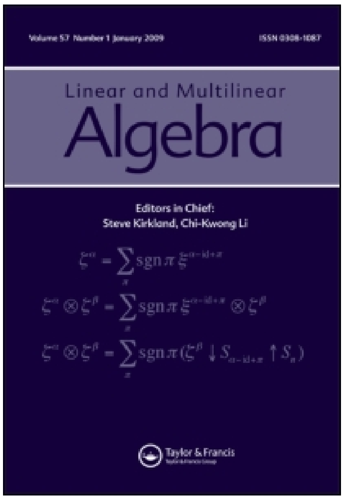

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Local differentiable pole assignment

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Local differentiable pole assignment

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Given a general local differentiable family of pairs of matrices, we obtain a local differentiable family of feedbacks solving the pole assignment problem, that is to say, shifting the spectrum into a prefixed one. We point out that no additional hypothesis is needed. In fact, simple approaches work in particular cases (controllable pairs, constancy of the dimension of the controllable subspace, and so on). Here the general case is proved by means of Arnold's techniques: the key point is to reduce the construction to a versal deformation of the central pair; in fact to a quite singular miniversal one for which the family of feedbacks can be explicitly constructed. As a direct application, a differentiable family of stabilizing feedbacks is obtained, provided that the central pair is stabilizable.

Keywords: pole assignment; differentiable families of systems; miniversal deformation

AMS Subject Classifications: 93B29; 93B50; 93D15

1. Introduction

There is a wide literature concerning the existence and construction of nicely parameterized solutions when families of linear systems are considered. In many cases (canonical form, pole assignment...), the pointwise approach does not guarantee properties such as continuity or differentiability for the family of solutions.

This article concerns the pole assignment problem, that is to say, the existence and the construction of suitable feedbacks in order to shift the set of poles of a system into a prefixed one. In the non-parameterized case, the solution is well-known for controllable systems or, more generally, for the 'modes' corresponding to the restriction to the controllability subspace. A classical and important application is the stabilization of a system provided that the 'unstable modes' are controllable.

Here we tackle the pole assignment problem for a family of perturbations of a system, that is to say, for a family of linear systems parameterized in a neighbourhood of a point corresponding to the given one, provided that the dependence on the parameters is differentiable. For such a local differentiable family of systems, we ask for a local differentiable family of feedbacks giving as poles, for

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any value of the parameters, those of a prescribed set of local differentiable functions (see the main Theorem 3.2). As said earlier, a direct application is derived to the stabilization of the systems of a given local differentiable family (Corollary 3.3).

The global case has been solved by several authors using quite different approaches (see e.g. [3], where the parameters vary in a contractible manifold). However, to our knowledge, there is no solution for the general local case. The classical non-parameterized construction easily generalizes to local differentiable families when the original system is controllable (or even when the dimension of the controllability subspace does not change), but it does not work in the general case. As in many other situations, the obstruction lies in the fact that the family is not a direct split of 'simple' ones.

In our case, this obstruction is removed by means of the use of Arnold's techniques, reducing the problem to a versal deformation of the central pair of matrices. More specifically, we use the quite singular miniversal deformation obtained in [2], for which an explicit family of suitable feedbacks can be constructed (Proposition 3.1). From this, we obtain a family of solutions for a general family of perturbations (Theorem 3.2).

We recall that versal deformations were introduced in [1] for square matrices, and generalized, for example, in [4]. In Section 2, we recall the definition of versal deformation (2.1) and the specific one to be used (2.2).

The symbols $\mathbb{R}$ and $\mathbb{C}$ denote the fields of real and complex numbers, respectively, and $\mathbb{C}^-$ is the open left-half complex plane. We write $M_{n,m}(\mathbb{R})$ for the vector space of matrices with n rows and m columns with entries in $\mathbb{R}$. We identify $M_{n,n+m}(\mathbb{R})$ with the set of pairs of matrices $M_n(\mathbb{R}) \times M_{n,m}(\mathbb{R})$. If A is a matrix, A^T is its transpose. I denotes the identity matrix of suitable size and I_k the $k \times k$ identity matrix.

