DECOMPOSITION SPACES, INCIDENCE ALGEBRAS AND MÖBIUS INVERSION II: COMPLETENESS, LENGTH FILTRATION, AND FINITENESS

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Abstract. This is part 2 of a trilogy of papers introducing and studying the notion of decomposition space as a general framework for incidence algebras and Möbius inversion, with coefficients in ∞-groupoids. A decomposition space is a simplicial ∞-groupoid satisfying an exactness condition weaker than the Segal condition. Just as the Segal condition expresses up-to-homotopy composition, the new condition expresses decomposition.

In this paper, we introduce various technical conditions on decomposition spaces. The first is a completeness condition (weaker than Rezk completeness), needed to control nondegeneracy. For complete decomposition spaces we establish a general Möbius inversion principle, expressed as an explicit equivalence of ∞-groupoids.

Next we analyse two finiteness conditions on decomposition spaces. The first, that of locally finite length, guarantees the existence of the important length filtration on the associated incidence coalgebra. We show that a decomposition space of locally finite length is actually the left Kan extension of a semi-simplicial space. The second finiteness condition, local finiteness, ensures we can take homotopy cardinality to pass from the level of ∞-groupoids to the level of $\mathbb{Q}$-vector spaces.

These three conditions — completeness, locally finite length and local finiteness — together define our notion of Möbius decomposition space, which extends Leroux's notion of Möbius category (in turn a common generalisation of the locally finite posets of Rota et al. and of the finite decomposition monoids of Cartier–Foata), but which also covers many coalgebra constructions which do not arise from Möbius categories, such as the Faà di Bruno and Connes–Kreimer bialgebras.

Note: The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov (arXiv:1212.3563) who call them unital 2-Segal spaces.

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0. Introduction

In Part 1 of this work [9], we introduced the notion of decomposition space as a general framework for incidence (co)algebras. The relevant main results are recalled in Section 1 below. A decomposition space is a simplicial ∞-groupoid X satisfying a certain exactness condition, weaker than the Segal condition. Just as the Segal condition expresses up-to-homotopy composition, the new condition expresses decomposition, and there is a rich supply of examples in combinatorics [12]. In the present paper we proceed to establish a Möbius inversion principle for what we call *complete* decomposition spaces, and analyse the associated finiteness issues.

Classically [27], the Möbius inversion principle states that the zeta function in any incidence algebra (of a locally finite poset, say, or more generally a Möbius category in the sense of Leroux [23]) is invertible for the convolution product; its inverse is by definition the Möbius function. The Möbius inversion formula is a powerful and versatile counting device, but since it is an equality stated at the vector-space level in the incidence algebra, it belongs to algebraic combinatorics rather than bijective combinatorics.

It is possible to give Möbius inversion a bijective meaning, by following the objective method, pioneered in this context by Lawvere and Menni [21], which seeks to lift algebraic identities to the 'objective level' of (finite) sets and bijections, working with certain categories spanned by the combinatorial objects instead of working with vector spaces spanned by isoclasses of these objects. The algebraic identity then appears as the cardinality of the established bijection at the objective level.

To illustrate this, observe that a vector in the free vector space on a set S is just a collection of scalars indexed by (a finite subset of) S. The objective counterpart is a family of sets indexed by S, i.e. an object in the slice category $\boldsymbol{Set}_{/S}$. Linear maps at this level are given by spans $S \leftarrow M \rightarrow T$. The Möbius inversion principle states an equality between certain linear maps (elements in the incidence algebra). At the objective level, such an equality can be expressed as a bijection between sets in the spans representing those linear functors. In this way, the algebraic identity is revealed to be just the cardinality of a bijection of sets, which carry much more structural information. Lawvere and Menni [21] established an objective version of the Möbius inversion principle for Möbius categories in the sense of Leroux [23].

Our discovery in [9] is that something considerably weaker than a category suffices to construct an incidence algebra, namely a decomposition space. This discovery is interesting even at the level of sets, but we work at the level of ∞-groupoids. Thus, the role of vector spaces is played by slices of the ∞-category of ∞-groupoids. In [11] we have developed the necessary 'homotopy linear algebra' and homotopy cardinality, extending and streamlining many results of Baez–Hoffnung–Walker [1] who worked with 1-groupoids.

The decomposition space axiom on a simplicial ∞-groupoid X is expressly the condition needed for a canonical coalgebra structure to be induced on the slice ∞-category $\boldsymbol{Grpd}_{/X_1}$, (where $\boldsymbol{Grpd}$ denotes ∞-category of ∞-groupoids). The comultiplication is the linear functor

$$\Delta : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}_{/X_1} \otimes \boldsymbol{Grpd}_{/X_1}$$

given by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

This can be read as saying that comultiplying an edge $f \in X_1$ returns the sum of all pairs of edges (a, b) that are the short edges of a triangle with long edge f. In the case that X is the nerve of a category, this is the sum of all pairs (a, b) of arrows with composite $b \circ a = f$.

The aims of this paper are to establish a Möbius inversion principle in the framework of *complete* decomposition spaces, and also to introduce the necessary *finiteness* conditions on a complete decomposition space to ensure that incidence (co)algebras and Möbius inversion descend to classical vector-space-level coalgebras on taking the homotopy cardinality of the objects involved.

We proceed to summarise the main results.

Definition. 2.1 We say that a decomposition space X is *complete* when $s_0 : X_0 \to X_1$ is a monomorphism. It then follows that all degeneracy maps are monomorphisms (Lemma 2.3).

The motivating feature of this notion is that all issues concerning degeneracy can then be settled in terms of the canonical projection maps $X_r \to (X_1)^r$ sending a simplex to its principal edges: a simplex in a complete decomposition space is nondegenerate precisely when all its principal edges are nondegenerate (Corollary 2.13). Let $\vec{X}_r \subset X_r$ denote the subspace of these nondegenerate simplices.

For any decomposition space X, the comultiplication on $\boldsymbol{Grpd}_{/X_1}$ yields a convolution product on the linear dual $\boldsymbol{Grpd}^{X_1}$, called the *incidence algebra* of X. This contains, in particular, the *zeta functor* ζ, given by the span $X_1 \xleftarrow{=} X_1 \to 1$, and the counit ε (a neutral element for convolution) given by $X_1 \leftarrow X_0 \to 1$. In a complete decomposition space X we can consider the spans $X_1 \leftarrow \vec{X}_r \to 1$ and the linear functors Φ_r they define in the incidence algebra of X. We can now establish the decomposition-space version of the Möbius inversion principle, in the spirit of [21]:

Theorem 3.8. *For a complete decomposition space,*

$$\zeta * \Phi_{\text{even}} \;=\; \varepsilon \;+\; \zeta * \Phi_{\text{odd}}, \qquad \Phi_{\text{even}} * \zeta \;=\; \varepsilon \;+\; \Phi_{\text{odd}} * \zeta.$$

It is tempting to read this a saying that "$\Phi_{\text{even}} - \Phi_{\text{odd}}$" is the convolution inverse of ζ, but the lack of additive inverses in $\boldsymbol{Grpd}$ necessitates our sign-free formulation. Upon taking homotopy cardinality, as we will later, this yields the usual Möbius inversion formula $\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$, valid in the incidence algebra with $\mathbb{Q}$-coefficients.

Having established the general Möbius inversion principle on the objective level, we proceed to analyse the finiteness conditions needed for this principle to descend to the vector-space level of $\mathbb{Q}$-algebras. There are two conditions: X should be of locally finite length (Section 6), and X should be locally finite (Section 7). The first is a numerical condition, like a chain condition; the second is a homotopy finiteness condition. Complete decomposition spaces satisfying both conditions are called *Möbius decomposition spaces* (Section 8). We analyse the two conditions separately.

Definition. The *length* of an arrow f is the greatest dimension of an nondegenerate simplex with long edge f. We say that a complete decomposition space is *of locally finite length* — we also say *tight* — when every arrow has finite length.

Although many examples coming from combinatorics do satisfy this condition, it is actually a rather strong condition, as witnessed by the following result:

Every tight decomposition space is the left Kan extension of a semi-simplicial space.

We can actually prove this theorem for more general simplicial spaces, and digress to establish this. A complete simplicial space is called *split* if all face maps preserve nondegenerate simplices. This condition, which is the subject of Section 5, is the analogue of the condition for categories that identities are indecomposable, enjoyed in particular by Möbius categories in the sense of Leroux [23]. We prove that a simplicial space is split if and only if it is the left Kan extension along $\Delta_{\text{inj}} \subset \Delta$ of a semi-simplicial space $\Delta_{\text{inj}}^{\text{op}} \to \boldsymbol{Grpd}$, and in fact we establish more precisely:

Theorem 5.7. *Left Kan extension along $\Delta_{\text{inj}} \subset \Delta$ induces an equivalence of ∞-categories*

$$\mathrm{Fun}(\Delta_{\text{inj}}^{\text{op}}, \boldsymbol{Grpd}) \simeq \boldsymbol{Split}^{cons},$$

where the right-hand side is the ∞-category of split simplicial spaces and conservative maps.

This has the following interesting corollary.

Proposition 5.8. *Left Kan extension along $\Delta_{\text{inj}} \subset \Delta$ induces an equivalence between the ∞-category of 2-Segal semi-simplicial spaces and ULF maps, and the ∞-category of split decomposition spaces and cULF maps.*

We show that a complete decomposition space X is tight if and only if it has a filtration

$$X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)} \hookrightarrow \cdots \hookrightarrow X$$

of cULF monomorphisms, the so-called *length filtration*. This is precisely the structure needed to get a filtration of the associated coalgebra (6.19).

In Section 7 we impose the finiteness condition needed to be able to take homotopy cardinality and obtain coalgebras and algebras at the numerical level of $\mathbb{Q}$-vector spaces (and pro-vector spaces).

Definition. A decomposition space X is called *locally finite* (7.4) when X_1 is a locally finite ∞-groupoid (i.e. has only finite homotopy groups, and an upper bound on the nontrivial ones) and $s_0 : X_0 \to X_1$ and $d_1 : X_2 \to X_1$ are finite maps.

The condition 'locally finite' extends the notion of locally finite for posets. The condition ensures that the coalgebra structure descends to finite-groupoid coefficients, and hence, via homotopy cardinality, to $\mathbb{Q}$-algebras. In Section 7 we calculate the section coefficients (structure constants for the (co)multiplication) in some easy cases.

Finally we introduce the Möbius condition:

Definition. A complete decomposition space is called *Möbius* when it is locally finite and of locally finite length (i.e. is tight).

This is the condition needed for the general Möbius inversion formula to descend to finite-groupoid coefficients and $\mathbb{Q}$-coefficients, giving the following formula for the Möbius function (convolution inverse to the zeta function):

$$|\mu| = |\Phi_{\text{even}}| - |\Phi_{\text{odd}}| .$$

Related work. The notion of decomposition space was discovered independently by Dyckerhoff and Kapranov [5], who call them unital 2-Segal spaces. While some of the basic results in [9] were also proved in [5], the present paper has no overlap with [5].

The results in this paper on Möbius inversion are in the tradition of Leroux et al. [23], [3], [24], Dür [4] and Lawvere–Menni [21]. There is a different notion of Möbius category, due to Haigh [14]. The two notions have been compared, and to some extent unified, by Leinster [22], who calls Leroux's Möbius inversion *fine* and

Haigh's *coarse* (as it only depends on the underlying graph of the category). We should mention also the K-theoretic Möbius inversion for quasi-finite EI categories of Lück and collaborators [25], [6].

