The spectral excess theorem for distance-regular graphs having distance-d graph with fewer distinct eigenvalues

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Abstract

Let Γ be a distance-regular graph with diameter d and Kneser graph $K = \Gamma_d$, the distance-d graph of Γ. We say that Γ is partially antipodal when K has fewer distinct eigenvalues than Γ. In particular, this is the case of antipodal distance-regular graphs (K with only two distinct eigenvalues), and the so-called half-antipodal distance-regular graphs (K with only one negative eigenvalue). We provide a characterization of partially antipodal distance-regular graphs (among regular graphs with $d+1$ distinct eigenvalues) in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as a more general version of the so-called spectral excess theorem, which allows us to characterize those distance-regular graphs which are half-antipodal, antipodal, bipartite, or with Kneser graph being strongly regular.

Keywords: Distance-regular graph; Kneser graph; Partial antipodality; Spectrum; Predistance polynomials.

AMS subject classifications: 05C50, 05E30.

1 Preliminaries

Let Γ be a distance-regular graph with adjacency matrix A and $d+1$ distinct eigenvalues. In the recent work of Brouwer and the author [2], we studied the situation where the distance-d graph Γ_d of Γ, or the Kneser graph K of Γ, with adjacency matrix $A_d = p_d(A)$ where p_d is the distance-d polynomial, has fewer distinct eigenvalues than Γ. In this case we say that Γ is *partially antipodal*. Examples are the so-called half antipodal (K with only one negative eigenvalue, up to multiplicity), and antipodal distance-regular graphs (K being disjoint copies of a complete graph). Here we generalize such a study to the case when Γ is a regular graph with $d+1$ distinct eigenvalues. The main result of this paper is a characterization of partially antipodal distance-regular graphs, among regular graphs with $d+1$ distinct eigenvalues, in terms of the spectrum and the mean number of vertices at maximal distance d from every vertex. This can be seen as a more general version of the so-called spectral excess theorem, and allows us to characterize those distance-regular graphs which are half antipodal, antipodal, bipartite, or with Kneser graph being strongly regular. Other related characterizations of some of these cases were given by the author in [8, 9, 10]. For background on distance-regular graphs and strongly regular graphs, we refer the reader to Brouwer, Cohen, and Neumaier [1], Brouwer and Haemers [3], and Van Dam, Koolen and Tanaka [6].

Let Γ be a regular (connected) graph with degree k, n vertices, and spectrum $\operatorname{sp}\Gamma = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0(=k) > \lambda_1 > \cdots > \lambda_d$, and $m_0 = 1$. In this work, we use the following scalar product on the $(d+1)$-dimensional vector space of real polynomials modulo $m(x) = \prod_{i=0}^{d}(x-\lambda_i)$, that is, the minimal polynomial of A.

$$\langle p, q\rangle_\Gamma = \frac{1}{n}\operatorname{tr}(p(A)q(A)) = \frac{1}{n}\sum_{i=0}^{d} m_i p(\lambda_i)q(\lambda_i), \qquad p,q \in \mathbb{R}_d[x]/(m(x)). \tag{1}$$

This is a special case of the inner product of symmetric $n\times n$ real matrices M, N, defined by $\langle M, N\rangle = \frac{1}{n}\operatorname{tr}(MN)$. The *predistance polynomials* $p_0, p_1, \ldots, p_d$, introduced by the author and Garriga [13], are a sequence of orthogonal polynomials with respect to the inner product (1), normalized in such a way that $\|p_i\|_\Gamma^2 = p_i(k)$ (this makes sense since it is known that $p_i(k) > 0$ for any $i = 0, \ldots, d$).

As every sequence of orthogonal polynomials, the predistance polynomials satisfy a three-term recurrence of the form

$$xp_i = \beta_{i-1}p_{i-1} + \alpha_i p_i + \gamma_{i+1}p_{i+1} \qquad (i = 0, 1, \ldots, d),$$

where the constants β_{i-1}, α_i, and γ_{i+1} are the Fourier coefficients of xp_i in terms of p_{i-1}, p_i, and p_{i+1}, respectively (and $\beta_{-1} = \gamma_{d+1} = 0$), initiated with $p_0 = 1$ and $p_1 = x$.

Then, it is known that Γ is distance-regular if and only if such polynomials satisfy $p_i(A) = A_i$ (the adjacency matrix of the distance-i graph Γ_i) for $i = 0, \ldots, d$, in which case they turn out to be the distance polynomials of Γ. Moreover, as expected, the constants α_i, β_i and γ_i become the intersection numbers a_i, b_i and c_i of Γ.

