DECOMPOSITION SPACES, INCIDENCE ALGEBRAS AND MÖBIUS INVERSION I: BASIC THEORY

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Abstract. This is the first in a series of papers devoted to the theory of decomposition spaces, a general framework for incidence algebras and Möbius inversion, where algebraic identities are realised by taking homotopy cardinality of equivalences of ∞-groupoids. A decomposition space is a simplicial ∞-groupoid satisfying an exactness condition, weaker than the Segal condition, expressed in terms of generic and free maps in Δ. Just as the Segal condition expresses up-to-homotopy composition, the new condition expresses decomposition, and there is an abundance of examples coming from combinatorics.

After establishing some basic properties of decomposition spaces, the main result of this first paper shows that to any decomposition space there is an associated incidence coalgebra, spanned by the space of 1-simplices, and with coefficients in ∞-groupoids. We take a functorial viewpoint throughout, emphasising conservative ULF functors; these induce coalgebra homomorphisms. Reduction procedures in the classical theory of incidence coalgebras are examples of this notion, and many are examples of decalage of decomposition spaces. We treat a few examples of decomposition spaces beyond Segal spaces, the most interesting being that of Hall algebras: the Waldhausen $S_\bullet$-construction of an abelian (or stable infinity) category is shown to be a decomposition space.

In the second paper in this series we impose further conditions on decomposition spaces to obtain a general Möbius inversion principle, and to ensure that the various constructions and results admit a homotopy cardinality. In the third paper we show that the Lawvere–Menni Hopf algebra of Möbius intervals is the homotopy cardinality of a certain universal decomposition space. Two further sequel papers deal with numerous examples from combinatorics.

Note: The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov [16] who call them unital 2-Segal spaces. Our theory is quite orthogonal to theirs: the definitions are different in spirit and appearance, and the theories differ in terms of motivation, examples and directions.

Contents

Note. This paper originally formed Sections 1 and 2 of the manuscript [21], which has now been split into six papers.

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0. Introduction

Background and motivation

Leroux's notion of Möbius category [47] generalises at the same time locally finite posets (Rota [58]) and Cartier–Foata finite-decomposition monoids [9], the two classical settings for incidence algebras and Möbius inversion. An important advantage of having these classical theories on the same footing is they may then be connected by an appropriate class of functors, the conservative ULF functors (ULF = unique lifting of factorisations; see Section 4). In particular it gives a nice explanation of the important process of reduction, to get the most interesting algebras out of posets, a process that was sometimes rather ad hoc. For the most classical example of this process, consider the divisibility poset $(\mathbb{N}^\times, |)$ as a category. It admits a conservative ULF functor to the multiplicative monoid $(\mathbb{N}^\times, \times)$, considered as a category with only one object. This functor induces a homomorphism of incidence coalgebras which is precisely the reduction map from the 'raw' incidence coalgebra of the divisibility poset to its reduced incidence coalgebra, which is isomorphic to the Cartier–Foata incidence coalgebra of the multiplicative monoid.

Shortly after Leroux's work, Dür [14] studied more involved categorical structures to extract further examples of incidence algebras and study their Möbius functions. In particular he realised what was later called the Connes–Kreimer Hopf algebra [10] as the reduced incidence coalgebra of a certain category of root-preserving forest embeddings, modulo the equivalence relation that identifies two root-preserving forest embeddings if their complement crowns are isomorphic forests. Another prominent example fitting Dür's formalism is the Faà di Bruno bialgebra, previously obtained in [31] from the category of surjections, which is however not a Möbius category.

Our work on Faà di Bruno formulae in bialgebras of trees [20] prompted us to look for a more general version of Leroux's theory, which would naturally realise the Faà di Bruno and Connes–Kreimer bialgebras as incidence coalgebras. A sequence of generalisations and simplifications of the theory led to the notion of decomposition space which is the central notion of our present work.

The first abstraction step is to follow the objective method, pioneered in this context by Lawvere and Menni [45], working directly with the combinatorial objects, using linear algebra with coefficients in $\boldsymbol{Set}$ rather than working with numbers and functions on the vector spaces spanned by the objects.

To illustrate this, observe that a vector in the free vector space on a set S is just a collection of scalars indexed by (a finite subset of) S. The objective counterpart is a family of sets indexed by S, i.e. an object in the slice category $\boldsymbol{Set}_{/S}$, and linear maps at this level are given by spans $S \leftarrow M \rightarrow T$. The Möbius inversion principle states an equality between certain linear maps (elements in the incidence algebra). At the objective level, such an equality can be expressed as a bijection between sets in the spans representing those linear functors. In this way, algebraic identities are revealed to be just the cardinality of bijections of sets, which carry much more information.

In the present work, the coefficients are ∞-groupoids, meaning that the role of vector spaces is played by slices of the ∞-category of ∞-groupoids. In [24] we have developed the necessary 'homotopy linear algebra' and homotopy cardinality, extending many results of Baez–Hoffnung–Walker [3] who worked with 1-groupoids. In order to be able to recover numerical or algebraic results by taking cardinality, suitable finiteness conditions must be imposed, but as long as we work at the objective level, where all results and proofs are naturally bijective, these finiteness conditions do not play an essential role. Outside of this introduction we are not concerned with finiteness conditions and cardinality in the present paper, but will return to them in Part II [22].

The price to pay for working at the objective level is the absence of additive inverses: in particular, Möbius functions cannot exist in the usual form of an alternating sum indexed by chains of different lengths. However, an explicit equivalence expressing the Möbius inversion principle can be obtained by splitting into even- and odd-length chains, and under the appropriate finiteness assumptions one can pass from the objective level to the numerical level by taking cardinality; the even-odd split version of Möbius inversion then yields the usual form of an alternating sum.

There are two levels of finiteness conditions needed in order to take cardinality and arrive at algebraic (numerical) results: namely, just in order to obtain a numerical coalgebra, for each arrow f and for each $n \in \mathbb{N}$, there should be only finitely many decompositions of f into a chain of n arrows. Second, in order to obtain also Möbius inversion, the following additional finiteness condition is needed: for each arrow f, there is an upper bound on the number of non-identity arrows in a chain of arrows composing to f. The latter condition is important in its own right, as it is the condition for the existence of a length filtration, useful in many applications.

The importance of chains of arrows naturally suggests a simplicial viewpoint, regarding a category as a simplicial set via its nerve. Leroux's theory can be formulated in terms of simplicial sets, and many of the arguments then rely on certain simple pullback conditions, the first being the Segal condition which characterises categories among simplicial sets.

The fact that combinatorial objects typically have symmetries prompted the upgrade from sets to groupoids, in fact a substantial conceptual simplification [20]. This upgrade is essentially straightforward, as long as the notions involved are taken in a correct homotopy sense: bijections of sets are replaced by equivalences of groupoids; the slices playing the role of vector spaces are homotopy slices, the pullbacks and fibres involved in the functors are homotopy pullbacks and homotopy fibres, and the sums are homotopy sums (i.e. colimits indexed by groupoids, just as classical sums are colimits indexed by sets). In this setting one may abandon also the strict notion of simplicial object in favour of a pseudo-functorial analogue. For example, the classifying space of $(\mathbb{B}, +, 0)$, the monoidal groupoid of finite sets and bijections under disjoint union, is actually only a pseudofunctor $\mathbf{B} : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$. This level of abstraction allows us to state for example that the incidence algebra of $\mathbf{B}$ is the category of species with the Cauchy product (suggested as an exercise by Lawvere and Menni [45]).

While it is doable to handle the 2-category theory involved to deal with groupoids, pseudo-functors, pseudo-natural isomorphisms, and so on, much conceptual clarity is obtained by passing immediately to ∞-groupoids: thanks to the monumental effort of Joyal [33], [34], Lurie [50] and others, ∞-groupoids can now be handled efficiently. At least at the elementary level where we work, all that is needed is some basic knowledge about (homotopy) pullbacks and (homotopy) sums, and everything looks very much like the category of sets. So we work throughout with certain simplicial ∞-groupoids. Weak categories in ∞-groupoids are precisely Rezk complete Segal spaces [56]. Our theory at this level says that for any Rezk complete Segal space there is a natural incidence coalgebra defined with coefficients in ∞-groupoids, and that the objective sign-free Möbius inversion principle holds.

The final abstraction step, which becomes the starting point for the paper, is to notice that in fact neither the Segal condition nor the Rezk condition is needed in full in order to get a (co)associative (co)algebra and a Möbius inversion principle. Coassociativity follows from (in fact is essentially equivalent to) the *decomposition space axiom* (see Section 3 for the axiom, and the discussion at the beginning of Section 5 for its derivation from coassociativity): a decomposition space is a simplicial ∞-groupoid sending generic-free pushout squares in Δ to pullbacks. Whereas the Segal condition is the expression of the ability to compose morphisms, the new condition is about the ability to decompose, which of course in general is easier to achieve than composability.

It is likely that all incidence (co)algebras can be realised directly (without imposing a reduction) as incidence (co)algebras of decomposition spaces. The decomposition space is found by analysing the reduction step. For example, Dür realises the q-binomial coalgebra as the reduced incidence coalgebra of the category of finite-dimensional vector spaces over a finite field and linear injections, by imposing the equivalence relation identifying two linear injections if their quotients are isomorphic. Trying to realise the reduced incidence coalgebra directly as a decomposition space immediately leads to the Waldhausen $S_\bullet$-construction, which is a general class of examples: we show that for any abelian category or stable ∞-category, the Waldhausen $S_\bullet$-construction is a decomposition space (which is not Segal). Under the appropriate finiteness conditions, the resulting incidence algebras include the (derived) Hall algebras.

Other examples of coalgebras that can be realised as incidence coalgebras of decomposition spaces but not of categories are Schmitt's Hopf algebra of graphs [61] and the Butcher–Connes–Kreimer Hopf algebra of rooted trees [10]. In a sequel paper [26], these examples are subsumed as examples of decomposition spaces induced from restriction species and directed restriction species.

The appropriate notion of morphism between decomposition spaces is that of conservative ULF functor. These induce coalgebra homomorphisms. Many relationships between incidence coalgebras, and in particular most of the reductions that play a central role in the classical theory (from Rota [58] and Dür [14] to Schmitt [61]), are induced from conservative ULF functors. The simplicial viewpoint taken in this work reveals furthermore that many of these conservative ULF functors are actually instances of the notion of decalage, which goes back to Lawvere [43] and Illusie [29]. Decalage is in fact an important ingredient in the theory to relate decomposition spaces to Segal spaces: we observe that the decalage of a decomposition space is a Segal space.

Throughout we have strived for deriving all results from elementary principles, such as pullbacks, factorisation systems and other universal constructions. It is also characteristic for our approach that we are able to reduce many technical arguments to simplicial combinatorics. The main notions are formulated in terms of the generic-free factorisation system in Δ. To establish coassociativity we explore also $\underline{\Delta}$ (the algebraist's Delta, including the empty ordinal) and establish and exploit a universal property of its twisted arrow category. Sequels to this paper further vindicate this philosophy: in [26], as a general method for establishing functoriality in free maps, we study a certain category ∇ of convex correspondences in $\underline{\Delta}$. Finally in [23], in order to construct the universal decomposition space of intervals, we study the category Ξ of finite strict intervals, yet another variation of the simplex category, related to it by an adjunction. These 'simplicial preliminaries' are likely to have applications also outside the theory of decomposition spaces.

Related work: 2-Segal spaces of Dyckerhoff and Kapranov

The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov [16]: a decomposition space is essentially the same thing as what they call a unital 2-Segal space. We hasten to give them full credit for having arrived at the notion first. Unaware of their work, we arrived at the same notion from a very different path, and the theory we have developed for it is mostly orthogonal to theirs.

The definitions are different in appearance: the definition of decomposition space refers to preservation of certain pullbacks, whereas the definition of 2-Segal space (reproduced in 3.1 below) refers to triangulations of convex polygons. The coincidence of the notions was noticed by Mathieu Anel because two of the basic results are the same: specifically, the characterisation in terms of decalage and Segal spaces (our Theorem 4.11) and the result that the Waldhausen $S_\bullet$-construction of a stable ∞-category is a decomposition space (our Theorem 10.14) were obtained independently (and first) in [16].

We were motivated by rather elementary aspects of combinatorics and quantum field theory, and our examples are all drawn from incidence algebras and Möbius inversion, whereas Dyckerhoff and Kapranov were motivated by representation theory, geometry, and homological algebra, and develop a theory with a much vaster range of examples in mind: in addition to Hall algebras and Hecke algebras they find cyclic bar construction, mapping class groups and surface geometry (see also [17] and [18]), construct a Quillen model structure and relate to topics of interest in higher category theory such as ∞-2-categories and operads.

In the end we think our contribution is just a little corner of a vast theory, but an important little corner, and we hope that our viewpoints and insights will prove useful also for the rest of the theory.

Related work on Möbius categories

Where incidence algebras and Möbius inversion are concerned, our work descends from Leroux et al. [47], [12], [48], Dür [14] and Lawvere–Menni [45].

There is a different notion of Möbius category, due to Haigh [28]. The two notions have been compared, and to some extent unified, by Leinster [46], who calls Leroux's Möbius inversion *fine* and Haigh's *coarse*, as it only depends on the underlying graph of the category. We should mention also the K-theoretic Möbius inversion for quasi-finite EI categories of Lück and collaborators [49], [19].

Outline of the present paper, section by section

We begin in Section 1 with a review of some elementary notions from the theory of ∞-categories, to render the paper accessible also to readers without prior experience of these notions. Section 2 contains a few preliminaries on simplicial objects and Segal spaces, and in Section 3 we introduce the main notion of this work, decomposition spaces:

Definition. A simplicial space $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$ is called a *decomposition space* when it takes generic-free pushouts in Δ to pullbacks.

We give some equivalent pullback characterisations, and observe that every Segal space is a decomposition space.

In Section 4 we turn to the relevant notion of morphism, that of conservative ULF functor (unique lifting of factorisations):

Definition. A simplicial map is called *ULF* if it is cartesian on generic face maps, and it is called *conservative* if cartesian on degeneracy maps. We write *cULF* for conservative and ULF, that is, cartesian on all generic maps.

After some variations, we come to decalage, and establish the following important relationship between Segal spaces and decalage:

Theorem 4.11. *A simplicial space X is a decomposition space if and only if both* $\mathrm{Dec}_\top(X)$ *and* $\mathrm{Dec}_\bot(X)$ *are Segal spaces, and the two comparison maps back to X are cULF.*

In Section 5 we introduce the incidence coalgebra associated to a decomposition space X. It is the slice ∞-category $\boldsymbol{Grpd}_{/X_1}$, with the comultiplication map given by the span

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

We explain how a naive view of coassociativity provided the motivation for the decomposition space axioms, but to formally establish the coassociativity result we first require more simplicial preliminaries, introduced in Section 6. In particular we introduce the twisted arrow category $\mathscr{D}$ of the category of finite ordinals, which is monoidal under *external sum*. We show that simplicial objects in a cartesian monoidal category can be characterised as monoidal functors on $\mathscr{D}$, and characterise decomposition spaces as those simplicial spaces whose extension to $\mathscr{D}$ preserves certain pullback squares.

In Section 7 the homotopy coassociativity of the incidence coalgebra is established in terms of the monoidal structure on $\mathscr{D}$:

Theorem 7.3. *For X a decomposition space, the slice ∞-category $\boldsymbol{Grpd}_{/X_1}$ has the structure of strong homotopy comonoid in the symmetric monoidal ∞-category **LIN**, with the comultiplication defined by the span*

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

In Section 8 we show first of all that cULF functors induce coalgebra homomorphisms. We also comment on a contravariant functoriality in object-equivalence relatively Segal functors.

In Section 9 we introduce the notion of *monoidal decomposition space*, as a monoid object in the monoidal ∞-category of decomposition spaces and cULF maps. The incidence algebra of a monoidal decomposition space is naturally a *bialgebra*.

In Section 10 we give some basic examples to provide a taste of the breadth of applications. Further examples are expounded in detail in [25] and [26]. We begin with the example of finite sets and injections (which leads to the binomial coalgebra), to illustrate how decalage formalises reduction processes, and how the convolution product at the objective level is the Cauchy product of species. Coming to examples of decomposition spaces which are not Segal, we take a short look at Schmitt's coalgebra of graphs and at the Connes–Kreimer Hopf algebra, one of the examples that motivated us. Finally we consider the example of finite vector spaces, which leads to the general case of Waldhausen's $S_\bullet$ construction and Hall algebras.

Brief summary of the four sequels to this paper

In the paper [22], which is Part 2 of the present paper and of the series, we introduce the notion of *complete* decomposition space necessary for the theory of Möbius inversion. In this context we can consider linear functors Φ_n defined by spans $X_1 \leftarrow \vec{X}_n \to 1$, where $\vec{X}_n \subset X_n$ is the subspace of nondegenerate n-simplices, and prove the general Möbius inversion principle on the objective level:

$$\zeta * \Phi_{\text{even}} \quad = \quad \varepsilon \quad + \quad \zeta * \Phi_{\text{odd}}.$$

Having established this, we analyse the finiteness conditions necessary to take cardinality and obtain numerical incidence algebras, and for the Möbius inversion principle to descend to these $\mathbb{Q}$-algebras.

