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Workshop

TEACHER TRAINING IN THE HISTORY OF MATHEMATICS

Fàtima Romero Vallhonesta[a], Maria Rosa Massa Esteve[a], Iolanda Guevara Casanova[b], Carles Puig-Pla[a] & Antoni Roca-Rosell[a]

[a]Universitat Politècnica de Catalunya, [b]Departament d'Ensenyament de la Generalitat de Catalunya & Departament de Didàctica de les Matemàtiques i les Ciències Experimentals de la UAB

The History of Mathematics could be a powerful tool for Mathematics teachers to improve their teaching, by offering to students a variety of ways to achieve mathematical concepts successfully. The Catalan Mathematics curriculum for secondary schools, published in June 2007, contains notions of the historical genesis of relevant mathematical subjects within the syllabus. However, there is no indication to develop the content associated with these subjects. We have designed a course for pre-service teachers of Mathematics with the aim of providing them with the knowledge needed to use historical materials in their classrooms. This contribution aims to analyze the implementation of these historical Mathematics activities.

INTRODUCTION [1]

By means of original sources and significant texts, it is possible to learn from the past and teach Mathematics through a historical and cultural approach (Fauvel & Maanen, 2000; Katz, & Tzanakis, 2011). Knowledge of the History of Mathematics provides teachers with an understanding of the foundations and the nature of Mathematics, and with a capacity for a better understanding of how and why the different branches of Mathematics have taken shape as well as their connection with other disciplines (Jankvist, 2009).

In fact, the History of Mathematics is a very useful tool to help in the comprehension of mathematical ideas and concepts (Demattè, 2010). It is also a very effective tool to help in the understanding of Mathematics as a useful, dynamic, humane, interdisciplinary and heuristic science.

On the one hand, the History of Mathematics can be used as an implicit resource in the design of activities to adapt some standard concepts to the teaching syllabus, to choose context, and to prepare problems and auxiliary sources.

On the other hand, the History of Mathematics can also be used in an explicit way to direct and propose research works at baccalaureate level using historical material, to design and impart elective subjects involving the History of Mathematics using ICT, to hold workshops, centenaries and conferences using historical subjects, and to use significant historical texts in order that students understand better mathematical concepts.

We have designed a course for teachers of Mathematics with the aim of providing them with the knowledge needed to use historical materials in their classrooms. In addition, the use of these historical materials allows the teacher to use a different approach in which at the start of the class the teacher sets the students a text to read and helps them to interpret it, by giving them some guidelines and questions to answer. The fact of having to locate the texts in their historical context also encourages an interdisciplinary approach and helps the students to understand Mathematics as a discipline which is linked to other disciplines. We supply the teachers with original sources on which the knowledge of mathematics in the past is based. They have to work with these sources, which consist of reading and interpreting a selection of classical mathematical texts as well as learning how to locate and use historical literature or historical online resources. The task of teachers also involves the recognition of the most significant changes in the discipline of Mathematics; those which have influenced its structure and classification; its methods, its fundamental concepts and its relation to other sciences. Some materials have also been chosen to emphasize the socio-cultural relations of mathematics with politics, religion, philosophy and culture in a given period, and most importantly to encourage teachers to reflect on the development of mathematical thought and the transformations of natural philosophy (Pestre, 1995). The final project is drawn up by teachers themselves and consists of designing an activity for the students based on the material with which they have worked throughout the course.

In this paper we present three of the activities carried out in the course:

1. Using Chinese problems and procedures from an ancient classical book for teaching Mathematics.
2. Introducing the quadratic equation using historical methods
3. Algebra and geometry in the Mathematics classroom

Most of these activities have been tried out in secondary schools and intended to inspire teachers to create their own activities. The criteria for selecting the specific texts consist of their relationship with the historical contexts in the Catalan curriculum (Catalunya. Decret 143/2007).

USING CHINESE PROBLEMS AND PROCEDURES FROM AN ANCIENT CLASSICAL BOOK FOR TEACHING MATHEMATICS

For the following activities we use Chinese problems and procedures from *The Nine Chapters* for teaching mathematics.

