'(\mathcal{G}_0) = 0.$$

It is easy to show that the solution to this problem has the following form,

$$F_2(\mathcal{G}) = c_5 \mathcal{G} + c_6 \mathcal{G}^{(1-2n)/4} + c_3 \frac{\sqrt{3} n^{3/2}(20n^3 + 12n^2 - 23n + 6)}{2\sqrt{2n-1}(2n+1)^3} \mathcal{G}^{(3-2n)/4} +$$
$$+ \frac{6\sqrt{3} n^{9/2}(32n^3 - 78n^2 + 51n - 20)}{\sqrt{2n-1}(2n+1)^5(2n+5)} \mathcal{G}^{3/2},$$

where c_5, c_6 are constants, which is easy to specify from the initial conditions given above. In principle, by continuing this iterative process, we can find all the higher order corrections of the perturbative ε-expansion.

In conclusion, for the case $n = 1$, the resulting approximate solution of Eq. (25), has the following form,

$$F(\mathcal{G}) = -\frac{2\sqrt{\mathcal{G}}}{\sqrt{3}} + \varepsilon^2 \left(\frac{2\sqrt{\mathcal{G}_0}}{45\sqrt{3}} \mathcal{G} - \frac{4\mathcal{G}_0^{7/4}}{315\sqrt{3}} \frac{1}{\mathcal{G}^{1/4}} - \frac{2}{63\sqrt{3}} \mathcal{G}^{3/2} \right) + o(\varepsilon^2). \tag{32}$$

where we took into account only the first two corrections of the perturbative ε-expansion. As a final task, we investigate the discrepancy between the approximate solution (32) and the one appearing in Eq. (25). For this purpose we substitute (32) in equation (25) for this model and as a result we obtain the function $g(\mathcal{G}, \varepsilon) = o(\varepsilon^2)$, $\varepsilon \to 0$. We have to note that the approximate solution is defined in the neighborhood of point $\mathcal{G} = \mathcal{G}_0$.

7 Bounce Cosmology from LQC $F(\mathcal{G})$ Gravity

One appealing cosmological scenario which is alternative to the standard inflationary description is the bounce cosmology scenario [10, 11, 12, 13, 14, 15, 16, 17, 18]. According to this, the Universe contracts until a minimal radius is reached, where it bounces off and starts to expand. In the bouncing cosmology context, the initial singularity is avoided and this feature is mainly what renders the bouncing cosmologies so appealing, in comparison to the inflationary picture. Also, it is possible to consistently describe early-time acceleration [20, 21, 22], so in conjunction with the singularity avoidance, we have a very appealing candidate theory for our Universe's evolution. In the context of modified gravity, bouncing cosmologies are easily realized [17, 18, 19], without the peculiar requirements that the standard Einstein-Hilbert imposes in order for a bounce to occur [10]. In this section we shall investigate which LQC-corrected $F(\mathcal{G})$ gravities can realize bouncing cosmologies, using well known bouncing cosmologies paradigms. Special emphasis shall be given on the exact form of the LQC-corrected $F(\mathcal{G})$ which describes the bounce near the bouncing point. We address the problem of finding the LQC-corrected $F(\mathcal{G})$, using the perturbative expansion we used earlier.

7.1 Exponential Bouncing Models

We start off our analysis by studying some bouncing cosmologies with exponential scale factor. Before we get into the details of each model, it is worth recalling in brief the conditions that define a bounce cosmology. For detailed accounts on these issues, see for example [10, 11, 12, 13, 14, 15].

As we already mentioned, a cosmological bounce is described by two evolutionary eras, a contraction and a subsequent expansion. In the contracting era, the Universe's scale factor $a(t)$ decreases up to a point, say at $t = t_s$, at which a minimal radius is reached. This means that during the contraction, the derivative of the scale factor is negative $\dot{a} < 0$, and at the minimal radius point the derivative of the scale factor is zero, that is $\dot{a} = 0$. At this point, the Universe bounces off and starts to expand, so that $\dot{a} > 0$, until a finite time singularity is probably reached. We shall study two bouncing models, the scale factor of which contains exponential functions of the cosmological time t.

7.1.1 Symmetric Bounce Model

In this section we study a symmetric bounce model, the scale factor of which is equal to,

$$a(t) = e^{\alpha t^2}\,, \tag{33}$$

where α is a positive arbitrary parameter. The scale factor (33) describes a cosmological bounce, for which the bouncing point is at $t = 0$. Indeed, this can be verified in Fig. 1, where we plotted the time-dependence of the derivative of the scale factor $\dot{a}$. As it can be seen in Fig. 1, the function $\dot{a}$ is negative before the bouncing point $t = 0$, equal to zero at the bouncing point and positive after the bouncing point. What interests us the most in this section, is to investigate which LQC-corrected $F(G)$ theory can generate the bounce (33), near the bouncing point $t = 0$. It is conceivable that what we need to examine is the limit $t \to 0$ of the differential equations (27) and (28) presented in the previous section. We shall be interested in finding the first two functions of the perturbative ϵ-expansion of the previous section, namely $F_0(\mathcal{G})$ and $F_1(\mathcal{G})$, assuming absence of any matter fluids. The Gauss-Bonnet invariant for the scale factor (33) reads,

$$\mathcal{G} = 192 t^2 \alpha^3 \left(1 + 2t^2\alpha\right)\,, \tag{34}$$

and in the limit $t \to 0$, this is approximately equal to,

$$\mathcal{G} \simeq 192 t^2 \alpha^3\,. \tag{35}$$

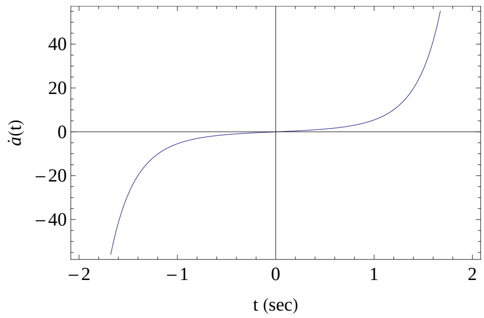

Figure 1: The derivative of the scale factor $\dot{a}(t)$ as a function of the cosmic time t, for the bouncing cosmology $a(t) = e^{\alpha t^2}$, with $\alpha = 1$.