2. Versal deformations

We consider differentiable perturbations $(A(\tau), B(\tau))$, where τ belongs to an open neighbourhood of the origin in $\mathbb{R}^k$, of a given pair (A, B) and we ask for the existence of a differentiable family of feedbacks $F(\tau)$ such that the matrices $A(\tau)+B(\tau)F(\tau)$ have a prescribed spectrum. We will use the techniques for reducing these perturbations to the so-called versal deformations [1]. We recall this definition in our case, which concerns pairs of matrices.

Thus, given a pair (A, B), we consider *local differentiable families of deformations* of it, that is to say, differentiable families of pairs $(A(\tau), B(\tau))$ defined in an open neighbourhood of the origin in $\mathbb{R}^k$ and satisfying $(A(0), B(0)) = (A, B)$.

Definition 2.1 [1] Given a pair $(A, B) \in M_{n,n+m}(\mathbb{R})$, a local differentiable family of deformations $(A(\alpha), B(\alpha))$, $\alpha \in U$, where U is an open neighbourhood of the origin in $\mathbb{R}^l$, is called a *versal deformation* if for any other local deformation $(A(\tau), B(\tau))$, $\tau \in V$, where V is an open neighbourhood of the origin in $\mathbb{R}^k$, there is a neighbourhood of the origin $V' \subset V$ and a differentiable map $\alpha(\tau) : V' \to U$ such that, for all $\tau \in V'$, the pair $(A(\tau), B(\tau))$ is block-equivalent to the one induced by $\alpha(\tau)$, that is to say,

$$(A(\tau), B(\tau)) = S^{-1}(\tau)(A(\alpha(\tau)), B(\alpha(\tau)))\begin{pmatrix} S(\tau) & 0 \\ R(\tau) & T(\tau) \end{pmatrix},$$

where the last factor is a local deformation in V' of the identity matrix.

Among versal deformations, those having a minimal number of parameters are called *miniversal*. We will use the following result.

THEOREM 2.2 [2] *Let* $(\bar{A}, \bar{B}) \in M_{n,n+m}(\mathbb{R})$ *be a pair in Brunovsky form, that is,*

$$\bar{A} = \begin{pmatrix} N & 0 \\ 0 & J \end{pmatrix}, \quad \bar{B} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

where

(i) $N = \operatorname{diag}(N_1, \ldots, N_r)$, *where* $N_i \in M_{k_i}(\mathbb{R})$ *is an upper nilpotent block for* $1 \le i \le r$ *and* $k_1 \ge \cdots \ge k_r$, $k_1 + \cdots + k_r = p$,
(ii) $J = \operatorname{diag}(J_1, \ldots, J_s)$, *where* $J_j \in M_{l_j}(\mathbb{R})$ *is an upper* λ_j*-Jordan block, for* $1 \le j \le s$,
(iii) $E = \operatorname{diag}(E_1, \ldots, E_r)$, *where* $E_i = (0 \cdots 0\ 1)^T \in M_{k_i,1}(\mathbb{R})$, *for* $1 \le i \le r$.

Let (X, Y) *be the linear variety in* $M_{n,\,n+m}(\mathbb{R})$ *defined by*

$$X = \begin{pmatrix} 0 & 0 \\ X_1^2 & X_2^2 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1^1 & Y_2^1 \\ 0 & Y_2^2 \end{pmatrix}, \quad X_1^2 \in M_{n-p,p}(\mathbb{R}), \quad Y_1^1 \in M_{p,r}(\mathbb{R}),$$

where

(i) *all the entries of* X_1^2 *are zero except*

$$x_1^{p+1}, \ldots, x_1^n, x_{k_1+1}^{p+1}, \ldots, x_{k_1+1}^n, \ldots, x_{k_1+\cdots+k_{r-1}+1}^{p+1}, \ldots, x_{k_1+\cdots+k_{r-1}+1}^n,$$

which are arbitrary;
(ii) $J + X_2^2$ *is the Arnold canonical form* [1]*;*
(iii) *all the entries of* Y_1^1 *are zero except*

$$y_2^{k_2+1}, \ldots, y_2^{k_1-1}, y_3^{k_1+k_3+1}, \ldots, y_k^{k_1+\cdots+k_{r-2}+k_r+1}, \ldots, y_r^{k_1+\cdots+k_{r-1}-1},$$

which are arbitrary;
(iv) Y_2^1 *is such that*

$$y_{r+1}^{k_1} = \cdots = y_m^{k_1} = y_{r+1}^{k_1+k_2} = \cdots = y_m^{k_1+k_2} = \cdots = y_{r+1}^{p} = y_m^{p} = 0;$$