We identify two conditions on complete decomposition spaces: having *locally finite length* and being *locally finite*. Complete decomposition spaces that satisfy both finiteness conditions are called *Möbius decomposition spaces*.

The first finiteness condition is equivalent to the existence of a certain *length filtration*, which is useful in applications. Although many examples coming from combinatorics do satisfy this condition, it is actually a rather strong condition, as witnessed by the following result:

Every decomposition space with length filtration is the left Kan extension of a semi-simplicial space.

This result holds for more general simplicial spaces that we term *stiff*, and we digress to establish this. We also consider an even weaker notion of *split* simplicial space, in which all face maps preserve nondegenerate simplices. This condition is the analogue of the condition for categories that identities are indecomposable, enjoyed in particular by Möbius categories in the sense of Leroux.

In paper [23] we come to what is perhaps the deepest theorem so far in our work. Lawvere showed in the 1980s that there is a Hopf algebra of Möbius intervals which contains the universal Möbius function, see [45]. This Hopf algebra, obtained from the collection of all isomorphisms classes of Möbius intervals, is universal for incidence coalgebras of Möbius categories X, by virtue of the canonical coalgebra homomorphism from the incidence coalgebra of X sending an arrow in X to its factorisation interval. The universal Hopf algebra is not, however, the incidence coalgebra of a Möbius category.

We show that it *is* a decomposition space. We construct a (large) complete decomposition space U of all 'subdivided intervals', together with a canonical cULF classifying functor $X \to U$ for any complete decomposition space X. We prove that the space of cULF maps from X to U is connected, and conjecture that it is contractible.

If we also impose the relevant finiteness conditions, we obtain the result that the space of all Möbius intervals is a Möbius decomposition space. It follows that it admits a Möbius inversion formula with coefficients in finite ∞-groupoids or in $\mathbb{Q}$, and since every Möbius decomposition space admits a canonical cULF functor to it, we conclude that Möbius inversion in every incidence algebra is induced from this master formula.

In the paper [25] we give examples from classical (and less classical) combinatorics. The first batch of examples, similar to the binomial posets of Doubilet–Rota–Stanley [13], are straightforward but serve to illustrate two key points: (1) the incidence algebra in question is realised directly from a decomposition space, without a reduction step, and reductions are typically given by cULF functors; (2) at the objective level, the convolution algebra is a monoidal structure of species (specifically: the usual Cauchy product of species, the shuffle product of $\mathbb{L}$-species, the Dirichlet product of arithmetic species, the Joyal–Street external product of q-species, and the Morrison 'Cauchy' product of q-species). In each of these cases, a power series representation results from taking cardinality.

The next class of examples includes the Faà di Bruno bialgebra, the Butcher–Connes–Kreimer bialgebra of trees, with several variations, and similar structures on directed graphs (cf. Manchon [53] and Manin [54]).

Another important class of examples is provided by showing that the Waldhausen $S_\bullet$-construction on an abelian category, or a stable ∞-category, is a decomposition space, as also explained in 10.6 below. This result was first proved by Dyckerhoff and Kapranov and constitutes a corner stone in their work [16], [17], [18], [15], to which we refer for the remarkable richness of this class of examples.

We conclude the paper by computing the Möbius function in a few cases, and commenting on certain cancellations that occur in the process of taking cardinality, substantiating that these cancellations are not possible at the objective level. This is related to the distinction between bijections and natural bijections.

In the paper [26] we show that Schmitt coalgebras of restriction species [60] (such as graphs, matroids, posets, etc.) naturally define decomposition spaces. We also introduce a new notion of *directed restriction species*: whereas ordinary restriction species are presheaves of the category of finite sets and injections, directed restriction species are presheaves on the category of finite posets and convex inclusions. Examples covered by this notion are the Butcher–Connes–Kreimer bialgebra and the Manchon–Manin bialgebra of directed graphs. Both ordinary and directed restriction species are shown to be examples of a construction of decomposition spaces from what we call sesquicartesian fibrations, certain cocartesian fibrations over the category of finite ordinals that are also cartesian over convex maps.

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1. Preliminaries on ∞-groupoids and ∞-categories

1.1. Groupoids and ∞-groupoids. Although most of our motivating examples can be naturally cast in the setting of 1-groupoids, we have chosen to work in the setting of ∞-groupoids. This is on one hand the natural generality of the theory, and on the other hand a considerable conceptual simplification: thanks to the monumental effort of Joyal [33], [34] and Lurie [50], the theory of ∞-categories has now reached a stage where it is just as workable as the theory of 1-groupoids — if not more! The philosophy is that, modulo a few homotopy *caveats*, one is allowed to think as if working in the category of sets. A recent forceful vindication of this philosophy is Homotopy Type Theory [55], in which a syntax that resembles set theory is shown to be a powerful language for general homotopy types.

A recurrent theme in the present work is to upgrade combinatorial constructions from sets to ∞-groupoids. To this end the first step consists in understanding the construction in abstract terms, often in terms of pullbacks and sums, and then the second step consists in copying over the construction to the ∞-setting. The ∞-category theory needed will be accordingly elementary, and it is our contention that it should be feasible to read this work without prior experience with ∞-groupoids or ∞-categories, simply by substituting the word 'set' for the word '∞-groupoid'. Even at the 0-level, our theory contributes interesting insight, revealing many constructions in the classical theory to be governed by very general principles proven useful also in other areas of mathematics.

The following short review of some basic aspects of ∞-categories should suffice for reading this paper and its sequels.

1.2. From posets to Rezk categories. A few remarks may be in order to relate these viewpoints with classical combinatorics. A 1-groupoid is the same thing as an ordinary groupoid, and a 0-groupoid is the same thing as a set. A (-1)-groupoid is the same thing as a truth value: up to equivalence there exist only two (-1)-groupoids, namely the contractible groupoid (a point) and the empty groupoid. A poset is essentially the same thing as a category in which all the mapping spaces are (-1)-groupoids. An ordinary category is a category in which all the mapping spaces are 0-groupoids. Hence the theory of incidence algebras of posets of Rota and collaborators can be seen as the (-1)-level of the theory. Cartier–Foata theory and Leroux theory take place at the 0-level. We shall see that in a sense the natural setting for combinatorics is the 1-level, since this level naturally takes into account that combinatorial structures can have symmetries. (From this viewpoint, it looks as if the classical theory compensates for working one level below the natural one by introducing reductions.) It is convenient to follow this ladder to infinity: the good notion of category with ∞-groupoids as mapping spaces is that of Rezk complete Segal space, also called Rezk category; this is the level of generality of the present work

1.3. ∞-categories and ∞-groupoids. By ∞-category we mean quasi-category [33]. These are simplicial sets satisfying the weak Kan condition: inner horns admit a filler. (An ordinary category is a simplicial set in which every inner horn admits a *unique* filler.) We refer to Joyal [33], [34] and Lurie [50]. The definition does not actually matter much in this work. The main point, Joyal's great insight, is that category theory can be generalised to quasi-categories, and that the results look the same, although to bootstrap the theory very different techniques are required. There are other implementations of ∞-categories, such as complete Segal spaces, see Bergner [7] for a survey. We will only use results that hold in all implementations, and for this reason we say ∞-category instead of referring explicitly to quasi-categories. Put another way, we shall only ever distinguish quasi-categories up to (categorical) equivalence, and most of the constructions rely on universal properties such as pullback, which in any case only determine the objects up to equivalence.

An ∞-groupoid is an ∞-category in which all morphisms are invertible. We often say *space* instead of ∞-groupoid, as they are a combinatorial substitute for topological spaces up to homotopy; for example, to each object x in an ∞-groupoid X, there are associated homotopy groups $\pi_n(X, x)$ for $n > 0$. In terms of quasi-categories, ∞-groupoids are precisely Kan complexes, i.e. simplicial sets in which every horn, not just the inner ones, admits a filler.

∞-groupoids play the role analogous to sets in classical category theory. In particular, for any two objects x, y in an ∞-category $\mathscr{C}$ there is (instead of a hom set) a mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$ which is an ∞-groupoid. ∞-categories form a (large) ∞-category denoted ***Cat***. ∞-groupoids form a (large) ∞-category denoted ***Grpd***; it can be described explicitly as the coherent nerve of the (simplicially enriched) category of Kan complexes. Given two ∞-categories $\mathscr{D}, \mathscr{C}$, there is a functor ∞-category $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$. Since $\mathscr{D}$ and $\mathscr{C}$ are objects in the ∞-category ***Cat*** we also have the ∞-groupoid $\mathrm{Map}_{\mathbf{Cat}}(\mathscr{D}, \mathscr{C})$, which can also be described as the maximal sub-∞-groupoid inside $\mathrm{Fun}(\mathscr{D}, \mathscr{C})$.

1.4. Defining ∞-categories and sub-∞-categories. While in ordinary category theory one can define a category by saying what the objects and the arrows are (and how they compose), this from-scratch approach is more difficult for ∞-categories, as one would have to specify the simplices in all dimensions and verify the filler condition (that is, describe the ∞-category as a quasi-category). In practice, ∞-categories are constructed from existing ones by general constructions that automatically guarantee that the result is again an ∞-category, although the construction typically uses universal properties in such a way that the resulting ∞-category is only defined up to equivalence. To specify a sub-∞-category of an ∞-category $\mathscr{C}$, it suffices to specify a subcategory of the homotopy category of $\mathscr{C}$ (i.e. the category whose hom sets are π_0 of the mapping spaces of $\mathscr{C}$), and then pull back along the components functor. What this amounts to in practice is to specify the objects (closed under equivalences) and specifying for each pair of objects x, y a full sub-∞-groupoid of the mapping space $\mathrm{Map}_{\mathscr{C}}(x, y)$, also closed under equivalences, and closed under composition.

1.5. Monomorphisms. A map of ∞-groupoids $f : X \to Y$ is a *monomorphism* when its fibres are (-1)-groupoids (i.e. are either empty or contractible). In other words, it is fully faithful as a functor: $\mathrm{Map}_X(a, b) \to \mathrm{Map}_Y(fa, fb)$ is an equivalence. In some respects, this notion behaves like for sets: for example, if f is a monomorphism, then there is a complement $Z := Y \smallsetminus X$ such that $Y \simeq X + Z$. Hence a monomorphism is essentially an equivalence from X onto some connected components of Y. On the other hand, a crucial difference between sets and ∞-groupoids is that diagonal maps of ∞-groupoids are not in general monomorphisms. In fact $X \to X \times X$ is a monomorphism if and only if X is discrete (i.e. equivalent to a set).

1.6. Diagram categories and presheaves. Every 1-category is also a quasi-category via its nerve. In particular we have the ∞-category Δ of non-empty finite ordinals, and for each $n \geq 0$ the ∞-category $\Delta[n]$ which is the nerve of the linearly ordered set $\{0 \leq 1 \leq \cdots \leq n\}$. As an important example of a functor ∞-category, for a given ∞-category I, we have the ∞-category of presheaves $\mathrm{Fun}(I^{\mathrm{op}}, \boldsymbol{Grpd})$, and there is a Yoneda lemma that works as in the case of ordinary categories. In particular we have the ∞-category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \boldsymbol{Grpd})$ of simplicial ∞-groupoids, which will be one of our main objects of study.

Since arrows in an ∞-category do not compose on the nose (one can talk about 'a' composite, not 'the' composite), the 1-categorical notion of commutative diagram does not make sense. Commutative triangle in an ∞-category $\mathscr{C}$ means instead 'object in the functor ∞-category $\mathrm{Fun}(\Delta[2], \mathscr{C})$': the 2-dimensional face of $\Delta[2]$ is mapped to a 2-cell in $\mathscr{C}$ mediating between the composite of the 01 and 12 edges and the long edge 02. Similarly, 'commutative square' means object in the functor ∞-category $\mathrm{Fun}(\Delta[1] \times \Delta[1], \mathscr{C})$. In general, 'commutative diagram of shape I' means object in

$\mathrm{Fun}(I, \mathscr{C})$, so when we say for example 'simplicial ∞-groupoid' it is not implied that the usual simplicial identities hold on the nose.

1.7. Adjoints, limits and colimits. There are notions of adjoint functors, limits and colimits, which behave in the same way as these notions in ordinary category theory, and are characterised by universal properties up to equivalence. For example, the singleton set $*$ (also denoted 1), or any contractible ∞-groupoid, is a terminal object in ***Grpd***.

1.8. Pullbacks and fibres. Central to this work is the notion of pullback: given two morphisms of ∞-groupoids $X \to S \leftarrow Y$, there is a square

$$\begin{array}{ccc} X \times_S Y & \longrightarrow & Y \\ \downarrow & \lrcorner & \downarrow \\ X & \longrightarrow & S \end{array}$$

called the pullback, an example of a limit. It is defined via a universal property, as a terminal object in a certain auxiliary ∞-category consisting of squares with sides $X \to S \leftarrow Y$. All formal properties of pullbacks of sets carry over to ∞-groupoids.

Given a morphism of ∞-groupoids, $p : X \to S$, and an object $s \in S$ (which in terms of quasi-categories can be thought of as a zero-simplex of S, but which more abstractly is encoded as a map $* \xrightarrow{s} S$ from the terminal ∞-groupoid $* = \Delta[0]$), the fibre of p over s is simply the pullback

$$\begin{array}{ccc} X_s & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ * & \xrightarrow[s]{} & S \end{array}$$

1.9. Working in the ∞-category of ∞-groupoids, versus working in the model category of simplicial sets. When working with ∞-categories in terms of quasi-categories, one often works in the Joyal model structure on simplicial sets (whose fibrant objects are precisely the quasi-categories). This is a very powerful technique, exploited masterfully by Joyal [34] and Lurie [50], and essential to bootstrap the whole theory. In the present work, we can benefit from their work, and since our constructions are generally elementary, we do not need to invoke model structure arguments, but can get away with synthetic arguments. To illustrate the difference, consider the following version of the Segal condition (see 2.10 for details): we shall formulate it and use it by simply saying *the natural square*

$$\begin{array}{ccc} X_2 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & X_0 \end{array}$$

is a pullback. This is a statement taking place in the ∞-category of ∞-groupoids. A Joyal–Lurie style formulation would rather take place in the category of simplicial sets with the Joyal model structure and say something like *the natural map* $X_2 \to X_1 \times_{X_0} X_1$ *is an equivalence*. Here $X_1 \times_{X_0} X_1$ refers to the actual 1-categorical pullback in the category of simplicial sets, which does not coincide with X_2 on the nose, but is only naturally equivalent to it.

The following Lemma is used many times in our work. It is a straightforward extension of a familiar result in 1-category theory:

Lemma. 1.10. *If in a prism diagram of* ∞*-groupoids*

$$\begin{array}{ccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot \end{array}$$

the outer rectangle and the right-hand square are pullbacks, then the left-hand square is a pullback.

A few remarks are in order. Note that we talk about a prism, i.e. a $\Delta[1] \times \Delta[2]$-diagram: although we have only drawn two of the squares of the prism, there is a third, whose horizontal sides are composites of the two indicated arrows. The triangles of the prism are not drawn either, because they are the fillers that exist by the axioms of quasi-categories. The proof follows the proof in the classical case, except that instead of saying 'given two arrows such and such, there exists a unique arrow making the diagram commute, etc.', one has to argue with equivalences of mapping spaces (or slice ∞-categories). See for example Lurie [50], Lemma 4.4.2.1 (for the dual case of pushouts).

1.11. Homotopy sums. In ordinary category theory, a colimit indexed by a discrete category (that is, a set) is the same thing as a sum (coproduct). For ∞-categories, the role of sets is played by ∞-groupoids. A colimit indexed by an ∞-groupoid is called a *homotopy sum*. In the case of 1-groupoids, these sums are ordinary sums weighted by inverses of symmetry factors. Their importance was stressed in [20]: by dealing with homotopy sums instead of ordinary sums, the formulae start to look very much like in the case of sets. For example, given a map of ∞-groupoids $X \to S$, we have that X is the homotopy sum of its fibres.

1.12. Slice categories. Maps of ∞-groupoids with codomain S form the objects of a slice ∞-category $\boldsymbol{Grpd}_{/S}$, which behaves very much like a slice category in ordinary category theory. For example, for the terminal object $*$ we have $\boldsymbol{Grpd}_{/*} \simeq \boldsymbol{Grpd}$. Again a word of warning is due: when we refer to the ∞-category $\boldsymbol{Grpd}_{/S}$ we only refer to an object determined up to equivalence of ∞-categories by a certain universal property (Joyal's insight of defining slice categories as adjoint to a join operation [33]). In the Joyal model structure for quasi-categories, this category is represented by an explicit simplicial set. However, there is more than one possibility, depending on which explicit version of the join operator is employed (and of course these are canonically equivalent). In the works of Joyal and Lurie, these different versions are distinguished, and each has some technical advantages. In the present work we shall only need properties that hold for both, and we shall not distinguish them.