Note. This paper is the second in a series, originally posted on the arXiv as a single manuscript *Decomposition spaces, incidence algebras and Möbius inversion* [8] but split for publication into:

(0) Homotopy linear algebra [11]
(1) Decomposition spaces, incidence algebras and Möbius inversion I: basic theory [9]
(2) Decomposition spaces, incidence algebras and Möbius inversion II: completeness, length filtration, and finiteness [this paper]
(3) Decomposition spaces, incidence algebras and Möbius inversion III: the decomposition space of Möbius intervals [10]
(4) Decomposition spaces in combinatorics [12]
(5) Decomposition spaces and restriction species [13]

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1. Preliminaries on decomposition spaces

1.1. ∞-groupoids. We work in the ∞-category $\boldsymbol{Grpd}$ of ∞-groupoids, and in closely related categories such as its slices. By ∞-category we mean quasi-category in the sense of Joyal [17], [18], but follow rather the terminology of Lurie [26]. Most of our arguments are elementary, though, and for this reason we can get away with model-independent reasoning. We avoid reference to the Joyal model structure on simplicial sets, a key aspect of the way Joyal and Lurie bootstrap the whole theory and prove many of the core results which we exploit.

In this connection, a word of caution is due in particular regarding slice ∞-categories. When we refer to the ∞-category $\boldsymbol{Grpd}_{/S}$ (whose objects are maps $X \to S$), we only refer to an ∞-category determined up to equivalence by a certain universal property (Joyal's insight of defining slice categories as adjoint to a join operation [17]). In the Joyal model structure for quasi-categories, this category is represented by an explicit simplicial set. However, there is more than one possibility, though of course they are canonically equivalent, depending on which explicit version of the join operator is employed. In the works of Joyal and Lurie, the different versions are distinguished, and each has some technical advantages, but in the present work we shall only need properties that hold for both and we shall not distinguish between them.

1.2. Pullbacks. Pullbacks play an essential role in many of our arguments. This notion enjoys a universal property which in the model-independent formulation is identical to the universal property of the pullback in ordinary categories. Again, we shall only ever need homotopy invariant properties, making it irrelevant which particular model is chosen for the notion of pullback in the Joyal model structure for quasi-categories. One property which we shall use repeatedly is the following elementary lemma (a proof can be found in [26, 4.4.2.1]).

Lemma. 1.3. *In any diagram of ∞-groupoids*

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

if the outer rectangle and the right-hand square are pullbacks, then the left-hand square is a pullback.

1.4. Monomorphisms. The notion of monomorphism of ∞-groupoids plays an important role throughout this paper, notably through the definition of complete decomposition space (2.1). A map of ∞-groupoids is a *monomorphism* when its fibres are (-1)-groupoids (i.e. are either empty or contractible). In some respects, this notion behaves like for sets: for example, if $f : X \to Y$ is a monomorphism, then there is a complement $Z := Y \smallsetminus X$ such that $Y \simeq X + Z$. Hence a monomorphism is essentially an equivalence from X onto some connected components of Y. On the other hand, a crucial difference from sets to ∞-groupoids is that diagonal maps of ∞-groupoids are not in general monomorphisms. In fact $X \to X \times X$ is a monomorphism if and only if X is discrete (i.e. equivalent to a set).

1.5. Linear algebra with coefficients in ∞-groupoids. [11] The ∞-categories of the form $\boldsymbol{Grpd}_{/S}$ form the objects of a symmetric monoidal ∞-category $\boldsymbol{LIN}$, described in detail in [11]: the morphisms are the linear functors, meaning that they preserve homotopy sums, or equivalently indeed all colimits. Such functors are given by spans: the span

$$S \xleftarrow{p} M \xrightarrow{q} T$$

defines the linear functor

$$q_! \circ p^* : \boldsymbol{Grpd}_{/S} \longrightarrow \boldsymbol{Grpd}_{/T}$$

given by pullback along p followed by composition with q. The ∞-category $\boldsymbol{LIN}$ can play the role of the category of vector spaces, although to be strict about that interpretation, finiteness conditions should be imposed, as we do later in this paper (Section 7).

The symmetric monoidal structure on $\boldsymbol{LIN}$ is easy to describe on objects:

$$\boldsymbol{Grpd}_{/S} \otimes \boldsymbol{Grpd}_{/T} = \boldsymbol{Grpd}_{S\times T}$$

just as the tensor product of vector spaces with bases indexed by sets S and T is the vector spaces with basis indexed by $S \times T$. The neutral object is $\boldsymbol{Grpd}$.

We briefly review the main notions and results from Part 1, and in particular the notion of decomposition space. This notion is equivalent to that of unital 2-Segal space, introduced by Dyckerhoff and Kapranov [5]. While Dyckerhoff and Kapranov formulate the condition in terms of triangulation of convex polygons, our formulation refers to the categorical notion of generic and free maps, which we recall:

1.6. Generic and free maps. The category Δ of nonempty finite ordinals and monotone maps has a generic-free factorisation system. An arrow $a : [m] \to [n]$ in Δ is *generic* when it preserves end-points, $a(0) = 0$ and $a(m) = n$; and it is *free* if it is distance preserving, $a(i + 1) = a(i) + 1$ for $0 \leq i \leq m - 1$. The generic maps are generated by the codegeneracy maps and the inner coface maps, while the free maps are generated by the outer coface maps. Every morphism in Δ factors uniquely as a generic map followed by a free map.

The notions of generic and free maps are general notions in category theory, introduced by Weber [30, 31], who extracted the notion from earlier work of Joyal [16]; a recommended entry point to the theory is Berger–Melliès–Weber [2].

Lemma. 1.7. *Generic and free maps in Δ admit pushouts along each other, and the resulting maps are again generic and free.*

1.8. Decomposition spaces. [9] A simplicial space $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ is called a *decomposition space* when it takes generic-free pushouts in Δ to pullbacks.

Every Segal space is a decomposition space. In a Segal space X, all the information is contained in X_0 and X_1. This cannot be said about a decomposition space, but we still have the following important property.

Lemma. 1.9. *In a decomposition space X, every generic face map is a pullback of $d_1 : X_2 \to X_1$, and every degeneracy map is a pullback of $s_0 : X_0 \to X_1$.*

The notion of decomposition space can be seen as an abstraction of coalgebra: it is precisely the condition required to obtain a counital coassociative comultiplication on $\boldsymbol{Grpd}_{/X_1}$. Precisely, the following is the main theorem of [9].

Theorem 1.10. [9] *For X a decomposition space, the slice ∞-category $\boldsymbol{Grpd}_{/X_1}$ has the structure of strong homotopy comonoid in the symmetric monoidal ∞-category $\boldsymbol{LIN}$, with the comultiplication defined by the span*

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

1.11. Conservative ULF functors. The relevant notion of morphism is that of conservative ULF functor: A simplicial map is called *ULF* (unique lifting of factorisations) if it is cartesian on generic face maps, and it is called *conservative* if cartesian on degeneracy maps. We write *cULF* for conservative and ULF, that is, cartesian on all generic maps.

When X is a Segal space, thought of as a category, the notion of conservative is the classical notion, i.e. only invertible maps become invertible, and ULF is unique lifting of factorisations.

The cULF maps induce homomorphisms of the associated incidence coalgebras.

2. Complete decomposition spaces

2.1. Complete decomposition spaces. A decomposition space X is called *complete* if $s_0 : X_0 \to X_1$ is a monomorphism.

2.2. Discussion. It is clear that a Rezk complete Segal space is complete in the sense of 2.1. It makes sense also to state the Rezk completeness condition for decomposition spaces, but the condition 2.1 covers some important examples, such as the nerve of a group, which are not Rezk complete. The classical incidence algebra of the nerve of a group is the group algebra — certainly an example worth covering. We shall see that if a *tight* decomposition space is a Segal space then it is also Rezk complete (6.5).

The completeness condition is necessary to define the Phi functors (the odd and even parts of the 'Möbius functor', see 3.4) and to establish the Möbius inversion principle at the objective level (3.8). The completeness condition is also needed to make sense of the notion of length (6.1), and to define the length filtration (6.15), which is of independent interest, and is also required to be able to take cardinality of Möbius inversion.

The following basic result follows immediately from Lemma 1.9.

Lemma. 2.3. *In a complete decomposition space, all degeneracy maps are monomorphisms.*

2.4. Completeness for simplicial spaces. We shall briefly need completeness also for general simplicial spaces, and the first batch of results hold in this generality. We shall say that a simplicial space $X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$ is complete if all degeneracy maps are monomorphisms. In view of Lemma 2.3, this agrees with the previous definition when X is a decomposition space.

Until otherwise stated, let X be a complete simplicial space.

2.5. Word notation. Since $s_0 : X_0 \to X_1$ is mono, we can identify X_0 with a full subgroupoid of X_1. We denote by X_a its complement, the full subgroupoid of *nondegenerate* 1*-simplices*:

$$X_1 = X_0 + X_a.$$

We extend this notation as follows. Consider the alphabet with three letters $\{0, 1, a\}$. Here 0 is to indicate degenerate edges $s_0(x) \in X_1$, the letter a denotes edges specified to be nondegenerate, and 1 denotes edges which are not specified to be degenerate or nondegenerate. For w a word in this alphabet $\{0, 1, a\}$, of length $|w| = n$, put

$$X^w := \prod_{i \in w} X_i \subset (X_1)^n.$$

This inclusion is full since $X_a \subset X_1$ is full by completeness. Denote by X_w the ∞-groupoid of n-simplices whose principal edges have the types indicated in the word w, or more explicitly, the full subgroupoid of X_n given by the pullback diagram

$$\begin{array}{ccc} X_w & \longrightarrow & X_n \\ \downarrow & \lrcorner & \downarrow \\ X^w & \longrightarrow & (X_1)^n. \end{array} \tag{1}$$

Lemma. 2.6. *If X and Y are complete simplicial spaces and $f : Y \to X$ is conservative, then Y_a maps to X_a, and the following square is a pullback:*

$$\begin{array}{ccc} Y_1 & \longleftarrow & Y_a \\ \downarrow & \llcorner & \downarrow \\ X_1 & \longleftarrow & X_a. \end{array}$$

Proof. This square is the complement of the pullback saying what conservative means. But it is general in extensive ∞-categories such as $\boldsymbol{Grpd}$, that in the situation

$$\begin{array}{ccccc} A' & \longrightarrow & A' + B' & \longleftarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ A & \longrightarrow & A + B & \longleftarrow & B, \end{array}$$

one square is a pullback if and only if the other is. $\square$

Corollary 2.7. *If X and Y are complete simplicial spaces and $f : Y \to X$ is conservative, then for every word $w \in \{0, 1, a\}^*$, the following square is a pullback:*

(2)
$$\begin{array}{ccc} Y_n & \longleftarrow & Y_w \\ \downarrow & \lrcorner & \downarrow \\ X_n & \longleftarrow & X_w. \end{array}$$

Proof. The square is connected to

(3)
$$\begin{array}{ccc} (Y_1)^n & \longleftarrow & Y^w \\ \downarrow & \lrcorner & \downarrow \\ (X_1)^n & \longleftarrow & X^w \end{array}$$

by two instances of pullback-square (1), one for Y and one for X. It follows from 2.6 that (3) is a pullback, hence also (2) is a pullback, by Lemma 1.3. ☐

Proposition 2.8. *If X and Y are complete simplicial spaces and $f : Y \to X$ is cULF, then for any word $w \in \{0, 1, a\}^*$ the following square is a pullback:*

$$\begin{array}{ccc} Y_1 & \longleftarrow & Y_w \\ \downarrow & \lrcorner & \downarrow \\ X_1 & \longleftarrow & X_w. \end{array}$$

Proof. Just compose the square of Corollary 2.7 with the square

$$\begin{array}{ccc} Y_1 & \longleftarrow & Y_n \\ \downarrow & \lrcorner & \downarrow \\ X_1 & \longleftarrow & X_n, \end{array}$$

which is a pullback since f is cULF. ☐

Lemma. 2.9. *Let X be a complete simplicial space. Then for any words $v, v' \in \{0, 1, a\}^*$, we have*
$$X_{v1v'} = X_{v0v'} + X_{vav'},$$
and hence
$$X_n = \sum_{w\in\{0,a\}^n} X_w.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} X_{v0v'} & \longrightarrow & X_{v1v'} & \longleftarrow & X_{vav'} \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ X^{v0v'} & \longrightarrow & X^{v1v'} & \longleftarrow & X^{vav'} \end{array}$$

The two squares are pullbacks, by an application of Lemma 1.3, since horizontal composition of either with the pullback square (1) for $w = v1v'$ gives again the pullback square (1), for $w = v0v'$ or $w = vav'$.