The Nine Chapters and the historical context

Ancient mathematical texts were compiled during the Qin dynasty (221–206 BC) and Han dynasty (206 BC–AD 220). The most influential of all Chinese mathematical books, *The Nine Chapters,* was probably compiled in early Han dynasty (Dauben,

2007, 227). The purpose of this practical manual of mathematics consisting of 246 problems was to provide methods to be used in solving everyday problems of engineering, surveying, trade, and also taxation (Lam, 1994). Scholars believe that *The Nine Chapters* has been the most important mathematical source in China for the past 2000 years, comparable in significance to Euclid's *Elements* in Western Culture. Along the centuries, some scholars made copious commentaries on the book to explain the implied mathematical concepts. Among those commentators are Liu Hui (ca. 220-280), one of the greatest mathematicians of ancient China and Li Chunfeng (602-670), an outstanding astronomer and mathematician.

As the name suggests, the book contains nine chapters, and we focus on Chapter 9 "*Gougu*" (or base and height), which deals with problems for solving right triangles, involving the Gougu procedure, the principle known in Western Culture as the Pythagorean Theorem.

Activities carried out by pupils of secondary education

We have proposed a sequence of activities carried out by pupils of secondary education, based on the problems in Chapter 9 of *The Nine Chapters.* The activities are designed according to the fundamental figures described by Liu Hui (263) and Li Chunfeng (656) in their commentaries on the classical text, analysed in the bilingual translation by Chemla & Shuchun (2005, 703-745) and following the suggestions about their pedagogic value by Siu Man-Keung (2000, 159-166).

In general, *The Nine Chapters* is organized as follows: first there is the classical text, dealing with the problem statement with specific numerical data; secondly, the questions; thirdly, the answers, then a brief description of the procedure to find the solution; then the commentaries by Liu Hui and Li Chunfeng, which provide the algorithms needed to solve the problems, and finally the explanations of how the algorithms work.

Going to the beginning of Chapter 9: The title *Gougu*, which means base (*gou)* and height *(gu)*, has a subtitle, "*Solving height and depth, width and length*". The chapter contains 24 problems on right triangles. Problems 1-12 (Chemla & Shuchun, 2005, 703-721) deal with the base and height procedure, and problems 13-24 deal with similar triangles (Chemla & Shuchun, 2005, 723-745). The following problems are related to situations in a real context where the initial geometric assumptions appear.

At the beginning of Chapter 9, the classical text states "*Base (gou) and height (gu) procedure*", but later Lui Hui adds the following:

> "The shorter side is called the base (gou), the longest side the height (gu) joining the corners with each another is called the hypotenuse (xian)" (Chemla & Shuchun, 2005, 705).

The classical text states the *Gougu* theorem like an algorithm:

> "If each is multiplied by itself and the results, once added, are divided by the square root extraction, the result is the hypotenuse" (Chemla & Shuchun, 2005, 705).

and later Lui Hui gives a geometrical proof :

> "The shorter leg multiplied by itself is the vermilion square, and the longer leg multiplied by itself is the blue-green square. Let them be moved about so as to patch each other, according to its type. Because the differences are completed, there is no instability. Together they form the area of the square on the hypotenuse; extracting the square root gives the hypotenuse" (Chemla & Shuchun, 2005, 705).

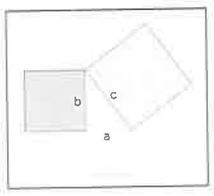

Figure 1: *Gougu* theorem

We propose an activity for the students in which they try to obtain a similar proof themselves. In order to prove that "the area of the square on side c is the sum of the areas of the squares on the sides a and b", they need to construct a square whose side is equal the hypotenuse from 2 squares of sides a and b, respectively. Then to prove the theorem, they have only to cut and paste figures.