Therefore, Eq. (35) can be explicitly solved with respect to the cosmic time t,

$$t = \frac{\sqrt{\mathcal{G}}}{8\sqrt{3}\alpha^{3/2}} \,. \tag{36}$$

Notice that the small t limit is equivalent to the small $\mathcal{G}$ limit, so we shall take this observation into account in the rest of this section. In view of the above approximations, the differential equation that yields the function $F_0(\mathcal{G})$, namely Eq. (27), at the vicinity of the bouncing point $t = 0$ is equal to,

$$\frac{1}{24}\mathcal{G}^2 \left(48 + \frac{\mathcal{G}}{\alpha^2}\right) F_0''(\mathcal{G}) - \mathcal{G}F_0'(\mathcal{G}) + F_0(\mathcal{G}) + \frac{\mathcal{G}}{8\alpha} = 0 \,, \tag{37}$$

which can be solved analytically, with the solution being of the following form,

$$F_0(\mathcal{G}) = \mathcal{G}C_1 - \frac{\sqrt{48\mathcal{G}}C_2}{24\alpha} + \frac{-\mathcal{G}\ln[\mathcal{G}] + 2\sqrt{48\mathcal{G}}\ln\left[\alpha\sqrt{48}\right]}{8} \,. \tag{38}$$

where we kept only the leading order terms in the limit $\mathcal{G} \to 0$. Having $F_0(\mathcal{G})$ at hand, we can easily obtain the function $F_1(\mathcal{G})$, by solving the second differential equation given in Eq. (28). In order to find an analytic approximation of the solution $F_1(\mathcal{G})$, we need to find an approximate form of the term,

$$\mathcal{C} = -18H^4(1 + 2HF_0''(\mathcal{G})\dot{\mathcal{G}})^2(1 + 6HF_0''(\mathcal{G})\dot{\mathcal{G}}) \,, \tag{39}$$

appearing in the $F_1(\mathcal{G})$ differential equation, where for notational simplicity we denoted the term with $\mathcal{C}$. The resulting expression of $\mathcal{C}$ is quite lengthy and can be found in the Appendix, so here we quote only the simplified expression for $\mathcal{C}$, where we will keep only the dominant terms in the limit $\mathcal{G} \to 0$. In this limit, the term $\mathcal{C}$ reads,

$$\mathcal{C} = \mathcal{A}_1\sqrt{\mathcal{G}} \,, \tag{40}$$

where $\mathcal{A}_1$ stands for,

$$\mathcal{A}_1 = -\frac{C_2^3}{128\sqrt{3}\alpha^2} + \frac{3\sqrt{3}C_2^2\ln\left[4\sqrt{3}\alpha\right]}{64\alpha} - \frac{9\sqrt{3}C_2\ln\left[4\sqrt{3}\alpha\right]^2}{32} + \frac{9}{16}\sqrt{3}\alpha\ln\left[4\sqrt{3}\alpha\right]^3 \tag{41}$$

Consequently, the differential equation that yields the solution for $F_1(\mathcal{G})$, namely Eq. (28), becomes,

$$\frac{1}{24}\mathcal{G}^2\left(48+\frac{\mathcal{G}}{\alpha^2}\right)F_1''(\mathcal{G})-\mathcal{G}F_1'(\mathcal{G})+F_1(\mathcal{G})-\mathcal{A}_1\sqrt{\mathcal{G}}=0\,, \tag{42}$$

which can analytically be solved to yield,

$$F_1(\mathcal{G})=C_1\mathcal{G}+\frac{e^{-\frac{24\alpha^2}{\mathcal{G}}}\mathcal{G}C_2}{24\alpha^2}\,. \tag{43}$$

In conclusion, the final form of the small ε limit of the $F(\mathcal{G})$, which generates the bounce (33) near the bouncing point, is approximately equal to,

$$\begin{aligned}&F(\mathcal{G})\simeq F_0(\mathcal{G})+\varepsilon F_1(\mathcal{G})\simeq \\ &\mathcal{G}C_1-\frac{\sqrt{48\mathcal{G}}C_2}{24\alpha}+\frac{-\mathcal{G}\ln[\mathcal{G}]+2\sqrt{48\mathcal{G}}\ln\left[\alpha\sqrt{48}\right]}{8}+\varepsilon\Big(C_1\mathcal{G}+\frac{e^{-\frac{24\alpha^2}{\mathcal{G}}}\mathcal{G}C_2}{24\alpha^2}\Big)\,.\end{aligned} \tag{44}$$

Recall that the variable ε is equal to $1/\rho_c$, so practically the limit $\varepsilon\to 0$ measures the differences of the resulting LQC-corrected $F(\mathcal{G})$, in the limit where the LQC effects fade away, and therefore the LQC effects are contained in the term which is linear to ε.