Since the bottom row is a sum diagram, it follows that the top row is also (since the ∞-category of ∞-groupoids is extensive). ☐

We now specialise to complete decomposition spaces, although the following result will be subsumed in Section 4 on 'stiff' simplicial spaces.

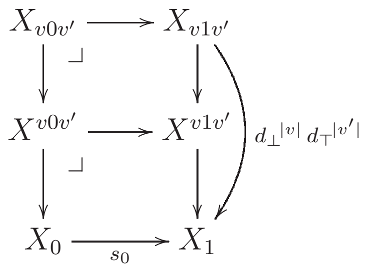
Proposition 2.10. *Let X be a complete decomposition space. Then for any words v, v' in the alphabet $\{0, 1, a\}$ we have*

$$X_{v0v'} = \mathrm{Im}(s_{|v|} : X_{vv'} \to X_{v1v'}).$$

That is, the kth principal edge of a simplex σ is degenerate if and only if $\sigma = s_{k-1}d_k\sigma$.

Recall that $|v|$ denotes the length of the word v and, as always, the notation Im refers to the essential image.

Proof. From (1) we see that (independent of the decomposition-space axiom) $X_{v0v'}$ is characterised by the top pullback square in the diagram



But the decomposition-space axiom applied to the exterior pullback diagram says that the top horizontal map is $s_{|v|}$, and hence identifies $X_{v0v'}$ with the image of $s_{|v|} : X_{vv'} \to X_{v1v'}$. For the final statement, note that if $\sigma = s_{k-1}\tau$ then $\tau = d_k\sigma$. $\square$

Combining this with Lemma 2.9 we obtain the following result.

Corollary 2.11. *Let X be a complete decomposition space. For any words v, v' in the alphabet $\{0, 1, a\}$ we have*

$$X_{v1v'} = s_{|v|}(X_{vv'}) + X_{vav'}.$$

2.12. Effective simplices. A simplex in a complete simplicial space X is called *effective* when all its principal edges are nondegenerate. We put

$$\vec{X}_n = X_{a\cdots a} \subset X_n,$$

the full subgroupoid of X_n consisting of the effective simplices. (Every 0-simplex is effective by convention: $\vec{X}_0 = X_0$.) It is clear that outer face maps $d_\perp, d_\top : X_n \to X_{n-1}$ preserve effective simplices, and that every effective simplex is nondegenerate, i.e. is not in the image of any degeneracy map. It is a useful feature of complete *decomposition* spaces that the converse is true too:

Corollary 2.13. *In a complete decomposition space X, a simplex is effective if and only if it is nondegenerate:*

$$\vec{X}_n = X_n \setminus \textstyle\bigcup_{i=0}^n \mathrm{Im}(s_i).$$

Proof. It is clear that $\vec{X}_n$ is the complement of $X_{01\cdots 1} \cup \cdots \cup X_{1\cdots 10}$ and by Proposition 2.10 we can identify each of these spaces with the image of a degeneracy map. $\square$

In fact this feature is enjoyed by a more general class of complete simplicial spaces, treated in Section 4.

Iterated use of 2.11 yields

Corollary 2.14. *For X a complete decomposition space we have*

$$X_n = \sum s_{j_k} \dots s_{j_1}(\vec{X}_{n-k}),$$

where the sum is over all subsets $\{j_1 < \cdots < j_k\}$ of $\{0, \dots, n-1\}$.

Lemma. 2.15. *If a complete decomposition space X is a Segal space, then $\vec{X}_n \simeq \vec{X}_1 \times_{X_0} \cdots \times_{X_0} \vec{X}_1$, the ∞-groupoid of strings of n composable nondegenerate arrows in $X_n \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$.*

This follows immediately from (1). Note that if furthermore X is Rezk complete, we can say non-invertible instead of nondegenerate.

3. Möbius inversion in the convolution algebra

3.1. Convolution. The ∞-category $\boldsymbol{Grpd}_{/S}$ plays the role of the vector space with basis S. Just as a linear functional is determined by its values on basis elements, linear functors $\boldsymbol{Grpd}_{/S} \to \boldsymbol{Grpd}$ correspond to arbitrary functors $S \to \boldsymbol{Grpd}$, hence the presheaf category $\boldsymbol{Grpd}^S$ can be considered the linear dual of the slice category $\boldsymbol{Grpd}_{/S}$ (see [11] for the precise statements and proofs).

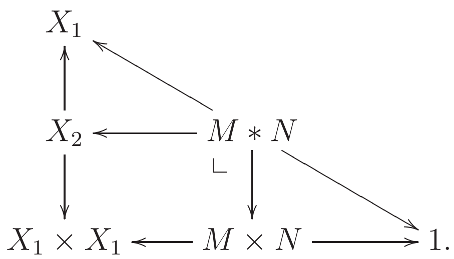
If X is a decomposition space, the coalgebra structure on $\boldsymbol{Grpd}_{/X_1}$ therefore induces an algebra structure on $\boldsymbol{Grpd}^{X_1}$. The convolution product of two linear functors

$$F, G : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd},$$

given by spans $X_1 \leftarrow M \rightarrow 1$ and $X_1 \leftarrow N \rightarrow 1$, is the composite of their tensor product $F \otimes G$ and the comultiplication,

$$F * G : \quad \boldsymbol{Grpd}_{/X_1} \xrightarrow{\Delta} \boldsymbol{Grpd}_{/X_1} \otimes \boldsymbol{Grpd}_{/X_1} \xrightarrow{F \otimes G} \boldsymbol{Grpd}.$$

Thus the convolution is given by the composite of spans



The neutral element for convolution is $\varepsilon : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$ defined by the span

$$X_1 \xleftarrow{s_0} X_0 \rightarrow 1\,.$$

3.2. The zeta functor. The *zeta functor*

$$\zeta : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}$$

is the linear functor defined by the span

$$X_1 \xleftarrow{=} X_1 \rightarrow 1\,.$$

We will see later in the locally finite situation (see Section 7.4) that on taking the homotopy cardinality of the zeta functor one obtains the constant function 1 on $\pi_0 X_1$, that is, the classical zeta function in the incidence algebra.

It is clear from the definition of the convolution product that the kth convolution power of the zeta functor is given by

$$\zeta^k : \ X_1 \xleftarrow{g} X_k \rightarrow 1,$$

where $g : [1] \to [k]$ is the unique generic map in degree k.

Consider also the elements δ^a and h^a of the incidence algebra given by the spans

$$\delta^a : \; X_1 \leftarrow (X_1)_{[a]} \to 1, \qquad h^a : \; X_1 \xleftarrow{\ulcorner a \urcorner} 1 \to 1$$

where $(X_1)_{[a]}$ denotes the component of X_1 containing $a \in X_1$. Then zeta is the sum of the elements δ^a, or the homotopy sum of h^a

$$\zeta \;=\; \sum_{a \in \pi_0 X_1} \delta^a \;=\; \int^a h^a.$$

3.3. The idea of Möbius inversion à la Leroux. We are interested in the invertibility of the zeta functor under the convolution product. Unfortunately, at the objective level it can practically *never* be convolution invertible, because the inverse μ should always be given by an alternating sum (cf. 3.8)

$$\mu = \Phi_{\text{even}} - \Phi_{\text{odd}}$$

(of the Phi functors defined below). We have no minus sign available, but following the idea of Content–Lemay–Leroux [3], developed further by Lawvere–Menni [21], we establish the sign-free equations

$$\zeta * \Phi_{\text{even}} = \varepsilon + \zeta * \Phi_{\text{odd}}, \qquad \Phi_{\text{even}} * \zeta = \varepsilon + \Phi_{\text{odd}} * \zeta.$$

In the category case (cf. [3] and [21]), Φ_{even} (resp. Φ_{odd}) are given by even-length (resp. odd-length) chains of non-identity arrows. (We keep the Φ-notation in honour of Content–Lemay–Leroux). In the general setting of decomposition spaces we cannot talk about chains of arrows, but in the complete case we can still talk about effective simplices and their principal edges.

From now on we assume again that X is complete decomposition space.

3.4. 'Phi' functors. We define Φ_n to be the linear functor given by the span

$$X_1 \longleftarrow \vec{X}_n \longrightarrow 1.$$

If $n = 0$ then $\vec{X}_0 = X_0$ by convention, and Φ_0 is given by the span

$$X_1 \longleftarrow X_0 \longrightarrow 1.$$

That is, Φ_0 is the linear functor ε. Note that $\Phi_1 = \zeta - \varepsilon$. The minus sign makes sense here, since X_0 (representing ε) is really a full subgroupoid of X_1 (representing ζ).

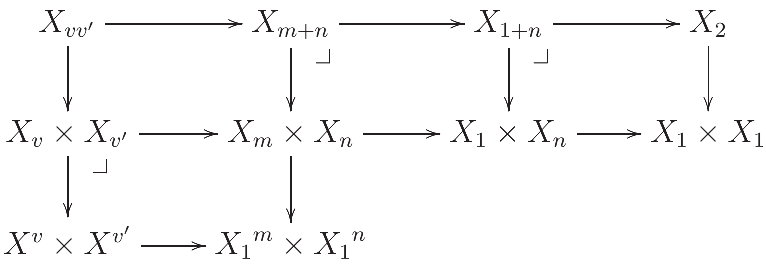
To compute convolution with Φ_n, a key ingredient is the following general lemma (with reference to the word notation of 2.5).

Lemma. 3.5. *Let X be a complete decomposition space. Then for any words v, v' in the alphabet $\{0, 1, a\}$, the square*

$$\begin{array}{ccc} X_{vv'} & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_v \times X_{v'} & \longrightarrow & X_1 \times X_1 \end{array}$$

is a pullback.

Proof. Let $m = |v|$ and $n = |v'|$. The square is the outer rectangle in the top row of the diagram



The left-hand outer rectangle is a pullback by definition of $X_{vv'}$, and the bottom square is a pullback by definition of X_v and $X_{v'}$. Hence the top-left square is a pullback. But the other squares in the top row are pullbacks because X is a decomposition space. ☐

Lemma. 3.6. *We have*

$$\Phi_n \quad = \quad (\Phi_1)^n \quad = \quad (\zeta - \varepsilon)^n,$$

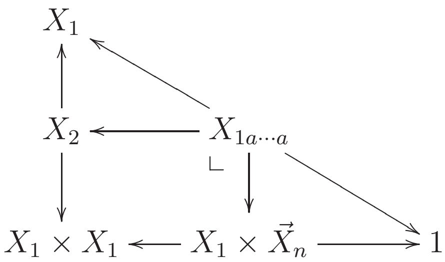
the nth convolution product of Φ_1 *with itself.*

Proof. This follows from the definitions and Lemma 3.5. ☐

Proposition 3.7. *The linear functors* Φ_n *satisfy*

$$\zeta * \Phi_n \quad = \quad \Phi_n + \Phi_{n+1} \quad = \quad \Phi_n * \zeta.$$

Proof. We can compute the convolution $\zeta * \Phi_n$ by Lemma 3.5 as



But Lemma 2.9 tells us that $X_{1a\cdots a} = X_{0a\cdots a} + X_{aa\cdots a} = \vec{X}_n + \vec{X}_{n+1}$, where the identification in the first summand is via s_0, in virtue of Proposition 2.10. This is an equivalence of ∞-groupoids over X_1 so the resulting span is $\Phi_n + \Phi_{n+1}$ as desired. The second identity claimed follows similarly. ☐

Put

$$\Phi_{\text{even}} := \sum_{n \text{ even}} \Phi_n, \qquad \Phi_{\text{odd}} := \sum_{n \text{ odd}} \Phi_n.$$

Theorem 3.8. *For a complete decomposition space, the following Möbius inversion principle holds:*

$$\begin{aligned} \zeta * \Phi_{\text{even}} \quad &= \quad \varepsilon \quad + \quad \zeta * \Phi_{\text{odd}}, \\ = \quad \Phi_{\text{even}} * \zeta \quad &= \quad \varepsilon \quad + \quad \Phi_{\text{odd}} * \zeta. \end{aligned}$$

Proof. This follows immediately from the proposition: all four linear functors are in fact equivalent to $\sum_{r \geq 0} \Phi_r$. ☐

We note the following immediate corollary of Proposition 2.8, which can be read as saying 'Möbius inversion is preserved by cULF functors':

Corollary 3.9. *If* $f : Y \to X$ *is cULF, then* $f^*\zeta = \zeta$ *and* $f^*\Phi_n = \Phi_n$ *for all* $n \geq 0$.