The instructions for students could consist of the following: **a)** cut any two squares; **b)** place the small square inside the large square so that the two have a common vertex and base; **c)** draw the triangle on the side of the small square base and the height of the larger square; **d)** cut a third square of a side equal to the hypotenuse of the right triangle **e)** draw below the triangle and draw three squares obtained where appropriate (see Figure 2).

a b & c d

Figure 2: The steps of the student's instructions

Figure 3 below shows the work by one student:

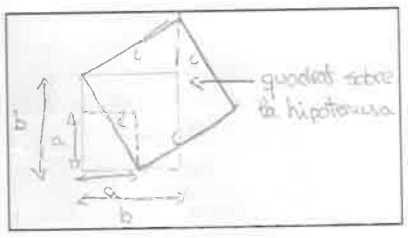

Figure 3: A student's production (14-15 years old)

Then, to prove the theorem (the largest area of the square is the sum of the areas of two other squares) it is only a matter of cutting and pasting (see Figure 4).

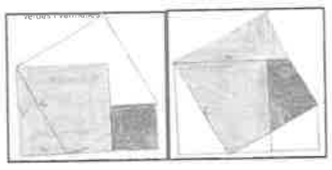

Figure 4: A student's production

Another activity we propose is based on problem 5, following the commentaries by Lui Hui. The wording is as follows:

> "Suppose we have a tree of 2 *zhang* as height and 3 *chi* as a perimeter. A climbing plant that grows from its base surrounds the tree seven times before reaching the top. One asks how long the climbing plant is" (Chemla & Shuchun, 2005, 709).

From the commentaries we can deduce that if we: a) roll up a sheet of paper forming a cylinder, simulating the trunk of the tree; b) draw the climbing plant around it; c) expand the sheet, we will obtain the solution, which is related to the *Gougu* theorem, as may be seen in Figure 5. Simply by adding seven times the hypotenuse, we will obtain the answer.

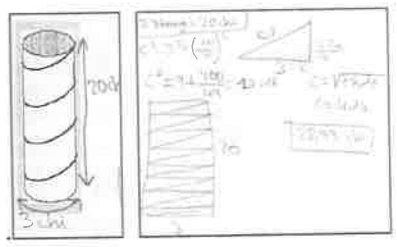

Figure 5: A student's production of problem 5

We also propose many interesting activities on the course by using special figures that the ancient Chinese employed to infer relationships between measures of the sides of the triangle, and sums and differences between them. They considered three geometrical figures, called "the fundamental figures", which helped them to solve problems with right triangles in a geometrical way, that is, with "visual aids".

The following table (Figure 6) shows these three fundamental figures and the relationships between their different measures to solve problems of right triangles:

Name and picture	Description	Relationships
1st	square of side a+b mark a in each side alternately	c, b-a ➡ a+b ➡ a, b
2n	square of side c within a square of side b, the gnomon has area a²	a, c-b ➡ c+b ➡ b, c
3rd	square of side c + a add a rectangle of area b². 2c is the side of the new rectangle	b, c+a ➡ c, a

Figure 6: The three fundamental figures and their relationships

These activities were conducted in the same way as the ancient Chinese, who in the absence of algebraic symbolism solved problems with reasoning based on geometry, and were very well accepted by the students. They were able to make sense of the rules of formal algebra, remarking that: "Now I understand it. These operations with letters are like the calculations we are doing with the figures!"

INTRODUCING THE QUADRATIC EQUATION USING HISTORICAL METHODS

In the following activities, we propose to solve equations using the al-Khwārizmī method (by completing squares); students can benefit from visual reasoning that combines algebra (in current notation) and geometry (Katz & Barton, 2007, 185-201).

Abu Ja'far Muhammad ibn Musa al-Khwārizmī (ca. 780-850)

His name indicates that he may have come from Khwarezm (Khiva), then in Greater Khorasan, which occupied the Eastern part of the Greater Iran, now the Xorazm Province in Uzbekistan.

He was a mathematician, astronomer and geographer during the Abbasid Empire, and a scholar at the House of Wisdom in Baghdad.