7.1.2 Hyperbolic Cosine Bounce Model

Another exponential bouncing model is described by the following scale factor,

$$a(t)=\cosh(\lambda t),\quad \lambda>0$$

For this model the Hubble rate is equal to,

$$H(t)=\lambda\tanh(\lambda t),\quad \mathcal{G}(t)=24\lambda^4\tanh^2(\lambda t).$$

Whence it follows immediately that,

$$H^2=\frac{\mathcal{G}}{24\lambda^4},\quad H\dot{\mathcal{G}}=2\lambda^2\mathcal{G}\left(1-\frac{\mathcal{G}}{24\lambda^4}\right).$$

Consider the following initial conditions for the Cauchy problem of Eq. (25),

$$F(\mathcal{G}_0)=\gamma_1,\quad F'(\mathcal{G}_0)=\gamma_2,$$

where γ_1, γ_2 are constants, and also $\mathcal{G}_0$ satisfies $0<\mathcal{G}_0<24\lambda^4$. Then, for the functions $F_k(\mathcal{G})$, the following boundary conditions hold true:

$$\begin{aligned}F_0(\mathcal{G}_0)=\gamma_1,\quad F_0'(\mathcal{G}_0)=\gamma_2,\\ F_k(\mathcal{G}_0)=0,\quad F_k'(\mathcal{G}_0)=0,\quad k=1,2,\ldots.\end{aligned} \tag{45}$$

The general solution of the unperturbed differential equation (27) has the form:

$$F_0(\mathcal{G}) = c_1\mathcal{G} + c_2\sqrt{\mathcal{G}}\sqrt{24\lambda^4 - \mathcal{G}} - \\ - \frac{1}{2\lambda^2}\mathcal{G}\ln\left(\frac{\sqrt{\mathcal{G}}}{2\sqrt{6}\lambda^2}\right) + \frac{1}{\lambda^2}\sqrt{\mathcal{G}}\sqrt{24\lambda^4 - \mathcal{G}}\arctan\left(\frac{\sqrt{\mathcal{G}} + 2\sqrt{6}\lambda^2}{\sqrt{24\lambda^4 - \mathcal{G}}}\right), \quad (46)$$

where c_1, c_2 are constants of integration. Note that this solution is defined for $0 < \mathcal{G} < 24\lambda^4$. It is not difficult to find the values of the constants c_1 and c_2:

$$c_1 = \gamma_1\frac{\mathcal{G}_0 - 12\lambda^4}{12\lambda^4\mathcal{G}_0} + \gamma_2\frac{24\lambda^4 - \mathcal{G}_0}{12\lambda^4} + \frac{1}{2\lambda^2}\ln\left(\frac{\sqrt{\mathcal{G}_0}}{2\sqrt{6}\lambda^2}\right),$$

$$c_2 = \gamma_1\frac{\sqrt{24\lambda^4 - \mathcal{G}_0}}{12\lambda^4\sqrt{\mathcal{G}_0}} - \gamma_2\frac{\sqrt{24\lambda^4 - \mathcal{G}_0}\sqrt{\mathcal{G}_0}}{12\lambda^4} - \frac{1}{\lambda^2}\arctan\left(\frac{2\sqrt{6}\lambda^2 + \sqrt{\mathcal{G}_0}}{\sqrt{24\lambda^4 - \mathcal{G}_0}}\right).$$

Note that in the limit $\mathcal{G}_0 \to 24\lambda^4$:

$$c_1 \to \frac{\gamma_1}{24\lambda^4}, \quad c_2 \to -\frac{\pi}{2\lambda^2}.$$

One can consider the first-order correction of the perturbative ε-expansion, namely $F_1(\mathcal{G})$. By considering Eqs. (45) and (46), the Cauchy problem for the differential equation (28) becomes:

$$2\mathcal{G}^2\left(1 - \frac{\mathcal{G}}{24\lambda^4}\right)F_1''(\mathcal{G}) - \mathcal{G}F_1'(\mathcal{G}) + F_1(\mathcal{G}) = \\ \frac{1}{2304\lambda^{10}}\mathcal{G}^2(\mathcal{G}(\mathcal{G} - 24\lambda^4)F_0''(\mathcal{G}) - 6\lambda^2)^2(\mathcal{G}(\mathcal{G} - 24\lambda^4)F_0''(\mathcal{G}) - 2\lambda^2), \quad (47)$$

$$F_1(\mathcal{G}_0) = 0, \quad F_1'(\mathcal{G}_0) = 0. \quad (48)$$

It is easy to show that the solutions of the homogeneous differential equation are:

$$y_1(\mathcal{G}) = \mathcal{G}, \quad y_2(\mathcal{G}) = \sqrt{\mathcal{G}}\sqrt{24\lambda^4 - \mathcal{G}}.$$

Then the solution of the inhomogeneous equation (47) can be written as:

$$F_1(\mathcal{G}) = c_3\mathcal{G} + c_4\sqrt{\mathcal{G}}\sqrt{24\lambda^4 - \mathcal{G}} + \\ + \frac{\mathcal{G}}{2304\lambda^{10}}\int(\mathcal{G}(\mathcal{G} - 24\lambda^4)F_0''(\mathcal{G}) - 2\lambda^2)(\mathcal{G}(\mathcal{G} - 24\lambda^4)F_0''(\mathcal{G}) - 6\lambda^2)^2 d\mathcal{G} - \\ - \frac{\sqrt{\mathcal{G}}\sqrt{24\lambda^4 - \mathcal{G}}}{2304\lambda^{10}}\int\frac{\sqrt{\mathcal{G}}}{\sqrt{24\lambda^4 - \mathcal{G}}}(\mathcal{G}(\mathcal{G} - 24\lambda^4)F_0''(\mathcal{G}) - 2\lambda^2)(\mathcal{G}(\mathcal{G} - 24\lambda^4)F_0''(\mathcal{G}) - 6\lambda^2)^2 d\mathcal{G},$$

where c_3, c_4 are constants of integration. We can explicitly calculate the integral by using a new variable,

$$s = \arctan\left(\frac{\sqrt{\mathcal{G}} + 2\sqrt{6}\lambda^2}{\sqrt{24\lambda^4 - \mathcal{G}}}\right), \quad \frac{\pi}{4} \le s < \frac{\pi}{2}.$$

