Perturbations of Discrete Elliptic operators

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Abstract. Given V a finite set, a self–adjoint operator on $\mathcal{C}(V)$, $\mathcal{K}$, is called elliptic if it is positive semi–definite and its lowest eigenvalue is simple. Therefore, there exists a unique, up to sign, unitary function $\omega \in \mathcal{C}(V)$ satisfying $\mathcal{K}(\omega) = \lambda\omega$ and then $\mathcal{K}$ is named (λ, ω)–*elliptic*. Clearly, a (λ, ω)–elliptic operator is singular iff $\lambda = 0$. Examples of elliptic operators are the so–called Schrödinger operators on finite connected networks, as well as the signless Laplacian of connected bipartite graphs.
If $\mathcal{K}$ is a (λ, ω)–elliptic operator, it defines an automorphism on $\omega^{\perp}$, whose inverse is called *orthogonal Green operator of* $\mathcal{K}$. We aim here at studying the effect of a perturbation of $\mathcal{K}$ on its orthogonal Green operator. The perturbation here considered is performed by adding a self–adjoint and positive semi–definite operator to $\mathcal{K}$. As particular cases we consider the effect of changing the conductances on semi–definite Schödinger operators on finite connected networks and on the signless Laplacian of connected bipartite graphs. The expression obtained for the perturbed networks is explicitly given in terms of the orthogonal Green function of the original network.

1. Introduction

In this paper we study discrete elliptic operators more general than the standard elliptic Schrödinger operators that were the main theme of [2]. As standard elliptic Schrödinger operator on a network we understand the positive semi–definite operator formed by the combinatorial Laplacian of the network plus a potential. We deal here with the elliptic operators whose principal part is the signless Laplacian on a bipartite network, but we also consider other elliptic operators defined on less specific networks. In addition, we also study some perturbations of these discrete elliptic operators. Moreover, since any discrete elliptic operator has associated an orthogonal Green operator we analyze the effect of the perturbation on this operator. The perturbations here considered are due to the addition of a self–adjoint and positive semi–definite operator or equivalently, by the addition of a sum of projectors. In [4], the authors showed that on standard Schrödinger operators this class of perturbations leads to standard Schrödinger operators on networks on the same set of vertices and whose conductances are non–negative perturbations of the former ones. In this work we extend this property to a more general class of discrete elliptic operators. Specifically, we pay attention on a generalization of discrete Schrödinger operators whose principal part is singular and positive semi–definite. This class of operators includes signless Laplacians on bipartite networks.

Let V be a finite set whose cardinality equals n. The space of real valued functions on V is denoted by $\mathcal{C}(V)$ and for any $x \in V$, $\varepsilon_x \in \mathcal{C}(V)$ stands for the Dirac function at x. More generally, for any $F \subset V$ the characteristic function of F is denoted by $\varepsilon_{_F}$. So, $\varepsilon_{_V}$ is the function whose value is 1 at each vertex.

2000 Mathematics Subject Classification:
Keywords and Prases: Elliptic operator, signless Laplacian, Green kernel.

A function $\chi \in \mathcal{C}(V)$ is called a *partition function* if it satisfies that $\chi^2 = \varepsilon_V$. Clearly, χ is a partition function iff there exists $\{V_0, V_1\}$ a partition of V; that is, $V_0 \cap V_1 = \emptyset$ and $V_0 \cup V_1 = V$, satisfying that $\chi = \varepsilon_{V_0} - \varepsilon_{V_1}$. Observe that $\chi = \varepsilon_V$ iff $V_1 = \emptyset$, whereas $\chi = -\varepsilon_V$ iff $V_0 = \emptyset$.

The standard inner product on $\mathcal{C}(V)$ and its associated norm are denoted by $\langle \cdot, \cdot \rangle$ and $||\cdot||$, respectively; that is, $\langle u, v \rangle = \sum\limits_{x \in V} u(x)\, v(x)$ and $||u|| = \sqrt{\langle u, u \rangle}$, for each $u, v \in \mathcal{C}(V)$. If $u \in \mathcal{C}(V)$ we say that u is *unitary* if $||u|| = 1$, whereas the subspace orthogonal to u is denoted by $u^\perp$.

A *weight* is an unitary function that is non null at each vertex, whereas a positive weight is a weight that is positive at each vertex. We denote by $\Omega(V)$ and by $\Omega_+(V)$ the set of weights and the set of positive weights, respectively.

If $\mathcal{K}$ is an endomorphism of $\mathcal{C}(V)$, it is called *self–adjoint* when $\langle \mathcal{K}(u), v \rangle = \langle u, \mathcal{K}(v) \rangle$, for any $u, v \in \mathcal{C}(V)$. Moreover, $\mathcal{K}$ is called *positive semi–definite* when $\langle \mathcal{K}(u), u \rangle \geq 0$ for any $u \in \mathcal{C}(V)$ and *positive definite* when $\langle \mathcal{K}(u), u \rangle > 0$ for any non–null $u \in \mathcal{C}(V)$.

A function $K \colon V \times V \longrightarrow \mathbb{R}$ is called *a kernel on* V and determines an endomorphism of $\mathcal{C}(V)$ by assigning to any $u \in \mathcal{C}(V)$ the function $\mathcal{K}(u) = \sum\limits_{y \in V} K(\cdot, y)\, u(y)$. Conversely, each endomorphism of $\mathcal{C}(V)$ is determined by the kernel given by $K(x, y) = \langle \mathcal{K}(\varepsilon_y), \varepsilon_x \rangle$ for any $x, y \in V$. Therefore, an endomorphism $\mathcal{K}$ is self–adjoint iff its kernel is a symmetric function. Notice that if we label the vertices of the network, then functions are equivalent to n–vectors and kernels are equivalent to matrices of order n.

Given $\sigma \in \mathcal{C}(V)$ non null, we denote by $\mathcal{P}_\sigma$ the *Projector* of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{P}_\sigma(u) = \langle \sigma, u \rangle\, \sigma$ and hence, its kernel is given by $(\sigma \otimes \sigma)(x, y) = \sigma(x)\sigma(y)$, $x, y \in V$. Clearly, the projector $\mathcal{P}_\sigma$ is self–adjoint and positive semi–definite.