(v) *all the parameters in* Y_2^2 *are arbitrary.*

Then $(\bar{A}, \bar{B}) + (X, Y)$ *is a miniversal deformation of* $(\bar{A}, \bar{B})$ *at* 0 *such that the non-zero entries of* (X, Y) *are the independent parameters.*

Remark 1 In the proof of Proposition 3.1 there is a graph where this deformation is represented.

3. The locally parameterized pole assignment

Let us consider that, for the miniversal deformation in 2.2, one can obtain an explicit differentiable family of feedbacks shifting the spectrum into a prescribed one.

From now on, we will use the following notations:

(i) We will write $(\bar{A}(\alpha), \bar{B}(\alpha))$ for the miniversal deformation of $(\bar{A}, \bar{B})$ in Theorem, 2.2. Furthermore, $J(\alpha) = J + X_2^2$ is the miniversal deformation of J in [1]. We write $\bar{\mu}_1(\alpha), \dots, \bar{\mu}_{n-p}(\alpha)$ for the differentiable families of complex numbers giving the eigenvalues of $J(\alpha)$.

(ii) $\sigma(\cdot)$ denotes the spectrum of the corresponding matrix or pair of matrices, each eigenvalue being repeated as many times as its multiplicity.

Notice that the spectrum of the pair $(\bar{A}, \bar{B})$ is exactly $\sigma(J)$. If $\alpha \neq 0$, the spectrum of $(\bar{A}(\alpha), \bar{B}(\alpha))$ is in general a subset of $\sigma(J(\alpha)) = \{\bar{\mu}_1(\alpha), \dots, \bar{\mu}_{n-p}(\alpha)\}$.

PROPOSITION 3.1 *Let* $(\bar{A}, \bar{B}) \in M_{n,n+m}(\mathbb{R})$ *be a Brunovsky pair and* $(\bar{A}(\alpha), \bar{B}(\alpha))$ *its miniversal deformation as in Theorem* 2.2. *For any set* $\lambda_1(\tau), \dots, \lambda_p(\tau)$, $\tau \in V$, *where* V *is an open neighbourhood of the origin in* $\mathbb{R}^k$, *of local differentiable families of complex numbers* (*not necessarily different*) *closed under conjugation, there is a local differentiable family of feedbacks* $F_0(\tau) \in M_{m,n}(\mathbb{R})$ *such that*

$$\sigma(\bar{A}(\alpha) + \bar{B}(\alpha)F_0(\tau)) = \{\lambda_1(\tau), \dots, \lambda_p(\tau)\} \cup \{\bar{\mu}_1(\alpha), \dots, \bar{\mu}_{n-p}(\alpha)\}.$$

Proof The miniversal deformation in Theorem 2.2 is given by

$$(\bar{A}(\alpha), \bar{B}(\alpha)) = \left(\left(\begin{array}{cccc|cccc|c|c} & N_1 & & & & & & & & \\ & & & & & & & & & \\ \hline & & & & & N_2 & & & & \\ & & & & & & & & & \\ \hline & & & & & & & & \ddots & \\ \hline * & 0 & \cdots & 0 & * & 0 & \cdots & 0 & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \cdots & J(\alpha) \\ * & 0 & \cdots & 0 & * & 0 & \cdots & 0 & & \end{array} \right), \right.$$

$$\left. \left(\begin{array}{c|c|c|ccc} 0 & 0 & & * & \cdots & * \\ \vdots & \vdots & \cdots & \vdots & & \vdots \\ 0 & (*) & & * & \cdots & * \\ 1 & 0 & & 0 & \cdots & 0 \\ \hline & 0 & & * & \cdots & * \\ & \vdots & \cdots & \vdots & & \vdots \\ & 0 & & * & \cdots & * \\ & 1 & & 0 & \cdots & 0 \\ \hline & & \ddots & & \vdots & \\ \hline & & & * & \cdots & * \\ & & & \vdots & & \vdots \\ & & & * & \cdots & * \end{array} \right) \right),$$

where * are parameters of the deformation which always appear and (*) are those which do not always appear.