1.13. Families. A map of ∞-groupoids $X \to S$ can be interpreted as a family of ∞-groupoids parametrised by S, namely the fibres X_s. Just as for sets, the same family can also be interpreted as a presheaf $S \to \boldsymbol{Grpd}$. Precisely, for each ∞-groupoid S, we have the fundamental equivalence

$$\boldsymbol{Grpd}_{/S} \xrightarrow{\simeq} \mathrm{Fun}(S, \boldsymbol{Grpd}),$$

which takes a family $X \to S$ to the functor sending $s \mapsto X_s$. In the other direction, given a functor $F : S \to \boldsymbol{Grpd}$, its colimit is the total space of a family $X \to S$.

1.14. Beck–Chevalley equivalence. Pullback along a morphism $f : T \to S$ defines an ∞-functor $f^* : \boldsymbol{Grpd}_{/S} \to \boldsymbol{Grpd}_{/T}$. This functor is right adjoint to the functor $f_! : \boldsymbol{Grpd}_{/T} \to \boldsymbol{Grpd}_{/S}$ given by post-composing with f. (The latter construction requires some care: as composition is not canonically defined, one has to choose composites. One can check that different choices yield equivalent functors.) The following Beck–Chevalley rule (push-pull formula) [27] holds for ∞-groupoids: given a pullback square

$$\begin{array}{ccc} \cdot & \xrightarrow{f} & \cdot \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle q} \\ \cdot & \xrightarrow[g]{} & \cdot \end{array}$$

there is a canonical equivalence of functors

$$p_! \circ f^* \simeq g^* \circ q_! . \tag{1}$$

1.15. Symmetric monoidal ∞-categories. There is a notion of symmetric monoidal ∞-category, but it is technically more involved than the 1-category case, since in general higher coherence data has to be specified beyond the 1-categorical associator and MacLane pentagon condition. This theory has been developed in detail by Lurie [51, Ch.2], subsumed in the general theory of ∞-operads. In the present work, a few monoidal structures play an important role, but since they are directly induced by cartesian product, we have preferred to deal with them in an informal (and possibly not completely rigorous) way, with the same freedom as one deals with cartesian products in ordinary category theory. The following case is the most important for our theory. It is defined rigorously in [24], as a straightforward consequence of results of Lurie.

1.16. The symmetric monoidal ∞-category *LIN*. The ∞-categories of the form $\boldsymbol{Grpd}_{/S}$ form the objects of a symmetric monoidal ∞-category $\boldsymbol{LIN}$, described in detail in [24]: the morphisms are the linear functors, meaning that they preserve homotopy sums, or equivalently indeed all colimits. Such functors are given by spans: the span

$$S \xleftarrow{p} M \xrightarrow{q} T$$

defines the linear functor

$$q_! \circ p^* : \boldsymbol{Grpd}_{/S} \longrightarrow \boldsymbol{Grpd}_{/T}.$$

The ∞-category $\boldsymbol{LIN}$ plays the role of the category of vector spaces (although to be strict about that interpretation, and in particular to entertain a notion of cardinality to embody the analogy, certain finiteness conditions should be imposed — these play no essential role in the present paper).

The symmetric monoidal structure on $\boldsymbol{LIN}$ is easy to describe on objects:

$$\boldsymbol{Grpd}_{/S} \otimes \boldsymbol{Grpd}_{/T} = \boldsymbol{Grpd}_{S\times T}$$

just as the tensor product of vector spaces with bases indexed by sets S and T is the vector spaces with basis indexed by $S \times T$. The neutral object is $\boldsymbol{Grpd}$.

2. Simplicial preliminaries and Segal spaces

Our work relies heavily on simplicial machinery. We briefly review the notions needed, to establish conventions and notation.

2.1. The simplex category (the topologist's Delta). Recall that the 'simplex category' Δ is the category whose objects are the nonempty finite ordinals

$$[k] := \{0, 1, 2, \dots, k\},$$

and whose morphisms are the monotone maps. These are generated by the coface maps $d^i : [n-1] \to [n]$, which are the monotone injective functions for which $i \in [n]$ is not in the image, and codegeneracy maps $s^i : [n+1] \to [n]$, which are monotone surjective functions for which $i \in [n]$ has a double preimage. We write $d^\perp := d^0$ and $d^\top := d^n$ for the outer coface maps.

2.2. Generic and free maps. The category Δ has a generic-free factorisation system. A morphism of Δ is termed *generic*, and written $g : [m] \twoheadrightarrow [n]$, if it preserves endpoints, $g(0) = 0$ and $g(m) = n$. A morphism is termed *free*, and written $f : [m] \rightarrowtail [n]$, if it is distance preserving, $f(i+1) = f(i) + 1$ for $0 \le i \le m-1$. The generic maps are generated by the codegeneracy maps and the inner coface maps, and the free maps are generated by the outer coface maps. Every morphism in Δ factors uniquely as a generic map followed by a free map, as detailed below.

2.3. Background remarks. The notions of generic and free maps are general notions in category theory, introduced by Weber [64, 65], who extracted the notion from earlier work of Joyal [32]; a recommended entry point to the theory is Berger–Melliès–Weber [6]. The notion makes sense for example whenever there is a cartesian monad on a presheaf category $\mathscr{C}$: in the Kleisli category, the free maps are those from $\mathscr{C}$, and the generic maps are those generated by the monad. In practice, this is restricted to a suitable subcategory of combinatorial nature. In the case at hand the monad is the free-category monad on the category of directed graphs, and Δ arises as the restriction of the Kleisli category to the subcategory of non-empty linear graphs. Other important instances of generic-free factorisation systems are found in the category of rooted trees [38] (where the monad is the free-operad monad), the category of Feynman graphs [35] (where the monad is the free-modular-operad monad), the category of directed graphs [41] (where the monad is the free-properad monad), and Joyal's cellular category Θ [5] (where the monad is the free-omega-category monad).

2.4. Amalgamated ordinal sum. The *amalgamated ordinal sum over* $[0]$ of two objects $[m]$ and $[n]$, denoted $[m] \pm [n]$, is given by the pushout of free maps

$$
\begin{array}{ccc}
[0] & \xrightarrow{(d^\top)^n} & [n] \\
{\scriptstyle (d^\perp)^m}\downarrow & & \downarrow{\scriptstyle (d^\perp)^m} \\
[m] & \xrightarrow[(d^\top)^n]{} & [m] \pm [n] = [m+n]
\end{array}
\tag{2}
$$

This operation is not functorial on all maps in Δ, but on the subcategory Δ_{gen} of generic maps it is functorial and defines a monoidal structure on Δ_{gen} (dual to ordinal sum (cf. Lemma 6.2)).

The free maps $f : [n] \rightarrowtail [m]$ are precisely the maps that can be written

$$f : [n] \rightarrowtail [a] \pm [n] \pm [b].$$

Every generic map with source $[a] \pm [n] \pm [b]$ splits as

$$(\ [a] \xrightarrow{g_1} [a'] \ ) \ \pm \ (\ [n] \xrightarrow{g} [k] \ ) \ \pm \ (\ [b] \xrightarrow{g_2} [b'] \ )$$

With these observations we can be explicit about the generic-free factorisation:

Lemma. 2.5. *With notation as above, the generic-free factorisation of a free map f followed by a generic map $g_1 \pm g \pm g_2$ is given by*

$$\begin{array}{ccc} [n] & \overset{f}{\rightarrowtail} & [a] \pm [n] \pm [b] \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_1 \pm g \pm g_2} \\ [k] & \rightarrowtail & [a'] \pm [k] \pm [b'] \end{array} \tag{3}$$

2.6. Identity-extension squares. A square (3) in which g_1 and g_2 are identity maps is called an *identity-extension square.*

Lemma. 2.7. *Generic and free maps in Δ admit pushouts along each other, and the resulting maps are again generic and free. In fact, generic-free pushouts are precisely the identity extension squares.*

$$\begin{array}{ccc} [n] & \rightarrowtail & [a] \pm [n] \pm [b] \\ \downarrow & & \downarrow \\ [k] & \rightarrowtail & [a] \pm [k] \pm [b] \end{array}$$

These pushouts are fundamental to this work. We will define decomposition spaces to be simplicial spaces $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$ that send these pushouts to pullbacks.

The previous lemma has the following easy corollary.

Corollary 2.8. *Every codegeneracy map is a pushout (along a free map) of $s^0 : [1] \to [0]$, and every generic coface maps is a pushout (along a free map) of $d^1 : [1] \to [2]$.*

2.9. Simplicial ∞-groupoids. Our main object of study will be simplicial ∞-groupoids subject to various exactness conditions, all formulated in terms of pullbacks. More precisely we work in the functor ∞-category

$$\text{Fun}(\Delta^{\text{op}}, \boldsymbol{Grpd}),$$

whose vertices are functors from the ∞-category Δ^{op} to the ∞-category $\boldsymbol{Grpd}$. In particular, the simplicial identities for $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$ are not strictly commutative squares; rather they are $\Delta[1] \times \Delta[1]$-diagrams in $\boldsymbol{Grpd}$, hence come equipped with a homotopy between the two ways around in the square. But this is precisely the setting for pullbacks.

Consider a simplicial ∞-groupoid $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$. We recall the *Segal maps*

$$(\partial_{0,1}, \ldots, \partial_{r-1,r}) : X_r \longrightarrow X_1 \times_{X_0} \cdots \times_{X_0} X_1 \qquad r \geq 0.$$

where $\partial_{k-1,k} : X_r \to X_1$ is induced by the map $[1] \rightarrowtail [r]$ sending 0,1 to $k-1, k$.

A *Segal space* is a simplicial ∞-groupoid satisfying the Segal condition, namely that the Segal maps are equivalences.

Lemma. 2.10. *The following conditions are equivalent, for any simplicial ∞-groupoid X:*

(1) *X satisfies the Segal condition,*

$$X_r \xrightarrow{\simeq} X_1 \times_{X_0} \cdots \times_{X_0} X_1 \qquad r \geq 0.$$

(2) *The following square is a pullback for all $p, q \geq r$*

$$\begin{array}{ccc} X_{p-r+q} & \xrightarrow{d_0{}^{p-r}} & X_q \\ {\scriptstyle d_{p+1}{}^{q-r}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_{r+1}{}^{q-r}} \\ X_p & \xrightarrow[d_0{}^{p-r}]{} & X_r \end{array}$$

(3) *The following square is a pullback for all $n > 0$*

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_\bot} & X_n \\ {\scriptstyle d_\top}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_\bot]{} & X_{n-1} \end{array}$$

(4) *The following square is a pullback for all $p, q \geq 0$*

$$\begin{array}{ccc} X_{p+q} & \xrightarrow{d_0{}^{p}} & X_q \\ {\scriptstyle d_{p+1}{}^{q}}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle d_1{}^{q}} \\ X_p & \xrightarrow[d_0{}^{p}]{} & X_0 \end{array}$$

Proof. It is straightforward to show that the Segal condition implies (2). Now (3) and (4) are special cases of (2). Also (3) implies (2): the pullback in (2) is a composite of pullbacks of the type given in (3). Finally one shows inductively that (4) implies the Segal condition (1). □

A map $f : Y \to X$ of simplicial spaces is *cartesian* on an arrow $[n] \to [k]$ in Δ if the naturality square for f with respect to this arrow is a pullback.

Lemma. 2.11. *If a simplicial map $f : Y \to X$ is cartesian on outer face maps, and if X is a Segal space, then Y is a Segal space too.*

2.12. Rezk completeness. Let J denote the (ordinary) nerve of the groupoid generated by one isomorphism $0 \to 1$. A Segal space X is *Rezk complete* when the natural map

$$\mathrm{Map}(*, X) \to \mathrm{Map}(J, X)$$

(obtained by precomposing with $J \to *$) is an equivalence of ∞-groupoids. It means that the space of identity arrows is equivalent to the space of equivalences. (See [56, Thm.6.2], [7] and [37].) A Rezk complete Segal space is also called a *Rezk category.*

2.13. Ordinary nerve. Let $\mathscr{C}$ be a small 1-category. The *nerve* of $\mathscr{C}$ is the simplicial set

$$\begin{array}{rcl} N\mathscr{C} : \Delta^{\mathrm{op}} & \longrightarrow & \textbf{\textit{Set}} \\ {[n]} & \longmapsto & \mathrm{Fun}([n], \mathscr{C}), \end{array}$$

where $\mathrm{Fun}([n], \mathscr{C})$ is the *set* of strings of n composable arrows. Subexamples of this are given by any poset or any monoid. The simplicial sets that arise like this are precisely those satisfying the Segal condition (which is strict in this context). If each set is regarded as a discrete ∞-groupoid, $N\mathscr{C}$ is thus a Segal space. In general it is not Rezk complete, since some object may have a nontrivial automorphism. As an example, if $\mathscr{C}$ is a one-object groupoid (i.e. a group), then inside $(N\mathscr{C})_1$ the space of equivalences is the whole set $(N\mathscr{C})_1$, but the degeneracy map $s_0 : (N\mathscr{C})_0 \to (N\mathscr{C})_1$ is not an equivalence (unless the group is trivial).

2.14. The fat nerve of an essentially small 1-category. In most cases it is more interesting to consider the *fat nerve*, defined as the simplicial *groupoid*

$$\begin{aligned} X : \Delta^{\mathrm{op}} &\longrightarrow \boldsymbol{Grpd} \\ [k] &\longmapsto \mathrm{Map}(\Delta[k], \mathscr{C}), \end{aligned}$$

where $\mathrm{Map}(\Delta[k], \mathscr{C})$ is the mapping space, defined as the maximal subgroupoid of the functor category $\mathrm{Fun}(\Delta[k], \mathscr{C})$. In other words, $(\mathbf{N}\mathscr{C})_n$ is the groupoid whose objects are strings of n composable arrows in $\mathscr{C}$ and whose morphisms are connecting isos between such strings:

$$\begin{array}{ccccccccc} \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdots & \longrightarrow & \cdot \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & & & \downarrow\wr \\ \cdot & \longrightarrow & \cdot & \longrightarrow & \cdot & \longrightarrow & \cdots & \longrightarrow & \cdot \end{array}$$

It is straightforward to check the Segal condition, remembering that the pullbacks involved are homotopy pullbacks. For instance, the pullback $X_1 \times_{X_0} X_1$ has as objects strings of 'weakly composable' arrows, in the sense that the target of the first arrow is isomorphic to the source of the second, and a comparison isomorphism is specified. The Segal map $X_2 \to X_1 \times_{X_0} X_1$ is the inclusion of the subgroupoid consisting of strictly composable pairs. But any weakly composable pair is isomorphic to a strictly composable pair, and the comparison isomorphism is unique, hence the inclusion $X_2 \hookrightarrow X_1 \times_{X_0} X_1$ is an equivalence. Furthermore, the fat nerve is Rezk complete. Indeed, it is easy to see that inside X_1, the equivalences are the invertible arrows of $\mathscr{C}$. But any invertible arrow is equivalent to an identity arrow.

Note that if $\mathscr{C}$ is a category with no non-trivial isomorphisms (e.g. any Möbius category in the sense of Leroux) then the fat nerve coincides with the ordinary nerve, and if $\mathscr{C}$ is just equivalent to such a category then the fat nerve is level-wise equivalent to the ordinary nerve of any skeleton of $\mathscr{C}$.

2.15. Joyal–Tierney $t^!$ — the fat nerve of an ∞-category. The fat nerve construction is just a special case of the general construction $t^!$ of Joyal and Tierney [37], which is a functor from quasi-categories to complete Segal spaces, meaning specifically certain simplicial objects in the category of Kan complexes: given a quasi-category $\mathscr{C}$, the complete Segal space $t^!\mathscr{C}$ is given by

$$\begin{aligned} \Delta^{\mathrm{op}} &\longrightarrow \boldsymbol{Kan} \\ [n] &\longmapsto \big[[k] \mapsto \boldsymbol{sSet}(\Delta[n] \times \Delta'[k], \mathscr{C}) \big] \end{aligned}$$

where $\Delta'[k]$ denotes the groupoid freely generated by a string of k invertible arrows. They show that $t^!$ constitutes in fact a (right) Quillen equivalence between the simplicial sets with the Joyal model structure, and bisimplicial sets with the Rezk model structure.