Then, the first correction $F_1(\mathcal{G})$ can be written as follows,

$$
\begin{aligned}
F_1(s) = 12\lambda^4(2c_3\cos^2 2s - c_4\sin 4s) + \\
+ \frac{9}{4}\lambda^4\left(1 + 4s^2 + 76(s + c_2\lambda^2)^2 + 72c_2^2\lambda^4\ln(24\lambda^4)\right)\cos^2 2s - \\
- \frac{9}{4}c_2\lambda^6\left(9\ln 2 + 11\ln(24\lambda^4)\right)\sin 4s - \frac{9}{8}\lambda^4\arctan(\cot 2s)(8c_2\lambda^2\cos^2 2s + \sin 4s) + \\
+ 18\lambda^4\left(4(7\mathcal{A}'(s) + 2\mathcal{B}'(s))\cos^2 2s + (18\mathcal{A}(s) - 18\mathcal{B}(s) + 5\mathcal{A}''(s))\sin 4s\right) - \\
- 18\lambda^4(s + c_2\lambda^2)\left((18\mathcal{A}'(s) - 18\mathcal{B}'(s) + 5\ln(\sin 2s))\sin 4s + 4(7\mathcal{A}''(s) + 2\mathcal{B}''(s))\cos^2 2s\right) + \\
+ 18\lambda^4(s + c_2\lambda^2)^2\left(9(\mathcal{A}''(s) - \mathcal{B}''(s))\sin 4s + 2(2\ln|\cos 2s| + 7\ln\sin 2s)\cos^2 2s\right) - \\
- 54\lambda^4(s + c_2\lambda^2)^3\left(\cos 6s\csc 2s + \sin 4s\ln|\tan 2s|\right), \quad (49)
\end{aligned}
$$

where the functions $\mathcal{A}(s)$ and $\mathcal{B}(s)$ are defined as follows:

$$
\mathcal{A}(s) = -\frac{11}{36}s^3 + \frac{1}{6}s^3\ln 2s + \sum_{n=1}^{\infty}(-1)^n\frac{2^{4n-1}}{k(2k+3)!}B_{2k}s^{2k+3},
$$

$$
\mathcal{B}(s) = \sum_{n=1}^{\infty}(-1)^n\frac{2^{2n-4}(1-2^{2k})}{k(2k+3)!}B_{2k}(\pi - 2s)^{2k+3},
$$

and B_{2k} are the Bernoulli numbers. Note that

$$
\mathcal{A}'''(s) = \ln\sin 2s, \quad \mathcal{B}'''(s) = \ln|\cos 2s|.
$$

From the initial conditions (48), it is easy to find the coefficients c_3 and c_4, but due to lengthy and complicated expressions, we do not present the final solution to the Cauchy problem.

Before closing this section, we need to note that the following limiting cases hold true,

$$
\lim_{\mathcal{G}\to 0^+} F_0(\mathcal{G}) = 0, \quad \lim_{\mathcal{G}\to 24\lambda^4 -} F_0(\mathcal{G}) = 24c_1\lambda^4.
$$

$$
\lim_{\mathcal{G}\to 0^+} F_1(\mathcal{G}) = 0, \quad \lim_{\mathcal{G}\to 24\lambda^4 -} F_1(\mathcal{G}) = \begin{cases} \frac{9}{4}\lambda^4\left(1 + 18\pi^2\ln(24\lambda^4) + 224\mathcal{A}'\left(\frac{\pi}{2}\right)\right), & c_2 = -\frac{\pi}{2\lambda^2}; \\ +\infty, & c_2 \neq -\frac{\pi}{2\lambda^2}. \end{cases}
$$

This implies that for the case $c_2 \neq -\frac{\pi}{2\lambda^2}$, the first order correction $F_1(\mathcal{G})$ is defined only in a neighborhood of the point $\mathcal{G} = \mathcal{G}_0$. However, the case $c_2 = -\frac{\pi}{2\lambda^2}$ should be considered a separate way. It can be obtained, for example, in the formulation of the boundary value problem of Eq. (25):

$$
\lim_{\mathcal{G}\to 0^+} F(\mathcal{G}) = 0, \quad \lim_{\mathcal{G}\to 24\lambda^4 -} F(\mathcal{G}) = 0.
$$

At the same time it is easy to show that,

$$
c_1 = 0, \quad c_2 = -\frac{\pi}{2\lambda^2}, \quad c_3 = -\frac{3}{32}\left(1 + 18\pi^2\ln(24\lambda^4) + 224\mathcal{A}'\left(\frac{\pi}{2}\right)\right), \quad c_4 \in \mathbb{R}.
$$

Note also that in this case, the first order correction $F_1(\mathcal{G})$ can take finite values on the whole interval, where the function $F_0(\mathcal{G})$ is defined, which is the following interval,

$$\left| \frac{F_1(\mathcal{G})}{F_0(\mathcal{G})} \right| < C, \quad 0 \leq \mathcal{G} \leq 24\lambda^4,$$

where C is a finite constant.

Similarly, one can build corrections any other order for the model $a(t) = \cosh(\lambda t)$, but the expressions can be more complicated so we confine ourselves to the first two orders.

