Dynamics close to a non semi-simple 1:-1 resonant periodic orbit

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Abstract

In this work, our target is to analyze the dynamics around the $1:-1$ resonance which appears when a family of periodic orbits of a real analytic three-degree of freedom Hamiltonian system changes its stability from elliptic to a complex hyperbolic saddle passing through degenerate elliptic. Our analytical approach consists of computing, in a constructive way and up to some given arbitrary order, the normal form around that resonant (or *critical*) periodic orbit.

Hence, dealing with the normal form itself and the differential equations related to it, we derive the generic existence of a two-parameter family of invariant 2D tori which bifurcate from the critical periodic orbit. Moreover, the coefficient of the normal form that determines the stability of the bifurcated tori is identified. This allows us to show the Hopf-like character of the unfolding: elliptic tori unfold "around" hyperbolic periodic orbits (case of *direct* bifurcation) while normal hyperbolic tori appear "around" elliptic periodic orbits (case of *inverse* bifurcation). Further, a global description of the dynamics of the normal form is also given.

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Contents

1 Introduction

In this paper, the main topic is the study of the dynamics close to $1:-1$ resonant periodic orbits of three-degree of freedom Hamiltonian systems. To be more precise: we consider a one-parameter family of periodic orbits of a real analytic three-degree of freedom Hamiltonian system, and assume that the orbits of the family are first linearly stable, for a *critical* value of the parameter the nontrivial (i. e., those different from one) characteristic multipliers of the corresponding periodic orbit collide on the unit circle (Krein collision) and then, if certain generic conditions are met, the characteristic multipliers leave out the unit circle to the complex plane. Hence the family looses its (linear) stability and the periodic orbits become *complex unstable.* In other words, the family changes from stable to complex-unstable by means of a passage through a a *critical*, $1:-1$ *resonant* or simply *resonant* periodic orbit (see figure 1).

These transitions are not a strange or uncommon phenomenon. Far beyond they are generic, so examples can be found in several fields of science, from astronomy –galactic dynamics [19, 27, 28, 23], planetary theory, e. g. in [13]– to particle accelerators, e. g. in [15]; and not only in three degrees of freedom Hamiltonian systems, but also in higher dimensional problems. For example, in [22] families of periodic orbits were found with transitions stable to complex-unstable for the spatial elliptic three body problem (three and a half degrees of freedom): two pair of characteristic multipliers collide, while the third stays on the unit circle.

Furthermore, this same mechanism of instabilization takes place in families of symplectic maps where a fixed point undergoes Krein collisions of its eigenvalues for some (critical) value of the parameter: see [26, 24] and recently [16], where the behaviour of two of such families are investigated numerically. This constitutes an alternative approach to the investigation of the phenomena, since these maps can be thought of as Poincaré or first return maps of the corresponding flows (see the Gorov's introduction to the English edition of [29] for historical references).

On the other hand, several analytical studies have been also carried out: [4] (for symplectic maps) and [33] for the Hamiltonian Hopf bifurcation at equilibrium points in two-degrees of freedom Hamiltonian systems (also see [21, 30, 20]).

Thus, with the above references on mind, the outcome of this work can be summarized as follows: We rely on normal forms as the key tool of our approach, deriving in a *constructive* way and up to *any* (arbitrary) order, a versal normal form of the Hamiltonian around the resonant periodic orbit. This involves the following steps: (i) We assume the Hamiltonian given in a suitable system of canonical coordinates which are adapted to the resonant periodic orbit; (ii) apply a canonical Floquet transformation to reduce the normal variational equations of the orbit to constant coefficients and (iii) proceed with the nonlinear reduction and describe, in some tricky and constructive way, the normal form.

Hence, dealing with the normal form itself (i.e., we compute the normal form up to a given order and we skip the remainder) we show the generic unfolding of a two-parameter family of 2D-invariant tori (*Hamiltonian Hopf bifurcation*) and identify the coefficients which govern not only the bifurcation, but also its character: *direct* or *inverse*. In the case of direct bifurcation, there appear elliptic tori around complex-unstable periodic orbits, while in the case of inverse bifurcation, hyperbolic tori (but also parabolic and elliptic tori) unfold around stable periodic orbits. This study is completed with a description of the global dynamics of the normal form. We remark that this is not a merely qualitative (i. e., formal) process for, in addition, accurate parametrizations of the families of invariant tori and even of the invariant manifolds of the hyperbolic periodic orbits and hyperbolic tori are derived in this way.