Taking a more invariant viewpoint, talking about ∞-groupoids abstractly, the Joyal–Tierney $t^!$ functor associates to an ∞-category $\mathscr{C}$ the Rezk complete Segal space

$$\begin{aligned} \Delta^{\mathrm{op}} &\longrightarrow \boldsymbol{Grpd} \\ [n] &\longmapsto \mathrm{Map}(\Delta[n], \mathscr{C}). \end{aligned}$$

2.16. Fat nerve of bicategories with only invertible 2-cells. From a bicategory $\mathscr{C}$ with only invertible 2-cells one can get a complete Segal bigroupoid by a construction analogous to the fat nerve. (In fact, this can be viewed as the $t^!$ construction applied to the so-called Duskin nerve of $\mathscr{C}$.) The *fat nerve* of a bicategory $\mathscr{C}$ is the simplicial bigroupoid

$$\begin{aligned} \Delta^{\mathrm{op}} &\longrightarrow \boldsymbol{2Grpd} \\ [n] &\longmapsto \boldsymbol{PsFun}(\Delta[n], \mathscr{C}), \end{aligned}$$

the 2-groupoid of normalised pseudofunctors.

2.17. Monoidal groupoids. Important examples of the previous situation come from monoidal groupoids $(\mathscr{M}, \otimes, I)$. The fat nerve construction applied to the classifying space $B\mathscr{M}$ yields in this case a complete Segal bigroupoid, with zeroth space $B\mathscr{M}^{\mathrm{eq}}$, the classifying space of the full subcategory $\mathscr{M}^{\mathrm{eq}}$ spanned by the tensor-invertible objects.

The fat nerve construction can be simplified considerably in the case that $\mathscr{M}^{\mathrm{eq}}$ is contractible. This happens precisely when every tensor-invertible object is isomorphic to the unit object I and I admits no non-trivial automorphisms.

Proposition 2.18. *If $(\mathscr{M}, \otimes, I)$ is a monoidal groupoid such that $\mathscr{M}^{\mathrm{eq}}$ is contractible, then the simplicial bigroupoid given by the classifying space is equivalent to the simplicial 1-groupoid*

$$\begin{aligned} \Delta^{\mathrm{op}} &\longrightarrow \boldsymbol{1\text{-}Grpd} \\ [n] &\longmapsto \mathscr{M} \times \mathscr{M} \times \cdots \times \mathscr{M} =: \mathscr{M}^n. \end{aligned}$$

where the outer face maps project away an outer factor, the inner face maps tensor together two adjacent factors, and the degeneracy maps insert a neutral object.

We have omitted the proof, to avoid going into 2-category theory. (Note that the simplicial 1-groupoid that we obtain is not *strictly* simplicial, unless the monoidal structure is strict.)

Examples of monoidal groupoids satisfying the conditions of the Proposition are the monoidal groupoid $(\boldsymbol{FinSet}, +, 0)$ of finite sets and bijections or the monoidal groupoid $(\boldsymbol{Vect}, \oplus, \mathbf{0})$ of vector spaces and linear isomorphisms under direct sum. In contrast, the monoidal groupoid $(\boldsymbol{Vect}, \otimes, \Bbbk)$ of vector spaces and linear isomorphisms under tensor product is not of this kind, as the unit object has many automorphisms. The assignment $[n] \mapsto \boldsymbol{Vect}^{\otimes n}$ does constitute a Segal 1-groupoid, but it is not Rezk complete.

3. Decomposition spaces

Recall from Lemma 2.7 that generic and free maps in Δ admit pushouts along each other.

Definition. A *decomposition space* is a simplicial ∞-groupoid

$$X : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$$

such that the image of any pushout diagram in Δ of a generic map g along a free map f is a pullback of ∞-groupoids,

$$X\left(\begin{array}{ccc} [p] & \xleftarrow{g'} & [m] \\ {\scriptstyle f'}\uparrow & \lrcorner & \uparrow{\scriptstyle f} \\ [q] & \xleftarrow[g]{} & [n] \end{array}\right) \quad = \quad \begin{array}{ccc} X_p & \xrightarrow{g'^*} & X_m \\ {\scriptstyle f'^*}\downarrow & \lrcorner & \downarrow{\scriptstyle f^*} \\ X_q & \xrightarrow[g^*]{} & X_n. \end{array}$$

Remark 3.1. The notion of decomposition space can be seen as an abstraction of coalgebra, cf. Section 5 below: it is precisely the condition required to obtain a counital coassociative comultiplication on $\boldsymbol{Grpd}_{/X_1}$.

The notion is equivalent to the notion of unital (combinatorial) 2-Segal space introduced by Dyckerhoff and Kapranov [16] (their Definition 2.3.1, Definition 2.5.2, Definition 5.2.2, Remark 5.2.4). Briefly, their definition goes as follows. For any triangulation T of a convex polygon with n vertices, there is induced a simplicial subset $\Delta^T \subset \Delta[n]$. A simplicial space X is called 2-Segal if, for every triangulation T of every convex n-gon, the induced map $\mathrm{Map}(\Delta[n], X) \to \mathrm{Map}(\Delta^T, X)$ is a weak homotopy equivalence. Unitality is defined in terms of pullback conditions involving degeneracy maps, similar to our (4) below. The equivalence between decomposition spaces and unital 2-Segal spaces follows from Proposition 2.3.2 of [16] which gives a pullback criterion for the 2-Segal condition.

3.2. Alternative formulations of the pullback condition. To verify the conditions of the definition, it will in fact be sufficient to check a smaller collection of squares. On the other hand, the definition will imply that many other squares of interest are pullbacks too. The formulation in terms of generic and free maps is preferred both for practical reasons and for its conceptual simplicity compared to the smaller or larger collections of squares.

Recall from Lemma 2.7 that the generic-free pushouts used in the definition are just the identity extension squares,

$$\begin{array}{ccc} [n] & \xrightarrow{\quad g \quad} & [k] \\ \downarrow & & \downarrow \\ [a] \pm [n] \pm [b] & \xrightarrow[\mathrm{id} \pm g \pm \mathrm{id}]{} & [a] \pm [k] \pm [b] \end{array}$$

Such a square can be written as a vertical composite of squares in which either $a = 1$ and $b = 0$, or vice-versa. In turn, since the generic map g is a composite of inner face maps $d^i : [m-1] \to [m]$ $(0 < i < m)$ and degeneracy maps $s^j : [m+1] \to [m]$, these squares are horizontal composites of pushouts of a single generic d^i or s^j along $d^\perp$ or $d^\top$. Thus, to check that X is a decomposition space, it is sufficient to check the following special cases are pullbacks, for $0 < i < n$ and $0 \leq j \leq n$:

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array}$$

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{s_{1+j}} & X_{1+n+1} \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[s_j]{} & X_{n+1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{s_j} & X_{n+1+1} \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[s_j]{} & X_{n+1}. \end{array} \tag{4}$$

The following proposition shows we can be more economic: instead of checking all $0 < i < n$ it is enough to check all $n \geq 2$ and *some* $0 < i < n$, and instead of checking all $0 \leq j \leq n$ it is enough to check the case $j = n = 0$.

Proposition 3.3. *A simplicial ∞-groupoid X is a decomposition space if and only if the following diagrams are pullbacks*

$$\begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1, \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_0 & \xrightarrow[s_0]{} & X_1, \end{array}$$

and the following diagrams are pullbacks for some choice of $i = i_n$, $0 < i < n$, for each $n \geq 2$:

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}. \end{array}$$

Proof. To see the non-necessity of the other degeneracy cases, observe that for $n > 0$, every degeneracy map $s_j : X_n \to X_{n+1}$ is the section of an *inner* face map d_i (where $i = j$ or $i = j + 1$). Now in the diagram

$$\begin{array}{ccccc} X_{1+n} & \xrightarrow{s_{1+j}} & X_{1+n+1} & \xrightarrow{d_{1+i}} & X_{1+n} \\ {\scriptstyle d_\perp}\downarrow & & \downarrow{\scriptstyle d_\perp} & & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[s_j]{} & X_{n+1} & \xrightarrow[d_i]{} & X_n, \end{array}$$

the horizontal composites are identities, so the outer rectangle is a pullback, and the right-hand square is a pullback since it is one of cases outer face with inner face. Hence the left-hand square, by Lemma 1.10, is a pullback too. The case $s_0 : X_0 \to X_1$ is the only degeneracy map that is not the section of an inner face map, so we cannot eliminate the two cases involving this map. The non-necessity of the other inner-face-map cases is the content of the following lemma. ☐

Lemma. 3.4. *The following are equivalent for a simplicial ∞-groupoid X.*

(1) *For each $n \geq 2$, the following diagram is a pullback for all $0 < i < n$:*

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \qquad \left(\textit{resp.} \; \begin{array}{ccc} X_{n+1} & \xrightarrow{d_i} & X_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[d_i]{} & X_{n-1}, \end{array} \right)$$

(2) *For each $n \geq 2$, the above diagram is a pullback for some $0 < i < n$.*

(3) *For each $n \geq 2$, the following diagram is a pullback:*

$$\begin{array}{ccc} X_{1+n} & \xrightarrow{{d_2}^{n-1}} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array} \qquad \left(\textit{resp.} \; \begin{array}{ccc} X_{n+1} & \xrightarrow{{d_1}^{n-1}} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_n & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array} \right)$$

Proof. The hypothesised pullback in (2) is a special case of that in (1), and that in (3) is a horizontal composite of those in (2), since there is a unique generic map

$[1] \to [n]$ in Δ for each n. The implication (3) $\Rightarrow$ (1) follows by Lemma 1.10 and the commutativity for $0 < i < n$ of the diagram

$$\begin{array}{ccccc} X_{1+n} & \xrightarrow{d_{1+i}} & X_n & \xrightarrow{{d_2}^{n-1}} & X_2 \\ {\scriptstyle d_\perp}\downarrow & \lrcorner & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_i]{} & X_{n-1} & \xrightarrow[{d_1}^{n-1}]{} & X_1 \end{array}$$

Similarly for the 'resp.' case. □

Proposition 3.5. *Any Segal space is a decomposition space.*

Proof. Let X be Segal space. In the diagram ($n \geq 2$)

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_n} & X_n & \xrightarrow{d_\top} & X_{n-1} \\ {\scriptstyle d_\perp}\downarrow & & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_n & \xrightarrow[d_{n-1}]{} & X_{n-1} & \xrightarrow[d_\top]{} & X_{n-2}, \end{array}$$

since the horizontal composites are equal to $d_\top \circ d_\top$, both the outer rectangle and the right-hand square are pullbacks by the Segal condition (2.10 (3)). Hence the left-hand square is a pullback. This establishes the third pullback condition in Proposition 3.3. In the diagram

$$\begin{array}{ccccc} X_1 & \xrightarrow{s_1} & X_2 & \xrightarrow{d_\top} & X_1 \\ {\scriptstyle d_\perp}\downarrow & & {\scriptstyle d_\perp}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\perp} \\ X_0 & \xrightarrow[s_0]{} & X_1 & \xrightarrow[d_\top]{} & X_0, \end{array}$$

since the horizontal composites are identities, the outer rectangle is a pullback, and the right-hand square is a pullback by the Segal condition. Hence the left-hand square is a pullback, establishing the first of the pullback conditions in Proposition 3.3. The remaining two conditions of Proposition 3.3, those involving $d_\top$ instead of $d_\perp$, are obtained similarly by interchanging the roles of $\perp$ and $\top$. □

Remark 3.6. This result was also obtained by Dyckerhoff and Kapranov [16] (Propositions 2.3.3, 2.5.3, and 5.2.6).

Corollary 2.8 implies the following important property of decomposition spaces.

Lemma. 3.7. *In a decomposition space X, every generic face map is a pullback of $d_1 : X_2 \to X_1$, and every degeneracy map is a pullback of $s_0 : X_0 \to X_1$.*

Thus, even though the spaces in degree ≥ 2 are not fibre products of X_1 as in a Segal space, the higher generic face maps and degeneracies are determined by 'unit' and 'composition',

$$X_0 \xrightarrow{s_0} X_1 \xleftarrow{d_1} X_2.$$

In Δ^{op} there are more pullbacks than those between generic and free. Diagram (2) in 2.2 is a pullback in Δ^{op} that is not preserved by all decomposition spaces, though it is preserved by all Segal spaces. On the other hand, certain other pullbacks in Δ^{op} are preserved by general decomposition spaces. We call them colloquially 'bonus pullbacks':

Lemma. 3.8. *For a decomposition space X, the following squares are pullbacks:*

$$\begin{array}{ccc} X_{n+1} & \xrightarrow{d_j} & X_n \\ {\scriptstyle s_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+2} & \xrightarrow[d_{j+1}]{} & X_{n+1} \end{array} \quad \text{for all } i < j, \quad \text{and} \quad \begin{array}{ccc} X_{n+1} & \xrightarrow{d_j} & X_n \\ {\scriptstyle s_{i+1}}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+2} & \xrightarrow[d_j]{} & X_{n+1} \end{array} \quad \text{for all } j \leq i.$$

Proof. We treat the case $i < j$; for the other case, interchange the roles of $\top$ and $\bot$. Postcompose horizontally with sufficiently many $d_\top$ to make the total composite free:

$$\begin{array}{ccccc} X_{n+1} & \xrightarrow{d_j} & X_n & \xrightarrow{{d_\top}^{n+1-j}} & X_{j-1} \\ {\scriptstyle s_i}\downarrow & & {\scriptstyle s_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+2} & \xrightarrow[d_{j+1}]{} & X_{n+1} & \xrightarrow[{d_\top}^{n+1-j}]{} & X_j. \end{array}$$

The horizontal composite maps are now ${d_\top}^{n+2-j}$, so the outer rectangle is a pullback, and the second square is a pullback. Hence by the basic lemma 1.10, also the first square is a pullback, as claimed. ☐

Lemma. 3.9. *For a decomposition space X, the following squares are pullbacks for all $i < j$:*

$$\begin{array}{ccc} X_n & \xrightarrow{s_{j-1}} & X_{n+1} \\ {\scriptstyle s_i}\downarrow & \lrcorner & \downarrow{\scriptstyle s_i} \\ X_{n+1} & \xrightarrow[s_j]{} & X_{n+2} \end{array}$$

Proof. Just observe that s_j is a section to d_{j+1}, and apply the standard argument: if d_{j+1} is an outer face map then the square is a basic generic-free pullback; if d_{j+1} is inner, we can use instead the previous lemma. ☐

4. Conservative ULF functors and decalage

Definition. A simplicial map $F : Y \to X$ is called *ULF (unique lifting of factorisations)* if it is a cartesian natural transformation on generic face maps of Δ. It is called *conservative* if it is cartesian on degeneracy maps. It is called *cULF* if it is both conservative and ULF.

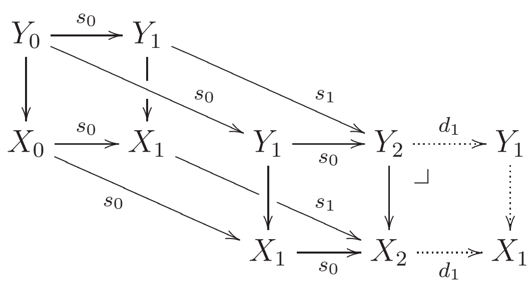
Lemma. 4.1. *For a simplicial map $F : Y \to X$, the following are equivalent.*

(1) *F is cartesian on all generic maps (i.e. cULF).*
(2) *F is cartesian on every inner face map and on every degeneracy map.*
(3) *F is cartesian on every generic map of the form $[1] \to [n]$.*

Proof. That (1) implies (2) is trivial. The implication (2) $\Rightarrow$ (3) is easy since the generic map $[1] \to [n]$ factors as a sequence of inner face maps (or is a degeneracy map if $n = 0$). For the implication (3) $\Rightarrow$ (1), consider a general generic map $[n] \to [m]$, and observe that if F is cartesian on the composite of generic maps $[1] \to [n] \to [m]$ and also on the generic map $[1] \to [n]$, then it is cartesian on $[n] \to [m]$ also. ☐

Proposition 4.2. *If X and Y are decomposition spaces then every ULF map $F : Y \to X$ is automatically conservative.*

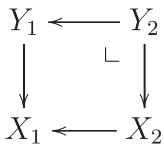
Proof. In the diagram



the front square is a pullback since it is a section to the dotted square, which is a pullback since F is ULF. The same argument shows that F is cartesian on all degeneracy maps that are sections to generic face maps. This includes all degeneracy maps except the one appearing in the back square of the diagram. But since the top and bottom slanted squares are bonus pullbacks (3.9), also the back square is a pullback. □

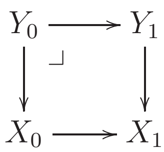
The following result is a consequence of Lemma 3.7 and Proposition 4.2.

Lemma. 4.3. *A simplicial map $F : Y \to X$ between decomposition spaces is cULF if and only if it is cartesian on the generic map* $[1] \to [2]$



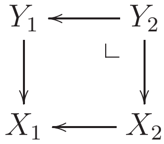
Remark 4.4. The notion of cULF can be seen as an abstraction of coalgebra homomorphism, cf. 8.2 below: 'conservative' corresponds to counit preservation, 'ULF' corresponds to comultiplicativity.