In fact, we have the following strongest proposition, which is a combination of results in [14, 7].

Proposition 1. *A regular graph Γ as above is distance-regular if and only if there exists a polynomial p of degree d such that $p(A) = A_d$, in which case $p = p_d$.* □

Many properties of the distance polynomials of distance-regular graphs hold also for the predistance polynomials. For instance, the sum of all predistance polynomials gives the Hoffman polynomial H:

$$H=\sum_{i=0}^{d}p_i=\frac{n}{\prod_{i=1}^{d}(\lambda_0-\lambda_i)}\prod_{i=1}^{d}(x-\lambda_i),$$

satisfying $H(\lambda_0)=n$ and $H(\lambda_i)=0$ for $i=1,\ldots,d$. This polynomial characterizes regular graphs by the condition $H(A)=J$, the all-1 matrix [16], and it can be used to show that $\alpha_i+\beta_i+\gamma_i=\lambda_0=k$ for all $i=0,\ldots,d$.

Also, as in the case of distance-regular graphs, the multiplicities of Γ can be obtained from the values of p_d since,

$$(-1)^i p_d(\lambda_i)\pi_i m_i=p_d(\lambda_0)\pi_0, \qquad i=1,\ldots,d. \tag{2}$$

where $\pi_i=\prod_{j\neq i}|\lambda_i-\lambda_j|$. Indeed, let $L_i(x)=\prod_{j\neq 0,i}(x-\lambda_j)/\prod_{j\neq 0,i}(\lambda_i-\lambda_j)$. Then, since the degree of each L_i is $d-1$, the equalities in (2) follow from $\langle L_i,p_d\rangle_\Gamma=0$ for $i=1,\ldots,d$. Some interesting consequences of the above, together with other properties of the predistance polynomials are the following (for more details, see [4]):

- The values of p_d at $\lambda_0,\lambda_1,\ldots,\lambda_d$ alternate in sign.
- Using the values of $p_d(\lambda_i)$, $i=0,\ldots,d$, given by (2), in the equality $\|p_d\|_\Gamma^2=p_d(\lambda_0)$, and solving for $p_d(\lambda_0)$ we get the so-called *spectral excess*

$$p_d(\lambda_0)=n\left(\sum_{i=0}^{d}\frac{\pi_0^2}{m_i\pi_i^2}\right)^{-1}. \tag{3}$$

- For every $i=0,\ldots,d$, (any multiple of) the sum polynomial $q_i=p_0+\cdots+p_i$ maximizes the quotient $r(\lambda_0)/\|r\|_\Gamma$ among the polynomials $r\in\mathbb{R}_i[x]$ (notice that $q_i(\lambda_0)^2/\|q_i\|_\Gamma^2=q_i(\lambda_0)$), and

$$(1=)q_0(\lambda_0)<q_1(\lambda_0)<\cdots<q_d(\lambda_0)(=H(\lambda_0)=n).$$

Let Γ have n vertices, $d+1$ distinct eigenvalues, and diameter $D(\leq d)$. For $i=0,\ldots,D$, let $k_i(u)$ be the number of vertices at distance i from vertex u. Let $s_i(u)=k_0(u)+\cdots+k_i(u)$. Of course, $s_0(u)=1$ and $s_D(u)=n$. The following result can be seen as a version of the spectral excess theorem, due to Garriga and the author [13] (for short proofs, see Van Dam [5], and Fiol, Gago and Garriga [12]):

Theorem 2. *Let Γ be a regular graph with spectrum $\operatorname{sp}\Gamma=\{\lambda_0,\lambda_1^{m_1},\ldots,\lambda_d^{m_d}\}$, where $\lambda_0>\lambda_1>\cdots>\lambda_d$. Let $\overline{s_i}=\frac{1}{n}\sum_{u\in V}s_i(u)$ be the average number of vertices at distance at most i from every vertex in Γ. Then, for any nonzero polynomial $r\in\mathbb{R}_{d-1}[x]$ we have*

$$\frac{r(\lambda_0)^2}{\|r\|_\Gamma^2}\leq\overline{s_{d-1}}, \tag{4}$$

with equality if and only if Γ is distance-regular and r is a multiple of q_{d-1}.