The contents of this paper are organized as follows. Section 2 tackles the computation of the normal form around the critical periodic orbit. The main result of this part, which is the normal form itself, is stated in theorem 2.5. Section 3 is devoted to the analysis of the dynamics of the normal form. In particular, theorem 3.5 establishes the unfolding of a two-parameter family of 2D invariant tori and proposition 3.7 states the normal stability of the bifurcated tori. Finally, in the appendix A, a study of the low (fourth) order normal form is carried out.

2 Analytic approach

The purpose of this section is to describe the normal form process around the critical periodic orbit. In section 2.1 we give a precise formulation of the problem and state the "normalization theorem" in which the normal form is described (see theorem 2.5). In section 2.2 we introduce (local) adapted coordinates around the resonant periodic orbit. The purpose of this change is to separate the dynamics along the periodic orbit (described now by an angular variable and its conjugate action), from the movement in the normal directions. Next, in section 2.3, a symplectic Floquet change is applied. The final goal is to arrive –through a symplectic 2π-periodic linear change–, to a "clean" Hamiltonian whose quadratic part is in Williamson's normal form with respect to its normal directions (see [1] and the references to the works of Galin and Williamson given there). Albeit it is not strictly necessary, the linearly reduced Hamiltonian is later complexified (section 2.4) to simplify the structure of the homological equations arising in the nonlinear normalization process, which begins in section 2.5. There, the Giorgilli-Galgani algorithm is introduced (proposition 2.26) as the canonical transformation device. From the recursive structure of the algorithm we: first, determine the form of the homological equations and second, prove their solvability in both, the generating function and the associated compatibility terms (proposition 2.48). These two last steps allow us to construct explicitly the normal form and hence prove theorem 2.5.

2.1 Formulation of the problem

Let $H(\zeta)$ with $\zeta^* = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3)$, be a real three degree of freedom analytic Hamiltonian (asterisk, in what follows will denote the transpose of a vector or a matrix), and consider its associated Hamiltonian system

$$\dot{\zeta} = J_3 \,\mathrm{grad}\, H(\zeta). \tag{1}$$

Henceforth, J_n will denote the matrix of the standard canonical n-form in $\mathbb{R}^{2n}$, i. e.,

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

being I_n the $n \times n$ identity matrix.

Suppose that this system has a nondegenerate family of periodic orbits, depending on a real parameter σ, $\{\mathcal{M}_\sigma\}_{\sigma \in \mathbb{R}}$ and such that for some value of the parameter, say $\sigma = 0$, the corresponding orbit $\mathcal{M}_0$ (from now on, the *critical* or *resonant* periodic orbit) has an *irrational* (see definition 2.2) collision of its nontrivial Floquet (characteristic) multipliers.

Remark 2.1. We recall that in Hamiltonian systems periodic orbits appear generically as one parameter families parametrized by the energy (see [31]).

To be more precise, in figure 1 we suppose that, for $\sigma < 0$, the nontrivial characteristic multipliers of $\mathcal{M}_\sigma$ lie on the unit circle, they approach pairwise as σ goes to $\sigma = 0$, for this value they collide and separate towards the complex plane when $\sigma > 0$. Collisions of characteristic multipliers as the one described are often referred in the literature as Krein collisions (see [3], appendix 29 and references therein).

Definition 2.2. Let $\lambda_0 \neq 1$ be a (double) nontrivial characteristic multiplier of the resonant periodic orbit $\mathcal{M}_0$ and let $\mu = 2\pi\kappa$ be its principal characteristic exponent (so $\lambda_0 = \mathrm{e}^{\mathrm{i}\mu}$). We say that the collision of characteristic multipliers on the unit circle is irrational if μ is not commensurable with 2π or, equivalently, if $\kappa \notin \mathbb{Q}$.

Moreover, we assume *genericity* of the collision, in the sense of the definition below.