Considering a block-diagonal matrix $F_0(\tau)$ of the form

$$F_0(\tau) = \left(\begin{array}{c|c|c|ccc} \beta_1(\tau)\cdots\beta_{k_1}(\tau) & & & & & \\ \hline & \gamma_1(\tau)\cdots\gamma_{k_2}(\tau) & & & & \\ \hline & & \ddots & & & \\ \hline & & & 0 & \cdots & 0 \\ & & & \vdots & & \vdots \\ & & & 0 & \cdots & 0 \end{array}\right)$$

we obtain that the matrix $\bar{A}(\alpha) + \bar{B}(\alpha)F_0(\tau)$ has the desired spectrum since its characteristic polynomial can be decomposed into the product of those of $J(\alpha)$ and an upper block-triangular matrix. □

Remark 1 Notice the quite singular property that if $\lambda_i(\tau)$, $1 \leq i \leq p$, are constant ($\lambda_i(\tau) = \lambda_i$), we can take it that $F_0(\tau)$ is also constant ($F_0(\tau) = F_0$).

Now, we can tackle the main result:

THEOREM 3.2 *Let $(A, B) \in M_{n,n+m}(\mathbb{R})$ be a pair and $(A(\tau), B(\tau))$, $\tau \in V$, where V is an open neighbourhood of the origin in $\mathbb{R}^k$, a local differentiable family of deformations. Let $\{\mu_1, \ldots, \mu_{n-p}\} = \sigma(A, B)$. For any set $\lambda_1(\tau), \ldots, \lambda_p(\tau)$, $\tau \in V$, of local differentiable families of complex numbers closed under conjugation, there is a local differentiable family of feedbacks $F(\tau) \in M_{m,n}(\mathbb{R})$, $\tau \in W \subset V$, and $\mu_1(\tau), \ldots, \mu_{n-p}(\tau)$, $\tau \in W$, local differentiable families of complex numbers with $\mu_i(0) = \mu_i$, $1 \leq i \leq n-p$, such that*

$$\begin{aligned} \sigma(A(\tau) + B(\tau)F(\tau)) &= \{\lambda_1(\tau), \ldots, \lambda_p(\tau)\} \cup \{\mu_1(\tau), \ldots, \mu_{n-p}(\tau)\}, \\ \sigma(A(\tau), B(\tau)) &\subset \{\mu_1(\tau), \ldots, \mu_{n-p}(\tau)\}. \end{aligned}$$

Proof Let $(\bar{A}, \bar{B})$ be the Brunovsky form of the pair (A, B) so that

$$\begin{aligned} \bar{A} &= S_0^{-1}(A + BR_0S_0^{-1})S_0, \\ \bar{B} &= S_0^{-1}BT_0. \end{aligned}$$

Let us consider $(\tilde{A}(\tau), \tilde{B}(\tau))$, $\tau \in V$, defined as

$$\begin{aligned} \tilde{A}(\tau) &= S_0^{-1}(A(\tau) + B(\tau)R_0S_0^{-1})S_0, \\ \tilde{B}(\tau) &= S_0^{-1}B(\tau)T_0. \end{aligned}$$

From 2.1, there exist differentiable $\alpha(\tau)$, $R(\tau)$, $S(\tau)$, $T(\tau)$ defined in a neighbourhood of the origin $V' \subset V$, such that

$$\begin{aligned} \tilde{A}(\tau) &= S^{-1}(\tau)[\bar{A}(\alpha(\tau)) + \bar{B}(\alpha(\tau))R(\tau)S^{-1}(\tau)]S(\tau), \\ \tilde{B}(\tau) &= S^{-1}(\tau)\bar{B}(\alpha(\tau))T(\tau). \end{aligned}$$