In the special case where X and Y are fat nerves of 1-categories, then the condition that the square



be a pullback is precisely the classical notion of conservative functor (i.e. if $f(a)$ is invertible then already a is invertible).

Similarly, the condition that the square



be a pullback is an up-to-isomorphism version of the classical notion of ULF functor, implicit already in Content–Lemay–Leroux [12], and perhaps made explicit first by Lawvere [44]; it is equivalent to the notion of discrete Conduché fibration [30]. See Street [62] for the 2-categorical notion. In the case of the Möbius categories of Leroux, where there are no invertible arrows around, the two notions of ULF coincide.

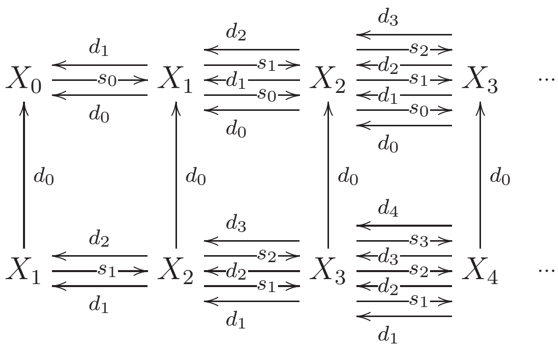
Example 4.5. Here is an example of a functor which is not cULF in Lawvere's sense (is not cULF on classical nerves), but which is cULF in the homotopical sense. Namely, let $\boldsymbol{OI}$ denote the category of finite ordered sets and monotone injections. Let $\boldsymbol{I}$ denote the category of finite sets and injections. The forgetful functor $\boldsymbol{OI} \to \boldsymbol{I}$ is not cULF in the classical sense, because the identity monotone map $\underline{2} \to \underline{2}$ admits a factorisation in $\boldsymbol{I}$ that does not lift to $\boldsymbol{OI}$, namely the factorisation into two nontrivial transpositions. However, it is cULF in our sense, as can easily be verified by checking that the square

$$\begin{array}{ccc} \boldsymbol{OI}_1 & \longleftarrow & \boldsymbol{OI}_2 \\ \downarrow & \lrcorner & \downarrow \\ \boldsymbol{I}_1 & \longleftarrow & \boldsymbol{I}_2 \end{array}$$

is a pullback by computing the fibres of the horizontal maps over a given monotone injection.

Lemma. 4.6. *If X is a decomposition space and $f : Y \to X$ is cULF then also Y is a decomposition space.*

4.7. Decalage. (See Illusie [29]). Given a simplicial space X as the top row in the following diagram, the *lower dec* $\mathrm{Dec}_\perp(X)$ is a new simplicial space (bottom row in the diagram) obtained by deleting X_0 and shifting everything one place down, deleting also all d_0 face maps and all s_0 degeneracy maps. It comes equipped with a simplicial map $d_\perp : \mathrm{Dec}_\perp(X) \to X$ given by the original d_0:



Similarly, the upper dec, denoted $\mathrm{Dec}_\top(X)$ is obtained by instead deleting, in each degree, the last face map $d_\top$ and the last degeneracy map $s_\top$.

4.8. Decalage in terms of an adjunction. (See Lawvere [43].) The functor $\mathrm{Dec}_\perp$ can be described more conceptually as follows. There is an 'add-bottom' endofunctor $b : \Delta \to \Delta$, which sends $[k]$ to $[k+1]$ by adding a new bottom element. This is in fact a monad; the unit $\varepsilon : \mathrm{Id} \Rightarrow b$ is given by the bottom coface map $d^\perp$. The lower dec is given by precomposition with b:

$$\mathrm{Dec}_\perp(X) = b^* X$$

Hence $\mathrm{Dec}_\perp$ is a comonad, and its counit is the bottom face map $d_\perp$.

Similarly, the upper dec is obtained from the 'add-top' monad on Δ. In [23] we shall exploit crucially the combination of the two comonads.

4.9. Slice interpretation. If X is the strict nerve of a category $\mathbb{C}$ then there is a close relationship between the upper dec and the slice construction. For the strict nerve, $X = N\mathbb{C}$, $\mathrm{Dec}_\top X$ is the disjoint union of all (the nerves of) the slice categories of $\mathbb{C}$:

$$\mathrm{Dec}_\top X = \sum_{x \in X_0} N(\mathbb{C}_{/x}).$$

(In general it is a homotopy sum.)

Any individual slice category can be extracted from the upper dec, by exploiting that the upper dec comes with a canonical augmentation given by (iterating) the bottom face map. The slice over an object x is obtained by pulling back the upper dec along the name of x:

$$\begin{array}{ccc} 1 & \longleftarrow & N\mathbb{C}_{/x} \\ {\scriptstyle \ulcorner x \urcorner}\downarrow & \llcorner & \downarrow \\ X_0 & \underset{d_\bot}{\longleftarrow} & \operatorname{Dec}_\top X \end{array}$$

There is a similar relationship between the lower dec and the coslices.

Proposition 4.10. *If X is a decomposition space then $\operatorname{Dec}_\top(X)$ and $\operatorname{Dec}_\bot(X)$ are Segal spaces, and the maps $d_\top : \operatorname{Dec}_\top(X) \to X$ and $d_\bot : \operatorname{Dec}_\bot(X) \to X$ are cULF.*

Proof. We put $Y = \operatorname{Dec}_\top(X)$ and check the pullback condition 2.10 (3),

$$\begin{array}{ccc} Y_{n+1} & \xrightarrow{d_\bot} & Y_n \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ Y_n & \underset{d_\bot}{\longrightarrow} & Y_{n-1}. \end{array}$$

This is the same as

$$\begin{array}{ccc} X_{n+2} & \xrightarrow{d_\bot} & X_{n+1} \\ {\scriptstyle d_{\top-1}}\downarrow & \lrcorner & \downarrow{\scriptstyle d_{\top-1}} \\ X_{n+1} & \underset{d_\bot}{\longrightarrow} & Y_n \end{array}$$

and since now the horizontal face maps that with respect to Y were outer face maps, now become inner face maps in X, this square is one of the decomposition square axiom pullbacks. The cULF conditions say that the various $d_\top$ form pullbacks with all generic maps in X. But this follows from the decomposition space axiom for X. □

Theorem 4.11. *For a simplicial ∞-groupoid $X : \Delta^{\mathrm{op}} \to$ **Grpd**, the following are equivalent*

(1) *X is a decomposition space*
(2) *both $\operatorname{Dec}_\top(X)$ and $\operatorname{Dec}_\bot(X)$ are Segal spaces, and the two comparison maps back to X are ULF and conservative.*
(3) *both $\operatorname{Dec}_\top(X)$ and $\operatorname{Dec}_\bot(X)$ are Segal spaces, and the two comparison maps back to X are conservative.*
(4) *both $\operatorname{Dec}_\top(X)$ and $\operatorname{Dec}_\bot(X)$ are Segal spaces, and the following squares are pullbacks:*

$$\begin{array}{ccc} X_1 & \xrightarrow{s_1} & X_2 \\ {\scriptstyle d_\bot}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\bot} \\ X_0 & \underset{s_0}{\longrightarrow} & X_1, \end{array} \qquad \begin{array}{ccc} X_1 & \xrightarrow{s_0} & X_2 \\ {\scriptstyle d_\top}\downarrow & \lrcorner & \downarrow{\scriptstyle d_\top} \\ X_0 & \underset{s_0}{\longrightarrow} & X_1. \end{array}$$

Proof. The implication (1) ⇒ (2) is just the preceding Proposition, and the implications (2) ⇒ (3) ⇒ (4) are specialisations. The implication (4) ⇒ (1) follows from Proposition 3.3. □

Remark 4.12. Dyckerhoff and Kapranov [16] (Theorem 6.3.2) obtain the result that a simplicial space is 2-Segal (i.e. a decomposition space except that there are no conditions imposed on degeneracy maps) if and only if both Decs are Segal spaces.

4.13. Right and left fibrations. A functor of Segal spaces $f : Y \to X$ is called a *right fibration* if it is cartesian on $d_\bot$ and on all generic maps, or a *left fibration* if it is cartesian on $d_\top$ and on generic maps. Here the condition on generic degeneracy maps is in fact a consequence of that on the face maps. These notions are most meaningful when the Segal spaces involved are Rezk complete.

Proposition 4.14. *If $f : Y \to X$ is a conservative ULF functor between decomposition spaces, then* $\mathrm{Dec}_\bot(f) : \mathrm{Dec}_\bot(Y) \to \mathrm{Dec}_\bot(X)$ *is a right fibration of Segal spaces, cf. 4.13. Similarly,* $\mathrm{Dec}_\top(f) : \mathrm{Dec}_\top(Y) \to \mathrm{Dec}_\top(X)$ *is a left fibration.*

Proof. It is clear that if f is cULF then so is $\mathrm{Dec}_\bot(f)$. The further claim is that $\mathrm{Dec}_\bot(f)$ is also cartesian on d_0. But d_0 was originally a d_1, and in particular was generic, hence has cartesian component. □

5. Incidence coalgebras

We now turn to the incidence coalgebra (with ∞-groupoid coefficients) associated to any decomposition space, and explain the origin of the decomposition space axioms.

The incidence coalgebra associated to a decomposition space X will be a comonoid object in the symmetric monoidal ∞-category $\boldsymbol{LIN}$, and the underlying object is $\boldsymbol{Grpd}_{/X_1}$. Since $\boldsymbol{Grpd}_{/X_1} \otimes \boldsymbol{Grpd}_{/X_1} = \boldsymbol{Grpd}_{/X_1 \times X_1}$, and since linear functors are given by spans, to define a comultiplication functor is to give a span

$$X_1 \leftarrow M \to X_1 \times X_1$$

For any simplicial space X, we can consider the following structure maps on $\boldsymbol{Grpd}_{/X_1}$.

5.1. Comultiplication and counit. The span

$$X_1 \xleftarrow[d_1]{m_X} X_2 \xrightarrow[(d_2,d_0)]{p_X} X_1 \times X_1$$

defines a linear functor, the *comultiplication*

$$\begin{array}{rcl} \Delta : \boldsymbol{Grpd}_{/X_1} & \longrightarrow & \boldsymbol{Grpd}_{/(X_1 \times X_1)} \\ (S \xrightarrow{s} X_1) & \longmapsto & p_{X!} \circ m_X^*(s). \end{array}$$

Likewise, the span

$$X_1 \xleftarrow[s_0]{u_X} X_0 \xrightarrow{t_X} 1$$

defines a linear functor, the *counit*

$$\begin{array}{rcl} \varepsilon : \boldsymbol{Grpd}_{/X_1} & \longrightarrow & \boldsymbol{Grpd} \\ (S \xrightarrow{s} X_1) & \longmapsto & t_{X!} \circ u_X^*(s). \end{array}$$

The desired coassociativity diagram (which should commute up to equivalence)

$$\begin{array}{ccc} \boldsymbol{Grpd}_{/X_1} & \xrightarrow{\Delta} & \boldsymbol{Grpd}_{/X_1 \times X_1} \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \times \mathrm{id}} \\ \boldsymbol{Grpd}_{/X_1 \times X_1} & \xrightarrow[\mathrm{id} \times \Delta]{} & \boldsymbol{Grpd}_{/X_1 \times X_1 \times X_1} \end{array}$$

is induced by the spans in the outline of the following diagram.

$$\begin{array}{ccccc}
X_1 & \xleftarrow{\;d_1\;} & X_2 & \xrightarrow{(d_2,d_0)} & X_1 \times X_1 \\
{\scriptstyle d_1}\uparrow & & {\scriptstyle d_1}\uparrow\;\lrcorner & & \uparrow{\scriptstyle d_1\times \mathrm{id}} \\
X_2 & \xleftarrow{\;d_2\;} & X_3 & \xrightarrow[(d_3,d_0d_0)]{} & X_2 \times X_1 \\
{\scriptstyle (d_2,d_0)}\downarrow & & \llcorner\;\downarrow{\scriptstyle (d_2^2,d_0)} & & \downarrow{\scriptstyle (d_2,d_0)\times \mathrm{id}} \\
X_1 \times X_1 & \xleftarrow[\mathrm{id}\times d_1]{} & X_1 \times X_2 & \xrightarrow[\mathrm{id}\times(d_2,d_0)]{} & X_1 \times X_1 \times X_1
\end{array}$$

Coassociativity will follow from Beck–Chevalley equivalences if the interior part of the diagram can be established, with pullbacks as indicated. Consider the upper right-hand square: it will be a pullback if and only if its composite with the first projection is a pullback:

$$\begin{array}{ccccc}
X_2 & \xrightarrow{(d_2,d_0)} & X_1 \times X_1 & \xrightarrow{\mathrm{pr}_1} & X_1 \\
{\scriptstyle d_1}\uparrow\;\lrcorner & & {\scriptstyle d_1\times\mathrm{id}}\uparrow\;\lrcorner & & \uparrow{\scriptstyle d_1} \\
X_3 & \xrightarrow[(d_3,d_0d_0)]{} & X_2 \times X_1 & \xrightarrow[\mathrm{pr}_1]{} & X_2
\end{array}$$

But demanding the outer rectangle to be a pullback is precisely one of the basic decomposition space axioms. This argument is the origin of the decomposition space axioms.

Just finding an equivalence is not enough, though. Higher coherence has to be established, which will be accounted for by the full decomposition space axioms. To establish coassociativity in a strong homotopy sense we must deal on an equal footing with all 'reasonable' spans

$$\prod X_{n_j} \leftarrow \prod X_{m_j} \rightarrow \prod X_{k_i}$$

which could arise from composites of products of the comultiplication and counit. We therefore take a more abstract approach, relying on some more simplicial machinery. This also leads to another characterisation of decomposition spaces, and is actually of independent interest.

6. More simplicial preliminaries

6.1. The category $\underline{\Delta}$ of finite ordinals (the algebraist's Delta). We denote by $\underline{\Delta}$ the category of all finite ordinals (including the empty ordinal) and monotone maps. Clearly $\Delta \subset \underline{\Delta}$ (presheaves on $\underline{\Delta}$ are augmented simplicial sets), but this is not the most useful relationship between the two categories. We will thus use a different notation for the objects of $\underline{\Delta}$, given by their cardinality, with an underline:

$$\underline{n} = \{1, 2, \ldots, n\}.$$

The category $\underline{\Delta}$ is monoidal under ordinal sum

$$\underline{m} + \underline{n} := \underline{m+n},$$

with $\underline{0}$ as the neutral object.

The cofaces $d^i : \underline{n{-}1} \to \underline{n}$ and codegeneracies $s^i : \underline{n{+}1} \to \underline{n}$ in $\underline{\Delta}$ are, as usual, the injective and surjective monotone maps which skip and repeat the ith element, respectively, but note that now the index is $1 \leq i \leq n$.

Lemma. 6.2. *There is a canonical equivalence of monoidal categories (an isomorphism, if we consider the usual skeleta of these categories)*

$$\begin{array}{ccc} (\underline{\Delta}, +, \underline{0}) & \simeq & (\Delta^{\mathrm{op}}_{\mathrm{gen}}, \pm, [0]) \\ \underline{k} & \leftrightarrow & [k] \end{array}$$

Proof. The map from left to right sends $\underline{k} \in \underline{\Delta}$ to

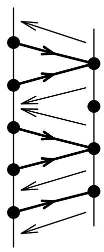
$$\mathrm{Hom}_{\underline{\Delta}}(\underline{k}, \underline{2}) \simeq [k] \in \Delta^{\mathrm{op}}_{\mathrm{gen}}.$$

The map in the other direction sends $[k]$ to the ordinal

$$\mathrm{Hom}_{\Delta_{\mathrm{gen}}}([k], [1]) \simeq \underline{k}.$$

In both cases, functoriality is given by precomposition. □

In both categories we can picture the objects as a line with some dots. The dots then represent the elements in $\underline{k}$, while the edges represent the elements in $[k]$; a map operates on the dots when considered a map in $\underline{\Delta}$ while it operates on the edges when considered a map in Δ_{gen}. Here is a picture of a certain map $\underline{5} \to \underline{4}$ in $\underline{\Delta}$ and of the corresponding map $[5] \leftarrow [4]$ in Δ_{gen}.



6.3. A twisted arrow category of $\underline{\Delta}$. Consider the category $\mathscr{D}$ whose objects are the arrows $\underline{n} \to \underline{k}$ of $\underline{\Delta}$ and whose morphisms (g, f) from $a : \underline{m} \to \underline{h}$ to $b : \underline{n} \to \underline{k}$ are commutative squares

$$\begin{array}{ccc} \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle a}\downarrow & (g,f) & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k}. \end{array} \tag{5}$$

That is, $\mathscr{D}^{\mathrm{op}}$ is the twisted arrow category [52, 4] of $\underline{\Delta}$.