Following the terminology of [4], a self–adjoint operator $\mathcal{K}$ is named *elliptic* if it is positive semi–definite and its lowest eigenvalue, $\lambda \geq 0$, is simple. Therefore, there exists a unique, up to sign, unitary function $\omega \in \mathcal{C}(V)$ satisfying $\mathcal{K}(\omega) = \lambda\omega$ and then $\mathcal{K}$ is named (λ, ω)*–elliptic*. Clearly, a (λ, ω)–elliptic operator is singular iff $\lambda = 0$ and in this case its nullity is the subspace generated by ω.

In [4] the authors established the main properties of (λ, ω)–elliptic operators where $\lambda \geq 0$ and $\omega \in \Omega_+(V)$. However, these properties remain true for general elliptic operators; that is for (λ, ω)–elliptic operators where $\lambda \geq 0$ and ω is an unitary function. Specifically, given $\mathcal{K}$ a (λ, ω)–elliptic operator, then $\mathcal{K}$ is an automorphism of $\omega^\perp$, see [4, Proposition 1], whose inverse is called *orthogonal Green operator* and it is denoted by $\mathcal{G}$. We can extend the orthogonal Green operator to $\mathcal{C}(V)$ by defining $\mathcal{G}(f) = \mathcal{G}\big(f - \mathcal{P}_\omega(f)\big)$ for any $f \in \mathcal{C}(V)$. Then, $\mathcal{G}$ is a singular elliptic operator such that $\mathcal{G}(\omega) = 0$; *i.e.*, it is a $(0, \omega)$–elliptic operator. Moreover it satisfies that

$$\mathcal{G} \circ \mathcal{F} = \mathcal{F} \circ \mathcal{G} = \mathcal{I} - \mathcal{P}_\omega$$

and when $\mathcal{K}$ is non–singular, then $\mathcal{K}^{-1} = \mathcal{G} + \lambda^{-1}\mathcal{P}_\omega$, see [4, Proposition 2]. The kernel of the orthogonal Green operator is called *orthogonal Green kernel* or *orthogonal Green function* and denoted by G.

Notice that if $\mathcal{K}$ is a (λ, ω)–elliptic operator, then $\mathcal{F} = \mathcal{K} - \lambda\mathcal{P}_\omega$ is a $(0, \omega)$–elliptic operator whose orthogonal Green function is the same that those corresponding to $\mathcal{K}$. Moreover if we label V, then the matrix identified with the orthogonal Green kernel is nothing else but the the Moore–Penrose inverse of the kernel associated with $\mathcal{F}$. Green kernels are diagonal positive as the following result shows.

Lemma 1.1. *If $\mathcal{K}$ is a (λ, ω)–elliptic operator on Γ and G is its orthogonal Green kernel, then $G(x, x) = 0$ iff $\lambda = 0$ and $\omega = \pm\varepsilon_x$. Therefore, when $\lambda = 0$, $G(x, x) > 0$ except for a vertex at most. In particular, $G(x, x) > 0$ for all $x \in V$ when $\omega \in \Omega(V)$.*

In [4] the authors also studied the perturbations of elliptic operators by adding a self–adjoint and positive semi–definite operator, or equivalently by adding a sum of projectors. Moreover, the mentioned work was focused on standard elliptic Schrödinger operators on a given network, and on their resulting perturbations after a nonnegative perturbation of the conductance of the network. In addition, the effect of the perturbation on the orthogonal Green operator was also analyzed.

Author's aim in this work is to extend the above results to a wider class of elliptic operators, including signless Laplacians on bipartite networks. In addition, we also study the effect of a positive perturbation of the conductance on the orthogonal Green kernel.

2. Schrödinger–like operators on a network

Now we consider a network whose set of vertices is the finite set V. Specifically, let $\Gamma=(V,E,c)$ be a finite network, whose *conductance* is the symmetric function $c\colon V\times V\longrightarrow[0,+\infty)$ satisfying that $c(x,x)=0$ for any $x\in V$ and moreover, x is adjacent to y; that is, the edge $e=\{x,y\}\in E$, iff $c(x,y)>0$. We always assume that Γ is *connected*; that is, that given $x,y\in V$ there exist $m\geq 1$ and $\{x_j\}_{j=0}^m\subset V$, such that $x=x_0$, $y=x_m$ and moreover $\prod\limits_{j=0}^{m-1}c(x_j,x_{j+1})>0$.

The network Γ is called *bipartite* if there exists a partition $\{V_0,V_1\}$ of V satisfying that $c(x,y)=0$ for any $x,y\in V_0$ and for any $x,y\in V_1$, see Figure 1. Then, V_0 and V_1 are called the *bipartite parts* of Γ and Γ

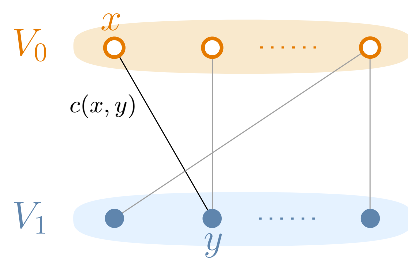

Figure 1. A (V_0,V_1)–bipartite network

is named (V_0,V_1)*–bipartite*. It is well known that a given network is bipartite iff has no cycles of odd order. Therefore, weighted stars, weighted paths, or more generally weighted trees are simple examples of bipartite networks, whereas a weighted cycle on n vertices is bipartite only when n is even.

A *Laplacian–like operator* on Γ is any endomorphism $\mathcal{F}$ of $\mathcal{C}(V)$ given by

$$\mathcal{F}(u)(x)=\sum_{y\in V}c(x,y)\big(u(x)\pm u(y)\big),\qquad x\in V. \tag{1}$$

The expression $\pm$ means that one of the two signs in each term of the right side of (1) is possible. More specifically, if for any $x\in V$ we consider the subsets of vertices

$$\begin{aligned}V^-(x)&=\big\{y\in V: c(x,y)>0 \text{ and the sign corresponding to the edge } \{x,y\} \text{ is minus}\big\}\\ V^+(x)&=\big\{y\in V: c(x,y)>0 \text{ and the sign corresponding to the edge } \{x,y\} \text{ is plus}\big\}\end{aligned} \tag{2}$$

then, for any $u\in\mathcal{C}(V)$ and any $x\in V$, we can express the operator $\mathcal{F}$ as

$$\mathcal{F}(u)(x)=\sum_{y\in V^-(x)}c(x,y)\big(u(x)-u(y)\big)+\sum_{y\in V^+(x)}c(x,y)\big(u(x)+u(y)\big) \tag{3}$$

Of course, given $x\in V$, any of the two subsets, $V^-(x)$ or $V^+(x)$, can be empty in which case the corresponding term in the Identity (3) does not appear.