"The Compendious Book on Calculation by Completion and Balancing" (*Kitāb al-Mukhtasar fī hisāb al-jabr wa'l-muqābala* (ca. 813) المختصر حساب الجبر والمقابلة) was the most famous and important of all of al-Khwārizmī 's works (Djebbar, 2005, 21; Toomer, 2008).

In Renaissance Europe, he was considered one of the inventors of algebra, although it is now known that his work was based on older Indian or Greek sources.

The treatise Hisāb al-jabr wa'l-muqābala (ca. 813)

The book was translated into Latin by Robert of Chester (Segovia, 1145) as *Liber algebrae et almucabala,* hence "algebra", and also by Gerard of Cremona (ca. 1170). A unique Arabic copy of manuscript from 1342 is kept at the Bodleian Library in Oxford, and was translated into English in 1831 by Frederic Rosen.

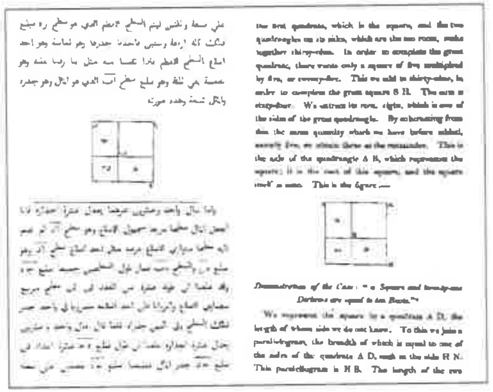

Figure 7. The text and diagrams in the Rosen edition (Rosen, 1831, 322/16)

We chose the text and the diagrams from the Rosen edition (see Figure 7) to design the activities for solving quadratic equations with visual reasoning (Rosen, 1831).

In the text, the author provided an exhaustive account of solving polynomial equations up to the second degree, and also discussed the fundamental methods of "reduction" and "balancing", which refers to the transposition of terms from one side of an equation to the other side, that is, the elimination of equal terms on both sides of the equation.

Al-Khwārizmī wanted to give his readers general rules for all kinds of equations and not just how to solve specific examples. His rules for solving linear and quadratic equations began by reducing the equation to one of six standard forms.

We will use the case: "a Square and ten Roots are equal to thirty-nine Dirhems", to design the activities. Al-Khwārizmī stated as follows (see Figure 8):

> "We proceed from the quadrate AB, which represents the square. It is our next business to add to it the ten roots of the same. We halve for this purpose the ten, so that it becomes five, and construct two quadrangles on two sides of the quadrate AB, namely, G and D, the length of each of them being five, as the moiety [half] of ten roots, whilst the breadth of each is equal to a side of the quadrate AB. Then a quadrate remains opposite the corner of the quadrate AB. This is equal to five multiplied by five: this five being half of the number of the roots, which we have

added to each of the two sides of the first quadrate. Thus we know that the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine in order to complete the great square SH. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity, which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle AB, which represents the square; it is the root of this square, and the square itself is nine."(Rosen, 1831, 15-16).

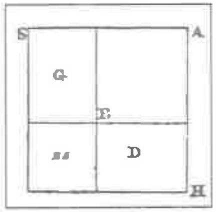

Figure 8. Geometrical justification by al-Khwārizmī (Rosen, 1831, 16)

We begin the activity by solving incomplete equations

a) The algebraic procedure

We use some sessions to solve incomplete equations with algebraic procedures, reduction and balancing. Students know how to solve linear equations and we apply this procedure to incomplete second-degree equations.

b) The geometrical procedure

We devote some sessions to the geometrical visualization of x^2 and, step by step, we introduce students to solving the second degree incomplete equations geometrically. They have to understand x and x^2 as the measure of the sides of the squares, and their areas, respectively (see Figure 9).