Definition 2.3. Let $M(\mathcal{M}_0)$ denote the monodromy matrix of the resonant periodic orbit $\mathcal{M}_0$. Hence, $\mathrm{Spec}(M(\mathcal{M}_0)) = \{1, \lambda_0, 1/\lambda_0\}$. The Krein collision will be called *generic* if the Jordan normal form of M_0, $J(M(\mathcal{M}_0))$ has the following (or an equivalent) block structure,

$$J(M(\mathcal{M}_0)) = \left(\begin{array}{cc|cc|cc} 1 & 0 & & & & \\ 1 & 1 & & & & \\ \hline & & \lambda_0 & 0 & & \\ & & 1 & \lambda_0 & & \\ \hline & & & & 1/\lambda_0 & 0 \\ & & & & 1 & 1/\lambda_0 \end{array}\right) \tag{2}$$

Remark 2.4. Thus, one is assuming that none of the Jordan blocks of the monodromy matrix at the resonance is trivial (diagonal). In particular, the nontrivial character of the first block –corresponding to the eigenvalue equal to 1–, follows from the nondegeneracy condition of the family of periodic orbits. This is precisely the generic condition which allows to parametrize the family using the energy as a parameter (see our previous remark 2.1).

The main result of section 2 is the following normal form theorem:

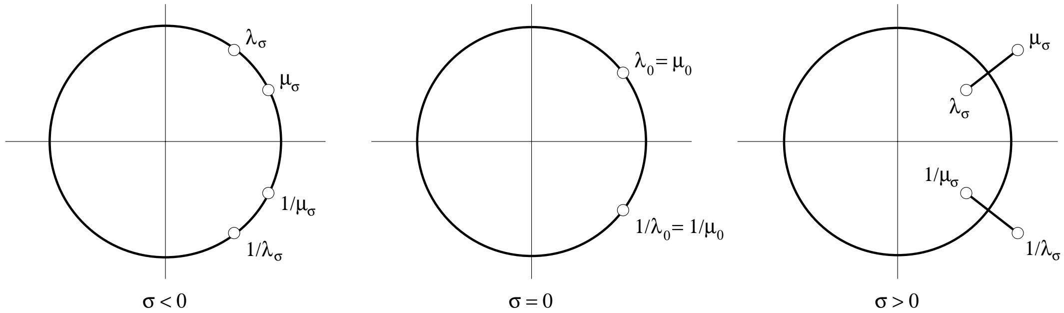


Figure 1: The transition from linear stability to complex instability for the family of periodic orbits $\{\mathcal{M}_\sigma\}_{\sigma\in\mathbb{R}}$ takes place through a collision of the nontrivial (i. e. different from 1) eigenvalues of the monodromy matrix corresponding to $\mathcal{M}_0$.

Theorem 2.5. *Consider the three-degree of freedom Hamiltonian system* (1)*. Let* $\{\mathcal{M}_\sigma\}_{\sigma\in\mathbb{R}}$ *be a one-parameter family of periodic orbits of this system with an irrational (so* $\kappa \notin \mathbb{Q}$*, see definition 2.2) and generic (in the sense of definition 2.3) Krein collision at* $\sigma = 0$*; also, let* ω_1 *denote the angular frequency of* $\mathcal{M}_0$ *and define* $\omega_2 := \kappa\omega_1$*.*

Then, given any $r \geq 3$*, there exists a real analytic symplectic change:* $(\xi,\eta) = \phi(\theta_1, x, I_1, y)$*, defined in* $\mathbb{S}^1 \times \mathfrak{W}$ *(*$\mathfrak{W}$ *a neighbourhood of the origin in* $\mathbb{R}^5$*) and taking values in a neighbourhood of* $\mathcal{M}_0$*, such that it casts the initial Hamiltonian into its normal form up to order* r*,*

$$H \circ \phi(\theta_1, x, I_1, y) = Z^{(r)}(x, I_1, y) + \mathfrak{R}^{(r)}(\theta_1, x, I_1, y).$$