7.2 Power-law Bouncing Model

As a final example, we shall study another bouncing cosmology which is described by the following scale factor,

$$a(t) = (t - t_s)^\alpha \,, \tag{50}$$

which is related to certain ekpyrotic models [17, 18]. The parameter α is a real positive number which for the purposes of this section we choose it to satisfy $1 < \alpha < 5$, for reasons to be clear later on in this section. The cosmological evolution described by the scale factor (50) perfectly describes a bounce, meaning that before the bouncing point, we have $\dot{a} < 0$, after the bouncing point $\dot{a} > 0$ and at the bounce $\dot{a} = 0$, as can easily be checked by looking at Fig. 2, where we plotted the time dependence of the function $\dot{a}$, for $\alpha = 4/3$ [1] and $t_s = 10^{-35}$sec. Since for

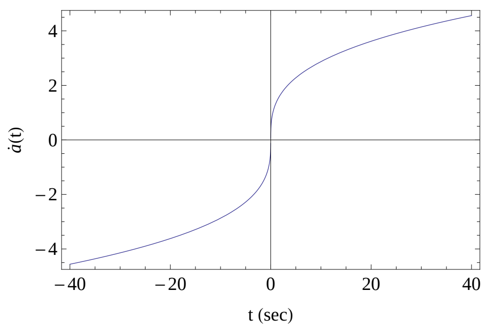

Figure 2: The derivative of the scale factor $\dot{a}(t)$ as a function of the cosmic time t, for the bouncing cosmology $a(t) = (t - t_s)^\alpha$, with $\alpha = 4/3$ and $t_s = 10^{-35}$sec.

general values of α, it is quite difficult to analytically solve the differential equations (28), as in the symmetric bounce case, we shall investigate here which LQC-corrected $F(\mathcal{G})$ gravity can describe the bounce (50) near the bouncing point $t = t_s$, which can be arbitrarily chosen. Notice that when $t \simeq t_s$, the expression $x = t - t_s$ tends to zero, and this observation shall be useful in the following analysis. The Gauss-Bonnet invariant $\mathcal{G}$, for the scale factor (50) reads,

$$\mathcal{G} = \frac{24(-1+\alpha)\alpha^3}{(t-t_s)^4} \,, \tag{51}$$

[1] Notice that in principle if α is not appropriately chosen, the scale factor might turn complex, so extra attention should be payed on this issue.

so as $t \to t_s$, the variable $\mathcal{G}$ increases. Therefore the limit $x \to 0$ corresponds to the limit $\mathcal{G} \to \infty$, and we shall use this correspondence in the analysis that follows. Solving Eq. (51), with respect to $t - t_s$, we obtain,

$$t - t_s = \frac{2^{3/4} 3^{1/4} \left(-\alpha^3 + \alpha^4\right)^{1/4}}{\mathcal{G}^{1/4}}, \tag{52}$$

and since $1 < \alpha < 5$, no inconsistency related to complex cosmological times occurs. The differential equation that yields the $F_0(\mathcal{G})$ gravity is therefore equal to,

$$\mathcal{B}_1 \mathcal{G}^2 F_0''(\mathcal{G}) - \mathcal{G} F_0'(\mathcal{G}) + F_0(\mathcal{G}) + \mathcal{D}\sqrt{\mathcal{G}} = 0\,, \tag{53}$$

where we have set for simplicity $\mathcal{D}$ and $\mathcal{B}_1$ to be equal to,

$$\mathcal{D} = \frac{\sqrt{\frac{3}{2}}\alpha^2}{\sqrt{(-1+\alpha)\alpha^3}} \quad \mathcal{B}_1 = -\frac{4}{\alpha - 1}. \tag{54}$$

The solution to the differential equation (53) is equal to,

$$F_0(\mathcal{G}) = \frac{4\mathcal{D}\sqrt{\mathcal{G}}}{-2 + \mathcal{B}_1} + \mathcal{G}^{\frac{1}{\mathcal{B}_1}} C_1 + \mathcal{G} C_2\,. \tag{55}$$

Finally, we calculate the first order correction of the LQC-corrected $F(\mathcal{G})$ gravity, namely the function $F_1(\mathcal{G})$, which easily follows if we use the analytic form of $F_0(\mathcal{G})$ given in Eq. (55). However, since the exact solution is quite complicated and lengthy, we can approximate the resulting expression by recalling that the limit $x \to 0$, corresponds to $\mathcal{G} \to \infty$, so the larger term dominates in the expressions. For $1 < \alpha < 5$, the most dominant term is of the order $\sim \mathcal{G}$, so $F(\mathcal{G})$ is approximately equal to $F_0(\mathcal{G}) \simeq C_2 \mathcal{G}$, and the resulting differential equation that yields the solution for $F_1(\mathcal{G})$, namely Eq. (28), becomes,

$$\mathcal{B}_1 \mathcal{G}^2 F_1''(\mathcal{G}) - \mathcal{G} F_1'(\mathcal{G}) + F_1(\mathcal{G}) - \frac{3\mathcal{G}\alpha^4}{4\left(-\alpha^3 + \alpha^4\right)} = 0\,, \tag{56}$$

which can analytically be solved to yield,

$$F_1(\mathcal{G}) = \mathcal{G}^{\frac{1}{\mathcal{B}_1}} C_1 + \mathcal{G} C_2 + \frac{3\mathcal{G}\alpha(-\mathcal{B}_1 - \ln[\mathcal{G}] + \mathcal{B}_1 \ln[\mathcal{G}])}{4(-1+\mathcal{B}_1)^2(-1+\alpha)}\,. \tag{57}$$