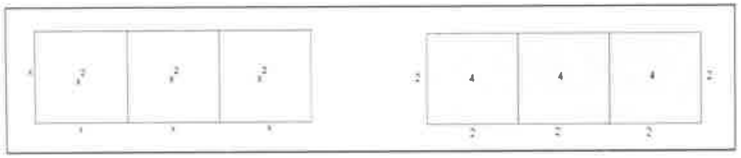

Figure 9: The geometrical interpretation of the incomplete equation $3x^2 = 12$

When students have discovered how to solve this kind of incomplete equation, we ask them to write equations knowing their solutions: For example:

$x = 3 \rightarrow x^2 = 9,\ 2x^2 = 18, \ldots$

$x = 0$ and $3 \rightarrow x^2 = 3x,\ 2x^2 = 6x, \ldots$

We solve equations like: $ax^2 = c$, and $ax^2 = bx$

Now we introduce the resolution of complete quadratic equations. The first example was the same as that by al-Khwārizmī: $x^2 + 10x = 39$ (Figure 10).

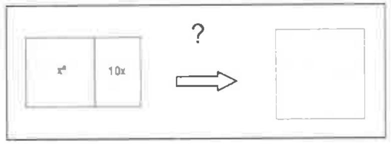

Figure 10: Is it possible to transform this rectangle of area 39 into a square?

As in the al-Khwārizmī procedure, we guide students to its solution with this idea (see Figure 11).

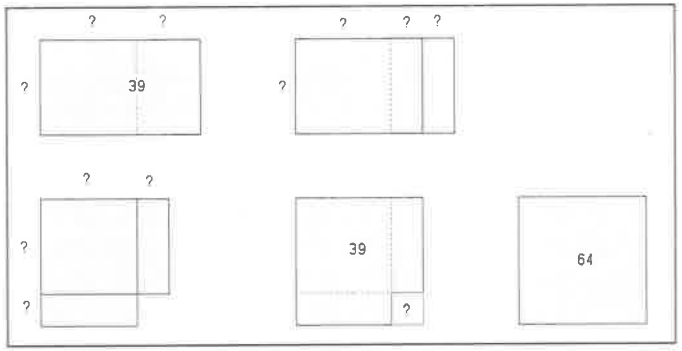

Figure 11: The diagrams in the sequence of activities

We also work with negative numbers, although al-Khwārizmī only worked with positive numbers.

Students know that 64 has two square roots, 8 and -8, and then we can obtain two solutions of the equation, the geometrical one x = 3 and -8 = x + 5 x = -13 (see Figure 12, when they solve $x^2+6x=40$)

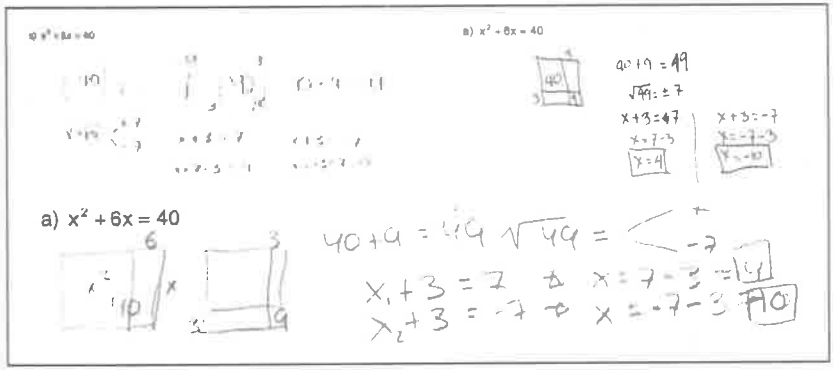

Figure 12: Some students' productions (14-15 years old) with $x^2 + 6x = 40$

By introducing the resolution of quadratic equations by completing squares throughout one school year, then waiting until the next course to introduce the resolution by using the usual formula, yielded to two relevant results about student learning. On the one hand, they discovered that the problems they were studying originated in ancient times and different cultures, while on the other they also realized that algebraic formulas could make more sense when interpreted in a geometrical manner.

ALGEBRA AND GEOMETRY IN THE MATHEMATICS CLASSROOM

This part consists of activities containing singular geometric constructions used for solving the quadratic equation in the seventeenth century. These analyses linking algebra to geometry provide students with a richer view of Mathematics and improve the teaching and learning processes. Thus, the reflection on these geometric constructions of algebraic expressions historical helps to develop the analytic and synthetic thought of students.