Proof. Let $S_{d-1} = I + A + \cdots + A_{d-1}$. As $\deg r \le d-1$, $\langle r(A), J\rangle = \langle r(A), S_{d-1}\rangle$. But $\langle r(A), J\rangle = \langle r, H\rangle_\Gamma = r(\lambda_0)$. Thus, Cauchy-Schwarz inequality gives

$$r^2(\lambda_0) \le \|r(A)\|^2 \|S_{d-1}\|^2 = \|r\|_\Gamma^2 \overline{s_{d-1}},$$

whence (4) follows. Besides, in case of equality we have that $r(A) = \alpha S_{d-1}$ for some nonzero constant α. Hence, the polynomial $p = H - (1/\alpha) r$ satisfies $p(A) = J - S_{d-1} = A_d$ and, from Proposition 1, Γ is distance-regular, $p = p_d$, and $r = \alpha q_{d-1}$. The converse is clear from $s_{d-1} = n - k_d = H(\lambda_0) - p_d(\lambda_0) = q_{d-1}(\lambda_0)$. □

In fact, as it was shown in [11], the above result still holds if we change the arithmetic mean of the numbers $s_{d-1}(u)$, $u \in V$, by its harmonic mean.

2 The results

As commented above, in [2] we studied the situation where the distance-d graph Γ_d, of a distance-regular graph Γ with diameter d, has fewer than $d+1$ distinct eigenvalues. Now, we are interested in the case when Γ is regular and with $d+1$ distinct eigenvalues. In this context, p_d is the highest degree predistance polynomial and, as $p_d(A)$ is not necessarily the distance-d matrix A_d (usually not even a 0-1 matrix), we consider the distinct eigenvalues of $p_d(A)$ vs. those of A. More precisely, given a set $\mathcal{I} \subset \{0, \ldots, d\}$, we give conditions for all $p_d(\lambda_i)$ with $i \in \mathcal{I}$ taking the same value. Notice that, because the values of p_d at the mesh $\lambda_0, \lambda_1, \ldots, \lambda_d$ alternate in sign, the feasible sets $\mathcal{I}$ must consist of either even or odd numbers.

2.1 The case $0 \notin \mathcal{I}$

We first study the more common case when $0 \notin \mathcal{I}$. For $i = 1, \ldots, d$, let $\phi_i(x) = \prod_{j \ne 0,i}(x - \lambda_j)$, and consider again the Lagrange interpolating polynomial $L_i(x) = \phi_i(x)/\phi_i(\lambda_i)$, satisfying $L_i(\lambda_j) = \delta_{ij}$ for $j \ne 0$, and $L_i(\lambda_0) = (-1)^{i+1}\frac{\pi_0}{\pi_i}$, where $\pi_i = |\phi_i(\lambda_i)|$.

Theorem 3. *Let Γ be a regular graph with degree k, n vertices, and spectrum $\operatorname{sp}\Gamma = \{\lambda_0, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where $\lambda_0 (= k) > \lambda_1 > \cdots > \lambda_d$. Let $\mathcal{I} \subset \{1, \ldots, d\}$. For every $i = 0, \ldots, d$, let $\pi_i = \prod_{j \ne i} |\lambda_i - \lambda_j|$. Let $\overline{k_d} = \frac{1}{n}\sum_{u\in V} k_d(u)$ be the average number of vertices at distance d from every vertex in Γ. Then,*

$$\overline{k_d} \le \frac{n \sum_{i\in\mathcal{I}} m_i}{\left(\sum_{i\in\mathcal{I}} \frac{\pi_0}{\pi_i}\right)^2 + \sum_{i\notin\mathcal{I}} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i\in\mathcal{I}} m_i}, \tag{5}$$

and equality holds if and only if Γ is a distance-regular graph with $k_d(u) = k_d$ for each $u \in V$, and constant

$$P_{id} = p_d(\lambda_i) = k_d \frac{\sum_{i\in\mathcal{I}} (-1)^{i+1} \frac{\pi_0}{\pi_i}}{\sum_{i\in\mathcal{I}} m_i} \qquad \text{for every } i \in \mathcal{I}. \tag{6}$$