The class of Laplacian–like operators on Γ, encompasses some well–known operators defined on the network Γ. In particular, when $V^+(x)=\emptyset$ for all $x\in V$, we have the *combinatorial Laplacian* of Γ, that we denote by $\mathcal{L}$, whereas when $V^-(x)=\emptyset$ for all $x\in V$, we have the so–called *signless Laplacian* of Γ, that we denote by $\mathcal{Q}$.

From the identity

$$\langle\mathcal{F}(u),v\rangle=\frac{1}{2}\sum_{x,y\in V}c(x,y)\big(u(x)\pm u(y)\big)\big(v(x)\pm v(y)\big),\quad u,v\in\mathcal{C}(V) \tag{4}$$

we have that any Laplacian–like operator is self–adjoint and positive semi–definite.

Notice that after labeling the vertex set, then the matrices equivalent to Laplacian–like operators are *weakly diagonally dominant*, in the sense that the sum of the absolute value of all row entries equals the diagonal value. Moreover, according with [1, Definition 6.2.8], the set of matrices equivalent to Laplacian–like operators is the set of *equimodular matrices* with positive diagonal entries associated with the matrix equivalent to the combinatorial Laplacian.

Given $\mathcal{F}$ a Laplacian–like operator and $q \in \mathcal{C}(V)$, the *Schrödinger–like operator* on Γ with *principal part* $\mathcal{F}$ and *potential* q is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{F}_q(u) = \mathcal{F}(u) + qu$. Those Schrödinger–like operators whose principal part is the combinatorial Laplacian are called *standard Schrödinger operators*.

In view of Identity (4), it is clear that each Schrödinger–like operator with nonnegative and non null potential is positive definite.

The application of the *Perron–Frobenius theorem*, see for instance [1, Theorem 2.1.4(b)], to the opposite of any standard Schrödinger operator, stablishes that the lowest eigenvalue of any of these operators is simple. Therefore, for standard Schrödinger operators ellipticity is equivalent to positive semi–definiteness and as straightforward consequence, the combinatorial Laplacian is always elliptic, in fact $(0, \omega)$–elliptic, where $\omega = \frac{1}{\sqrt{n}}\,\varepsilon_V$. However, this is not true for other Schrödinger–like operators, since Perron–Frobenius theorem is not applicable, except to show that for those operators whose principal part is the signless Laplacian, its largest eigenvalue is also simple. For instance, the eigenvalues of the signless Laplacian of a complete network on $n \geq 3$ vertices and with constant conductance, $c > 0$, are $(n+1)c$, that is simple and $(n-2)c$ whose multiplicity is $n-1$. Therefore, the signless Laplacian of any complete network with constant conductance is not elliptic. On the other hand, the eigenvalues of the signless Laplacian of the cycle network on n vertices with constant conductance $c > 0$ are $\lambda_j = 2c\big(1 + \cos(\frac{2\pi j}{n})\big)$, $j = 1, \ldots, n$. So, when n is even $\lambda = \lambda_{\frac{n}{2}}$ is the lowest eigenvalue, and hence it is simple, whereas when n is odd $\lambda = \lambda_{\frac{n-1}{2}} = \lambda_{\frac{n+1}{2}}$ is the lowest eigenvalue, that is double. Therefore, the signless Laplacian for a cycle on n vertices with constant conductance is elliptic iff n is even. As we show below, the fact that precisely these cycles are bipartite networks is far of be casual.

Our aim is to determine the ellipticity of a class of Schrödinger–like operators, broader that the standard one. For those operators whose principal part is the signless Laplacian some results are already known. Notice that the matrix version of these operators are the irreducible, symmetric and *essentially nonnegative matrices*. The following result, adapted to our terminology, is due to R. Roth.

Lemma 2.1. [8, Theorem 4] *Let Γ be a (V_0, V_1)–bipartite network and consider any potential $q \in C(V)$. Then each non null eigenfunction $u \in C(V)$ corresponding to λ, the lowest eigenvalue of $\mathcal{Q}_q$, satisfies that $u > 0$ on V_0 and $u < 0$ on V_1, or vice versa. In particular λ is simple.*

Observe that Lemma 2.1 stablished that for Schrödinger–like operators on bipartite networks whose principal part is the signless Laplacian, ellipticity is equivalent to positive semi–definiteness. Therefore, on a bipartite network the operator $\mathcal{Q}_q$, where q is nonnegative, is elliptic. Moreover, if Γ is (V_0, V_1)–bipartite and $\mathcal{Q}_q$ is (λ, ω)–elliptic, then $\omega > 0$ on V_0 and $\omega < 0$ on V_1, or vice versa. As a straightforward consequence, we obtain that the signless Laplacian of a bipartite networks is elliptic. In fact, if Γ is (V_0, V_1)–bipartite it is easy to show that in this case, $\mathcal{Q}$ is $(0, \omega)$–elliptic, where $\omega = \varepsilon_{V_0} - \varepsilon_{V_1}$.

Recently, F. Goldberg and S. Kirkland have generalized the result of Lemma 2.1 to non bipartite networks but maintaining the potential equals to 0; that is, for the signless Laplacian. In [7, Definition 1.3], given $H \subset V$ a maximal independent set, they say the network Γ is *H–Roth* if any non null eigenfunction $u \in C(V)$ corresponding to λ, the lowest eigenvalue of $\mathcal{Q}$, satisfies that $u > 0$ on H and $u < 0$ on $V \setminus H$, or vice versa. Newly, this property of the eigenfunctions implies that λ is simple and hence that $\mathcal{Q}$ is elliptic, since it is positive–semidefinite. Clearly, if Γ is (V_0, V_1)–bipartite, then it is V_0–Roth and also V_1–Roth.