There is a canonical factorisation system on $\mathscr{D}$: any morphism (5) factors uniquely as

$$\begin{array}{ccccc} \underline{m} & \xrightarrow{=} & \underline{m} & \xrightarrow{g} & \underline{n} \\ {\scriptstyle a=fbg}\downarrow & \varphi & \downarrow{\scriptstyle bg} & \gamma & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k} & \xleftarrow[=]{} & \underline{k} \end{array}$$

The maps $\varphi = (\mathrm{id}, f) : fb \to b$ in the left-hand class of the factorisation system are termed *segalic*,

$$\begin{array}{ccc} \underline{m} & \xrightarrow{=} & \underline{m} \\ {\scriptstyle fb}\downarrow & \varphi & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[f]{} & \underline{k}. \end{array} \tag{6}$$

The maps $\gamma = (g, \mathrm{id}) : bg \to b$ in the right-hand class are termed *ordinalic* and may be identified with maps in the slice categories $\underline{\Delta}_{/\underline{h}}$

$$\begin{array}{ccc} \underline{m} & \xrightarrow{\ g\ } & \underline{n} \\ {\scriptstyle bg}\downarrow & \gamma & \downarrow{\scriptstyle b} \\ \underline{h} & \xleftarrow[=]{} & \underline{h}. \end{array} \tag{7}$$

Observe that $\underline{\Delta}$ is isomorphic to the subcategory of objects with target $\underline{h} = \underline{1}$, termed the *connected objects* of $\mathscr{D}$,

$$\underline{\Delta} \xrightarrow{\ =\ } \underline{\Delta}_{/\underline{1}} \xrightarrow{\ \subseteq\ } \mathscr{D}. \tag{8}$$

The ordinal sum operation in $\underline{\Delta}$ induces a monoidal operation in $\mathscr{D}$: the *external sum* $(\underline{n} \to \underline{k}) \oplus (\underline{n}' \to \underline{k}')$ of objects in $\mathscr{D}$ is their ordinal sum $\underline{n} + \underline{n}' \to \underline{k} + \underline{k}'$ as morphisms in $\underline{\Delta}$. The neutral object is $\underline{0} \to \underline{0}$. The inclusion functor (8) is not monoidal, but it is easily seen to be oplax monoidal by means of the codiagonal map $\underline{1} + \underline{1} \to \underline{1}$.

Each object $\underline{m} \xrightarrow{a} \underline{k}$ of $\mathscr{D}$ is an external sum of connected objects,

$$a \quad = \quad a_1 \oplus a_2 \oplus \cdots \oplus a_k \quad = \quad \bigoplus_{i \in \underline{k}} \left(\underline{m}_i \xrightarrow{a_i} \underline{1} \right), \tag{9}$$

where $\underline{m}_i$ is (the cardinality of) the fibre of a over $i \in \underline{k}$.

Any segalic map (6) and any ordinalic map (7) in $\mathscr{D}$ may be written uniquely as external sums

$$\varphi \quad = \quad \varphi_1 \oplus \varphi_2 \oplus \cdots \oplus \varphi_h \quad = \quad \bigoplus_{j \in \underline{h}} \left(\begin{array}{ccc} \underline{m}_j & \xrightarrow{\ =\ } & \underline{m}_j \\ \downarrow & \varphi_j & \downarrow{\scriptstyle b_j} \\ \underline{1} & \longleftarrow & \underline{k}_j \end{array} \right) \tag{10}$$

$$\gamma \quad = \quad \gamma_1 \oplus \gamma_2 \oplus \cdots \oplus \gamma_h \quad = \quad \bigoplus_{j \in \underline{h}} \left(\underline{m}_j \xrightarrow{\gamma_j} \underline{n}_j \right) \tag{11}$$

where each γ_j is a map in $\underline{\Delta}_{/\underline{1}} = \underline{\Delta}$.

In fact $\mathscr{D}$ is a universal monoidal category in the following sense.

Proposition 6.4. *For any cartesian category $(\mathscr{C}, \times, 1)$, there is an equivalence*

$$\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{C}) \;\simeq\; \mathrm{Fun}^{\otimes}((\mathscr{D}, \oplus, 0), (\mathscr{C}, \times, 1))$$

between the categories of simplicial objects X in $\mathscr{C}$ and of monoidal functors $\overline{X} : \mathscr{D} \to \mathscr{C}$. The correspondence between X and $\overline{X}$ is determined by following properties.

(a) The functors $X : \Delta^{\mathrm{op}} \to \mathscr{C}$ and $\overline{X} : \mathscr{D} \to \mathscr{C}$ agree on the common subcategory $\Delta^{\mathrm{op}}_{\mathrm{gen}} \cong \underline{\Delta}$,

$$\begin{array}{ccccc} \Delta^{\mathrm{op}}_{\mathrm{gen}} & \hookrightarrow & \Delta^{\mathrm{op}} & & \\ {\scriptstyle \cong}\downarrow & & & \searrow^{X} & \\ & & & & \mathscr{C}. \\ & & & \nearrow_{\overline{X}} & \\ \underline{\Delta} & \hookrightarrow & \mathscr{D} & & \end{array}$$

(b) Let $(\underline{m} \xrightarrow{a} \underline{k}) = \bigoplus_i (\underline{m}_i \xrightarrow{a} \underline{1})$ be the external sum decomposition (9) of any object of $\mathscr{D}$, and denote by $f_i : [m_i] \rightarrowtail [m_1] \pm \cdots \pm [m_k] = [m]$ the canonical free map in Δ,

for $i \in \underline{k}$. Then

$$\overline{X}\left(\begin{array}{ccc} \underline{m} & \xrightarrow{=} & \underline{m} \\ \downarrow & \varphi & \downarrow a \\ \underline{1} & \longleftarrow & \underline{k} \end{array}\right) = (X(f_1), \dots, X(f_k)) \ : \ X_m \longrightarrow \prod_{i \in \underline{k}} X_{m_i}$$

and each $X(f_i)$ is the composite of $\overline{X}(\varphi)$ with the projection to X_i.

Proof. Given $\overline{X}$, property (a) says that there is a unique way to define X on objects and generic maps. Conversely, given X, then for any object $a : \underline{m} \to \underline{k}$ in $\mathscr{D}$ we have

$$\overline{X}_a = \prod_{i \in \underline{k}} \overline{X}_{a_i} = \prod_{i \in \underline{k}} X_{m_i}$$

using (9), and for any ordinalic map γ we have

$$\overline{X}(\gamma) = \prod_{i \in \underline{k}} \overline{X}(\gamma_i) = \prod_{i \in \underline{k}} X(g_i)$$

using (11), where $g_i \in \Delta^{\mathrm{op}}_{\mathrm{gen}}$ corresponds to $\gamma_i \in \underline{\Delta}$.

Thus we have a bijection between functors X defined on $\Delta^{\mathrm{op}}_{\mathrm{gen}}$ and monoidal functors $\overline{X}$ defined on the ordinalic subcategory of $\mathscr{D}$. Now we consider the free and segalic maps. Given $\overline{X}$, property (b) says that for any free map $f_r : [m_r] \to [m]$ we may define

$$X(f_r) = \left(X_m \xrightarrow{\overline{X}(\varphi)} \prod_{i \in \underline{k}} X_{m_i} \twoheadrightarrow X_{m_r} \right)$$

We may assume $k = 3$: given the factorisation

$$\varphi = \left(\begin{array}{ccccc} \underline{m} & \xrightarrow{=} & \underline{m}_{<r} + \underline{m}_r + \underline{m}_{>r} & \xrightarrow{=} & \sum_{i \in \underline{k}} m_i \\ \downarrow & \varphi_2 & \downarrow & \varphi_1 \oplus \mathrm{id} \oplus \varphi_3 & \downarrow \\ \underline{1} & \longleftarrow & \underline{3} & \longleftarrow & \underline{k} \end{array}\right)$$

one sees the value $X(f_r)$ is well defined from the following diagram

$$\begin{array}{ccccc} X_m & \xrightarrow{\overline{X}(\varphi_2)} & X_{m_{<r}} \times X_{m_r} \times X_{m_{>r}} & \xrightarrow{\overline{X}(\varphi_1) \times \mathrm{id} \times \overline{X}(\varphi_3)} & \prod_{i \in \underline{k}} X_{m_i} \\ & {\scriptstyle X(f_r)} \searrow & & \searrow & \downarrow \\ & & & & X_{m_r} \ . \end{array}$$

Functoriality of X on a composite of free maps, say $[m_3] \rightarrowtail [\sum_2^4 m_i] \rightarrowtail [\sum_1^5 m_i]$, now follows from the diagram

$$\begin{array}{ccccc} X_{\sum_1^5 m_i} & \longrightarrow & \prod_1^5 X_{m_i} & \longrightarrow\!\!\!\!\rightarrow & X_{m_3} \\ \searrow & \nearrow & \searrow & \nearrow & \\ & X_{m_1} \times X_{\sum_2^4 m_i} \times X_{m_5} & & \prod_2^4 X_{m_i} & \\ & \searrow & & \nearrow & \\ & & X_{\sum_2^4 m_i} & & \end{array}$$

in which the first triangle commutes by functoriality of $\overline{X}$.

Conversely, given X, then property (b) says how to define $\overline{X}$ on segalic maps with connected domain and hence, by (10), on all segalic maps. Functoriality of $\overline{X}$ on a composite of segalic maps, say $(\mathrm{id}, \underline{1} \leftarrow \underline{h} \leftarrow \underline{k})$, follows from functoriality of X:

$$X_m \xrightarrow{(X([m_j]\rightarrowtail[m]))_{j\in\underline{h}}} \prod_{j\in\underline{h}} X_{m_j} \xrightarrow{\prod_{j\in\underline{h}}(X([m_i]\rightarrowtail[m_j]))_{i\in\underline{k}_j}} \prod_{j\in\underline{h}}\prod_{i\in\underline{k}_j} X_{m_i}, \qquad X_m \xrightarrow{(X([m_i]\rightarrowtail[m]))_{i\in\underline{k}}} \prod_{j\in\underline{h}}\prod_{i\in\underline{k}_j} X_{m_i}$$

It remains only to check that the construction of $\overline{X}$ from X (and of X from $\overline{X}$) is well defined on composites of ordinalic followed by segalic (free followed by generic) maps. One then has the mutually inverse equivalences required. Consider the factorisations in $\mathscr{D}$,

$$\begin{array}{ccccc} \underline{m} & \xrightarrow{=} & \underline{m} & \xrightarrow{g} & \underline{m}' \\ \downarrow & \varphi & \downarrow & \gamma & \downarrow \\ \underline{1} & \longleftarrow & \underline{k} & \xleftarrow[=]{} & \underline{k} \end{array} \quad = \quad \begin{array}{ccccc} \underline{m} & \xrightarrow{g} & \underline{m}' & \xrightarrow{=} & \underline{m}' \\ \downarrow & \gamma' & \downarrow & \varphi' & \downarrow \\ \underline{1} & \xleftarrow[=]{} & \underline{1} & \longleftarrow & \underline{k}. \end{array}$$

To show that $\overline{X}$ is well defined, we must show that the diagrams

$$\begin{array}{ccccc} X_m & \xrightarrow{\overline{X}(\varphi)=(X(f_1),\dots,X(f_k))} & \prod X_{m_i} & \twoheadrightarrow & X_{m_r} \\ {\scriptstyle \overline{X}(\gamma')=X(\tilde{g})}\downarrow & & \downarrow{\scriptstyle \overline{X}(\gamma')=\prod X(\tilde{g}_i)} & & \downarrow{\scriptstyle X(\tilde{g}_r)} \\ X_{m'} & \xrightarrow[\overline{X}(\varphi')=(X(f'_1),\dots,X(f'_k))]{} & \prod X_{m'_i} & \twoheadrightarrow & X_{m'_r}, \end{array}$$

commute for each r, where $\tilde{g}$, $\tilde{g}_i$ in Δ_{gen} correspond to g, g_i in $\underline{\Delta}$. This follows by functoriality of X, since $\tilde{g}$ restricted to m'_r is the corestriction of $\tilde{g}_r$. Finally we observe that this diagram, with $k = 3$ and $r = 2$, also serves to show that the construction of X from $\overline{X}$ is well defined on

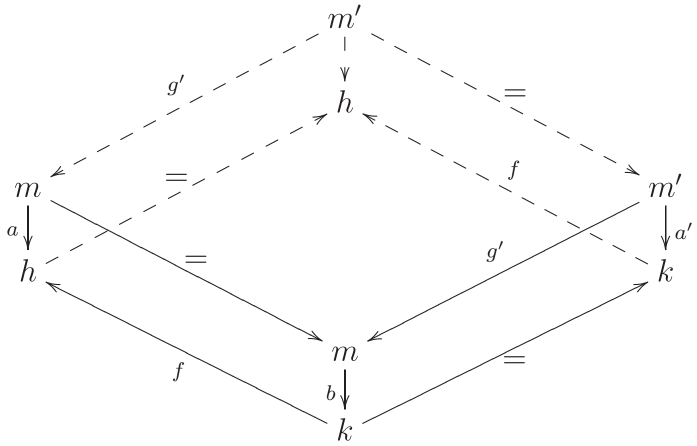
$$\begin{array}{ccc} [m_1 + m_2 + m_3] & \xleftarrow{f_2} & [m_2] \\ {\scriptstyle \tilde{g}}\uparrow & & \uparrow{\scriptstyle \tilde{g}_2} \\ [m'_1 + m'_2 + m'_3] & \xleftarrow[f'_2]{} & m'_2 \end{array}$$

□

Lemma. 6.5. *In the category $\mathscr{D}$, ordinalic and segalic maps admit pullback along each other, and the result is again maps of the same type.*

Proof. This is straightforward: in the diagram below, the map from a to b is segalic (given essentially by the bottom map f) and the map from a' to b is ordinalic (given essentially by the top map g'):

(12)



To construct the pullback, we are forced to repeat f and g', completing the squares with the corresponding identity maps. The connecting map in the resulting object is $fbg' : m' \to h$. It is clear from the presence of the four identity maps that this is a pullback. ☐

We now have the following important characterisation of decomposition spaces.

Proposition 6.6. *A simplicial space $X : \Delta^{\mathrm{op}} \to$ **Grpd** is a decomposition space if and only if the corresponding monoidal functor $\overline{X} : \mathscr{D} \to$ **Grpd** preserves pullback squares of the kind described in 6.5.*

Proof. Since an ordinalic map is a sum, it can be decomposed into a sequence of maps in which each map has only one nontrivial summand. This means that a pullback diagram like (12) is a sum of diagrams of the form in which $\underline{h} = \underline{1}$. So to prove that these pullbacks are preserved, it is enough to treat the case $h = 1$. In this case, the map g' in the square is just a map in $\underline{\Delta}$, so it can be decomposed into face and degeneracy maps. The X-image is then a diagram of the form

$$\begin{array}{ccc} X_m & \longrightarrow & X_{m_1} \times \cdots \times X_{m_k} \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n_1} \times \cdots \times X_{n_k}, \end{array}$$

where the map on the left is a face map or a degeneracy map. It follows that the map on the right is a product of maps in which all factors are identity maps except one, say the ith factor (which is again a face or a degeneracy map). Now whether or not this is a pullback can be checked on the projections onto the nontrivial factor:

$$\begin{array}{ccccc} X_m & \longrightarrow & X_{m_1} \times \cdots \times X_{m_k} & \longrightarrow & X_{m_i} \\ \downarrow & & \downarrow & & \downarrow \\ X_n & \longrightarrow & X_{n_1} \times \cdots \times X_{n_k} & \longrightarrow & X_{n_i} \end{array}$$

But by construction of $\overline{X}$, the composite horizontal maps are precisely free maps in the sense of the simplicial space X, and the vertical maps are precisely generic maps in the sense that it is an arbitrary map in $\underline{\Delta}$ and hence (in the other direction) a generic map in Δ, under the duality in 6.2. Since the right-hand square is always a pullback, by the standard pullback argument 1.10, the total square is a pullback (i.e. we have a decomposition space) if and only if the left-hand square is a pullback (i.e. the pullback condition on $\overline{X}$ is satisfied). □

7. Proof of coassociativity

We proceed to establish that, if X is a decomposition space, then the comultiplication and counit defined in sections 5.1 make $\boldsymbol{Grpd}_{/X_1}$ a coassociative and counital coalgebra in a strong homotopy sense.

We have more generally, for any $n \geq 0$, the generalised comultiplication maps

$$\Delta_n : \boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}_{/X_1\times\cdots\times X_1} \tag{13}$$

given by the spans

$$X_1 \leftarrow X_n \rightarrow X_1 \times \cdots \times X_1. \tag{14}$$

The case $n = 0$ is the counit map, $n = 1$ gives the identity, and $n = 2$ is the comultiplication we considered above. The coassociativity will say that all combinations (composites and tensor products) of these agree whenever they have the same source and target. For this we exploit the category $\mathscr{D}$ introduced in 6, designed exactly to encode also cartesian powers of the various spaces X_k.