4. Stiff simplicial spaces

We saw that in a complete decomposition space, degeneracy can be detected on principal edges. In Section 5 we shall come to split simplicial spaces, which share this property. A common generalisation is that of stiff simplicial spaces, which we now introduce.

4.1. Stiffness. A simplicial space $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$ is called *stiff* if it sends degeneracy/free pushouts in Δ to pullbacks in $\boldsymbol{Grpd}$. These pushouts are examples of generic-free pushouts, so in particular every decomposition space is stiff.

Lemma. 4.2. *A simplicial space X is stiff if and only if the following diagrams are pullbacks for all $0 \leq i \leq n$*

$$\begin{array}{ccc} X_n & \xrightarrow{s_i} & X_{n+1} \\ \downarrow & \lrcorner & \downarrow {\scriptstyle d_\bot{}^i\, d_\top{}^{n-i}} \\ X_0 & \xrightarrow[s_0]{} & X_1 \end{array}$$

Proof. The squares in the lemma are special cases of the degeneracy/free squares. On the other hand, every degeneracy/free square sits in between two of the squares of the lemma in such a way that Lemma 1.3 forces it to be a pullback too. ☐

The following four results for stiff simplicial spaces are proved in the same way as for decomposition spaces, cf. [9, Lemmas 3.7, 3.8, 3.9, 4.3].

Lemma. 4.3. *For a stiff simplicial space X, the following squares are pullbacks:*

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_j} & X_n \\ {\scriptstyle s_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+2} & \xrightarrow[d_{j+1}]{} & X_{n+1} \end{array} \quad \text{for all } i < j, \quad \text{and} \quad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_j} & X_n \\ {\scriptstyle s_{i+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+2} & \xrightarrow[d_j]{} & X_{n+1} \end{array} \quad \text{for all } j \leq i.$$

Lemma. 4.4. *For a stiff simplicial space X, the following squares are pullbacks for all $i < j$:*

$$\begin{array}{ccc} X_n & \xrightarrow{s_{j-1}} & X_{n+1} \\ {\scriptstyle s_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+1} & \xrightarrow[s_j]{} & X_{n+2}. \end{array}$$

Lemma. 4.5. *In a stiff simplicial space X, every degeneracy map is a pullback of $s_0 : X_0 \to X_1$. In particular, if just $s_0 : X_0 \to X_1$ is mono then all degeneracy maps are mono.*

Lemma. 4.6. *A simplicial map $f : Y \to X$ between stiff simplicial spaces is conservative if and only if it is cartesian on the first degeneracy map*

$$\begin{array}{ccc} Y_0 & \xrightarrow{s_0} & Y_1 \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \xrightarrow[s_0]{} & X_1. \end{array}$$

Lemma. 4.7. *A stiff simplicial space X is complete if and only if the canonical map from the constant simplicial space X_0 is conservative.*

Proof. Suppose X is complete. Then any $s_i : X_k \to X_{k+1}$ is mono, and hence in the following diagram the bottom square is a pullback:

$$\begin{array}{ccc} X_0 & \xrightarrow{=} & X_0 \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ X_k & \xrightarrow{=} & X_k \\ {\scriptstyle =}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_k & \xrightarrow[s_i]{} & X_{k+1}. \end{array}$$

Hence $X_0 \to X$ is cartesian on s_i. Since this is true for any degeneracy map s_i, altogether $X_0 \to X$ is conservative. Conversely, if $X_0 \to X$ is conservative, then in particular we have the pullback square

$$\begin{array}{ccc} X_0 & \xrightarrow{=} & X_0 \\ {\scriptstyle =}\downarrow & \lrcorner & \downarrow{\scriptstyle s_0} \\ X_0 & \xrightarrow[s_0]{} & X_1 \end{array}$$

which means that $s_0 : X_0 \to X_1$ is a monomorphism. □

For complete simplicial spaces, we can characterise stiffness also in terms of degeneracy:

Proposition 4.8. *The following are equivalent for a complete simplicial space X*

(1) *X is stiff.*
(2) *Outer face maps $d_\bot, d_\top : X_n \to X_{n-1}$ preserve nondegenerate simplices.*
(3) *Any nondegenerate simplex is effective. More precisely,*

$$\vec{X}_n = X_n \setminus \bigcup_{i=0}^n \mathrm{Im}(s_{i-1}).$$

(4) *If the ith principal edge of $\sigma \in X_n$ is degenerate, then $\sigma = s_{i-1}d_{i-1}\sigma = s_{i-1}d_i\sigma$, that is*

$$X_{1\ldots101\ldots1} = \mathrm{Im}(s_{i-1} : X_{n-1} \to X_n)$$

(5) *For each word $w \in \{0, a\}^n$ we have*

$$X_w = \mathrm{Im}(s_{j_k-1} \ldots s_{j_1-1} : \vec{X}_{n-k} \to X_n).$$

where $\{j_1 < \cdots < j_k\} = \{j : w_j = 0\}$.

Proof. (1) ⇒ (2): Suppose $\sigma \in X_n$ and that $d_\top\sigma$ is degenerate. Then $d_\top\sigma$ is in the image of some $s_i : X_{n-2} \to X_{n-1}$, and hence by (1) already σ is in the image of $s_i : X_{n-1} \to X_n$.

(2) ⇒ (3): The principal edges of a simplex are obtained by applying outer face maps, so nondegenerate simplices are also effective. For the more precise statement, just note that both subspaces are full, so are determined by the properties characterising their objects.

(3) ⇒ (4): As σ is not effective, we have $\sigma = s_j\tau$. If $j > i-1$ then the ith principal edge is of σ is also that of τ, so by induction $\tau \in \mathrm{Im}(s_{i-1})$. Therefore $\sigma \in \mathrm{Im}(s_{i-1})$ also, and $\sigma = s_{i-1}d_{i-1}\sigma = s_{i-1}d_i\sigma$ as required. If $j < i-1$ the argument is similar.

(4) ⇔ (1): To show that X is stiff, by Lemma 4.2 it is enough to check that this is a pullback:

$$\begin{array}{ccc} X_n & \xrightarrow{s_i} & X_{n+1} \\ \downarrow & & \downarrow {\scriptstyle d_\bot{}^i\, d_\top{}^{n-i}} \\ X_0 & \xrightarrow[s_0]{} & X_1 \end{array}$$

But the pullback is by definition $X_{1\cdots 101\cdots 1} \subset X_{n+1}$. But by assumption this is canonically identified with the image of $s_i : X_n \to X_{n+1}$, establishing the required pullback.

(4) ⇔ (5): This is clear, using Lemma 2.9. □

In summary, an important feature of stiff complete simplicial spaces is that all information about degeneracy is encoded in the principal edges. We exploit this to characterise conservative maps between stiff complete simplicial spaces:

Proposition 4.9. *For X and Y stiff complete simplicial spaces, and $f : Y \to X$ a simplicial map, the following are equivalent.*

1. *f is conservative.*
2. *f preserves the word splitting, i.e. for every word $w \in \{0, a\}^*$, f sends Y_w to X_w.*
3. *f_1 maps Y_a to X_a.*

Proof. We already saw (2.7) that conservative maps preserve the word splitting (independently of X and Y being stiff), which proves (1) ⇒ (2). The implication (2) ⇒ (3) is trivial. Finally assume that f_1 maps Y_a to X_a. To check that f is conservative, it is enough (by 4.6) to check that the square

$$\begin{array}{ccc} Y_0 & \xrightarrow{s_0} & Y_1 \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[s_0]{} & X_1 \end{array}$$

is a pullback. But since X and Y are complete, this square is just

$$\begin{array}{ccc} Y_0 & \xrightarrow{s_0} & Y_0 + Y_a \\ \downarrow & & \downarrow \\ X_0 & \xrightarrow[s_0]{} & X_0 + X_a, \end{array}$$

which is clearly a pullback when f_1 maps Y_a to X_a. □

This proposition can be stated more formally as follows. For X and Y stiff complete simplicial spaces, the space of conservative maps $\mathrm{Cons}(Y, X)$ is given as the pullback

$$\begin{array}{ccc} \mathrm{Cons}(Y,X) & \longrightarrow & \prod_{n\in\mathbb{N}} \prod_{w\in\{0,a\}^n} \mathrm{Map}(Y_w, X_w) \\ \downarrow & & \downarrow \\ \mathrm{Nat}(Y,X) & \longrightarrow & \prod_{n\in\mathbb{N}} \mathrm{Map}(Y_n, X_n). \end{array}$$

The vertical arrow on the right is given as follows. We have

$$\mathrm{Map}(Y_n, X_n) \;=\; \mathrm{Map}\Big(\sum_{w\in\{0,a\}^n} Y_w, \sum_{v\in\{0,a\}^n} X_v\Big) \;=\; \prod_{w\in\{0,a\}^n} \mathrm{Map}\Big(Y_w, \sum_{v\in\{0,a\}^n} X_v\Big).$$

For fixed $w \in \{0,a\}^n$, the space $\mathrm{Map}(Y_w, \sum_{v\in\{0,a\}^n} X_v)$ has a distinguished subobject, namely consisting of those maps that map into X_w for that same word w.

5. Split decomposition spaces

5.1. Split simplicial spaces. In a complete simplicial space X, by definition all degeneracy maps are monomorphisms, so in particular it makes sense to talk about nondegenerate simplices in degree n: these form the full subgroupoid of X_n given as the complement of the degeneracy maps $s_i : X_{n-1} \to X_n$. A complete simplicial space is called *split* if the face maps preserve nondegenerate simplices.

By Proposition 4.8, a split simplicial space is stiff, so the results from the previous section are available for split simplicial spaces. In particular, nondegeneracy can be measured on principal edges, and we have

Corollary 5.2. *If X is a split simplicial space, then the sum splitting*

$$X_n \;=\; \sum_{w\in\{0,a\}^n} X_w$$

is realised by the degeneracy maps.

5.3. Non-example. The strict nerve of any category with a non-trivial section-retraction pair of arrows, $r \circ s = \mathrm{id}$, constitutes an example of a complete decomposition space which is not split. Indeed, the nondegenerate simplices are the chains of composable non-identity arrows, but we have $d_1(s,r) = \mathrm{id}$.

In this way, splitness can be seen as an abstraction of the condition on a 1-category that its identity arrows be indecomposable. (Corollary 6.7 below will generalise the classical fact that in a Möbius category, the identity arrows are indecomposable [23].)

5.4. Semi-decomposition spaces. Let $\Delta_{\mathrm{inj}} \subset \Delta$ denote the subcategory consisting of all the objects and only the injective maps. A *semi-simplicial* space is an object in the functor ∞-category $\mathrm{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \boldsymbol{Grpd})$. A *semi-decomposition* space is a semi-simplicial space preserving generic-free pullbacks in $\Delta_{\mathrm{inj}}^{\mathrm{op}}$. Since there are no degeneracy maps in Δ_{inj}, this means that we are concerned only with pullbacks between generic face maps and free face maps.