Whereas Roth's theorem is a generalization to some Schrödinger–like operators, of the behavior of the signless Laplacian on bipartite networks, Goldberg and Kirkland show that Roth's theorem can be generalized to the signless Laplacian on networks obtained from a bipartite network by adding edges whose ends are in the same bipartite component. Specifically, [7, Corollary 6.5] says that if $K_{s,t}$ is the complete bipartite graph and $s \geq t$, then the graph Γ obtained from $K_{s,t}$ by adding edges between vertices in H, the bipartite component of cardinality s, is H–Roth. Some results concerning to the case $s < t$ are also studied in [7].

Y.-Z. Fan, Y. Wang and H. Guo have showed in [6] that for any $k \geq 1$, $g \geq 3$ and $\ell \geq 1$, the signless Laplacian of graph $U_n^k(g)$, where $n = g + \ell + k$ is elliptic and moreover each non null eigenfunction corresponding to the lowest eigenvalue has all its entries non null. The graph $U_n^k(g)$ is obtained by coalescing a path on $\ell + 1$ vertices with a cycle on g vertices by identifying one of its end vertices with some vertex of the cycle and also coalescing the path with k digons by identifying the other end of the path with one of the ends vertices of the digons, see Figure 2 Since when g is even, the graph $U_n^k(g)$ is bipartite the case g odd

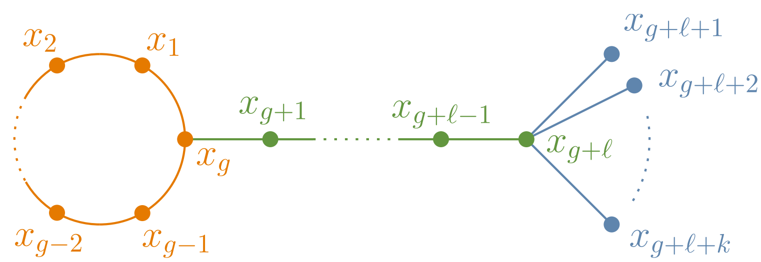

FIGURE 2. The graph $U_n^k(g)$

determines a nice example of non–bipartite graph whose signless Laplacian is elliptic (and invertible). The ellipticity of $U_n^k(g)$ for odd g is a consequence of [6, Lemma 3.3]. Moreover, it is not difficult to show that, in both cases g odd or even, there exists a maximal independent set H, such that $U_n^k(g)$ is H–Roth.

In this work we are only interested in Schrödinger–like operators whose principal part is a singular Laplacian–like operator. The remaining Schrödinger–like operators, including those studied in [7] will be treated in future works.

Clearly, the class of Schrödinger–like operators with singular principal part includes the standard ones and those whose principal part is the signless Laplacian of a bipartite network. Our next aim is to characterize when one of this operators is elliptic. To do this, we need to consider a special class of potentials determined by positive weights. Specifically, if $\sigma \in \Omega_+(V)$ is a positive weight on V, the function $q_\sigma = -\dfrac{1}{\sigma}\,\mathcal{L}(\sigma)$ is named *the potential determined by* σ. Since $\langle \sigma, q_\sigma \rangle = 0$, either $q_\sigma = 0$, and this occurs iff $\sigma = \frac{1}{\sqrt{n}}\,\varepsilon_V$, or q_σ takes positive and negative values. For standard Schrödinger operators, some of the authors proved the following key result.

Lemma 2.2. [2, Corollary 3.4] *A standard Schrödinger operator is elliptic iff its potential is given by* $q = q_\sigma + \lambda$, *where* $\lambda \geq 0$ *and* $\sigma \in \Omega_+(V)$, *in which case it is* (λ, σ)*–elliptic.*

Notice that a consequence of the above Lemma, if $q \geq 0$, then there exists $\lambda \geq 0$ and $\sigma \in \Omega_+(V)$ such that $q = q_\sigma + \lambda$. On the other hand, in [2] it was also proved that the value λ and the positive weight σ such that $q = q_\sigma + \lambda$ are uniquely determined by q.

The conclusion of Lemma 2.2 is closely related with the fact that, after labeling the set of vertices, the matrix equivalent to a standard Schrödinger operator is a Z–matrix. However, since a Schrödinger–like operator is not necessarily a Z–matrix, it is not true that for general Schrödinger–like operator the hypothesis on the potential assures their ellipticity as the previous example of complete graphs shows.

The key result in this work is the following characterization of singular Laplacian–like operators.

Theorem 2.3. *The endomorphism* $\mathcal{F}$ *is a singular Laplacian–like operator iff there exists a partition function* $\chi \in \mathcal{C}(V)$ *such that*

$$\mathcal{F}(u) = \chi \mathcal{L}(\chi u), \quad \textit{for any } u \in \mathcal{C}(V).$$

Therefore, $\mathcal{F}$ is $(0,\omega)$–elliptic where $\omega = \frac{1}{\sqrt{n}}\,\chi$.

Proof. Clearly, if $\mathcal{K}(u) = \chi\mathcal{L}(\chi u)$ for any $u \in \mathcal{C}(V)$, then f$\mathcal{K}$ is self–adjoint and positive semi–definite. Moreover, $\mathcal{K}(u) = 0$ iff $\mathcal{L}(\chi u) = 0$; that is, iff $\chi u = a\varepsilon_{_V}$, $a \in \mathbb{R}$ or, equivalently, iff $u = a\chi$, since $\chi^2 = \varepsilon_{_V}$. Therefore, $\mathcal{K}$ is singular and $(0,\omega)$–elliptic.