Here, $Z^{(r)}$ *is the normal form up to order* r *and* $\mathfrak{R}^{(r)}$ *is the remainder (carrying higher order terms). The normal form is given by the sum*

$$Z^{(r)} = \sum_{s=2}^{r} Z_s,$$

with

$$Z_2 = \omega_1 I_1 + \omega_2(y_1x_2 - y_2x_1) \pm \frac{1}{2}(y_1^2 + y_2^2),$$

where the sign $\pm$ *in* Z_2 *is a characteristic of the collision and, for* $s \geq 3$*,* $Z_s = 0$ *if* s *is odd or an homogeneous polynomial of degree* $s/2$ *in*

$$\frac{1}{2}(x_1^2 + x_2^2), \quad I_1, \quad y_1x_2 - y_2x_1,$$

when s *is even.*

Remark 2.6. Although it is not pointed out in theorem 2.5, the change $(\xi,\eta) = \phi(\theta_1, x, I_1, y)$ yielding the normal form, depends on the order r. This will become clear through the proof of the theorem, where the transformation is constructed explicitly.

The remaining of the section is devoted to describe in detail the proof of this theorem.

2.2 Suitable coordinates around the critical periodic orbit

As a first step, we shall introduce (local) adapted coordinates around the *critical* periodic orbit $\mathcal{M}_0$ through an analytic 2π-periodic in $\widetilde{\theta}$ change of variables,

$$\xi_i = \xi_i(\widetilde{\theta}, \widetilde{\xi}, \widetilde{I}, \widetilde{\eta}), \qquad \eta_i = \eta_i(\widetilde{\theta}, \widetilde{\xi}, \widetilde{I}, \widetilde{\eta}), \tag{3}$$

$i = 1, 2, 3$ and with $\widetilde{\xi} = (\widetilde{\xi}_1, \widetilde{\xi}_2)$, $\widetilde{\eta} = (\widetilde{\eta}_1, \widetilde{\eta}_2)$. Furthermore, we shall ask the change (3) to satisfy the following properties (see [7, 6] and references therein):

P1. It maps the product set $\mathfrak{V} = \mathbb{T}^1 \times \Omega$, where Ω is a five-dimensional open set around the origin, onto some (possibly small) neighbourhood, $\mathfrak{U}$, of $\mathcal{M}_0$. We shall denote $\mathbb{T}^n := (\mathbb{R}/2\pi\mathbb{Z})^n$, $n \in \mathbb{N}$, the standard n-torus. In particular $\mathbb{T}^1 \equiv \mathbb{S}^1$.

P2. The orbit $\mathcal{M}_0$ is given by $\widetilde{\xi} = \widetilde{\eta} = 0$ and $\widetilde{I} = 0$ (and parametrized by $\widetilde{\theta}$).

P3. The change (3) is symplectic with $\widetilde{\theta}$, $\widetilde{\xi}$ and $\widetilde{I}$, $\widetilde{\eta}$ the new conjugate positions and momenta respectively. So in this coordinates, the system (1) is transformed into another Hamiltonian system,

$$\begin{aligned} \dot{\widetilde{\theta}} &= \frac{\partial \widetilde{H}}{\partial \widetilde{I}}, & \dot{\widetilde{I}} &= -\frac{\partial \widetilde{H}}{\partial \widetilde{\theta}}. \\ \dot{\widetilde{\xi}}_i &= \frac{\partial \widetilde{H}}{\partial \widetilde{\eta}_i}, & \dot{\widetilde{\eta}}_i &= -\frac{\partial \widetilde{H}}{\partial \widetilde{\xi}_i}, \qquad i = 1, 2. \end{aligned} \tag{4}$$

For an example with an explicit construction of *local* canonical coordinates such like the ones just described, see [18].