Every simplicial space has an underlying semi-simplicial space obtained by restriction along $\Delta_{\mathrm{inj}} \subset \Delta$. The forgetful functor $\mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd}) \to \mathrm{Fun}(\Delta_{\mathrm{inj}}^{\mathrm{op}}, \boldsymbol{Grpd})$ has a left adjoint given by left Kan extension along $\Delta_{\mathrm{inj}} \subset \Delta$:

$$\begin{array}{ccc} \Delta_{\mathrm{inj}}^{\mathrm{op}} & \xrightarrow{Z} & \boldsymbol{Grpd} \\ \downarrow & \nearrow_{\overline{Z}} & \\ \Delta^{\mathrm{op}} & & \end{array}$$

The left Kan extension has the following explicit description:

$$\begin{aligned}
\overline{Z}_0 =& Z_0 \\
\overline{Z}_1 =& Z_1 + Z_0 \\
\overline{Z}_2 =& Z_2 + Z_1 + Z_1 + Z_0 \\
&\vdots \\
\overline{Z}_k =& \sum_{w \in \{0,a\}^k} Z_{|w|_a}
\end{aligned}$$

For $w \in \{0,a\}^k$ and $\sigma \in Z_{|w|_a}$ the corresponding element of $\overline{Z}_k$ is denoted

$$s_{i_r} \dots s_{i_2} s_{i_1} \sigma$$

where $r = k - |w|_a$ and $i_1 < i_2 < \cdots < i_r$ with $w_{i_j} = 0$. The faces and degeneracies of such elements are defined in the obvious way.

Proposition 5.5. *A a simplicial space is split if and only if it is the left Kan extension of a semi-simplicial space.*

Proof. Given $Z : \Delta_{\text{inj}}^{\text{op}} \to \boldsymbol{Grpd}$, it is clear from the construction that the new degeneracy maps in $\overline{Z}$ are monomorphisms. Hence $\overline{Z}$ is complete. On the other hand, to say that $\sigma \in \overline{Z}_n$ is nondegenerate is precisely to say that it belongs to the original component Z_n, and the face maps here are the original face maps, hence map σ into Z_{n-1} which is precisely the nondegenerate component of $\overline{Z}_{n-1}$. Hence $\overline{Z}$ is split.

For the other implication, given a split simplicial space X, we know that nondegenerate is the same thing as effective (4.8), so we have a sum splitting

$$X_n \;\; = \sum_{w \in \{0,a\}^n} X_w .$$

Now by assumption the face maps restrict to the nondegenerate simplices to give a semi-simplicial space $\vec{X} : \Delta_{\text{inj}}^{\text{op}} \to \boldsymbol{Grpd}$. It is now clear from the explicit description of the left Kan extension that $\overline{(\vec{X}_n)} = X_n$, from where it follows readily that X is the left Kan extension of $\vec{X}$. □

Proposition 5.6. *A simplicial space is a split decomposition space if and only if it is the left Kan extension of a semi-decomposition space.*

Proof. It is clear that if X is a split decomposition space then $\vec{X}$ is a semi-decomposition space. Conversely, if Z is a semi-decomposition space, then one can check by inspection that $\overline{Z}$ satisfies the four pullback conditions in [9, Proposition 3.3]: two of these diagrams concern only face maps, and they are essentially from Z, with degenerate stuff added. The two diagrams involving degeneracy maps are easily seen to be pullbacks since the degeneracy maps are sum inclusions. □

Theorem 5.7. *The left adjoint from before,* $\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \boldsymbol{Grpd}) \to \text{Fun}(\Delta^{\text{op}}, \boldsymbol{Grpd})$, *induces an equivalence of ∞-categories*

$$\text{Fun}(\Delta_{\text{inj}}^{\text{op}}, \boldsymbol{Grpd}) \simeq \boldsymbol{Split}^{cons},$$

the ∞-category of split simplicial spaces and conservative maps.

Proof. Let X and Y be split simplicial spaces, then $\vec{X}$ and $\vec{Y}$ are semi-simplicial spaces whose left Kan extensions are X and Y again. The claim is that

$$\mathrm{Cons}(Y, X) \simeq \mathrm{Nat}(\vec{Y}, \vec{X}).$$

Intuitively, the reason this is true can be seen in the first square as in the proof of Lemma 4.9: to give a pullback square

$$\begin{array}{ccc} Y_0 & \xrightarrow{s_0} & Y_0 + Y_a \\ \downarrow & \lrcorner & \downarrow \\ X_0 & \xrightarrow[s_0]{} & X_0 + X_a, \end{array}$$

amounts to giving $Y_0 \to X_0$ and $Y_a \to X_a$ (and of course, in both cases this data is required to be natural in face maps), that is to give a natural transformation $\vec{Y} \to \vec{X}$. To formalise this idea, note first that $\mathrm{Nat}(\vec{Y}, \vec{X})$ can be described as a limit

$$\mathrm{Nat}(\vec{Y}, \vec{X}) \longrightarrow \prod_{n \in \mathbb{N}} \mathrm{Map}(\vec{Y}_n, \vec{X}_n) \to \dots$$

where the rest of the diagram contains vertices indexed by all the face maps, expressing naturality. Similarly $\mathrm{Nat}(Y, X)$ is given as a limit

$$\mathrm{Nat}(Y, X) \longrightarrow \prod_{n \in \mathbb{N}} \mathrm{Map}(Y_n, X_n) \to \dots$$

where this time the rest of the diagram furthermore contains vertices corresponding to degeneracy maps. The full subspace of conservative maps is given instead as

$$\mathrm{Cons}(Y, X) \longrightarrow \prod_{w \in \{0,a\}*} \mathrm{Map}(Y_w, X_w) \to \dots$$

as explained in connection with Lemma 4.9. Now for each degeneracy map $s_i : X_n \to X_{n+1}$, there is a vertex in the diagram. For ease of notation, let us consider $s_0 : X_n \to X_{n+1}$. The corresponding vertex sits in the limit diagram as follows: for each word $v \in \{0, a\}^n$, we have

$$\begin{array}{ccc} \prod_{w \in \{0,a\}*} \mathrm{Map}(Y_w, X_w) & \xrightarrow{\text{proj}} & \mathrm{Map}(Y_{0v}, X_{0v}) \\ {\scriptstyle \text{proj}}\downarrow & & \downarrow{\scriptstyle \text{pre } s_0} \\ \mathrm{Map}(Y_v, X_v) & \xrightarrow[\text{post } s_0]{} & \mathrm{Map}(Y_n, X_{n+1}). \end{array}$$

Now both the pre and post composition maps are monomorphisms with essential image $\mathrm{Map}(Y_v, X_{0v})$, so the two projections coincide, which is to say that the limit factors through the corresponding diagonal. Applying this argument for every degeneracy map $s_i : X_n \to X_{n+1}$, and for all words, we conclude that the limit factors through the product indexed only over the words without degeneracies,

$$\prod_{n \in \mathbb{N}} \mathrm{Map}(\vec{Y}_n, \vec{X}_n).$$

Having thus eliminated all the vertices of the limit diagram that corresponded to degeneracy maps, the remaining diagram has precisely the shape of the diagram computing $\mathrm{Nat}(\vec{Y}, \vec{X})$, and we have already seen that the 'starting vertex' is the same, $\prod_{n \in \mathbb{N}} \mathrm{Map}(\vec{Y}_n, \vec{X}_n)$. For the remaining vertices, those corresponding to face maps, it is readily seen that in each case the space is that of the $\mathrm{Nat}(\vec{Y},\vec{X})$ diagram, modulo some constant factors that do not play any role in the limit calculation. In conclusion, the diagram calculating $\mathrm{Cons}(Y,X)$ as a limit is naturally identified with the diagram calculating $\mathrm{Nat}(\vec{Y},\vec{X})$ as a limit. □

Proposition 5.8. *This equivalence restricts to an equivalence between semi-decomposition spaces and all maps and split decomposition spaces and conservative maps, and it restricts further to an equivalence between semi-decomposition spaces and ULF maps and split decomposition spaces and cULF maps.*

5.9. Dyckerhoff–Kapranov 2-Segal semi-simplicial spaces. Dyckerhoff and Kapranov's notion of 2-Segal space [5] does not refer to degeneracy maps at all, and can be formulated already for semi-simplicial spaces: a 2-Segal space is precisely a simplicial space whose underlying semi-simplicial space is a semi-decomposition space. We get the following corollary to the results above.

Corollary 5.10. *Every split decomposition space is the left Kan extension of a 2-Segal semi-simplicial space.*

6. The length filtration

The *long edge* of a simplex $\sigma \in X_n$ in a simplicial space is the element $g(\sigma) \in X_1$, where $g : X_n \to X_1$ is the unique generic map.

6.1. Length. Let $a \in X_1$ be an edge in a complete decomposition space X. The *length* of a is defined to be the biggest dimension of an effective simplex with long edge a:

$$\ell(a) := \sup\{\dim \sigma \mid \sigma \in \vec{X}, g(\sigma) = a\},$$

where as usual $g : X_r \to X_1$ denotes the unique generic map. More formally: the length is the greatest r such that the pullback

$$\begin{array}{ccc} (\vec{X}_r)_a & \longrightarrow & \vec{X}_r \\ \big\downarrow & \lrcorner & \big\downarrow g \\ 1 & \xrightarrow[\ulcorner a \urcorner]{} & X_1 \end{array}$$

is nonempty (or ∞ if there is no such greatest r). Length zero can happen only for degenerate edges.

6.2. Decomposition spaces of locally finite length. A complete decomposition space X is said to have *locally finite length* when every edge $a \in X_1$ has finite length. That is, the pullback

$$\begin{array}{ccc} (\vec{X}_r)_a & \longrightarrow & \vec{X}_r \\ \big\downarrow & \lrcorner & \big\downarrow g \\ 1 & \xrightarrow[\ulcorner a \urcorner]{} & X_1 \end{array}$$

is empty for $r \gg 0$. We shall also use the word *tight* as synonym for 'of locally finite length', to avoid confusion with the notion of 'locally finite' introduced in Section 7.

Example 6.3. For posets, the notion of locally finite length coincides with the classical notion (see for example Stern [29]), namely that for every $x \leq y$, there is an upper bound on the possible lengths of chains from x to y. When X is the strict (resp. fat) nerve of a category, locally finite length means that for each arrow a, there is an upper bound on the length of factorisations of a containing no identity (resp. invertible) arrows.

A paradigmatic non-example is given by the strict nerve of a category containing an idempotent non-identity endo-arrow, $e = e \circ e$: clearly e admits arbitrarily long decompositions $e = e \circ \cdots \circ e$.

Proposition 6.4. *If $f : Y \to X$ is cULF and X is a tight decomposition space, then also Y is tight.*

Proof. Since X is a decomposition space and since f is cULF, also Y is a decompositions space ([9, Lemma 4.6]), and the cULF condition ensures that Y is furthermore complete, because the s_0 of Y is the pullback of the s_0 of X. Finally, Y is also tight by Proposition 2.8. □

Proposition 6.5. *If a tight decomposition space X is a Segal space, then it is Rezk complete.*

Proof. If X is not Rezk complete, then there exists a nondegenerate invertible arrow $a \in X_1$. Since for Segal spaces we have

$$\vec{X}_n \simeq \vec{X}_1 \times_{X_0} \cdots \times_{X_0} \vec{X}_1$$

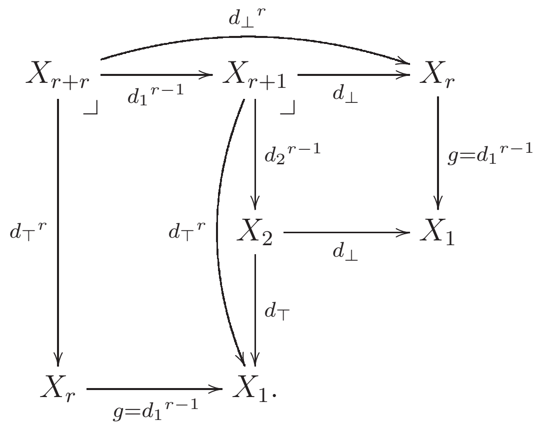
(by 2.15), we can use the arrow $a : x \to y$ and its inverse to go back and forth any number of times to create nondegenerate simplices of any length (subdivisions of id_x or id_y). □

Lemma. 6.6. *Let X be a tight decomposition space. Then for every $r \geq 1$ we have a pullback square*

$$\begin{array}{ccc} \varnothing & \longrightarrow & \vec{X}_r \\ \downarrow & \lrcorner & \downarrow g \\ X_0 & \xrightarrow[s_0]{} & X_1. \end{array}$$

More generally an effective simplex has all of its 1-dimensional faces non-degenerate, so all faces of an effective simplex are effective.