On the other hand, since $\chi = \varepsilon_{_{V_0}} - \varepsilon_{_{V_1}}$, for any $x \in V$ we have that

$$\mathcal{K}(u)(x) = \chi(x)\mathcal{L}(\chi u)(x) = \sum_{y\in V} c(x,y)\big(\chi^2(x)u(x) - \chi(x)\chi(y)u(y)\big) = \sum_{y\in V} c(x,y)\big(u(x) - \chi(x)\chi(y)u(y)\big),$$

which implies that

$$\mathcal{K}(u)(x) = \begin{cases} \displaystyle\sum_{y\in V_0} c(x,y)\big(u(x)-u(y)\big) + \sum_{y\in V_1} c(x,y)\big(u(x)+u(y)\big), & \text{if } x \in V_0,\\ \displaystyle\sum_{y\in V_0} c(x,y)\big(u(x)+u(y)\big) + \sum_{y\in V_1} c(x,y)\big(u(x)-u(y)\big), & \text{if } x \in V_1. \end{cases}$$

Therefore, $\mathcal{K}$ is a Laplacian–like operator. Conversely, if $\mathcal{F}$ is the Laplacian–like operator, defined for any $u \in \mathcal{C}(V)$ as

$$\mathcal{F}(u)(x) = \sum_{y\in V} c(x,y)\,\big(u(x) \pm u(y)\big), \qquad x \in V,$$

then $\mathcal{F}(u) = 0$ iff

$$0 = \langle \mathcal{F}(u), u\rangle = \frac{1}{2}\sum_{x,y\in V} c(x,y)\,\big(u(x) \pm u(y)\big)^2;$$

that is, iff $u(x) \pm u(y) = 0$ when $c(x,y) > 0$. This implies that $u^2(x) = u^2(y)$ when $c(x,y) > 0$ and hence u^2 is constant, since Γ is connected. Therefore, either $u = 0$ or there exists $a > 0$ such that $u^2(x) = a^2$ for any $x \in V$.

So, if $u \in \mathcal{C}(V)$ is non null and $\mathcal{F}(u) = 0$, there exists $a > 0$ such that if we consider the vertex subsets $V_0 = \{y \in V : u(y) = a\}$ and $V_1 = \{y \in V : u(y) = -a\}$, then $\{V_0, V_1\}$ is a partition of V and $u = a\chi$ where $\chi = \varepsilon_{_{V_0}} - \varepsilon_{_{V_1}}$. In addition, if $x \in V_0$, then $V^-(x) \subset V_0$ and $V^+(x) \subset V_1$, whereas if $x \in V_1$, then $V^-(x) \subset V_1$ and $V^+(x) \subset V_0$; that is,

$$\mathcal{F}(u)(x) = \begin{cases} \displaystyle\sum_{y\in V_0} c(x,y)\big(u(x)-u(y)\big) + \sum_{y\in V_1} c(x,y)\big(u(x)+u(y)\big), & \text{if } x \in V_0,\\ \displaystyle\sum_{y\in V_0} c(x,y)\big(u(x)+u(y)\big) + \sum_{y\in V_1} c(x,y)\big(u(x)-u(y)\big), & \text{if } x \in V_1, \end{cases}$$

or, equivalently, $\mathcal{F}(u) = \chi\mathcal{L}(\chi u)$. □

As an easy application, we obtain the following well–known property about signless laplacians, that we have already mentioned before.

Corollary 2.4. *The signless Laplacian is elliptic iff the network Γ is bipatite. In fact, if Γ is (V_0, V_1)–bipartite, then $\mathcal{Q}$ is $(0,\omega)$–elliptic, where $\omega = \frac{1}{\sqrt{n}}\big(\varepsilon_{_{V_0}} - \varepsilon_{_{V_1}}\big)$.*

The followin claim generalizes to singular Laplacian–like operators a well–known property of signless laplacians on bipartite networks.

Corollary 2.5. *The eigenvalues of any singular Laplacian–like operator coincide with those of the combinatorial Laplacian.*

The Lemma 2.2 Theorem 2.3 together lead to a complete characterization of the elliptic Schrödinger–like operators whose principal part is singular.

Proposition 2.6. *Consider $\mathcal{F}_q$ a Schrödinger–like operator whose principal part is singular. Then $\mathcal{F}_q$ is elliptic iff its potential is given by $q = q_\sigma + \lambda$, where $\lambda \geq 0$ and $\sigma \in \Omega_+(V)$. Moreover, if $\chi \in \mathcal{C}(V)$ is a partition function and $\mathcal{F}(u) = \chi\mathcal{L}(\chi u)$ for any $u \in \mathcal{C}(V)$, then $\mathcal{F}_q$ is (λ,ω)–elliptic, where $\omega = \sigma\chi$.*

Proof. Since $\mathcal{F}(u) = \chi\mathcal{L}(\chi u)$ for any $u \in \mathcal{C}(V)$, for any potential $q \in \mathcal{C}(V)$, we have that $\mathcal{F}_q(u) = \chi\mathcal{L}_q(\chi u)$ for any $u \in \mathcal{C}(V)$. Therefore, $\mathcal{F}_q$ is positive semi–definite iff $\mathcal{L}_q$ is positive semi–definite and moreover $\mathcal{F}_q(u) = \lambda u$ iff $\mathcal{L}_q(\chi u) = \lambda\chi u$, since $\chi^2 = \varepsilon_V$. So, $\mathcal{F}_q$ is elliptic iff $\mathcal{L}_q$ is elliptic, that from Lemma 2.2 is equivalent to be $q = q_\sigma + \lambda$ where $\lambda \geq 0$ and $\sigma \in \Omega_+(V)$. In this case, $\mathcal{L}_q$ is (λ, σ)–elliptic, which from identity $\mathcal{L}_q(u) = \lambda u$ iff $\mathcal{F}_q(\chi u) = \lambda\chi u$, in turns implies that $\mathcal{F}_q$ is (λ, ω)–elliptic. □

Now we particularize the above result to the case of Schrödinger–like operators, whose principal part is the signless Laplacian (compare with Roth's Theorem).

Corollary 2.7. *Assume that Γ is (V_0, V_1)–bipartite. Then, for any $\lambda \geq 0$ and $\sigma \in \Omega_+(V)$ if we consider $q = q_\sigma + \lambda$, the Schrödinger–like operator $\mathcal{Q}_q$ is (λ, ω)–elliptic where $\omega = \sigma\big(\varepsilon_{V_0} - \varepsilon_{V_1}\big)$.*

Given $\lambda \geq 0$, $\sigma \in \Omega_+(V)$ and the corresponding potential $q = q_\sigma + \lambda$, we denote by $G_{\lambda,\sigma}$ the orthogonal Green kernel of the standard Schrödinger operator $\mathcal{L}_q$. In view of the relation between elliptic Schrödinger–like operators whose principal part is singular and the standard elliptic Schrödinger operators, we have the following result.