Definition. A *reasonable span* in $\mathscr{D}$ is a span $a \stackrel{g}{\leftarrow} m \stackrel{f}{\rightarrow} b$ in which g is ordinalic and f is segalic. Clearly the external sum of two reasonable spans is reasonable, and the composite of two reasonable spans is reasonable (by Lemma 6.5).

Let $X : \Delta^{\text{op}} \to \boldsymbol{Grpd}$ be a fixed decomposition space, and interpret it also as a monoidal functor $\overline{X} : \mathscr{D} \to \boldsymbol{Grpd}$. A span in $\boldsymbol{Grpd}$ of the form

$$\overline{X}_a \leftarrow \overline{X}_m \rightarrow \overline{X}_b$$

is called reasonable if it is induced by a reasonable span in $\mathscr{D}$.

A linear map between slices of $\boldsymbol{Grpd}$ is called reasonable if it is given by a reasonable span. That is, it is a pullback along a ordinalic map followed by a lowershriek along a segalic map.

Lemma. 7.1. *Tensor products of reasonable linear maps are reasonable. For a decomposition space, composites of reasonable linear maps are reasonable.*

Proof. Cartesian products of reasonable spans in $\boldsymbol{Grpd}$ are reasonable since $\overline{X}$ is monoidal. For decomposition spaces, a composite of reasonable linear maps is induced by the composite reasonable span in $\mathscr{D}$, using Proposition 6.6. □

The interest in these notions is of course that the generalised comultiplication maps Δ_n are reasonable, see (13,14) above. In conclusion:

Proposition 7.2. *Any reasonable linear map*

$$\boldsymbol{Grpd}_{/X_1} \longrightarrow \boldsymbol{Grpd}_{/X_1\times\cdots\times X_1}, \quad n \geq 0$$

is canonically equivalent to the nth comultiplication map.

Proof. We have to show that the only reasonable span of the form $X_1 \leftarrow \prod X_{m_i} \rightarrow X_1 \times \cdots \times X_1$ is (14). Indeed, the left leg must come from an ordinalic map, so since X_1 has only one factor, the middle object has also only one factor, i.e. is the image of $\underline{m} \to \underline{1}$. On the other hand, the right leg must be segalic, which forces $m = n$. □

Theorem 7.3. *For X a decomposition space, the slice ∞-category $\mathbf{Grpd}_{/X_1}$ has the structure of strong homotopy comonoid in the symmetric monoidal ∞-category* **LIN**, *with the comultiplication defined by the span*

$$X_1 \xleftarrow{d_1} X_2 \xrightarrow{(d_2,d_0)} X_1 \times X_1.$$

8. Functorialities and coalgebra homomorphisms

8.1. Covariant functoriality. An important motivation for the notion of decomposition space is that they induce coalgebras. Correspondingly, it is an important feature of cULF maps that they induce coalgebra homomorphisms:

Lemma. 8.2. *If $F : X \to Y$ is a conservative ULF map between decomposition spaces then $F_! : \mathbf{Grpd}_{/X_1} \to \mathbf{Grpd}_{/Y_1}$ is a coalgebra homomorphism.*

Proof. In the diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{g} & X_n & \xrightarrow{f} & X_1^n \\ {\scriptstyle F_1}\downarrow & \lrcorner & \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_1^n} \\ Y_1 & \xleftarrow[g']{} & Y_n & \xrightarrow[f']{} & Y_1^n \end{array}$$

the left-hand square is a pullback since F is conservative (case $n = 0$) and ULF (cases $n > 1$). Hence by the Beck–Chevalley condition we have an equivalence of functors $g'^* \circ F_{1!} \simeq F_{n!} \circ g^*$, and by postcomposing with $f'_!$ we arrive at the coalgebra homomorphism condition $\Delta'_n F_{1!} \cong F_{1!}^{\,n} \Delta_n$ □

Remark 8.3. If Y is a Segal space, then the statement can be improved to an if-and-only-if statement.

8.4. Example. An important class of cULF maps are counits of decalage, cf. 4.10:

$$d_\bot : \mathrm{Dec}_\bot X \to X \qquad \text{and} \qquad d_\top : \mathrm{Dec}_\top X \to X.$$

Many coalgebra maps in the classical theory of incidence coalgebras, notably reduction maps, are induced from decalage in this way, as we shall see in the Examples Section 10, and as further amplified in [25].

8.5. Contravariant functoriality. There is also a contravariant functoriality for certain simplicial maps, which we briefly explain, although it will not be needed elsewhere in this paper.

A functor between decomposition spaces $F : X \to Y$ is called *relatively Segal* when for all 'spines' (i.e. inclusion of a string of principal edges into a simplex)

$$\Delta^1 \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^1 \longrightarrow \Delta^n$$

the space of fillers in the diagram

$$\begin{array}{ccc} \Delta^1 \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^1 & \longrightarrow & X \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & Y \end{array}$$

is contractible. Note that the precise condition is that the following square is a pullback:

$$\begin{array}{ccc}
\mathrm{Map}(\Delta^n, X) & \longrightarrow & \mathrm{Map}(\Delta^1 \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^1, X) \\
\downarrow \lrcorner & & \downarrow \\
\mathrm{Map}(\Delta^n, Y) & \longrightarrow & \mathrm{Map}(\Delta^1 \coprod_{\Delta^0} \cdots \coprod_{\Delta^0} \Delta^1, Y)
\end{array}$$

This can be rewritten

$$\begin{array}{ccc}
X_n & \longrightarrow & X_1 \times_{X_0} \cdots \times_{X_0} X_1 \\
\downarrow \lrcorner & & \downarrow \\
Y_n & \longrightarrow & Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1.
\end{array} \tag{15}$$

(Hence the ordinary Segal condition for a simplicial space X is the case where Y is a point.)

Proposition 8.6. *If $F : X \to Y$ is relatively Segal and $F_0 : X_0 \to Y_0$ is an equivalence, then*

$$F^* : \boldsymbol{Grpd}_{/Y_1} \to \boldsymbol{Grpd}_{/X_1}$$

is naturally a coalgebra homomorphism.

Proof. In the diagram

$$\begin{array}{ccccc}
X_1 & \xleftarrow{g} & X_n & \xrightarrow{f} & X_1^n \\
{\scriptstyle F_1}\downarrow & & {\scriptstyle F_n}\downarrow \lrcorner & & \downarrow {\scriptstyle F_1^n} \\
Y_1 & \xleftarrow[g']{} & Y_n & \xrightarrow[f']{} & Y_1^n
\end{array}$$

we claim that the right-hand square is a pullback for all n. Hence by the Beck–Chevalley condition we have an equivalence of functors $f_! \circ F_n{}^* \simeq F_1^{n*} \circ f'{}_!$, and by postcomposing with $g'{}^*$ we arrive at the coalgebra homomorphism condition

$$\Delta_n F_1{}^* \cong F_1{}^{*n} \Delta'_n.$$

The claim for $n = 0$ amounts to

$$\begin{array}{ccc}
X_0 & \xrightarrow{f} & 1 \\
{\scriptstyle F_0}\downarrow \lrcorner & & \downarrow \\
Y_0 & \xrightarrow[f']{} & 1
\end{array}$$

which is precisely to say that F_0 is an equivalence. For $n > 1$ we can factor the square as

$$\begin{array}{ccccc}
X_n & \xrightarrow{f} & X_1 \times_{X_0} \cdots \times_{X_0} X_1 & \longrightarrow & X_1 \times \cdots \times X_1 \\
{\scriptstyle F_n}\downarrow \lrcorner & & \downarrow {\scriptstyle F_1^n} & & \downarrow {\scriptstyle F_1^n} \\
Y_n & \xrightarrow[f']{} & Y_1 \times_{Y_0} \cdots \times_{Y_0} Y_1 & \longrightarrow & Y_1 \times \cdots \times Y_1
\end{array}$$

Here the left-hand square is a pullback since F is relatively Segal. It remains to prove that the right-hand square is a pullback. For the case $n = 2$, this whole square is the pullback of the square

$$\begin{array}{ccc} X_0 & \longrightarrow & X_0 \times X_0 \\ \downarrow & \lrcorner & \downarrow \\ Y_0 & \longrightarrow & Y_0 \times Y_0 \end{array}$$

which is a pullback precisely when F_0 is mono. But we have assumed it is even an equivalence. The general case $n > 2$ is easily obtained from the $n = 2$ case by an iterative argument. ☐

8.7. Remarks. It should be mentioned that in order for contravariant functoriality to preserve finiteness as in [22], and hence restrict to the coefficients in ***grpd***, it is necessary furthermore to require that F is finite.

When both X and Y are Segal spaces, then the relative Segal condition is automatically satisfied, because the horizontal maps in (15) are then equivalences. In this case, we recover the classical results on contravariant functoriality by Content–Lemay–Leroux [12, Prop. 5.6] and Leinster [46], where the only condition is that the functor be bijective on objects (in addition to requiring F finite, necessary since they work on the level of vector spaces).

9. Monoidal decomposition spaces

The ∞-category of decomposition spaces (as a full subcategory of simplicial ∞-groupoids), has finite products. Hence there is a symmetric monoidal structure on the ∞-category $\boldsymbol{Dcmp}^{\mathrm{cULF}}$ of decomposition spaces and cULF maps. We still denote this product as $\times$, although of course it is not the cartesian product in $\boldsymbol{Dcmp}^{\mathrm{cULF}}$.

Definition. A *monoidal decomposition space* is a monoid object (X, m, e) in $(\boldsymbol{Dcmp}^{\mathrm{cULF}}, \times, 1)$. A *monoidal functor* between monoidal decomposition spaces is a monoid homomorphism in $(\boldsymbol{Dcmp}^{\mathrm{cULF}}, \times, 1)$.

9.1. Remark. By this we mean a monoid in the homotopy sense, that is, an algebra in the sense of Lurie [51]. We do not wish at this point to go into the technicalities of this notion, since in our examples, the algebra structure will be given simply by sums (or products).

Example 9.2. Recall that a category $\mathscr{E}$ with finite sums is *extensive* [8] when the natural functor $\mathscr{E}_{/A} \times \mathscr{E}_{/B} \to \mathscr{E}_{/A+B}$ is an equivalence. The fat nerve of an extensive 1-category is a monoidal decomposition space. The multiplication is given by taking sum, the neutral object by the initial object, and the extensive property ensures precisely that, given a factorisation of a sum of maps, each of the maps splits into a sum of maps in a unique way.

A key example is the category of sets, or of finite sets. Certain subcategories, such as the category of finite sets and surjections, or the category of finite sets and injections, inherit the crucial property $\mathscr{E}_{/A} \times \mathscr{E}_{/B} \simeq \mathscr{E}_{/A+B}$. They fail, however, to be extensive in the strict sense, since the monoidal structure $+$ in these cases is not the categorical sum. Instead they are examples of *monoidal extensive* categories, meaning a monoidal category $(\mathscr{E}, \boxplus, 0)$ for which $\mathscr{E}_{/A} \times \mathscr{E}_{/B} \to \mathscr{E}_{/A \boxplus B}$ is an equivalence (and it should then be required separately that also $\mathscr{E}_{/0} \simeq 1$). The fat nerve of a monoidal extensive 1-category is a monoidal decomposition space.

Lemma. 9.3. *The dec of a monoidal decomposition space has again a natural monoidal structure, and the counit is a monoidal functor.*

9.4. Bialgebras. For a monoidal decomposition space the resulting coalgebra is also a bialgebra. Indeed, the fact that the monoid multiplication is cULF means that it induces a coalgebra homomorphism, and similarly with the unit. Note that this notion of bialgebra is not symmetric: while the comultiplication is induced from internal, simplicial data in X, the multiplication is induced by extra structure (the monoid structure). In the applications, the monoid structure will typically be given by categorical sum, and hence is associative up to canonical isomorphisms, something that seems much stricter than the comultiplication.

Proposition 9.5. *If $f : X \to Y$ is a cULF monoidal functor between monoidal decomposition spaces, then $f_! : \boldsymbol{Grpd}_{/X_1} \to \boldsymbol{Grpd}_{/Y_1}$ is a bialgebra homomorphism.*

10. Examples

10.1. Injections and the monoidal groupoid of sets under sum. Let $\mathbf{I}$ be the nerve of the category of finite sets and injections, and let $\mathbf{B}$ be the nerve of the monoidal groupoid $(\mathbb{B}, +, 0)$ of finite sets and bijections, or of the corresponding 1-object bicategory (see Proposition 2.18). Dür [14] noted that imposing the equivalence relation 'having isomorphic complements' on the incidence coalgebra of $\mathbf{I}$ gives the binomial coalgebra. We can see this reduction map as induced by a conservative ULF functor from a decalage:

Lemma. 10.2. *There is a levelwise equivalence of simplicial groupoids*

$$\mathrm{Dec}_\perp(\mathbf{B}) \xrightarrow{\simeq} \mathbf{I}$$

given in degree k by

$$(x_0, \ldots, x_k) \longmapsto [x_0 \subseteq x_0 + x_1 \subseteq \cdots \subseteq x_0 + \cdots + x_k]$$

and a conservative ULF functor $r : \mathbf{I} \to \mathbf{B}$ is given by

$$d_\perp : \mathrm{Dec}_\perp(\mathbf{B}) \to \mathbf{B}, \qquad (x_0, \ldots, x_k) \mapsto (x_1, \ldots, x_k).$$

The equivalence may also be represented using diagrams reminiscent of those in Waldhausen's $S_\bullet$-construction. As an example, both groupoids $\mathbf{I}_3$ and $\mathrm{Dec}_\perp(\mathbf{B})_3 = \mathbf{B}_4$ are equivalent to the groupoid of diagrams

$$\begin{array}{ccccccc}
 & & & & & & x_3 \\
 & & & & & & \downarrow \\
 & & & & x_2 & \longrightarrow & x_2 + x_3 \\
 & & & & \downarrow & & \downarrow \\
 & & x_1 & \longrightarrow & x_1 + x_2 & \longrightarrow & x_1 + x_2 + x_3 \\
 & & \downarrow & & \downarrow & & \downarrow \\
x_0 & \longrightarrow & x_0 + x_1 & \longrightarrow & x_0 + x_1 + x_2 & \longrightarrow & x_0 + x_1 + x_2 + x_3
\end{array}$$

The face maps $d_i : \mathbf{I}_3 \to \mathbf{I}_2$ and $d_{i+1} : \mathbf{B}_4 \to \mathbf{B}_3$ both act by deleting the column beginning x_i and the row beginning x_{i+1}. In particular $d_\perp : \mathbf{I} \to \mathbf{B}$ deletes the bottom row, sending a string of injections to the sequence of successive complements (x_1, x_2, x_3). We will revisit this theme in the treatment of the Waldhausen $S_\bullet$ construction in Section 10.6 below.

Both $\mathbf{I}$ and $\mathbf{B}$ are monoidal decomposition spaces under disjoint union, and $\mathbf{I} \simeq \mathrm{Dec}_\perp(\mathbf{B}) \to \mathbf{B}$ is a monoidal functor by Lemma 9.3, inducing a (surjective) homomorphism of bialgebras $\boldsymbol{Grpd}_{/\mathbf{I}_1} \to \boldsymbol{Grpd}_{/\mathbf{B}_1}$ by Proposition 9.5, which is the reduction map described by Dür.

The comultiplication on $\boldsymbol{Grpd}_{/\mathbf{B}_1}$ is given by

$$\Delta(\ulcorner S \urcorner) \;=\; \sum_{A+B=S} \ulcorner A \urcorner \otimes \ulcorner B \urcorner$$

(where the sum is more specifically over all $A, B \subset S,\;\; A \cup B = S,\;\; A \cap B = \emptyset$). The decomposition space $\mathbf{B}$ is *locally finite* (see [22]), and taking cardinality (as in [24]) gives the classical binomial coalgebra, spanned by symbols δ_n (the cardinality of $\ulcorner n \urcorner : 1 \to \mathbf{B}$), with

$$\Delta(\delta_n) = \sum_{a+b=n} \frac{n!}{a!\,b!}\,\delta_a \otimes \delta_b.$$

As a bialgebra we have $(\delta_1)^n = \delta_n$ and one recovers the comultiplication from $\Delta(\delta_n) = \big(\delta_0 \otimes \delta_1 + \delta_1 \otimes \delta_0\big)^n$.

The objective level is much richer. The linear dual [22] of $\boldsymbol{Grpd}_{/\mathbf{B}_1}$ is $\boldsymbol{Grpd}^{\mathbf{B}_1}$, the category of groupoid-valued species [2], [39], and its multiplication is the monoidal structure given by the convolution formula

$$(F * G)[S] = \sum_{A+B=S} F[A] \times G[B],$$

which is precisely the Cauchy product of species (see [1]). The cardinality of this monoidal category is the profinite-dimensional vector space $\mathbb{Q}^{\pi_0 \mathbf{B}}$ with pro-basis given by the symbols δ^n (dual to δ_n), with convolution product

$$\delta^a * \delta^b = \frac{n!}{a!\,b!}\,\delta^{a+b}.$$

This is isomorphic to the algebra $\mathbb{Q}[[z]]$, where δ^n corresponds to $z^n/n!$ and the cardinality of a species F corresponds precisely to its exponential generating series [31].