From Proposition 3.1, there exists a feedback $F_0(\tau)$, $\tau \in V$, such that

$$\sigma(\bar{A}(\alpha) + \bar{B}(\alpha)F_0(\tau)) = \{\lambda_1(\tau), \ldots, \lambda_p(\tau)\} \cup \{\bar{\mu}_1(\alpha), \ldots, \bar{\mu}_{n-p}(\alpha)\}.$$

Taking, for $\tau \in V'$,

$$\tilde{F}(\tau) = T^{-1}(\tau)(F_0(\tau) - R(\tau)S^{-1}(\tau))S(\tau),$$

it turns out that

$$\begin{aligned}\sigma(\tilde{A}(\tau) + \tilde{B}(\tau)\tilde{F}(\tau)) &= \sigma(S^{-1}(\tau)[\bar{A}(\alpha(\tau)) + \bar{B}(\alpha(\tau))R(\tau)S^{-1}(\tau)\\ &\quad + (\tau)[\bar{B}(\alpha(\tau))T(\tau)\tilde{F}(\tau)S^{-1}(\tau)]S(\tau))\\ &= \sigma(\bar{A}(\alpha(\tau)) + \bar{B}(\alpha(\tau))F_0(\tau))\\ &= \{\lambda_1(\tau), \ldots, \lambda_p(\tau)\} \cup \{\bar{\mu}_1(\alpha(\tau)), \ldots, \bar{\mu}_{n-p}(\alpha(\tau))\}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\sigma(\tilde{A}(\tau) + \tilde{B}(\tau)\tilde{F}(\tau)) &= \sigma(S_0^{-1}[A(\tau) + B(\tau)R_0S_0^{-1} + B(\tau)T_0\tilde{F}(\tau)S_0^{-1}]S_0)\\ &= \sigma(A(\tau) + B(\tau)[R_0S_0^{-1} + T_0\tilde{F}(\tau)S_0^{-1}])\\ &= \sigma(A(\tau) + B(\tau)F(\tau)),\end{aligned}$$

where $F(\tau) = R_0S_0^{-1} + T_0\tilde{F}(\tau)S_0^{-1}$, $\tau \in V'$.

Moreover, since block similarity preserves the eigenvalues of a pair, we have, for any $\tau \in V'$,

$$\begin{aligned}\sigma(A(\tau), B(\tau)) &= \sigma(\tilde{A}(\tau), \tilde{B}(\tau))\\ &= \sigma(\bar{A}(\alpha(\tau)), \bar{B}(\alpha(\tau)))\\ &\subset \{\bar{\mu}_1(\alpha(\tau)), \ldots, \bar{\mu}_{n-p}(\alpha(\tau))\}.\end{aligned}$$

Finally, we take $\mu_i(\tau) = \bar{\mu}_i(\alpha(\tau))$, $1 \le i \le n-p$, $\tau \in V'$. ■

Remark 2 Notice that in the above theorem $\mu_i(\tau) = \bar{\mu}_i(\alpha(\tau))$, $1 \le i \le n-p$.

As a direct application, let us consider the particular case when (A, B) is stabilizable. Since the set of stabilizable pairs is open, the pairs $(A(\tau), B(\tau))$ will also be stabilizable for any τ small enough. However, it is clear that a pointwise construction of stabilizing feedbacks $F(\tau)$ does not guarantee the differentiability of the resulting family. Let us see that the above theorem gives a differentiable family of suitable feedbacks.

COROLLARY 3.3 *Let $(A, B) \in M_{n,n+m}(\mathbb{R})$ be a stabilizable pair and $(A(\tau), B(\tau))$, $\tau \in V$, a local differentiable family of deformations. Then, there exists a local differentiable family of feedbacks $F(\tau) \in M_{m,n}(\mathbb{R})$, $\tau \in W \subset V$, such that*

$$\sigma(A(\tau) + B(\tau)F(\tau)) \subset \mathbb{C}^-.$$

Proof Since $\{\mu_1, \ldots, \mu_{n-p} \subset \mathbb{C}^-$, it is also true that

$$\{\mu_1(\tau), \ldots, \mu_{n-p}(\tau)\} \subset \mathbb{C}^-$$

for τ small enough. Then it suffices to take

$$\{\lambda_1(\tau), \ldots, \lambda_p(\tau)\} \subset \mathbb{C}^-.$$

■

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