Corollary 2.8. *Consider $\chi \in \mathcal{C}(V)$ a partition function, $\lambda \geq 0$, $\sigma \in \Omega_+(V)$ and the elliptic Schrödinger–like operator $\mathcal{F}_q$ where $\mathcal{F}(u) = \chi\mathcal{L}(\chi u)$ for any $u \in \mathcal{C}(V)$ and $q = q_\sigma + \lambda$. Then the orthogonal Green kernel of $\mathcal{F}_q$ is given by $(\chi \otimes \chi)G_{\lambda,\sigma}$.*

Proof. From Proposition 2.6 we know that $\mathcal{F}_q(u) = \chi\mathcal{L}_q(\chi u)$ for any $u \in \mathcal{C}(V)$ and that $\mathcal{F}_q$ is (λ, ω)–elliptic, where $\omega = \sigma\chi$.

If $f \in \omega^\perp$ then, $0 = \langle f, \sigma\chi\rangle = \langle f\chi, \sigma\rangle$, and hence, $f\chi \in \sigma^\perp$. So, if $\mathcal{G}_{\lambda,\sigma}$ denotes the orthogonal Green operator for $\mathcal{L}_q$, $u = \mathcal{G}_{\lambda,\sigma}(f\chi) \in \sigma^\perp$ and $\mathcal{L}_q(u) = f\chi$. Moreover, $u\chi \in \omega^\perp$ and

$$\mathcal{F}_q(u\chi) = \chi\mathcal{L}_q(u) = \chi^2 f = f.$$

Therefore, the orthogonal Green operator for $\mathcal{F}_q$ is given by $\chi\mathcal{G}_{\lambda,\sigma}(\chi u)$ for any $u \in \mathcal{C}(V)$, which implies the claimed relation between the orthogonal Green kernels. □

We end this section applying Corollary 2.8 to obtain the orthogonal Green kernel for the singular Schrödinger–like operators whose principal part is the signless Laplacian of a weighted path and whose potential is those determined by a positive weight. We make use of the expression for the Green kernel for the standard singular Schrödinger operator on a weighted path given in [3, Corollary 5.2]. So, let Γ be the weighted path where $V = \{x_1, \dots, x_n\}$ and the conductance is given by $c(x_j, x_{j+1}) > 0$, $j = 1, \dots, n-1$ and $c(x, y) = 0$, otherwise. Then, Γ is (V_0, V_1)–bipartite, where $V_0 = \{x_{2j-1}\}_{j=1}^{\lceil \frac{n}{2}\rceil}$ and $V_1 = \{x_{2j}\}_{j=1}^{\lfloor \frac{n}{2}\rfloor}$, see Figure 3, and hence $\chi(x_j) = (-1)^{j+1}$, $j = 1, \dots, n$.

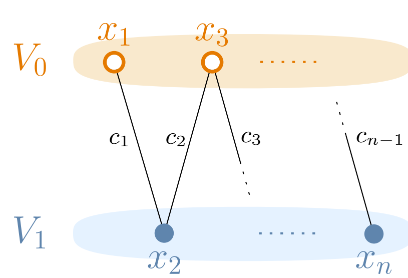

FIGURE 3. A path as bipartite network

Corollary 2.9. *If Γ is the weighted path with vertex set $V = \{x_1, \dots, x_n\}$ and whose conductance is given by $c(x_j, x_{j+1}) > 0$, $j = 1, \dots, n-1$ and $c(x, y) = 0$, otherwise, then for any $\sigma \in \Omega_+(V)$ the orthogonal Green* *kernel of $\mathcal{Q}_{q_\sigma}$ is given by*

$$G(x_i,x_j)=(-1)^{i+j}\sigma(x_i)\sigma(x_j)\left[\sum_{k=1}^{\min\{i,j\}-1}\frac{W_k^2}{C_k}+\sum_{k=\max\{i,j\}}^{n-1}\frac{(1-W_k)^2}{C_k}-\sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{W_k(1-W_k)}{C_k}\right],$$

for any $i,j=1,\dots,n$, where $W_k=\sum\limits_{\ell=1}^{k}\sigma^2(x_\ell)$ and $C_k=c(x_\ell,x_{\ell+1})\sigma(x_\ell)\sigma(x_{\ell+1})$, $k=1,\dots,n-1$.

3. Perturbation of elliptic Schrödinger–like operators

Our last aim in this work is to study the effect of a non negative perturbation of the conductance of the network $\Gamma=(V,E,c)$ on the orthogonal Green kernel of any elliptic Schrödinger–like operator whose principal part is singular.

Consider $\epsilon\colon V\times V\longrightarrow[0,+\infty)$ a symmetric function such that $\epsilon(x,x)=0$ for any $x\in V$, the sets

$$E^\epsilon=\Big\{e=\{x,y\}\subset V:\epsilon(x,y)>0\Big\}\quad\text{and}\quad E_0^\epsilon=\Big\{e=\{x,y\}\in E^\epsilon:c(x,y)=0\Big\}$$

and the *perturbed network* $\Gamma^\epsilon=(V,E\cup E_0^\epsilon,c+\epsilon)$ on the same set of vertices and whose conductance is $c+\epsilon$. Therefore, we can understand the perturbed network as a new network built from Γ by introducing new edges, the edges in E_0^ϵ, and by increasing the conductance of some old edges, the edges in $E^\epsilon\setminus E_0^\epsilon$. Clearly the connectivity of Γ implies that the perturbed network Γ^ϵ is also connected. In other words, Γ^ϵ is obtained from Γ by perturbing a total of $m=|E^\epsilon|$ edges.