Proof. The clue is to apply Theorem 2 with a polynomial $r \in \mathbb{R}_{d-1}[x]$ having the desired properties of q_{d-1}. To this end, let us assume that $p_d(\lambda_i) = t$ for any $i \in \mathcal{I}$, where t is a constant number. Moreover, as $q_{d-1} = H - p_d$, we have $q_{d-1}(\lambda_i) = -p_d(\lambda_i)$ for any $i \neq 0$. Thus, we take the polynomial r with values $r(\lambda_i) = -t$ for $i \in \mathcal{I}$, and $r(\lambda_i) = -p_d(\lambda_i)$ for $i \notin \mathcal{I}$, $i \neq 0$. Then, using (2),

$$\begin{aligned} r(x) &= -t \sum_{i\in\mathcal{I}} L_i(x) - \sum_{i\notin\mathcal{I}, i\neq 0} p_d(\lambda_i) L_i(x), \\ r(\lambda_0) &= -t \sum_{i\in\mathcal{I}} (-1)^{i+1} \frac{\pi_0}{\pi_i} - \sum_{i\notin\mathcal{I}, i\neq 0} p_d(\lambda_i)(-1)^{i+1} \frac{\pi_0}{\pi_i} \\ &= -t \sum_{i\in\mathcal{I}} (-1)^{i+1} \frac{\pi_0}{\pi_i} + p_d(\lambda_0) \sum_{i\notin\mathcal{I}, i\neq 0} \frac{\pi_0^2}{m_i \pi_i^2}, \\ n\|r\|_\Gamma^2 &= r(\lambda_0)^2 + t^2 \sum_{i\in\mathcal{I}} m_i + \sum_{i\notin\mathcal{I}, i\neq 0} m_i p_d(\lambda_i)^2. \end{aligned}$$

Thus, (4) yields

$$\Phi(t) = \frac{r(\lambda_0)^2}{\|r\|_\Gamma^2} = \frac{n(\alpha t + \beta)^2}{(\alpha t + \beta)^2 + \sigma t^2 + \gamma} \leq \overline{s_{d-1}} \tag{7}$$

where

$$\alpha = \sum_{i\in\mathcal{I}} (-1)^{i+1} \frac{\pi_0}{\pi_i}, \qquad \beta = -p_d(\lambda_0) \sum_{i\notin\mathcal{I}, i\neq 0} \frac{\pi_0^2}{m_i \pi_i^2}, \tag{8}$$

$$\gamma = \sum_{i\notin\mathcal{I}, i\neq 0} m_i p_d(\lambda_i)^2 = \sum_{i\notin\mathcal{I}, i\neq 0} \frac{p_d(\lambda_0)^2}{m_i} \frac{\pi_0^2}{\pi_i^2} = -p_d(\lambda_0)\beta, \qquad \sigma = \sum_{i\in\mathcal{I}} m_i. \tag{9}$$

Now, to have the best result in (7) (and since we are mostly interested in the case of equality), we have to find the maximum of the function $\Phi(t)$, which is attained at $t_0 = \alpha\gamma/\beta\sigma$. Then,

$$\Phi_{\max} = \Phi(t_0) = \frac{n(\alpha^2\gamma + \beta^2\sigma)}{\alpha^2\gamma + \beta^2\sigma + \gamma\sigma} \leq \overline{s_{d-1}} = n - \overline{k_d}.$$

Thus, using (8)–(9) and simplifying we get (5). In case of equality, we know, by Theorem 2, that Γ is distance-regular with $r(x) = \alpha q_{d-1}(x)$ for some constant α. If $i \notin \mathcal{I}, i \neq 0$, $r(\lambda_i) = -p_d(\lambda_i) = \alpha q_{d-1}(\lambda_i) = -\alpha p_d(\lambda_i)$, so that $\alpha = 1$ since $p_d(\lambda_i) \neq 0$. Then, for every $i \in \mathcal{I}$, we get

$$P_{id} = p_d(\lambda_i) = H(\lambda_i) - q_{d-1}(\lambda_i) = -r(\lambda_i) = t_0.$$

Conversely, if Γ is distance-regular, we have that $\overline{k_d} = k_d$, and, if P_{id} is a constant, say, τ for every $i \in \mathcal{I}$, we obtain, from (2), that $\sigma = \frac{k_d}{\tau} \sum_{i\in\mathcal{I}} (-1)^i \frac{\pi_0}{\pi_i} = -\frac{k_d}{\tau}\alpha$, whence $\tau = -k_d \frac{\alpha}{\sigma}$, which corresponds to (6). Moreover,

$$nk_d = \|p_d\|_\Gamma^2 = \sum_{i\notin\mathcal{I}} m_i p_d(\lambda_i)^2 + \sum_{i\in\mathcal{I}} m_i \tau^2 = k_d^2 \sum_{i\notin\mathcal{I}} \frac{\pi_0^2}{m_i \pi_i^2} + k_d^2 \frac{\left(\sum_{i\in\mathcal{I}} \frac{\pi_0}{\pi_i}\right)^2}{\sum_{i\in\mathcal{I}} m_i},$$

and equality in (5) holds. □

As mentioned above, when Γ is already a distance-regular graph, Brouwer and the author [2] gave parameter conditions for partial antipodality, and surveyed known examples. The examples listed here are taken from [2].