8 Conclusions

In this paper, we extended the holonomy corrections formalism of LQC to Gauss-Bonnet $F(\mathcal{G})$ modified gravity theories. Specifically, upon using the method of Lagrange multipliers, we constructed the classical dynamical cosmological equations in Gauss-Bonnet gravity, for a flat FLRW geometry. In addition, we extended LQC to Gauss-Bonnet gravity obtaining a holonomy corrected Friedmann equation, which contains all the dynamical information of the system. Then after explaining the reconstruction method in holonomy corrected Gauss-Bonnet gravity, we applied our formalism to certain cosmological scenarios, focusing in the realization of these cosmologies in the context of LQC-corrected $F(\mathcal{G})$ gravity. The cosmological scenarios on which we emphasized are the bouncing cosmologies and the reason for that is that bouncing cosmologies provide a quite elegant alternative scenario to the inflationary paradigm. Specifically we studied some well known bouncing cosmologies, with exponential and power- law scale factors. In some cases, we provided full analytic results for the first two functions $F_0(\mathcal{G})$ and $F_1(\mathcal{G})$ of the perturbative $F(\mathcal{G})$-expansion. As we evinced, the resulting LQC-corrected $F(\mathcal{G})$ dynamical equations are quite complicated, so we performed a perturbative expansion, using as a perturbation parameter, the parameter ε which is related to a very important physical quantity, the critical density ρ_c, as $\varepsilon = 1/\rho_c$. Practically, the parameter ρ_c measures how quantum is the theory, and the smaller it is, the theory "stretches" in the more quantum era. In the perturbation series we used, we assumed that $\varepsilon \to 0$, so the leading order corrections we found, practically quantify the way that the LQC-corrected theory deviates from the classical $F(\mathcal{G})$ theory. So iteratively, one can recover higher and higher corrections, being though less significant, depending on how fast ρ_c tends to infinity.

Moreover, the possibility of having finite time singularities [37] and specifically mild types of singularities [38] is also quite interesting and should be extensively discussed.

It is well-known that usually, the theories with accelerating epoch may develop future singularities [37], which occur not only after the dark energy era but also after the inflationary era [38] (so-called mild singularities). It was expected that bouncing cosmologies do not exhibit such finite-time singularities in their evolution. Unfortunately, as it was indicated recently [39] there is still possibility for bouncing cosmology with mild singularity of so-called Type IV. Hence, a bounce Universe maybe consistently combined with Type IV singularity and survive passing through it (due to completeness of geodesics on these time-like singularities). The extension to bouncing cosmology with singularity for the theory we discussed in this paper and for general $F(\mathcal{G})$-theories, will be discussed elsewhere.

Also, the viability of the theories should also be checked and confronted with current observational data, but this task extends the purpose of this paper, which was the presentation of the method of LQC-corrected $F(\mathcal{G})$ gravity, and its usefulness towards realizing cosmological scenarios. We hope however to discuss these topics in the near future.

Finally, a remark is in order. The holonomy corrections extension of $F(\mathcal{G})$ gravity in the context we presented these in this paper, can be easily generalized for other models, like non-local gravity, string-inspired gravity, etc. These cases will be considered in a future publication.

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Appendix

In this appendix, we give the exact form of the parameter $\mathcal{C}$ appearing in Eq. (39). It's detailed form is, and consequently, in the small $\mathcal{G}$ limit, it can be approximated by Eq. (40).

References

[1] R. Penrose, [Gen. Rel. Grav. **34**, 1141 (2002)].

[2] A. Ashtekar and P. Singh, Class. Quant. Grav. **28** (2011) 213001 [arXiv:1108.0893 [gr-qc]];
A. Ashtekar, Nuovo Cim. B **122** (2007) 135 [gr-qc/0702030];
M. Bojowald, Class. Quant. Grav. **26** (2009) 075020 [arXiv:0811.4129 [gr-qc]];
T. Cailleteau, A. Barrau, J. Grain and F. Vidotto, Phys. Rev. D **86** (2012) 087301 [arXiv:1206.6736 [gr-qc]];
A. Ashtekar, P. Singh, Class. Quant. Grav. 28, 213001 (2011) [arXiv:1108.0893];
A. Corichi, P. Singh, Phys.Rev. D80 (2009) 044024 [arXiv:0905.4949];
P. Singh, Class.Quant.Grav. 26 (2009) 125005 [arXiv:0901.2750];
A. Ashtekar, T. Pawlowski, P. Singh, Phys.Rev. D74 (2006) 084003 [gr-qc/0607039];
M. Bojowald, Class.Quant.Grav. 26 (2009) 075020 [arXiv:0811.4129]

[3] X. Zhang and Y. Ma, Phys. Rev. Lett. **106**, 171301 (2011) [arXiv:1101.1752].

[4] X. Zhang and Y. Ma, Phys. Rev. **D84**, 064040 (2011) [arXiv:1107.4921].
X. Zhang and Y. Ma, Front. Phys. **8**, 80 (2013) [arXiv:1211.5024].

[5] G. De Risi, R. Maartens and P. Singh, Phys. Rev. **D76**, 103531 (2007) [arXiv:0706.3586].

[6] J. Amorós, J. de Haro and S.D. Odintsov Phys. Rev. **D89**, 104010 (2014) [arXiv:1402.3071].

[7] J. de Haro, EPL **107**, 29001 (2014) [arXiv:1403.4529].

[8] K. Bamba, A.N. Makarenko, A.N. Myagki and S.D. Odintsov, Phys. Lett. **B732**, 349 (2014) [arXiv:1403.3242].