Indeed, the study of the origins of polynomials and their associated equations gives us a history of the geometric construction of the solution of the quadratic equation with instructive and suggestive passages for students, whether at high school or college degree level. We focus on the process of algebraization of mathematics, which took place from the late sixteenth century to the early eighteenth century (Mancosu, 1996, 84-91). This was mainly the result of the introduction of algebraic procedures for solving geometrical problems.

First geometrical justifications

In his treatise *Kitāb al-Mukhtasar fī hisāb al-jabr wa'l-muqābala* (ca. 813), Mohammed Ben Musa Al-Khwārizmī (ca. 780-850) describes different kinds of equations using rhetorical explanations, and without symbols. His geometrical justifications of the solutions of equations are given by squares and rectangles, as we have shown in the previous activity. Later, when Leonardo de Pisa (1170-1240) (known as Fibonacci) expresses these Arabic rules in his *Liber Abaci* (1202), he uses "radix" to represent the "thing" or unknown quantity (also called "res" by other authors) and the word "census" to represent the square power. This rhetorical language continued to be used in several algebraic works in the early Italian Renaissance, such as *Summa de Arithmetica, Geometria, Proportioni e Proportionalità* (1494) by Luca Pacioli (1445-1514), *Ars Magna Sive de Regulis Algebraicis* (1545) by Girolamo Cardano (1501-1576) and *Quesiti et Invenzioni Diversa* (1546) by Niccolò Tartaglia (1500-1557). All these writers used geometric squares, rectangles and cubes to represent or justify algebraic manipulations (Stedall, 2011, 1-49).

One of the firsts to question these geometrical justifications was Pedro Nunes or Núñez (1502-1578) in his book *Libro de algebra en arithmética y geometría* (1567). After showing the classic geometrical justifications by completing squares, he claims:

"While these demonstrations of the last three rules are very clear, by saying that in the demonstration of the first rule it is presupposed that a censo with the things of whatever number can equal any number, number being what we have defined at the beginning of this book, the adversary will be able to state that this presupposition is not true. Therefore, it will be necessary to demonstrate it." (Núñez, 1567, fo. 14r).

After this statement, Nuñez proceeds to introduce new geometrical constructions of the solutions to the quadratic equation. Although Nuñez was a pioneer in introducing new geometrical constructions, the more singular ones will occur later, as we analyse in an activity implemented in the classroom described below.

Geometrical justifications in the seventeenth century

The publication in 1591 of *In Artem Analyticen Isagoge* by François Viète (1540-1603) constituted a step forward in the development of a symbolic language. Viète used symbols to represent both known and unknown quantities and was thus able to investigate equations in a completely general form ($ax^2+bx=c$). He introduced a new analytical method for solving problems in the context of Greek analysis. This algebraic method of analysis allowed problems of any magnitude to be dealt with, and his symbolic language was the tool he used to develop this program. Viète showed the usefulness of algebraic procedures for solving equations in arithmetic, geometry and trigonometry (Bos, 2001, 145-154). He solved equations geometrically using the Euclidean idea of proportion: proportions can be converted into equations by setting the product of the medians equal to the product of the extremes. In 1593, Viète published *Effectionum Geometricarum canonica recensio*, in which he geometrically constructed the solutions of second-and fourth-degree equations. Later, in 1646, F. A. Schooten edited this book in Viète's *Opera Mathematica*. We have used this edition to design the activity for the classroom. Viète claims:

> "Proposition XII Given the mean of three proportional magnitudes and the difference between the extremes, find the extremes. [This involves] the geometrical solution of a square affected by a [plane based on a] root [$A^2 + BA = D^2$]. Let FD be the mean of three proportionals [= D] and let GF be the difference between the extremes [= B]. The extremes are to be found. Let GF and FD stand at right angles and let GF be cut in half at A. Describe a circle around the centre A at the distance AD and extend AG and AF to the circumference at the points B and C. I say that what was to be done has been done, for the extremes are found to be BF [$A + B$] and FC [=A], between which FD [=D] is the mean proportional. Moreover, BF and FC differ by FG, since AF and AG are equal by construction and AC and AB are also equal by construction. Thus, subtracting the equals AG and AF from the equals AB and AC, there remain the equals BG and FC. GF, in addition, is the difference between BF and BG or FC, as was to be demonstrated." (Viète, 1646, 234).