Example 4. The Odd graph $\Gamma = O_5$, with $n = 126$ vertices and diameter $d = 4$, has intersection array $\{5,4,4,3;1,1,2,2\}$, so that $k_d = 60$, and spectrum $5^1, 3^{27}, 1^{42}, -2^{48}, -4^8$. Then, with $\mathcal{I} = \{2,4\}$, the function $\Phi(t)$ is depicted in Fig. 1. Its maximum is attained for $t_0 = 6$, and its value is $\Phi(6) = 66 = s_{d-1}$. Then, $P_{24} = P_{44}$. Indeed, its distance-4 polynomial is $p_4(x) = \frac{1}{4}(x^4 - 17x^2 + 40)$ with values $p_4(5) = 60$, $p_4(3) = -8$, $p_4(1) = 6$, $p_4(-2) = -3$, and $p_4(-4) = 6$. Hence, the spectrum of Γ_4 is $60^1, 6^{50}, -3^{48}, -8^{27}$.

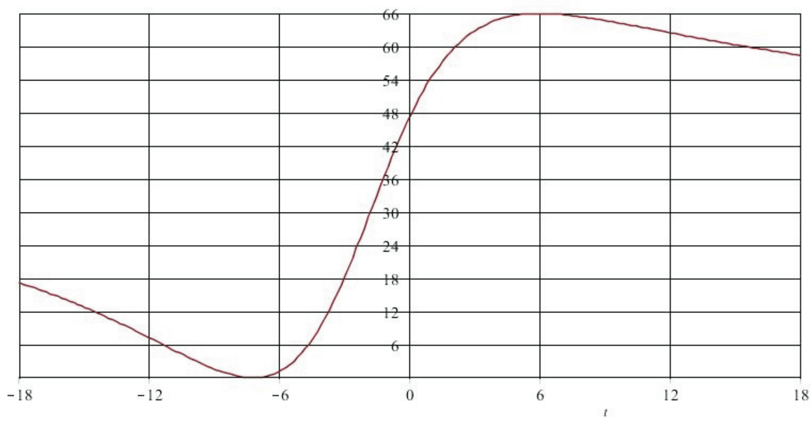

Figure 1: The function $\Phi(t)$ for O_5 with $\mathcal{I} = \{2,4\}$.

Notice that if, in the above result, $\mathcal{I}$ is a singleton, there is no restriction for the values of p_d, and then we get the so-called spectral excess theorem (originally proved by Garriga and the author [13]).

Corollary 5 (The spectral excess theorem)**.** *Let Γ be a regular graph with spectrum $\operatorname{sp}\Gamma$ and average number $\overline{k_d}$ as above. Then Γ is distance-regular if and only if*

$$\overline{k_d} = p_d(\lambda_0) = n\left(\sum_{i=0}^{d} \frac{\pi_0^2}{m_i \pi_i^2}\right)^{-1}.$$

Proof. Take $\mathcal{I} = \{i\}$ for some $i \neq 0$ in Theorem 3. □

As mentioned before, in [2, Th. 9–10] a distance-regular graph Γ was said to be *half antipodal* if the distance-d graph has only one negative eigenvalue (i.e., P_{id} is a constant for every $i = 1, 3, \ldots$). Then, a direct consequence of Theorem 3 by taking $\mathcal{I} = \mathcal{I}_{\text{odd}} = \{1, 3, \ldots\}$ is the following characterization of half antipodality.