Proof. For $r = 1$ the first statement is simply that $s_0 X_0$ and $\vec{X}_1$ are disjoint in X_1, which is true by construction, so we can assume $r \geq 2$. Suppose that $\sigma \in \vec{X}_r$ has degenerate long edge $u = g\sigma$. The idea is to exploit the decomposition-space axiom to glue together two copies of σ, called σ_1 and σ_2, to get a bigger simplex $\sigma_1 \# \sigma_2 \in \vec{X}_{r+r}$ again with long edge u. By repeating this construction we obtain a contradiction to the finite length of u. It is essential for this construction that u is degenerate, say $u = s_0 x$, because we glue along the 2-simplex $\tau = s_0 u = s_1 u = s_0 s_0 x$ which has the property that all three edges are u. Precisely, consider the diagram



The two squares are pullbacks since X is a decomposition space, and the triangles are simplicial identities. In the right-hand square we have $\sigma_2 \in X_r$ and $\tau \in X_2$, with $g\sigma_2 = u = d_\bot\tau$. Hence we get a simplex $\rho \in X_{r+1}$. This simplex has $d_\top{}^r\rho = d_\top\tau = u$, which means that in the left-hand square it matches $\sigma_1 \in X_r$, to produce altogether the desired simplex $\sigma_1\#\sigma_2 \in X_{r+r}$. By construction, this simplex belongs to $\vec{X}_{r+r}$: indeed, its first r principal edges are the principal edges of σ_1, and its last r principal edges are those of σ_2. Its long edge is clearly the long edge of τ, namely u again, so we have produced a longer decomposition of u than the one given by σ, thus contradicting the finite length of u.

Now the final statement follows since any 1-dimensional face of an effective simplex σ is the long edge of an effective simplex $d_\bot{}^i d_\top{}^j\sigma$. □

Corollary 6.7. *A tight decomposition space is split.*

For the next couple of corollaries, we shall need the following general lemma.

Lemma. 6.8. *Suppose X is a complete decomposition space and $\sigma \in X_n$ has at least $n-1$ of its principal edges degenerate. Then the following are equivalent:*

(1) *the long edge $g(\sigma)$ is degenerate,*
(2) *all principal edges of σ are degenerate,*
(3) *σ is totally degenerate, $\sigma \in s_0{}^n(X_0)$.*

Proof. Proposition 2.10 says that (2) and (3) are equivalent. Moreover it says that if all principal edges of σ except the jth are known to be degenerate then σ is an $(n-1)$-fold degeneracy of its jth principal edge. Therefore the long edge of σ is equal to its jth principal edge, and so (1) and (2) are equivalent. □

Corollary 6.9. *For any $\sigma \in X_2$ in a tight decomposition space X, we have that $d_1\sigma$ is degenerate if and only if both $d_0\sigma$ and $d_2\sigma$ are degenerate.*

Proof. By Lemma 6.6, if $d_1\sigma$ is degenerate then at least one of the two principal edges is degenerate. The result now follows from 6.8. □

Corollary 6.10. *In a tight decomposition space, if the long edge of a simplex is degenerate then all its edges are degenerate, and indeed the simplex is totally degenerate.*

Proof. Let σ be an n-simplex of a decomposition space X and consider the 2-dimensional faces τ_j of σ defined by the vertices $j-1 < j < n$. Applying Corollary 6.9 to each τ_j, $j = 1\ldots,n-1$, shows that all principal edges of σ are degenerate. Lemma 6.8 then says that σ is in the image of $s_0{}^n$. □

We can now give alternative characterisations of the length of an arrow in a tight decomposition space:

Proposition 6.11. *Let X be a tight decomposition space, and $f \in X_1$. Then the following conditions on $r \in \mathbb{N}$ are equivalent:*

1. *For all words w in the alphabet $\{0, a\}$ in which the letter a occurs at least $r+1$ times, the fibre $(X_w)_f$ is empty,*

$$\begin{array}{ccc} \varnothing & \longrightarrow & X_w \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow[\ulcorner f \urcorner]{} & X_1. \end{array}$$

2. *For all $k \geq r+1$, the fibre $(\vec{X}_k)_f$ is empty.*
3. *The fibre $(\vec{X}_{r+1})_f$ is empty.*

The length $\ell(f)$ of an arrow in a tight decomposition space is the least $r \in \mathbb{N}$ satisfying these equivalent conditions.

Proof. Clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and, by definition, the length of f is the least integer r satisfying (2). It remains to show that (3) implies (1). Suppose (1) is false, that is, we have $w \in \{0, a\}^n$ with $k \geq r+1$ occurrences of a and an element $\sigma \in X_w$ with $g(\sigma) = f$. Then by Corollary 2.14 we know that σ is an $(n-k)$-fold degeneracy of some $\tau \in \vec{X}_k$, and σ and τ will have the same long edge f. Finally we see that (3) is false by considering the element ${d_1}^{k-r-1}\tau \in X_{r+1}$, which has long edge f, and is effective by Lemma 6.6. □

6.12. The length filtration of the space of 1-simplices. Let X be a tight decomposition space. We define the kth stage of the *length filtration* for 1-simplices to consist of all the arrows of length at most k:

$$X_1^{(k)} := \{a \in X_1 \mid \ell(a) \leq k\}.$$

Corollary 6.13. *For a tight decomposition space X we have $X_1^{(0)} = X_0$.* □

Then $X_1^{(k)}$ is the full subgroupoid of X_1 given by any of the following equivalent definitions:

1. the complement of $\mathrm{Im}(\vec{X}_{k+1} \to X_1)$.
2. the complement of $\mathrm{Im}(\coprod_{|w|_a > k} X_w \to X_1)$.
3. the full subgroupoid of X_1 whose objects f satisfy $(X_{k+1})_f \subset \bigcup s_i X_k$
4. the full subgroupoid of X_1 whose objects f satisfy $(\vec{X}_{k+1})_f = \varnothing$
5. the full subgroupoid of X_1 whose objects f satisfy $(X_w)_f = \varnothing$ for all $w \in \{0, a\}^r$ such that $|w|_a > k$

It is clear from the definition of length that we have a sequence of monomorphisms

$$X_1^{(0)} \hookrightarrow X_1^{(1)} \hookrightarrow X_1^{(2)} \hookrightarrow \ldots \hookrightarrow X_1.$$

The following is now clear.

Proposition 6.14. *A complete decomposition space is tight if and only if the $X_1^{(k)}$ constitute a filtration, i.e.*

$$X_1 = \bigcup_{k=0}^{\infty} X_1^{(k)}.$$

6.15. Length filtration of a tight decomposition space. Now define the length filtration for all of X: the length of a simplex σ with longest edge $g\sigma = a$ is defined to be the length of a:

$$\ell(\sigma) := \ell(a).$$

In other words, we are defining the filtration in X_r by pulling it back from X_1 along the unique generic map $X_r \to X_1$. This automatically defines the generic maps in each filtration degree, yielding a generic-map complex

$$X_\bullet^{(k)} : \Delta_{\text{gen}}^{\text{op}} \to \boldsymbol{Grpd}.$$

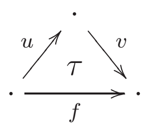
To get the outer face maps, the idea is simply to restrict (since by construction all the maps $X_1^{(k)} \hookrightarrow X_1^{(k+1)}$ are monos). We need to check that an outer face map applied to a simplex in $X_n^{(k)}$ again belongs to $X_{n-1}^{(k)}$. This will be the content of Proposition 6.16 below. Once we have done that, it is clear that we have a sequence of cULF maps

$$X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)} \hookrightarrow \cdots \hookrightarrow X$$

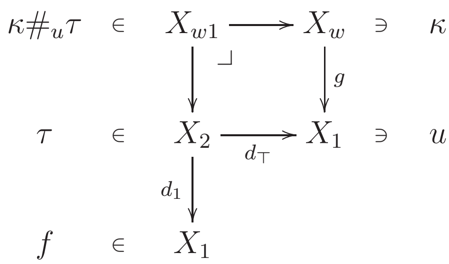
and we shall see that $X_\bullet^{(0)}$ is the constant simplicial space X_0.

Proposition 6.16. *In a tight decomposition space X, face maps preserve length: precisely, for any face map $d : X_{n+1} \to X_n$, if $\sigma \in X_{n+1}^{(k)}$, then $d\sigma \in X_n^{(k)}$.*

Proof. Since the length of a simplex only refers only to its long edge, and since a generic face map does not alter the long edge, it is enough to treat the case of outer face maps, and by symmetry it is enough to treat the case of $d_\top$. Let f denote the long edge of σ. Let τ denote the triangle ${d_1}^{n-1}\sigma$. It has long edge f again. Let u and v denote the short edges of τ,

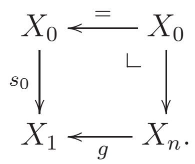


that is $v = d_\bot\tau = {d_\bot}^n\sigma$ and $u = d_\top\tau$, the long edge of $d_\top\sigma$. The claim is that if $\ell(f) \leq k$, then $\ell(u) \leq k$. If we were in the category case, this would be true since any decomposition of u could be turned into a decomposition of f of at least the same length, simply by postcomposing with v. In the general case, we have to invoke the decomposition-space condition to glue with τ along u. Precisely, for any simplex $\kappa \in X_w$ with long edge u we can obtain a simplex $\kappa \#_u \tau \in X_{w1}$ with long edge f: since X is a decomposition space, we have a pullback square



and $d_\top\tau = u = g(\kappa)$, giving us the desired simplex in X_{w1}. With this construction, any simplex κ of length $> k$ violating $\ell(u) = k$ (cf. the characterisation of length given in (1) of Proposition 6.11) would also yield a simplex $\kappa \#_u \tau$ (of at least the same length) violating $\ell(f) = k$. □

Proposition 6.17. *In a tight decomposition space X, for any generic map $g : X_n \to X_1$ we have*



Proof. By Corollary 6.10, if the long edge of $\sigma \in X_n$ is degenerate, then σ is in the image of the maximal degeneracy map $X_0 \to X_n$. ☐

Corollary 6.18. *For a tight decomposition space, $X_n^{(0)} = X_0$, $\forall n$.*

6.19. Coalgebra filtration. If X is a tight decomposition space, the sequence of cULF maps

$$X_\bullet^{(0)} \hookrightarrow X_\bullet^{(1)} \hookrightarrow \cdots \hookrightarrow X$$

defines coalgebra homomorphisms

$$\boldsymbol{Grpd}_{/X_1^{(0)}} \to \boldsymbol{Grpd}_{/X_1^{(1)}} \to \cdots \to \boldsymbol{Grpd}_{/X_1}$$

which clearly define a coalgebra filtration of $\boldsymbol{Grpd}_{/X_1}$.

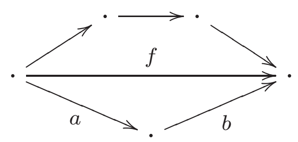
Recall that a filtered coalgebra is called connected if its 0-stage coalgebra is the trivial coalgebra (the ground ring). In the present situation the 0-stage is $\boldsymbol{Grpd}_{/X_1^{(0)}} \simeq \boldsymbol{Grpd}_{/X_0}$, so we see that $\boldsymbol{Grpd}_{/X_1}$ is connected if and only if X_0 is contractible.

On the other hand, the 0-stage elements are precisely the degenerate arrows, which almost tautologically are group-like. Hence the incidence coalgebra of a tight decomposition space will always have the property that the 0-stage is spanned by group-like elements. For some purposes, this property is nearly as good as being connected (cf. [19], [20] for this viewpoint in the context of renormalisation).

6.20. Grading. Given a 2-simplex $\sigma \in X_2$ in a complete decomposition space X, it is clear that we have

$$\ell(d_2\sigma) + \ell(d_0\sigma) \leq \ell(d_1\sigma)$$

generalising the case of a category, where $f = ab$ implies $\ell(a) + \ell(b) \leq \ell(f)$. In particular, the following configuration of arrows illustrates that one does not in general have equality:



Provided none of the arrows can be decomposed further, we have $\ell(f) = 3$, but $\ell(a) = \ell(b) = 1$. For the same reason, the length filtration is not in general a grading: $\Delta(f)$ contains the term $a \otimes b$ of degree splitting $1 + 1 < 3$. Nevertheless, it is actually common in examples of interest to have a grading: this happens when all maximal chains composing to a given arrow f have the same length, $\ell(f)$. Many examples from combinatorics have this property [12].