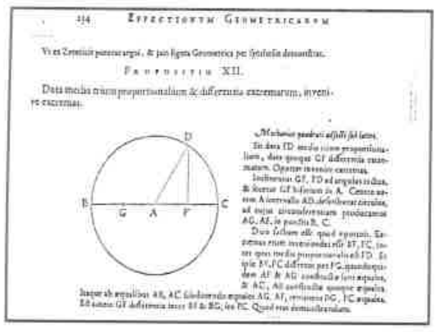

Figure 13: Viète's construction of three proportional (Viète, 1646, 234)

He sets up the equation $A^2 + BA = D^2$ by means of a proportion, which can be expressed in modern notation as $(A + B) : D = D : A$. Viète's geometric construction of the lines A, B, D satisfying this equality is set out in Figure 13. Viète draws FD = D and GF = B, making a right angle, and divides B by half AF = $B/2$. He describes a circle whose radius is equal to AC, which we can identify with the hypotenuse of the triangle formed by $B/2$ and D, AD = AC=$[(B/2)^2 + D^2]^{1/2}$. The solutions are then the segments FC = AC - AF and BF = BA + AF, which take BA = AC = radius (Massa, 2008, 295).

In the classroom, after finishing the lesson of quadratic equation, we carry out an activity taking into account these geometrical constructions in order to highlight the algebraic solution of the quadratic equation from another perspective. The procedure is as follows: after describing historical context, including Nuñez's quotation, and analysing Viète's geometrical construction, the teacher could pose the students some questions to clarify the ideas.

1) Reproduce Viète's geometrical construction and give an explanation of the procedure. 2) Could this geometrical construction be used for any quadratic equation?Give reasons. 3) What about negative solutions? 4) How are the Pythagorean and the altitude Theorem used? Explain their relationship to the solution of the equation. 5) What is the main difference between this geometrical construction and the classical construction by completing squares?

After analysing and discussing students' answers, the teacher continues by presenting a new historical text with another geometrical construction. Indeed, as Viète's work came to prominence at the beginning of the seventeenth century, mathematicians began to consider the utility of algebraic procedures for solving all kinds of problems. Thus, the other singular example is the geometrical construction in a quadratic equation found in the influential work *La Géométrie* (1637) by René Descartes (1596-1650). Descartes begins Book I by developing an algebra of segments and shows how to add, multiply, divide segments, and calculate the square root of segments with geometrical constructions (Bos, 2001, 293-305). Next, Descartes shows how a quadratic equation may be solved geometrically (see Figure 14):

"For example, if I have $z^2 = az + bb$, I construct a right triangle NLM with one side LM, equal to b, the square root of the known quantity b^2, and the other side, LN equal to ½ a, that is, to half the other known quantity which was multiplied by z, which I assumed to be the unknown line. Then prolonging MN, the hypotenuse of this triangle, to O, so that NO is equal to NL, the whole line OM is the required line z." (Descartes, 1637, 302-303).

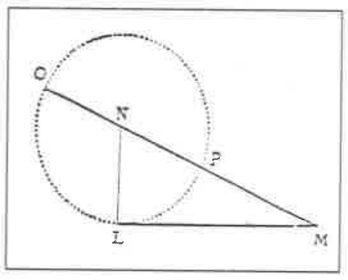

Figure 14: Geometrical construction (Descartes, 1637, 302)

In the classroom, after drawing and analysing Descartes' geometrical construction, we could hold a discussion with the students. It is important to point out that symbolic formula appears explicitly in Descartes' work. His geometrical construction corresponds to the construction of an unknown line in terms of some given lines; hence, the solution of the equation is given by the sum of a line and a square root, which has been obtained using the Pythagorean Theorem. However, Descartes ignores the second root, which is negative, and he did not quote that this geometrical construction could be justified by Euclid III, 36, where the power of a point is proved with respect to a circle.