Corollary 6. *A regular graph* Γ *as above is a half antipodal distance-regular graph if and only if the following equality holds:*

$$\overline{k_d} = \frac{n \sum_{i\ odd} m_i}{\left(\sum_{i\ odd} \frac{\pi_0}{\pi_i}\right)^2 + \sum_{i\ even} \frac{\pi_0^2}{m_i \pi_i^2} \sum_{i\ odd} m_i}. \tag{10}$$

□

Example 7. The Coxeter graph $\Gamma = C$, on $n = 28$ vertices, has diameter $d = 4$, intersection array $\{3, 2, 2, 1; 1, 1, 1, 2\}$, $k_4 = 6$, and spectrum $3^1, 2^8, (\sqrt{2}-1)^6, -1^7, (-1-\sqrt{2})^6$. Then, with $\mathcal{I} = \{1, 3\}$, the equality in (10) holds and, then $P_{14} = P_{34}$. In fact, the distance-4 polynomial is $p_4(x) = \frac{1}{2}(x^4 - x^3 - 7x^2 + 5x + 6)$ with values $p_4(3) = 6$, $p_4(2) = -2$, $p_4(\sqrt{2}-1) = 2+\sqrt{2}$, $p_4(-1) = -2$, and $p_4(-1-\sqrt{2}) = 2-\sqrt{2}$. Thus, Γ is half antipodal since the spectrum of Γ_4 is $6^1, (2+\sqrt{2})^6, (2-\sqrt{2})^6, -2^{15}$.

Recall that a regular graph Γ is strongly regular if and only if it has, either three (when Γ is connected), or two (when Γ is the disjoint union of several copies of a complete graph) distinct eigenvalues (see e.g. Godsil [15]). Then, we have the following characterization of those distance-regular graphs having strongly regular distance-d graph.

Corollary 8. *A regular graph* Γ *as above is distance-regular with strongly regular distance-*d *graph* Γ_d *if and only if the following equality holds:*

$$\overline{k_d} = \frac{n(n-1)}{\left(\sum_{\substack{i\ even \\ i\neq 0}} \frac{\pi_0}{\pi_i}\right)^2 + \left(\sum_{i\ odd} \frac{\pi_0}{\pi_i}\right)^2 + \left(1 + \sum_{\substack{i\ even \\ i\neq 0}} \frac{\pi_0^2}{m_i \pi_i^2}\right) \sum_{i\ odd} m_i + \left(1 + \sum_{i\ odd} \frac{\pi_0^2}{m_i \pi_i^2}\right) \sum_{\substack{i\ even \\ i\neq 0}} m_i}. \tag{11}$$

Proof. Apply Theorem 3 with equality for $\mathcal{I}_{\text{even}} = \{2, 4, \ldots\}$, and $\mathcal{I}_{\text{odd}} = \{1, 3, \ldots\}$, to obtain the values of $\sum_{i \text{ odd}} m_i$ and $\sum_{\substack{i \text{ even} \\ i\neq 0}} m_i$, add up both equalities and solve for $\overline{k_d}$. □

Example 9. The Wells graph $\Gamma = W$, on $n = 32$ vertices, has intersection array $\{5, 4, 1, 1; 1, 1, 4, 5\}$ and spectrum $5^1, \sqrt{5}^8, 1^{10}, -\sqrt{5}^8, -3^5$. This graph is 2-antipodal, so that $k_d = 1$. Then, Fig. 2 shows the functions $\Phi_0(t)$ with $\mathcal{I}_0 = \{2, 4\}$, and $\Phi_1(t)$ with $\mathcal{I}_1 = \{1, 3\}$. Their (common) maximum value is attained for $t_0 = 1$ and $t_1 = -1$, respectively, and it is $\Phi_0(1) = \Phi_1(-1) = 31 = s_{d-1}$. Then, $P_{24} = P_{44}$ and $P_{14} = P_{34}$. Indeed, the distance-4 polynomial is $p_4(x) = \frac{1}{20}(x^4 - 3x^3 - 13x^2 + 15x + 20)$ with values $p_4(5) = 1$, $p_4(\sqrt{5}) = -1$, $p_4(1) = 1$, $p_4(-\sqrt{5}) = -1$, and $p_4(-3) = 1$. Hence, the spectrum of Γ_4 is $1^{16}, -1^{16}$ since it is constituted by 16 disjoint copies of K_2.