The combinatorial Laplacian of the perturbed network is denoted $\mathcal{L}^P$. Then $\mathcal{L}^P=\mathcal{L}+\mathcal{L}^\epsilon$ when $\mathcal{L}^\epsilon$ is the endomorphism on $\mathcal{C}(V)$ defined for any $u\in\mathcal{C}(V)$ as

$$\mathcal{L}^\epsilon(u)(x)=\sum_{y\in V}\epsilon(x,y)\big(u(x)-u(y)\big),\quad x\in V.$$

Therefore, if for any $\sigma\in\Omega(V)$ we denote by q_σ^P and by q_σ^ϵ the potentials $-\sigma^{-1}\mathcal{L}^P(\sigma)$ and $-\sigma^{-1}\mathcal{L}^\epsilon(\sigma)$, respectively, then

$$q_\sigma^P=q_\sigma+q_\sigma^\epsilon.$$

Consequently, if we consider $\lambda\ge 0$, $\sigma\in\Omega_+(V)$ and the potentials $p=q_\sigma^P+\lambda$ and $q=q_\sigma+\lambda$, then $p=q+q_\sigma^\epsilon$. Then, we denote by $\mathcal{G}_{\lambda,\sigma}$ and by $G_{\lambda,\sigma}$ the orthogonal Green operator and kernel, respectively, for the standard Schrodinger operator $\mathcal{L}_q$, whereas G^P is the orthogonal Green kernel corresponding to $\mathcal{L}_p^P$.

In [4] the authors showed that $\mathcal{L}_p^P$ is a perturbation of $\mathcal{L}_q$ obtained by adding a sum of projectors, which leads to obtain G^P expressed only in terms of $G_{\lambda,\sigma}$. To do this, consider for any $e=\{z,t\}\in E^\epsilon$ the *dipole*

$$\tau_e=\sqrt{\epsilon(z,t)\sigma(z)\sigma(t)}\left(\frac{\varepsilon_z}{\sigma(z)}-\frac{\varepsilon_t}{\sigma(t)}\right).\tag{5}$$

Then, the $m\times m$–matrix $\mathsf{M}=\mathsf{I}+\mathsf{G}$ where I is the Identity and $\mathsf{G}=\Big(\langle\mathcal{G}_{\lambda,\omega}(\tau_e),\tau_{\hat e}\rangle\Big)_{e,\hat e\in E^\epsilon}$ is symmetric, positive definite and hence invertible. Observe that if $e=\{z,t\},\hat e=\{\hat z,\hat t\}\in E^\epsilon$, then the $e,\hat e$–entry of matrix G is given by

$$\mathsf{G}_{e,\hat e}=\sqrt{\epsilon(z,t)\epsilon(\hat z,\hat t)\sigma(z)\sigma(\hat z)\sigma(t)\sigma(\hat t)}\left[\frac{G_{\lambda,\sigma}(\hat z,z)}{\sigma(\hat z)\sigma(z)}+\frac{G_{\lambda,\sigma}(\hat t,t)}{\sigma(\hat t)\sigma(t)}-\frac{G_{\lambda,\sigma}(\hat z,t)}{\sigma(\hat z)\sigma(t)}-\frac{G_{\lambda,\sigma}(\hat t,z)}{\sigma(\hat t)\sigma(z)}\right]\tag{6}$$

In addition, for any $x\in V$ we consider the m–vector whose e–entry, where $e=\{z,t\}\in E^\epsilon$, is given by

$$\mathsf{v}_x(e)=\sigma^{-1}(x)\mathcal{G}_{\lambda,\sigma}(\tau_e)(x)=\sqrt{\epsilon(z,t)\sigma(z)\sigma(t)}\left[\frac{G_{\lambda,\sigma}(x,z)}{\sigma(x)\sigma(z)}-\frac{G_{\lambda,\sigma}(x,t)}{\sigma(x)\sigma(t)}\right]\tag{7}$$

Lemma 3.1. [4, Theorem 4.2] *With the above notations, for any $x,y\in V$, we have*

$$G^P(x,y)=G_{\lambda,\omega}(x,y)-\sigma(x)\sigma(y)\,\langle\mathsf{M}^{-1}\mathsf{v}_x,\mathsf{v}_y\rangle.$$

If $\mathcal{F}_p$ is a Schrödinger–like operator on the perturbed network Γ^ϵ, whose principal part, $\mathcal{F}$, is singular, then from Proposition 2.6, $\mathcal{F}_p$ is elliptic iff its potential is given by $p = q_\sigma^P + \lambda$, where $\lambda \geq 0$ and $\sigma \in \Omega_+(V)$. Moreover, if χ is the partition function such that $\mathcal{F}(u) = \chi \mathcal{L}^P(\chi u)$ for any $u \in \mathcal{C}(V)$, then $\mathcal{F}_p$ is (λ, ω)–elliptic where $\omega = \sigma\chi$. The next result, determines the orthogonal Green kernel of $\mathcal{F}_p^P$ in terms of $G_{\lambda,\sigma}$.

Theorem 3.2. *Consider $\mathcal{F}_p$ a Schrödinger–like operator on the perturbed network Γ^ϵ whose principal part is given by $\mathcal{F}(u) = \chi\mathcal{L}^P(\chi u)$ for any $u \in \mathcal{C}(V)$, where χ is a partition function, and whose potential is given by $p = q_\sigma^P + \lambda$. Then $\mathcal{F}_p$ is (λ, ω)–elliptic, where $\omega = \sigma\chi$ and its orthogonal Green kernel is given by*

$$G(x, y) = \chi(x)\chi(y)\Big[G_{\lambda,\omega}(x, y) - \sigma(x)\sigma(y)\left\langle \mathsf{M}^{-1}\mathsf{v}_x, \mathsf{v}_y\right\rangle\Big], \quad x, y \in V.$$

Proof. From Corollary 2.8 we know that $G = (\chi \otimes \chi)G^P$ and hence it suffices to apply Lemma 3.1. □

Notice that

$$\mathcal{F}_p(u) = \chi\mathcal{L}^P(\chi u) + p = \chi\mathcal{L}(\chi u) + qu + \chi\mathcal{L}^\epsilon(\chi u) + q_\sigma^\epsilon u \tag{8}$$

and hence, $\mathcal{F}_p$ appears as a perturbation of the (λ, σ)–elliptic operator $\mathcal{K}$, where $\mathcal{K}(u) = \chi\mathcal{L}(\chi u) + qu$ and $q = q_\sigma + \lambda$. The operator given by $\chi\mathcal{L}^\epsilon(\chi u) + q_\sigma^\epsilon u$ represent the amount of the perturbation. Notice that if $\chi = \varepsilon_{V_0} - \varepsilon_{V_1}$, then for any $u \in \mathcal{C}(V)$ we have

$$\chi(x)\mathcal{L}^\epsilon(\chi u)(x) = \begin{cases} \displaystyle\sum_{y\in V_0} \epsilon(x, y)\big(u(x) - u(y)\big) + \sum_{y\in V_1} \epsilon(x, y)\big(u(x) + u(y)\big), & \text{if } x \in V_0,\\ \displaystyle\sum_{y\in V_0} \epsilon(x, y)\big(u(x) + u(y)\big) + \sum_{y\in V_1} \epsilon(x, y)\big(u(x) - u(y)\big), & \text{if } x \in V_1. \end{cases} \tag{9}$$