The questions posed to students are similar to those by Viète. Moreover, students may also reflect on the meaning of both constructions. The differences from Viète are relevant because Descartes explicitly writes in the margin "how to solve" the equation, while Viète, by contrast, solves a geometric problem with a geometric figure in which a proportion is identified with an equation. Another relevant subject to consider with the students is the analytical and/or synthetic approach used in each construction.

Other possible questions: What geometrical reasoning did the author use? What is the role of the Pythagorean Theorem in solving the equation of second degree? What relation is there between this geometrical construction and the algebraic solution of the second degree equation?

All these questions enable teachers to consider the solution of quadratic equations from a geometrical point of view, as well as prompting thought about the relation between algebra and geometry through history.

SOME REMARKS

These kinds of activities are very rich in terms of competency-based learning, since they allow students to apply their knowledge in different situations rather than to reproduce exactly what they have learned. In addition, they help students to appreciate the contribution of different cultures to knowledge, which is especially important in classrooms today, where students often come from different countries and cultures.

The design of these activities also allows different levels of development and in some cases the distribution of tasks among students according to their individual skills.

The activities, based on the analysis of historical texts connected to the curriculum, contribute to improving the students' overall formation by giving them additional knowledge of the social and scientific context of the periods involved. Students achieve a vision of Mathematics not as a final product but as a science that has been developed on the basis of trying to answer the questions that mankind has been asking throughout history about the world around us.

All these activities devote an important part to geometry, which is a standard in the syllabus that students should improve, as recommended in the results of PISA assessment.

Geometry has a great visual and aesthetic value and offers a beautiful way of understanding the world. The elegance of its constructions and proofs makes it a part of Mathematics that is very suitable for developing the standard process of *reasoning and proof* of the students.

In addition, geometrical proofs have a great potential for relating geometry and algebra; that is to say, establishing connections between figures and formulas; geometric constructions and calculations.

NOTES

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Workshop
HISTORICAL MATHEMATICAL MODELS IN TEACHER EDUCATION - WORKSHOP ON THE DEVELOPMENT OF QUESTIONS AND CRITICAL QUESTIONING

Ysette Weiss-Pidstrygach

Johannes Gutenberg-University Mainz

We present the conception and design of a seminar in the master's programme of mathematics education. The seminar starts with students' personal experiences relating to mathematical experiments, models and visualisations of mathematical objects, followed by a historical excursion around historical collections of mathematical models. On the basis of that, students undertake project work on models of drawing instruments and simple curves in historical, socio-cultural or mathematical contexts.

INTRODUCTION

The paper deals with a workshop held in the afternoon of the last day of the 7th European Summer University on History and Epistemology in Mathematics Education at Aarhus University, Campus Copenhagen. In spite of the time of the event, the workshop was well attended and met the interests of mathematics educators with various backgrounds. Our aim is to give the participants a memory of the event and to outline the concept and design of a seminar, which uses history as a tool to awaken awareness and understanding of individual development and societal change in a mathematical context. We use a comparative view on the everyday world and its past to disturb widespread routines, approaching development from an output-orientated perspective and in normative terms.

THE MAIN MOTIVATION FOR THE CONCEPT AND DESIGN OF THE SEMINAR

The aim of the presented seminar concept is to deal with normative perspectives, evaluative assessments and output-orientated categorizations of our students on education and development. Since the so-called PISA shock, the German education system has undergone subtly comprehensive restructuring, the concept of "Bildung" (usually translated as education) being replaced by the notion of "Ausbildung" (training). At least since Klafki (1994, 2000) formal education approaches in German educational sciences are believed to have been overcome. However current educational policy and real school life tells a different story. Global testing, outcome-oriented learning, competence-based redesigning of curricula and measuring and evaluating by educational standards led to a predominance of normative approaches to learning and development (Jahnke & Meyerhöfer, 2007).