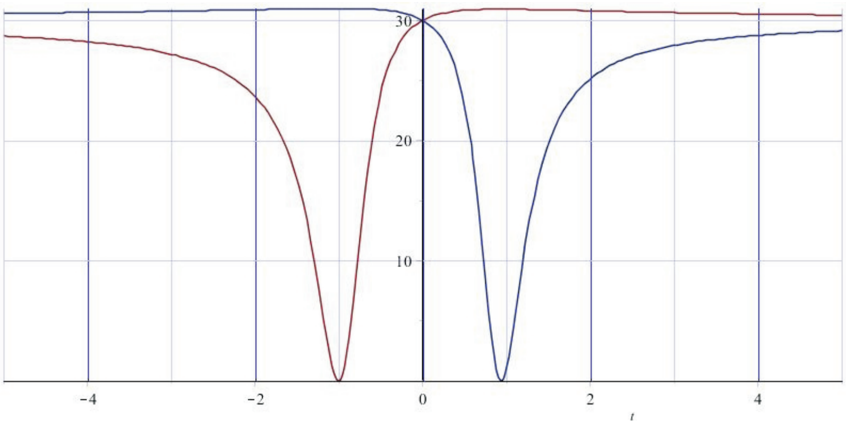

Figure 2: The functions $\Phi_0(t)$ (in red) with $\mathcal{I}_0 = \{2,4\}$, and $\Phi_1(t)$ (in blue) with $\mathcal{I}_1 = \{1,3\}$ of the Wells graph.

In fact, the above expression can be simplified because $\sum_{i \text{ even}} m_i + \sum_{i \text{ odd}} m_i = n$ (with $m_0 = 1$), $\sum_{i \text{ even}} \frac{\pi_0}{\pi_i} = \sum_{i \text{ odd}} \frac{\pi_0}{\pi_i}$ (see [9]), and, from (3), $\sum_{i \text{ even}} \frac{\pi_0^2}{m_i \pi_i^2} + \sum_{i \text{ odd}} \frac{\pi_0^2}{m_i \pi_i^2} = n/p_d(\lambda_0)$. Anyway, we have written (11) as it is to emphasize the 'symmetries' between even and odd terms.

As in the case of Theorem 3, the equalities in Corollaries 6 and 8 also hold as inequalities, but, as one of the referees pointed out, the best inequality for general graphs is the one that would come with Corollary 5. Namely $\overline{k_d} \le p_d(\lambda_0)$, where $p_d(\lambda_0)$ is the spectral excess given by (3). This follows from the mentioned property that $q_{d-1}(x) = n - p_d(x)$ is the polynomial $r \in \boldsymbol{R}_{d-1}[x]$ that maximizes the quotient $r(\lambda_0)/\|r\|_\Gamma$.

2.2 The case $0 \in \mathcal{I}$

To deal with this case, we could proceed as above by defining conveniently a degree $d-1$ polynomial r. Then the proof is similar to the one for Theorem 3. If $0 \in \mathcal{I}$ then $p(\lambda_i) = p(\lambda_0)$ for any $i \in \mathcal{I}$. Moreover, the odd indexes, cannot belong to $\mathcal{I}$. In particular $1 \notin \mathcal{I}$. For instance, a possible choice for $r \in \mathbb{R}_{d-1}[x]$ is:

- $r(\lambda_0) = n - p_d(\lambda_0)$, $r(\lambda_i) = -p_d(\lambda_0)$ for $i \in \mathcal{I}$, $i \neq 0$.
- $r(\lambda_i) = -tp_d(\lambda_i)$ for $i \notin \mathcal{I}$, $i \neq 1$,

However, we can follow a more direct approach by using (7). First, the following result was proved in [2]:

Proposition 10 ([2, Prop. 8])**.** *Let Γ be a distance-regular graph with diameter d. If $P_{0d} = P_{id}$ then i is even. Let $i > 0$ be even. Then $P_{0d} = P_{id}$ if and only if Γ is antipodal, or $i = d$ and Γ is bipartite.* □

Notice that, in this case, the Kneser graph is disconnected. Thus, the above proposition can be seen as a spectral characterization of the so-called *imprimitive* distance-regular graphs (see Smith [17]).

Theorem 11. *Let* Γ *be a regular graph with* n *vertices, spectrum* $\operatorname{sp}\Gamma$ *as above, and mean excess* $\overline{k_d}$*. Then, for every* $i = 1, \ldots, d$,

$$\overline{k_d} \leq \frac{n\left(m_i + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j \pi_j^2}\right)}{\left(\frac{\pi_0}{\pi_i} + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j \pi_j^2}\right)^2 + m_i + \sum_{j\neq 0,i} \frac{\pi_0^2}{m_j \pi_j^2}}. \tag{12}$$

Moreover:

(a) *Equality holds for some* $i \neq d$ *if and only if it holds for any* $i = 1, \ldots, d$ *and* Γ *is an antipodal distance-regular graph.*

(b) *Equality holds only for* $i = d$ *if and only if* Γ *is a bipartite, but not antipodal, distance-regular graph.*