10.3. Graphs and restriction species. The following coalgebra of graphs has been studied by Schmitt [61, §12]. For a graph G with vertex set V (admitting multiple edges and loops), and a subset $U \subset V$, define $G|U$ to be the graph whose vertex set is U, and whose edges are those edges of G both of whose incident vertices belong to U. On the vector space spanned by isoclasses of graphs, define a comultiplication by the rule

$$\Delta(G) = \sum_{A+B=V} G|A \otimes G|B.$$

This coalgebra is the cardinality of the coalgebra of a decomposition space but not directly of a category. Indeed, define a simplicial groupoid with $\mathbf{G}_1$ the groupoid of graphs, and more generally let $\mathbf{G}_k$ be the groupoid of graphs with an ordered partition of the vertex set into k (possibly empty) parts. In particular, $\mathbf{G}_0$ is the contractible groupoid consisting only of the empty graph. The outer face maps delete the first or last part of the graph, and the inner face maps join adjacent parts. The degeneracy maps insert an empty part. It is clear that this is not a Segal space: a graph structure on a given set cannot be reconstructed from knowledge of the graph structure of the parts of the set, since chopping up the graph and restricting to the parts throws away all information about edges going from one part to another. One can easily check that it is a decomposition space. It is clear that the resulting coalgebra is Schmitt's coalgebra of graphs. Note that disjoint union of graphs makes this into a bialgebra.

10.4. Butcher–Connes–Kreimer Hopf algebra. Dür [14, Ch.IV, §3] constructed what was later called the Connes–Kreimer Hopf algebra of rooted trees, after [10]: he starts with the notion of (combinatorial) tree (i.e. connected and simply connected graphs with a specified root vertex); then a forest is a disjoint union of rooted trees. He then considers the category of root-preserving inclusions of forests. A coalgebra is induced from this (in our language it is given by the simplicial groupoid $\mathbf{R}$, where $\mathbf{R}_k$ is the groupoid of strings of k root-preserving forest inclusions) but it is not the most interesting one. The Connes–Kreimer coalgebra is obtained by the reduction that identifies two root-preserving forest inclusions if their complement crowns are isomorphic forests.

We can obtain this coalgebra directly from a decomposition space: let $\mathbf{H}_1$ denote the groupoid of forests, and let $\mathbf{H}_2$ denote the groupoid of forests with an admissible cut. More generally, $\mathbf{H}_0$ is defined to be a point, and $\mathbf{H}_k$ is the groupoid of forests with $k-1$ compatible admissible cuts. These form a simplicial groupoid in which the inner face maps forget a cut, and the outer face maps projects away either the crown or the bottom layer (the part of the forest below the bottom cut). The notion of admissible cut is standard, see for example [11]. One convenient way to define what it means is to say that it is a root-preserving inclusion of forests: then the cut is interpreted as the division between the included forest and its complement. In this way we see that $\mathbf{H}_k$ is the groupoid of $k-1$ consecutive root-preserving inclusions.

There is a natural conservative ULF functor from $\mathbf{R}$ to $\mathbf{H}$: on $\mathbf{R}_1 \to \mathbf{H}_1$ it sends a root-preserving forest inclusion to its crown. More generally, on $\mathbf{R}_k \to \mathbf{H}_k$ it deletes the first inclusion in the string. In close analogy with Example 10.1, we see that $\mathbf{R} \simeq \mathrm{Dec}_\perp(\mathbf{H})$, and that the reduction is just the counit of decalage.

It is clear that $\mathbf{H}$ is not a Segal space: a tree with a cut cannot be reconstructed from its crown and its bottom tree, which is to say that $\mathbf{H}_2$ is not equivalent to $\mathbf{H}_1 \times_{\mathbf{H}_0} \mathbf{H}_1$. It is straightforward to check that it *is* a decomposition space. On the other hand, there are important variations on the Connes–Kreimer Hopf algebra which *do* form Segal spaces, namely with operadic trees instead of combinatorial trees [40], [20], [42].

Just as Example 10.1, the examples with graphs and trees are naturally bialgebras, with the monoidal structure given by disjoint union.

The graph example is an example of a decomposition space coming from a restriction species in the sense of Schmitt [60] (see also [1]). The examples with trees and directed graphs are examples of decomposition spaces coming from directed restriction species, a new notion we develop in a separate publication [26].

10.5. q-binomials: $\mathbb{F}_q$-vector spaces. Consider the finite field $\mathbb{F}_q$ with q elements. The q-binomial coalgebra (see Dür [14, 1.54]) may be obtained as a certain reduction of the incidence coalgebra of the category ***vect***, of finite-dimensional $\mathbb{F}_q$-vector spaces and $\mathbb{F}_q$-linear injections, by identifying two injections if their cokernels are isomorphic.

The same coalgebra can be obtained without reduction as follows. Put $\mathbf{V}_0 = *$, let $\mathbf{V}_1$ be the maximal groupoid of ***vect***, and let $\mathbf{V}_2$ be the groupoid of short exact sequences. The span

$$\begin{array}{ccccc} \mathbf{V}_1 & \longleftarrow & \mathbf{V}_2 & \longrightarrow & \mathbf{V}_1 \times \mathbf{V}_1 \\ E & \longleftarrow\!\!\dashv & [E' \to E \to E''] & \longmapsto & (E', E'') \end{array}$$

(together with the span $\mathbf{V}_1 \leftarrow \mathbf{V}_0 \to 1$) defines a coalgebra on $\boldsymbol{Grpd}_{/\mathbf{V}_1}$ which (after taking cardinality) is the q-binomial coalgebra, without further reduction. The groupoids and maps involved are part of a simplicial groupoid $\mathbf{V} : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, namely the Waldhausen S-construction of $\boldsymbol{vect}$, which is a decomposition space but not a Segal space (cf. 10.6). The lower dec of $\mathbf{V}$ is naturally equivalent to the fat nerve of $\boldsymbol{vect}$, and the comparison map d_0 is the reduction map of Dür.

Although we have postponed notion of the dual incidence *algebra* to [22], we wish to mention that in this case the incidence algebra is $\boldsymbol{Grpd}^{\mathbf{V}_1}$, which is the category of groupoid-valued q-species, and the convolution tensor product resulting from our constructions is the *external product of q-species* of Joyal–Street [36] (except that they work with vector-space valued q-species). A main contribution of [36] is to show that this monoidal structure carries a non-trivial braiding. This is a very interesting structure, which cannot be seen after taking cardinality.

One can compute explicitly (see [25]) the section coefficients of the comultiplication (or the convolution product) to find the Hall numbers

$$\frac{|\mathrm{SES}_{k,n,n-k}|}{\left|\mathrm{Aut}(\mathbb{F}_q^k)\right| \left|\mathrm{Aut}(\mathbb{F}_q^{n-k})\right|} = \binom{n}{k}_q,$$

where $\mathrm{SES}_{k,n,n-k}$ denotes the groupoid of short exact sequence with specified vector spaces of dimensions k, n, and $n-k$.

This example is a special case of the following general construction with wide-ranging ramifications and consequences.

10.6. Waldhausen S-construction of an abelian category. We follow Lurie [51, Subsection 1.2.2] for the account of Waldhausen S. For I a linearly ordered set, let $\mathrm{Ar}(I)$ denote the category of arrows in I: the objects are pairs of elements $i \leq j$ in I, and the morphisms are relations $(i,j) \leq (i',j')$ whenever $i \leq i'$ and $j \leq j'$. A *gap complex* in an abelian category $\mathscr{A}$ is a functor $F : N(\mathrm{Ar}(I)) \to \mathscr{A}$ such that

(1) For each $i \in I$, the object $F(i,i)$ is zero.
(2) For every $i \leq j \leq k$, the associated diagram

$$\begin{array}{ccc} 0 = F(j,j) & \rightarrowtail & F(j,k) \\ \uparrow\!\!\uparrow & & \uparrow\!\!\uparrow \\ F(i,j) & \rightarrowtail & F(i,k) \end{array}$$

is a pushout (or equivalently a pullback).

Remark: since the pullback of a monomorphism is always a monomorphism, and the pushout of an epimorphism is always an epimorphism, it follows that automatically the horizontal maps are monomorphisms and the vertical maps are epimorphisms, as already indicated with the arrow typography. Altogether, it is just a fancy but very convenient way of saying ‘short exact sequence’ or ‘(co)fibration sequence’.

Let $\mathrm{Gap}(I, \mathscr{A})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ar}(I), \mathscr{A})$ consisting of the gap complexes. This is a 1-category, since $\mathscr{A}$ was assumed to be an abelian 1-category.

The assignment

$$[n] \mapsto \mathrm{Gap}([n], \mathscr{A})^{\mathrm{eq}}$$

defines a simplicial space $S\mathscr{A} : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, which by definition is the Waldhausen S-construction on $\mathscr{A}$. Intuitively (or essentially), the groupoid $\mathrm{Gap}([n], \mathscr{A})^{\mathrm{eq}}$ has as objects staircase diagrams like the following (picturing $n = 4$):

$$\begin{array}{ccccccc}
 & & & & & & A_{34} \\
 & & & & & & \uparrow \\
 & & & & A_{23} & \rightarrowtail & A_{24} \\
 & & & & \uparrow & & \uparrow \\
 & & A_{12} & \rightarrowtail & A_{13} & \rightarrowtail & A_{14} \\
 & & \uparrow & & \uparrow & & \uparrow \\
A_{01} & \rightarrowtail & A_{02} & \rightarrowtail & A_{03} & \rightarrowtail & A_{04}
\end{array}$$

The face map d_i deletes all objects containing an i index. The degeneracy map s_i repeats the ith row and the ith column.

A string of composable monomorphisms $(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n)$ determines, up to canonical isomorphism, short exact sequences $A_{ij} \rightarrowtail A_{ik} \twoheadrightarrow A_{jk} = A_{ij}/A_{ik}$ with $A_{0i} = A_i$. Hence the whole diagram can be reconstructed up to isomorphism from the bottom row. (Similarly, since epimorphisms have uniquely determined kernels, the whole diagram can also be reconstructed from the last column.)

We have $s_0(*) = 0$, and

$$d_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (A_2/A_1 \rightarrowtail \cdots \rightarrowtail A_n/A_1)$$
$$s_0(A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n) = (0 \rightarrowtail A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n)$$

The simplicial maps d_i, s_i for $i \geq 1$ are more straightforward: the simplicial set $\mathrm{Dec}_\bot(S\mathscr{A})$ is just the nerve of $\mathrm{mono}(\mathscr{A})$.

Lemma. 10.7. *The projection $S_{n+1}\mathscr{A} \to \mathrm{Map}([n], \mathrm{mono}(\mathscr{A}))$ is an equivalence. Similarly the projection $S_{n+1}\mathscr{A} \to \mathrm{Map}([n], \mathrm{epi}(\mathscr{A}))$.*

More precisely (with reference to the fat nerve):

Proposition 10.8. *These equivalences assemble into levelwise simplicial equivalences*

$$\mathrm{Dec}_\bot(S\mathscr{A}) \simeq N(\mathrm{mono}(\mathscr{A}))$$
$$\mathrm{Dec}_\top(S\mathscr{A}) \simeq N(\mathrm{epi}(\mathscr{A})).$$

Theorem 10.9. *The Waldhausen S-construction of an abelian category $\mathscr{A}$ is a decomposition space.*

Proof. For convenience we write $S\mathscr{A}$ simply as S. The previous proposition already implies that the two Decs of S are Segal spaces. By Theorem 4.11, it is therefore enough to establish that the squares

$$\begin{array}{ccc}
S_1 & \xrightarrow{s_1} & S_2 \\
{\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\
S_0 & \xrightarrow[s_0]{} & S_1
\end{array}
\qquad\qquad
\begin{array}{ccc}
S_1 & \xrightarrow{s_0} & S_2 \\
{\scriptstyle d_1}\downarrow & & \downarrow{\scriptstyle d_2} \\
S_0 & \xrightarrow[s_0]{} & S_1
\end{array}$$

are pullbacks. Note that we have $S_0 = *$ and $S_1 = \mathscr{A}^{\mathrm{iso}}$, and that $s_0 : S_0 \to S_1$ picks out the zero object, and since the zero object has no nontrivial automorphisms, this map is fully faithful. The map $d_0 : S_2 \to S_1$ sends a monomorphism to its quotient object. We need to compute the fibre over the zero object, but since s_0 is fully faithful, we are just asking for the full subgroupoid of S_2 consisting of those monomorphisms whose cokernel is zero. Clearly these are precisely the isos, so the fibre is just $\mathscr{A}^{\text{iso}} = S_1$. The other pullback square is established similarly, but arguing with epimorphisms instead of monomorphisms. □

Remark 10.10. Waldhausen's S-construction was designed for more general categories than abelian categories, namely what are now called Waldhausen categories, where the cofibrations play the role of the monomorphisms, but where there is no stand-in for the epimorphisms. The theorem does not generalise to Waldhausen categories in general, since in that case $\mathrm{Dec}_\top(S)$ is not necessarily a Segal space of any class of arrows.

10.11. Waldhausen S of a stable ∞-category. The same construction works in the ∞-setting, by considering stable ∞-categories instead of abelian categories. Let $\mathscr{A}$ be a stable ∞-category (see Lurie [51]). Just as in the abelian case, the assignment

$$[n] \mapsto \mathrm{Gap}([n], \mathscr{A})^{\mathrm{eq}}$$

defines a simplicial space $S\mathscr{A} : \Delta^{\mathrm{op}} \to \boldsymbol{Grpd}$, which by definition is the Waldhausen S-construction on $\mathscr{A}$. Note that in the case of a stable ∞-category, in contrast to the abelian case, every map can arise as either horizontal or vertical arrow in a gap complex. Hence the role of monomorphisms (cofibrations) is played by all maps, and the role of epimorphisms is also played by all maps.

Lemma. 10.12. *For each $k \in \mathbb{N}$, the two projection functors $S_{k+1}\mathscr{A} \to \mathrm{Map}(\Delta[k], \mathscr{A})$ are equivalences.*

From the description of the face and degeneracy maps, the following more precise result follows readily, comparing with the fat nerves:

Proposition 10.13. *We have natural (levelwise) simplicial equivalences*

$$\mathrm{Dec}_\perp(S\mathscr{A}) \simeq N(\mathscr{A})$$
$$\mathrm{Dec}_\top(S\mathscr{A}) \simeq N(\mathscr{A}).$$

Theorem 10.14. *Waldhausen's S-construction of a stable ∞-category $\mathscr{A}$ is a decomposition space.*

Proof. The proof is exactly the same as in the abelian case, relying on the following three facts:

(1) The Decs are Segal spaces.
(2) $s_0 : S_0 \to S_1$ is fully faithful.
(3) A map (playing the role of monomorphisms) is an equivalence if and only if its cofibre is the zero object, and a map (playing the role of epimorphism) is an equivalence if and only if its fibre is the zero object.

□

Remark 10.15. This theorem was proved independently (and first) by Dyckerhoff and Kapranov [16], Theorem 7.3.3. They prove it more generally for exact ∞-categories, a notion they introduce. Their proof that Waldhausen's S-construction of an exact ∞-category is a decomposition space is somewhat more complicated than ours above. In particular their proof of unitality (the pullback condition on degeneracy maps) is technical and involves Quillen model structures on certain marked simplicial sets à la Lurie [50]. We do not wish to go into exact ∞-categories here, and refer instead the reader to [16], but we wish to point out that our simple proof above works as well for exact ∞-categories. This follows since the three points in the proof hold also for exact ∞-categories, which in turn is a consequence of the definitions and basic results provided in [16, Sections 7.2 and 7.3].

10.16. Hall algebras. The finite-support incidence algebra of a decomposition space X is defined in [22, Section 7.15]. In order for it to admit a cardinality, the required assumption is that X_1 be locally finite, and that $X_2 \to X_1 \times X_1$ be a finite map. In the case of $X = S(\mathscr{A})$ for an abelian category $\mathscr{A}$, this translates into the condition that Ext^0 and Ext^1 be finite (which in practice means 'finite dimension over a finite field'). The finite-support incidence algebra in this case is the *Hall algebra* of $\mathscr{A}$ (cf. Ringel [57]; see also [59], although these sources twist the multiplication by the so-called Euler form).

For a stable ∞-category $\mathscr{A}$, with mapping spaces assumed to be locally finite ([24, 3.1]), the finite-support incidence algebra of $S(\mathscr{A})$ is the *derived Hall algebra*. These were introduced by Toën [63] in the setting of dg-categories.

Hall algebras were one of the main motivations for Dyckerhoff and Kapranov [16] to introduce 2-Segal spaces. We refer to their work for development of this important topic; see in particular the lecture notes of Dyckerhoff [15].

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