The abstract formulation of the condition for the length filtration to be a grading is this: For every k-simplex $\sigma \in X_k$ with long edge a and principal edges $e_1, \ldots, e_k$, we have

$$\ell(a) = \ell(e_1) + \cdots + \ell(e_k).$$

Equivalently, for every 2-simplex $\sigma \in X_2$ with long edge a and short edges e_1, e_2, we have

$$\ell(a) = \ell(e_1) + \ell(e_2).$$

The length filtration is a grading if and only if the functor $\ell : X_1 \to \mathbb{N}$ extends to a simplicial map to the nerve of the monoid $(\mathbb{N}, +)$ (this map is rarely cULF though).

If X is the nerve of a poset P, then the length filtration is a grading if and only if P is *ranked*, i.e. for any $x, y \in P$, every maximal chain from x to y has the same length [28].

7. Locally finite decomposition spaces

In order to be able to take cardinality of the $\boldsymbol{Grpd}$-coalgebra obtained from a decomposition space X to get a coalgebra at the numerical level (vector spaces), we need to impose certain finiteness conditions. Firstly, just for the coalgebra structure to have a cardinality, we need X to be *locally finite* (7.4) but it is not necessary that X be complete. Secondly, in order for Möbius inversion to descend, what we need in addition is precisely the filtration condition (which in turn assumes completeness). We shall define a *Möbius decomposition space* to be a locally finite tight decomposition space (8.3).

We begin with a few reminders on finiteness of ∞-groupoids.

7.1. Finiteness conditions for ∞-groupoids. (Cf. [11]) An ∞-groupoid S is *locally finite* if at each base point x the homotopy groups $\pi_i(S, x)$ are finite for $i \geq 1$ and are trivial for i sufficiently large. It is called *finite* if furthermore it has only finitely many components. We denote by $\boldsymbol{grpd}$ the ∞-category of finite groupoids.

The role of vector spaces is played by finite-groupoid slices $\boldsymbol{grpd}_{/S}$ (where S is a locally finite ∞-groupoid), while the role of profinite-dimensional vector spaces is played by finite-presheaf categories $\boldsymbol{grpd}^S$. Linear maps are given by spans of *finite type*, meaning $S \xleftarrow{p} M \xrightarrow{q} T$ in which p is a finite map. Prolinear maps are given by spans of *profinite type*, where q is a finite map. Inside the ∞-category $\boldsymbol{LIN}$, we have two ∞-categories: $\underrightarrow{\boldsymbol{lin}}$ whose objects are the finite-groupoid slices $\boldsymbol{grpd}_{/S}$ and whose mapping spaces are ∞-groupoids of finite-type spans, and the ∞-category $\underleftarrow{\boldsymbol{lin}}$ whose objects are finite-presheaf categories $\boldsymbol{grpd}^S$, and whose mapping spaces are ∞-groupoids of profinite-type spans.

Lemma. 7.2. *Cf.* [11, Lemma 4.3] *For a span $S \xleftarrow{p} M \xrightarrow{q} T$ defining a linear map $F : \boldsymbol{Grpd}_{/S} \to \boldsymbol{Grpd}_{/T}$, the following are equivalent:*

(1) *p is finite,*
(2) *F restricts to*

$$\boldsymbol{grpd}_{/S} \xrightarrow{p^*} \boldsymbol{grpd}_{/M} \xrightarrow{q_!} \boldsymbol{grpd}_{/T}$$

(3) *F restricts to*

$$\boldsymbol{Grpd}^{\textit{rel.fin.}}_{/T} \xrightarrow{q^*} \boldsymbol{Grpd}^{\textit{rel.fin.}}_{/M} \xrightarrow{p_!} \boldsymbol{Grpd}^{\textit{rel.fin.}}_{/S}$$

7.3. Cardinality. (Cf. [11]) The cardinality of a finite ∞-groupoid S is by definition

$$|S| := \sum_{x \in \pi_0 S} \prod_{i>0} |\pi_i(S, x)|^{(-1)^i}.$$

Here the norm signs on the right refer to order of homotopy groups.

For a locally finite ∞-groupoid S, there is a notion of cardinality $| \, | : \boldsymbol{grpd}_{/S} \to \mathbb{Q}_{\pi_0 S}$, sending a basis element $\ulcorner s \urcorner$ to the basis element $\delta_s := |\ulcorner s \urcorner|$. The delta notation for these basis elements is useful to keep track of the level of discourse.

Dually, there is a notion of cardinality $| \, | : \boldsymbol{grpd}^S \to \mathbb{Q}^{\pi_0 S}$. The profinite-dimensional vector space $\mathbb{Q}^{\pi_0 S}$ is spanned by the characteristic functions $\delta^t = \frac{|h^t|}{|\Omega(S,t)|}$, the cardinality of the representable functor h^t divided by the cardinality of the loop space.

7.4. Locally finite decomposition spaces. A decomposition space $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$ is called *locally finite* if X_1 is locally finite and both $s_0 : X_0 \to X_1$ and $d_1 : X_2 \to X_1$ are finite maps.

Lemma. 7.5. *Let X be a decomposition space.*

1. *If $s_0 : X_0 \to X_1$ is finite then so are all degeneracy maps $s_i : X_n \to X_{n+1}$.*
2. *If $d_1 : X_2 \to X_1$ is finite then so are all generic face maps $d_j : X_n \to X_{n-1}$, $j \neq 0, n$.*
3. *X is locally finite if and only if X_n is locally finite for every n and $g : X_m \to X_n$ is finite for every generic map $g : [n] \to [m]$ in Δ.*

Proof. Since finite maps are stable under pullback [11, lem:finitemaps], both (1) and (2) follow from Lemma 1.9.

Re (3): If X is locally finite, then by definition X_1 is locally finite, and for each $n \in \mathbb{N}$ the unique generic map $X_n \to X_1$ is finite by (1) or (2). It follows that X_n is locally finite [11, locfinbase]. The converse implication is trivial. □

Remark 7.6. If X is the nerve of a poset P, then it is locally finite in the above sense if and only if it is locally finite in the usual sense of posets [28], viz. for every $x, y \in P$, the interval $[x, y]$ is finite. The points in this interval parametrise precisely the two-step factorisations of the unique arrow $x \to y$, so this condition amounts to $X_2 \to X_1$ having finite fibre over $x \to y$. (The condition X_1 locally finite is void in this case, as any discrete set is locally finite; the condition on $s_0 : X_0 \to X_1$ is also void in this case, as it is always just an inclusion.)

For posets, 'locally finite' implies 'locally finite length'. (The converse is not true: take an infinite set, considered as a discrete poset, and adjoin a top and a bottom element: the result is of locally finite length but not locally finite.) Already for categories, it is not true that locally finite implies locally finite length: for example the strict nerve of a finite group is locally finite but not of locally finite length.

7.7. Numerical incidence algebra. It follows from 7.2 that, for any locally finite decomposition space X, the comultiplication maps

$$\Delta_n : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}_{/X_1 \times X_1 \times \cdots \times X_1}$$

given for $n \geq 0$ by the spans

$$X_1 \xleftarrow{m} X_n \xrightarrow{p} X_1 \times X_1 \times \cdots \times X_1$$

restrict to linear functors

$$\Delta_n : \boldsymbol{grpd}_{/X_1} \longrightarrow \boldsymbol{grpd}_{/X_1 \times X_1 \times \cdots \times X_1}.$$

Now we can take cardinality of the linear functors

$$\boldsymbol{grpd} \xleftarrow{\varepsilon} \boldsymbol{grpd}_{/X_1} \xrightarrow{\Delta} \boldsymbol{grpd}_{/X_1 \times X_1}$$

to obtain a coalgebra structure,

$$\mathbb{Q} \xleftarrow{|\varepsilon|} \mathbb{Q}_{\pi_0 X_1} \xrightarrow{|\Delta|} \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1}$$

termed the *numerical incidence coalgebra* of X.

7.8. Morphisms. It is worth noticing that for *any* conservative ULF functor $F : Y \to X$ between locally finite decomposition spaces, the induced coalgebra homomorphism $F_! : \boldsymbol{Grpd}_{/Y_1} \to \boldsymbol{Grpd}_{/X_1}$ restricts to a functor $\boldsymbol{grpd}_{/Y_1} \to \boldsymbol{grpd}_{/X_1}$. In other words, there are no further finiteness conditions to impose on morphisms.

7.9. Numerical convolution product. By duality, if X is locally finite, the convolution product descends to the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 X_1}$ obtained by taking cardinality of $\boldsymbol{grpd}^{X_1}$. It follows from the general theory of homotopy linear algebra (see [11]) that the cardinality of the convolution product is the linear dual of the cardinality of the comultiplication. Since it is the same span that defines the comultiplication and the convolution product, it is also the exact same matrix that defines the cardinalities of these two maps. It follows that the structure constants for the convolution product (with respect to the pro-basis $\{\delta^x\}$) are the same as the structure constants for the comultiplication (with respect to the basis $\{\delta_x\}$). These are classically called the section coefficients, and we proceed to derive formulae for them in simple cases.

Let X be a locally finite decomposition space. The comultiplication at the objective level

$$\begin{array}{rcl} \boldsymbol{grpd}_{/X_1} & \longrightarrow & \boldsymbol{grpd}_{/X_1 \times X_1} \\ \ulcorner f \urcorner & \longmapsto & \big[R_f : (X_2)_f \to X_2 \to X_1 \times X_1 \big] \end{array}$$

yields a comultiplication of vector spaces by taking cardinality (remembering that $|\ulcorner f \urcorner| = \delta_f$):

$$\begin{array}{rcl} \mathbb{Q}_{\pi_0 X_1} & \longrightarrow & \mathbb{Q}_{\pi_0 X_1} \otimes \mathbb{Q}_{\pi_0 X_1} \\ \delta_f & \longmapsto & |R_f| \\ & = & \displaystyle\int^{(a,b) \in X_1 \times X_1} |(X_2)_{f,a,b}| \, \delta_a \otimes \delta_b \\ & = & \displaystyle\sum_{a,b} \big|(X_1)_{[a]}\big| \, \big|(X_1)_{[b]}\big| \, |(X_2)_{f,a,b}| \, \delta_a \otimes \delta_b. \end{array}$$

where $(X_2)_{f,a,b}$ is the fibre over the three face maps. The integral sign is a sum weighted by homotopy groups. These weights together with the cardinality of the triple fibre are called the *section coefficients*, denoted

$$c^f_{a,b} := |(X_2)_{f,a,b}| \cdot \big|(X_1)_{[a]}\big| \, \big|(X_1)_{[b]}\big| \, .$$

In the case where X is a Segal space (and even more, when X_0 is a 1-groupoid), we can be very explicit about the section coefficients. For a Segal space we have $X_2 \simeq X_1 \times_{X_0} X_1$, which helps to compute the fibre of $X_2 \to X_1 \times X_1$:

Lemma. 7.10. *The pullback*

$$\begin{array}{ccc} S & \longrightarrow & X_1 \times_{X_0} X_1 \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow[\ulcorner a,b \urcorner]{} & X_1 \times X_1 \end{array}$$

is given by

$$S \;=\; \begin{cases} \Omega(X_0, y) & \text{if } d_0 a \simeq y \simeq d_1 b \\ 0 & \text{else.} \end{cases}$$

Proof. We can compute the pullback as

$$\begin{array}{ccccc} S & \longrightarrow & X_1 \times_{X_0} X_1 & \longrightarrow & X_0 \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \text{diag} \\ 1 & \xrightarrow[\ulcorner a,b \urcorner]{} & X_1 \times X_1 & \xrightarrow[d_0 \times d_1]{} & X_0 \times X_0, \end{array}$$

and the result follows since in general

$$\begin{array}{ccc} A \times_C B & \longrightarrow & C \\ \downarrow & \lrcorner & \downarrow \text{diag} \\ A \times B & \longrightarrow & C \times C. \end{array}$$