Proof. The inequality (12) follows from (7) by taking $\mathcal{I} = \{i\}$ for some even $i \neq 0$, and choosing $t = p_d(\lambda_0)$. Then, in case of equality, Theorem 3 tells us that Γ is distance-regular. Then, Γ_d is a regular graph with equal eigenvalues P_{0d} and P_{id}. So, the result follows from Proposition 10. □

Example 12. For the Wells graph the right hand expression of (12) gives $1(= k_4)$ for any $i = 1, \ldots, 4$, in concordance with its antipodal character. In contrast, the folded 10-cube FQ_{10}, on $n = 512$ vertices, has intersection array $\{10, 9, 8, 7, 6; 1, 2, 3, 4, 10\}$ and spectrum $10^1, 6^{45}, 2^{210}, -2^{210}, -6^{45}, -10^1$. Then, the right hand expression of (12) gives 234.16, 293.36, 293.36, 234.16 for $i = 1, 2, 3, 4$, respectively, and $126(= k_5)$ for $i = 5$, showing that FQ_{10} is a bipartite distance-regular graph, but not antipodal.

Another characterization of antipodal distance-regular graphs was given by the author in [8] by assuming that the distance-d graph of a regular graph is already a disjoint union of cliques.

Acknowledgments

The author sincerely acknowledges the three anonymous referees for their detailed and helpful comments on the first version of this paper, which led to significant improvements. Research supported by the *Ministerio de Ciencia e Innovación*, Spain, and the *European Regional Development Fund* under project MTM2011-28800-C02-01.

References

[1] A.E. Brouwer, A.M. Cohen, and A. Neumaier, *Distance-Regular Graphs*, Springer-Verlag, Berlin-New York, 1989.

[2] A.E. Brouwer and M.A. Fiol, Distance-regular graphs where the distance-d graph has fewer distinct eigenvalues, *Linear Algebra Appl.* **480** (2015), 115–126.

[3] A.E. Brouwer and W.H. Haemers, *Spectra of Graphs*, Springer, 2012; available online at `http://homepages.cwi.nl/~aeb/math/ipm/`.

[4] M. Cámara, J. Fàbrega, M.A. Fiol, and E. Garriga, Some families of orthogonal polynomials of a discrete variable and their applications to graphs and codes, *Electron. J. Combin.* **16(1)** (2009), #R83.

[5] E.R. van Dam, The spectral excess theorem for distance-regular graphs: a global (over)view, *Electron. J. Combin.* **15(1)** (2008), #R129.

[6] E.R. van Dam, J.H. Koolen, and H. Tanaka, Distance-regular graphs, preprint (2014); `arXiv:1410.6294 [math.CO]`.

[7] C. Dalfó, E.R. van Dam, M.A. Fiol, E. Garriga, and B.L. Gorissen, On almost distance-regular graphs, *J. Combin. Theory, Ser. A* **118** (2011), 1094–1113.

[8] M.A. Fiol, An eigenvalue characterization of antipodal distance-regular graphs, *Electron. J. Combin.* **4** (1997), #R30.

[9] M.A. Fiol, A quasi-spectral characterization of strongly distance-regular graphs, *Electron. J. Combin.* **7** (2000), #R51.

[10] M.A. Fiol, Some spectral characterization of strongly distance-regular graphs, *Combin. Probab. Comput.* **10** (2001), no. 2, 127–135.

[11] M.A. Fiol, Algebraic characterizations of distance-regular graphs, *Discrete Math.* **246** (2002), 111–129.

[12] M.A. Fiol, S. Gago, and E. Garriga, A simple proof of the spectral excess theorem for distance-regular graphs, *Linear Algebra Appl.* **432** (2010), 2418–2422.

[13] M.A. Fiol and E. Garriga, From local adjacency polynomials to locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **71** (1997), 162–183.

[14] M.A. Fiol, E. Garriga, and J.L.A. Yebra, Locally pseudo-distance-regular graphs, *J. Combin. Theory Ser. B* **68** (1996), 179–205.

[15] C.D. Godsil, *Algebraic Combinatorics*, Chapman and Hall, NewYork, 1993.

[16] A.J. Hoffman, On the polynomial of a graph, *Amer. Math. Monthly* 70 (1963), 30–36.

[17] D.H. Smith, Primitive and imprimitive graphs, *Quart. J. Math. Oxford* 22 (1971), no. 2, 551–557.