Therefore, the sign plus on the perturbed edges only appears associated with the edges whose extremes are in different parts of the partition $V = V_0 \cup V_1$, whereas edges with extremes on the same set of the partition always appears with sign minus. We end this work exemplifying this characteristic of this class of perturbation by considering the weighted cycle obtained by join the extremes of a weighted path. Since the cycle is obtained from the path by adding a new edge, we can see it as a perturbation of the path. Specifically, we analyze here the perturbation of the signless Laplacian of the path, since the perturbation of the combinatorial Laplacian has been treated in other works, see [5]. Recall that the path is always bipartite, and hence its signless Laplacian is singular and elliptic, but the cycle is elliptic only when the number of vertices is even. Therefore, for a cycle on an even number of vertices, its signless Laplacian appears as a perturbation of the signless Laplacian of the path, this does not occur for the signless Laplacian of a cycle on an odd number of vertices.

So, let Γ be the weighted path consider in the above section, where $V = \{x_1, \dots, x_n\}$ and the conductance is given by $c(x_j, x_{j+1}) > 0$, $j = 1, \dots, n-1$ and $c(x, y) = 0$, otherwise. Consider also $\sigma \in \Omega_+(V)$ and Q_{q_σ} the elliptic operator whose principal part is the signless Laplacian and whose potential is the one associated with σ. Recall that Q_{q_σ} is $(0, \omega)$–elliptic where $\omega(x_j) = (-1)^{j+1}\sigma(x_j)$, $j = 1, \dots, n$. We add a new edge with conductance ϵ between the vertices x_1 and x_n. In Figure 4, we show both cases, even and odd.

Proposition 3.3. *Let Γ be the weighted cycle with vertex set $V = \{x_1, \dots, x_n\}$ and whose conductance is given by $c(x_j, x_{j+1}) > 0$, $j = 1, \dots, n-1$, $c(x_1, x_n) = \epsilon > 0$ and $c(x, y) = 0$, otherwise. Consider $\mathcal{F}$ the endomorphism on $\mathcal{C}(V)$ given for any $u \in \mathcal{C}(V)$ by*

$$\begin{aligned} \mathcal{F}(u)(x_1) &= c(x_1, x_2)\big(u(x_1) + u(x_2)\big) + \epsilon\big(u(x_1) \pm u(x_n)\big),\\ \mathcal{F}(u)(x_j) &= c(x_j, x_{j+1})\big(u(x_j) + u(x_{j+1})\big) + c(x_j, x_{j-1})\big(u(x_j) + u(x_{j-1})\big), \quad j = 2, \dots, n-1\\ \mathcal{F}(u)(x_n) &= c(x_n, x_{n-1})\big(u(x_n) + u(x_{n-1})\big) + \epsilon\big(u(x_n) \pm u(x_1)\big), \end{aligned}$$

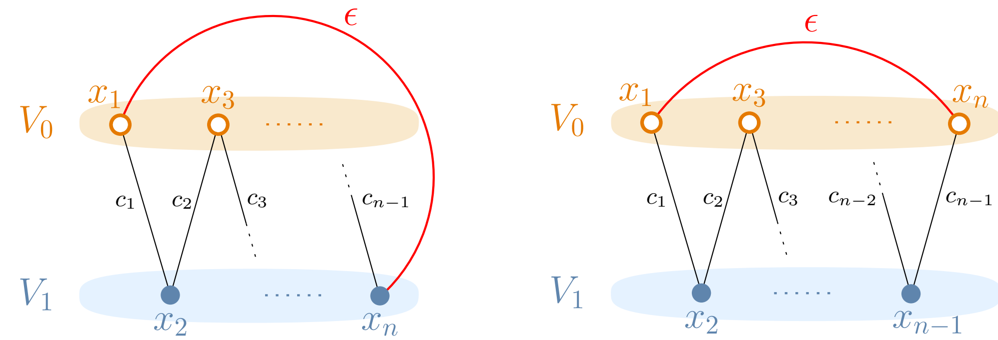

FIGURE 4. The cycle as a perturbation of the path. Cases even (left) and odd (right)

where in the last terms of the first and the third identities the sign $+$ appears when n is even and the sign $-$ appears when n is odd. Then for any $\sigma \in \Omega_+(V)$, the orthogonal Green kernel of $\mathcal{F}_{q_\sigma}$ is given by

$$G(x_i,x_j) = (-1)^{i+j}\sigma(x_i)\sigma(x_j)\left[\sum_{k=1}^{\min\{i,j\}-1}\frac{W_k^2}{C_k} + \sum_{k=\max\{i,j\}}^{n}\frac{(1-W_k)^2}{C_k} - \sum_{k=\min\{i,j\}}^{\max\{i,j\}-1}\frac{W_k(1-W_k)}{C_k}\right]$$
$$+ (-1)^{i+j+1}\sigma(x_i)\sigma(x_j)\left[\sum_{k=1}^{n}\frac{1}{C_k}\right]^{-1}\left[\sum_{k=1}^{n}\frac{W_k}{C_k} - \sum_{k=i}^{n}\frac{1}{C_k}\right]\left[\sum_{k=1}^{n}\frac{W_k}{C_k} - \sum_{k=j}^{n}\frac{1}{C_k}\right]$$

for any $i,j = 1,\dots,n$, where $W_k = \sum_{\ell=1}^{k}\sigma^2(x_\ell)$, $k = 1,\dots,n$, $C_k = c(x_\ell,x_{\ell+1})\sigma(x_\ell)\sigma(x_{\ell+1})$, $k = 1,\dots,n-1$ and $C_n = \epsilon\sigma(x_1)\sigma(x_n)$.

Acknowledgments. This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología,) under projects MTM2010-19660 and MTM2011-28800-C02-01/MTM.

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