[9] M. Novello and S. E. P. Bergliaffa, Phys. Rept. **463**, 127 (2008) [arXiv:0802.1634].
J. L. Lehners, Phys. Rept. **465**, 223 (2008) [arXiv:0806.1245].
R. H. Brandenberger, Int. J. Mod. Phys. Conf. Ser. **01**, 67 (2011) [arXiv:0902.4731].
R. H. Brandenberger, AIP Conf. Proc. **1268**, 3 (2010) [arXiv:1003.1745].
D. Battefeld and P. Peter, [arXiv:1406.2790].

[10] M. Novello, S.E.Perez Bergliaffa, Phys.Rept. **463** (2008) 127 [arXiv:0802.1634];
C. Li, R. H. Brandenberger and Y. K. E. Cheung, Phys. Rev. D **90** (2014) 12, 123535 [arXiv:1403.5625 [gr-qc]].;
Yi-Fu Cai, E. McDonough, F. Duplessis, R. H. Brandenberger, JCAP 1310 (2013) 024 [arXiv:1305.5259];
Yi-Fu Cai, E. Wilson-Ewing, JCAP 1403 (2014) 026 [arXiv:1402.3009];
J. Haro, J. Amoros, JCAP 08(2014)025 [arXiv:1403.6396];
T. Qiu and K. C. Yang, JCAP **1011**, 012 (2010) [arXiv:1007.2571 [astro-ph.CO]].;
T. Qiu, Class. Quant. Grav. **27**, 215013 (2010) [arXiv:1007.2929 [hep-ph]].

[11] Jean-Luc Lehners, Class.Quant.Grav. **28** (2011) 204004 [arXiv:1106.0172]

[12] Yi-Fu Cai, J. Quintin, E. N. Saridakis, E. Wilson-Ewing, JCAP **1407** (2014) 033 [arXiv:1404.4364]
J. Khoury, P. J. Steinhardt, Phys.Rev.Lett. **104** (2010) 091301 [arXiv:0910.2230]
J. Khoury, B. A. Ovrut, J. Stokes, JHEP **1208** (2012) 015 [arXiv:1203.4562]

[13] M. Koehn, Jean-Luc Lehners, B. A. Ovrut, Phys.Rev. D **90** (2014) 025005 [arXiv:1310.7577]

[14] Yi-Fu Cai, D. A. Easson, R. Brandenberger, JCAP **1208** (2012) 020 [arXiv:1206.2382];
S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **91** (2015) 6, 064036 [arXiv:1502.06125 [gr-qc]] '

[15] Y. F. Cai, T. Qiu, Y. S. Piao, M. Li and X. Zhang, JHEP **0710** (2007) 071 [arXiv:0704.1090 [gr-qc]].

[16] R. H. Brandenberger, arXiv:1206.4196;
J. Quintin, Y. F. Cai and R. H. Brandenberger, Phys. Rev. D **90** (2014) 6, 063507 [arXiv:1406.6049 [gr-qc]];
Y. F. Cai, R. Brandenberger and X. Zhang, Phys. Lett. B **703** (2011) 25 [arXiv:1105.4286 [hep-th]];
Y. F. Cai, R. Brandenberger and X. Zhang, JCAP **1103** (2011) 003 [arXiv:1101.0822 [hep-th]];
K. Bamba, J. de Haro and S. D. Odintsov, JCAP **1302** (2013) 008 [arXiv:1211.2968 [gr-qc]];
J. de Haro, JCAP **1211** (2012) 037 [arXiv:1207.3621 [gr-qc]].;
S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **90** (2014) 12, 124083 [arXiv:1410.8183 [gr-qc]];
J. Haro, Europhys. Lett. **107** (2014) 29001 [arXiv:1403.4529];
Jaume de Haro, Yi-Fu Cai, arXiv:1502.03230;
K. Bamba, S. Nojiri, S. D. Odintsov, JCAP **0810** (2008) 045 [arXiv:0807.2575];
K. Bamba, A. N. Makarenko, A. N. Myagky, S. Nojiri and S. D. Odintsov, JCAP **1401** (2014) 008 [arXiv:1309.3748 [hep-th]]. ;
C. Barragan, G. J. Olmo and H. Sanchis-Alepuz, Phys. Rev. D **80** (2009) 024016 [arXiv:0907.0318 [gr-qc]].

[17] S. D. Odintsov, V. K. Oikonomou and E. N. Saridakis, arXiv:1501.06591 [gr-qc].

[18] V. K. Oikonomou, arXiv:1412.4343 [gr-qc]

[19] S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **90** (2014) 12, 124083 [arXiv:1410.8183 [gr-qc]].

[20] J. Khoury, B. A. Ovrut , N. Seiberg, P. J. Steinhardt, N. Turok, Phys.Rev. D **65** (2002) 086007 [hep-th/0108187];
J. K. Erickson, D. H. Wesley, P. J. Steinhardt, N. Turok, Phys.Rev. D **69** (2004) 063514 [hep-th/0312009];
Yi-Fu Cai, Shih-Hung Chen, J. B. Dent, S. Dutta, E. N. Saridakis, Class. Quantum Grav. **28** (2011) 215011 [arXiv:1104.4349]

[21] J. Khoury, B. A. Ovrut, P. J. Steinhardt, N. Turok, Phys.Rev. D **66** (2002) 046005 [hep-th/0109050]

[22] E. Wilson-Ewing, JCAP **1303** (2013) 026 [arXiv:1211.6269]

[23] D. Wands, Phys. Rev. D **60**, 023507 (1999) [arXiv:9809062].
F. Finelli and R. Brandenberger, Phys. Rev. **D65**, 103522 (2002) [arXiv:0112249].
J. Haro and J. Amoros, JCAP **1412**, no. 12, 031 (2014) [arXiv:1406.0369].
Y. F. Cai, D. A. Easson and R. Brandenberger, JCAP **1208**, 020 (2012) [arXiv:1206.2382].
J. Haro, JCAP **1311**, 068 (2013) [Erratum-ibid. **1405**, E01 (2014)] [arXiv:1309.0352].
E. Wilson-Ewing, JCAP **1303**, 026 (2013) [arXiv:1211.6269].
Y. F. Cai and E. Wilson-Ewing, JCAP **1403**, 026 (2014) [arXiv:1402.3009].
Y. F. Cai, Sci. China Phys. Mech. Astron. **57**, 1414 (2014) [arXiv:1405.1369].
J. de Haro and J. Amoros, JCAP **1408**, 025 (2014) [arXiv:1403.6396].