□

Corollary 7.11. *Suppose X is a Segal space, and that X_0 is a 1-groupoid. Given $a, b, f \in X_1$ such that $d_0 a \cong y \cong d_1 b$ and $ab = f$, then we have*

$$(X_2)_{f,a,b} = \Omega(X_0, y) \times \Omega(X_1, f).$$

Proof. In this case, since X_0 is a 1-groupoid, the fibres of the diagonal map $X_0 \to X_0 \times X_0$ are 0-groupoids. Thus the fibre of the previous lemma is the discrete space $\Omega(X_0, y)$. When now computing the fibre over f, we are taking that many copies of the loop space of f. □

Corollary 7.12. *With notation as above, the section coefficients for a locally finite Segal 1-groupoid are*

$$c^{ab}_{a,b} = \frac{|\mathrm{Aut}(y)|\,|\mathrm{Aut}(ab)|}{|\mathrm{Aut}(a)|\,|\mathrm{Aut}(b)|}.$$

Coassociativity of the incidence coalgebra says that the section coefficients $\{c^{ab}_{a,b}\}$ form a *2-cocycle*,

$$c^{ab}_{a,b} c^{abc}_{ab,c} = c^{bc}_{b,c} c^{abc}_{a,bc}.$$

In fact this cocycle is cohomologically trivial, given by the coboundary of a 1-cochain,

$$c^{ab}_{a,b} = \partial(\phi)(a,b) = \phi(a)\phi(ab)^{-1}\phi(b),$$

In fact, if one fixes s, t such that $s + t = 1$, the 1-cochain may be taken to be

$$\phi(x \stackrel{a}{\to} y) = \frac{|\operatorname{Aut}(x)|^s |\operatorname{Aut}(y)|^t}{|\operatorname{Aut}(a)|}$$

.

7.13. 'Zeroth section coefficients': the counit. Let us also say a word about the zeroth section coefficients, i.e. the computation of the counit: the main case is when X is complete (in the sense that s_0 is a monomorphism). In this case, clearly we have

$$\varepsilon(f) = \begin{cases} 1 & \text{if } f \text{ degenerate} \\ 0 & \text{else.} \end{cases}$$

If X is Rezk complete, the first condition is equivalent to being invertible.

The other easy case is when $X_0 = *$. In this case

$$\varepsilon(f) = \begin{cases} \Omega(X_1, f) & \text{if } f \text{ degenerate} \\ 0 & \text{else.} \end{cases}$$

7.14. Example. The strict nerve of a 1-category $\mathscr{C}$ is a decomposition space which is discrete in each degree. The resulting coalgebra at the numerical level (assuming the due finiteness conditions) is the coalgebra of Content–Lemay–Leroux [3], and if the category is just a poset, that of Rota et al. [15].

For the fat nerve X of $\mathscr{C}$, we find

$$h^a * h^b = \begin{cases} \Omega(X_0, y)\, h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ 0 & \text{else,} \end{cases}$$

as follows from 7.10. Note that the cardinality of the representable h^a is generally different from the canonical basis element δ^a.

7.15. Finite support. It is also interesting to consider the subalgebra of the incidence algebra consisting of functions with finite support, i.e. the full subcategory $\boldsymbol{grpd}^{X_1}_{fin.sup} \subset \boldsymbol{grpd}^{X_1}$, and numerically $\mathbb{Q}^{\pi_0 X_1}_{fin.sup} \subset \mathbb{Q}^{\pi_0 X_1}$. Of course we have canonical identifications $\boldsymbol{grpd}^{X_1}_{fin.sup} \simeq \boldsymbol{grpd}_{/X_1}$, as well as $\mathbb{Q}^{\pi_0 X_1}_{fin.sup} \simeq \mathbb{Q}_{\pi_0 X_1}$, but it is important to keep track of which side of duality we are on.

That the decomposition space is locally finite is not the appropriate condition for these subalgebras to exist. Instead the requirement is that X_1 be locally finite and the functor

$$X_2 \to X_1 \times X_1$$

be finite. (This is always the case for a locally finite Segal 1-groupoid, by Lemma 7.10.) Similarly, one can ask for the convolution unit to have finite support, which is to require $X_0 \to 1$ to be a finite map.

Dually, the same conditions ensure that comultiplication and counit extend from $\boldsymbol{grpd}_{/X_1}$ to $\boldsymbol{Grpd}^{rel.fin}_{/X_1}$, which numerically is some sort of vector space of summable infinite linear combinations. An example of this situation is given by the bialgebra of P-trees, whose comultiplication does extend to $\boldsymbol{Grpd}^{rel.fin}_{/X_1}$. Importantly, this is the home for the Green function, an infinite (homotopy) sum of trees, and for the Faà di Bruno formula it satisfies, which does not hold for any finite truncation. See [7] for these results.

7.16. Examples. If X is the strict nerve of a 1-category $\mathscr{C}$, then the finite-support convolution algebra is precisely the *category algebra* of $\mathscr{C}$. (For a finite category, of course the two notions coincide.)

Note that the convolution unit is

$$\varepsilon = \sum_x \delta^{\mathrm{id}_x} = \begin{cases} 1 & \text{for id arrows} \\ 0 & \text{else,} \end{cases}$$

the sum of all indicator functions of identity arrows, so it will be finite if and only if the category has only finitely many objects.

In the case of the fat nerve of a 1-category, the finiteness condition for comultiplication is implied by the condition that every object has a finite automorphism group (a condition implied by local finiteness). On the other hand, the convolution unit has finite support precisely when there is only a finite number of isoclasses of objects, already a more drastic condition. Note the 'category algebra' interpretation: compared to the usual category algebra there is a symmetry factor (cf. 7.14):

$$h^a * h^b = \begin{cases} \Omega(X_0, y)\, h^{ab} & \text{if } a \text{ and } b \text{ composable at } y \\ 0 & \text{else.} \end{cases}$$

Finally, the finite-support incidence algebras are important in the case of the Waldhausen S-construction: they are the Hall algebras (see [9]. The finiteness conditions are then homological, namely finite Ext^0 and Ext^1.

8. Möbius decomposition spaces

Lemma. 8.1. *If X is a complete decomposition space then the following conditions are equivalent*

(1) $d_1 : X_2 \to X_1$ *is finite.*
(2) $d_1 : \vec{X}_2 \to X_1$ *is finite.*
(3) $d_1^{r-1} : \vec{X}_r \to X_1$ *is finite for all* $r \geq 2$.

Proof. We show the first two conditions are equivalent; the third is similar. Using the word notation of 2.5 we consider the map

$$\vec{X}_2 + \vec{X}_1 + \vec{X}_1 + X_0 \xrightarrow{\simeq} \vec{X}_2 + X_{0a} + X_{a0} + X_{00} \xrightarrow{=} X_2 \xrightarrow{d_1} X_1$$

Thus $d_1 : X_2 \to X_1$ is finite if and only if the restriction of this map to the first component, $d_1 : \vec{X}_2 \to X_1$, is finite. By completeness the restrictions to the other components are finite (in fact, mono). □

Corollary 8.2. *A complete decomposition space X is locally finite if and only if X_1 is locally finite and $d_1^{r-1} : \vec{X}_r \to X_1$ is finite for all $r \geq 2$.*

8.3. Möbius condition. A complete decomposition space X is called *Möbius* if it is locally finite and tight (i.e. of locally finite length). It then follows that the restricted composition map

$$\sum_r {d_1}^{r-1} : \sum_r \vec{X}_r \to X_1$$

is finite. In other words, the spans defining Φ_{even} and Φ_{odd} are of finite type, and hence descend to the finite groupoid-slices $\boldsymbol{grpd}_{/X_1}$. In fact we have:

Lemma. 8.4. *A complete decomposition space X is Möbius if and only if X_1 is locally finite and the restricted composition map*

$$\sum_r {d_1}^{r-1} : \sum_r \vec{X}_r \to X_1$$

is finite.

Proof. 'Only if' is clear. Conversely, if the map $m : \sum_r {d_1}^{r-1} : \sum_r \vec{X}_r \to X_1$ is finite, in particular for each individual r the map $\vec{X}_r \to X_1$ is finite, and then also $X_r \to X_1$ is finite, by Lemma 8.1. Hence X is altogether locally finite. But it also follows from finiteness of m that for each $a \in X_1$, the fibre $(\vec{X}_r)_a$ must be empty for big enough r, so the filtration condition is satisfied, so altogether X is Möbius. □

Remark 8.5. If X is a Segal space, the Möbius condition says that for each arrow $a \in X_1$, the factorisations of a into nondegenerate $a_i \in \vec{X}_1$ have bounded length. In particular, if X is the strict nerve of a 1-category, then it is Möbius in the sense of the previous definition if and only if it is Möbius in the sense of Leroux. (Note however that this would also have been true if we had not included the condition that X_1 be locally finite (as obviously this is automatic for any discrete set). We insist on including the condition X_1 locally finite because it is needed in order to have a well-defined cardinality.)

8.6. Filtered coalgebras in vector spaces. A Möbius decomposition space is in particular length-filtered. The coalgebra filtration (6.19) at the objective level

$$\boldsymbol{Grpd}_{/X_1^{(0)}} \to \boldsymbol{Grpd}_{/X_1^{(1)}} \to \cdots \to \boldsymbol{Grpd}_{/X_1}$$

is easily seen to descend to the finite-groupoid coalgebras:

$$\boldsymbol{grpd}_{/X_1^{(0)}} \to \boldsymbol{grpd}_{/X_1^{(1)}} \to \cdots \to \boldsymbol{grpd}_{/X_1},$$

and taking cardinality then yields a coalgebra filtration at the numerical level too. From the arguments in 6.19, it follows that this coalgebra filtration

$$C_0 \hookrightarrow C_1 \hookrightarrow \cdots \hookrightarrow C$$

has the property that C_0 is generated by group-like elements. (This property is found useful in the context of perturbative renormalisation [19], [20], where it serves as basis for recursive arguments, as an alternative to the more common assumption of connectedness.) Finally, if X is a graded Möbius decomposition space, then the resulting coalgebra at the algebraic level is furthermore a graded coalgebra.

The following is an immediate corollary to 6.5. It extends the classical fact that a Möbius category in the sense of Leroux does not have non-identity invertible arrows [21, Lemma 2.4].

Corollary 8.7. *If a Möbius decomposition space X is a Segal space, then it is Rezk complete.*

8.8. Möbius inversion at the algebraic level. Assume X is a locally finite complete decomposition space. The span $X_1 \xleftarrow{=} X_1 \longrightarrow 1$ defines the zeta functor (cf. 3.2), which as a presheaf is $\zeta = \int^t h^t$, the homotopy sum of the representables. Its cardinality is the usual zeta function in the incidence algebra $\mathbb{Q}^{\pi_0 X_1}$.

The spans $X_1 \longleftarrow \vec{X}_r \longrightarrow 1$ define the Phi functors

$$\Phi_r : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd},$$

with $\Phi_0 = \varepsilon$. By Lemma 8.1, these functors descend to

$$\Phi_r : \boldsymbol{grpd}_{/X_1} \longrightarrow \boldsymbol{grpd},$$

and we can take cardinality to obtain functions $|\zeta| : \pi_0(X_1) \to \mathbb{Q}$ and $|\Phi_r| : \pi_0(X_1) \to \mathbb{Q}$, elements in the incidence algebra $\mathbb{Q}^{\pi_0 X_1}$.

Finally, when X is furthermore assumed to be Möbius, we can take cardinality of the abstract Möbius inversion formula of 3.8:

Theorem 8.9. *If X is a Möbius decomposition space, then the cardinality of the zeta functor, $|\zeta| : \mathbb{Q}_{\pi_0 X_1} \to \mathbb{Q}$, is convolution invertible with inverse $|\mu| := |\Phi_{\mathrm{even}}| - |\Phi_{\mathrm{odd}}|$:*

$$|\zeta| * |\mu| = |\varepsilon| = |\mu| * |\zeta| .$$

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