[24] E. Elizalde, J. Haro and S. D. Odintsov, Phys. Rev. **D91**, 063522 (2015) [arXiv:1411.3475].

[25] P. A. R. Ade *et al.* [Planck Collaboration], Astron. Astrophys. **571**, A16 (2014) [arXiv:1303.5076].
P. A. R. Ade *et al.* [Planck Collaboration], Astron. Astrophys. **571**, A22 (2014) [arXiv:1303.5082].
P. A. R. Ade *et al.* [Planck Collaboration], [arXiv:1502.01589].
P. A. R. Ade *et al.* [Planck Collaboration], [arXiv:1502.02114].

[26] N. Deruelle, Y. Sendouda and A. Youssef, Phys. Rev. **D80**, 084032 (2009) [arXiv:0906.4983].

[27] S. Nojiri and S.D. Odintsov, Phys. Lett. **B631**, 1 (2005) [arXiv:0508049].

[28] P. Singh, Class. Quant. Grav. **26**, 125005 (2009) [arXiv:0901.2750].
A. Corichi and P. Singh, Phys. Rev. D **78**, 024034 (2008) [arXiv:0805.0136];

[29] J. de Haro, JCAP **11**, 037 (2012) [arXiv:1207.3621].

[30] A. Ashtekar and P. Singh, Class. Quant. Grav. **28**, 213001 (2011) [arXiv:1108.0893].

[31] A. Ashtekar, T. Pawlowski and P. Singh , Phys. Rev. **D74**, 084003 (2006).
A. Ashtekar, T. Pawlowski and P. Singh, Phys. Rev. **D73**, 124038 (2006) [arXiv:0604013].
A. Ashtekar, M. Bojowald and J. Lewandowski, Adv. Theor. Math. **7**, 233 (2003).
T. Thiemann, Phys. Lett. B **380**, 257 (1996) [arXiv:9606088].
T. Thiemann, Class. Quant. Grav. **15**, 839 (1998) [arXiv:9606089].
T. Thiemann, Cambridge, UK: Cambridge Univ. Press. (2007) [arXiv:0110034].

[32] M. Bojowald, Class. Quant. Grav. **26**, 075020 (2009) [arXiv:0811.4129].

[33] J. Haro and E. Elizalde, EPL **89**, 69001 (2010).

[34] P. Dzierzak, P. Malkiewicz and W. Piechocki, Phys. Rev. **D80**, 104001 (2009).

[35] P. Singh, K. Vandersloot and G. V. Vereshchagin, Phys. Rev. **D74**, 043510 (2006) [arXiv:0606032].
P. Singh, Phys. Rev. **D73**, 063508 (2006) [arXiv:0603043].
C. Rovelli and E. Wilson-Ewing, Phys. Rev. D **D90**, 023538 (2014) [arXiv:1310.8654].

[36] Nojiri S., Odintsov S.D., Phys. Lett. **B631**, 1 (2005) [arxiv:0508049].
Nojiri S., Odintsov S.D., Sasaki M., Gauss-Bonnet dark energy. Phys. Rev. **D71**, 123509 (2005) [arXiv:0504052].
E. Elizalde, R. Myrzakulov, V. V. Obukhov and D. Saez-Gomez, Class. Quant. Grav. **27**, 095007 (2010) [arXiv:1001.3636].
A.N. Makarenko, V.V. Obukhov, I.V. Kirnos, Astrophys.Space Sci. **343**, 481-488 (2013) [arXiv:1201.4742].

[37] S. Nojiri, S. D. Odintsov and S. Tsujikawa, Phys. Rev. D **71** (2005) 063004 [hep-th/0501025].

[38] J. D. Barrow and A. A. H. Graham, Phys. Rev. D **91**, no. 8, 083513 (2015) [arXiv:1501.04090 [gr-qc]].
S. Nojiri, S. D. Odintsov and V. K. Oikonomou, Phys. Rev. D **91** 084059 (2015) [arXiv:1502.07005 [gr-qc]].
S. Nojiri, S. D. Odintsov, V. K. Oikonomou and E. N. Saridakis, arXiv:1503.08443 [gr-qc].

[39] S. D. Odintsov and V. K. Oikonomou, arXiv:1504.06866 [gr-qc], PRD in Press

[40] M. Bojowald and G.M. Hossain, Phys. Rev. **D77** 023508 (2008) [arXiv:0709.2365].
T. Cailleteau, J. Mielczarek, A. Barrau and J. Grain, Class. Quant. Grav. **29** 095010 (2012) [arXiv:1111.3535].
T. Cailleteau, A. Barrau, J. Grain and F. Vidotto, Phys. Rev. **D86** 087301 (2012) [arXiv:1206.6736].
T. Cailleteau, L. Linsefors and A. Barrau, Class.Quant.Grav. **31** 125011 (2014) [arXiv:1307.5238].