The transformed Hamiltonian, $\widetilde{H}$, defined in $\mathfrak{V}$, is analytic and 2π-periodic in $\widetilde{\theta}$, so it can be expanded in a convergent Taylor series,

$$\widetilde{H}(\widetilde{\theta}, \widetilde{\xi}, \widetilde{I}, \widetilde{\eta}) = \sum_{k,l,m} \widetilde{h}_{k,l,m}(\widetilde{\theta}) \widetilde{I}^k \widetilde{\xi}^l \widetilde{\eta}^m, \tag{5}$$

with $\xi^* = (\xi_1, \xi_2), \eta^* = (\eta_1, \eta_2)$ and the standard multi-index notation $\widetilde{\xi}^l \widetilde{\eta}^m = \widetilde{\xi}_1^{l_1} \widetilde{\xi}_2^{l_2} \widetilde{\eta}_1^{m_1} \widetilde{\eta}_2^{m_2}$, which we shall use throughout the text. The index k, and the components of $l^* = (l_1, l_2)$, $m^* = (m_1, m_2)$ range over the nonnegative integers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, while the coefficients $\widetilde{h}_{k,l,m}(\widetilde{\theta})$ are analytic 2π-periodic functions and can be expanded in Fourier series. If we restrict the system (4) to the periodic orbit $\mathcal{M}_0$, and take into account the expansion (5), we get

$$0 = \widetilde{h}'_{0,0,0,0,0}(\widetilde{\theta}), \quad \dot{\widetilde{\theta}} = \widetilde{h}_{1,0,0,0,0}(\widetilde{\theta}), \quad 0 = \widetilde{h}_{0,e_i}(\widetilde{\theta}), \quad i = 1, 2, 3, 4, \tag{6}$$

(e_i is the i-th unit vector in $\mathbb{R}^4$), since by the condition *P2* is $\dot{\widetilde{\xi}} = \dot{\widetilde{\eta}} = 0$ and $\dot{\widetilde{I}} = 0$, on the periodic orbit $\mathcal{M}_0$. Then, from equations above, it follows that $\widetilde{h}_{0,0,0,0,0}(\widetilde{\theta}) \equiv$ const., so we can set $\widetilde{h}_{0,0,0,0,0} = 0$.

Finally, one may get rid of the angular dependence of $\widetilde{h}_{1,0,0,0,0}$ by an additional change.

Lemma 2.7. *Let ϖ denote the mean value of $1/\widetilde{h}_{1,0,0,0,0}(\widetilde{\theta})$, i. e.,*

$$\varpi = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\widetilde{\theta}}{\widetilde{h}_{1,0,0,0,0}(\widetilde{\theta})}.$$

If we introduce the function,

$$f(\widetilde{\theta}) := \int_0^{\widetilde{\theta}} \frac{\varpi}{\widetilde{h}_{1,0,0,0,0}(\psi)} \, d\psi$$

then, the symplectic change in the action-angle coordinates:

$$I = \frac{\widetilde{I}}{\varpi} \widetilde{h}_{1,0,0,0,0}(\widetilde{\theta}), \quad \theta = f(\widetilde{\theta}), \tag{7}$$

transforms $\widetilde{H}$ into a new Hamiltonian, $\widehat{H}$, which can be expanded as

$$\widehat{H}(\theta, I, \widetilde{\xi}, \widetilde{\eta}) = \sum_{k,l,m} \widehat{h}_{k,l,m}(\theta) I^k \widetilde{\xi}^l \widetilde{\eta}^m \tag{8}$$

($k \in \mathbb{Z}_+; l, m \in \mathbb{Z}_+^2$), with the coefficients

$$\widehat{h}_{0,0,0,0,0}(\theta) = 0, \quad \widehat{h}_{1,0,0,0,0}(\theta) = \varpi, \quad \widehat{h}_{0,e_i}(\theta) = 0, \quad i = 1, 2, 3, 4. \tag{9}$$

Furthermore, $\varpi = \omega_1$, the angular frequency of $\mathcal{M}_0$.

Remark 2.8. Note that $\widetilde{h}_{1,0,0,0,0}(\widetilde{\theta}) \neq 0$ for $\mathcal{M}_0$ is a periodic solution and there are not stationary points on it (see (6)).

Proof (of lemma 2.7). One checks immediately that $dI \wedge d\theta = d\widetilde{I} \wedge d\widetilde{\theta}$, then (9) follows straightforward (see the references [7, 6] quoted at the beginning). The last point comes obviously from the fact that, in view of (7), on $\mathcal{M}_0$, one has $\dot{\theta} = \varpi$ and hence $\theta = \varpi t + \text{const.}$, but $\mathcal{M}_0$ is a periodic orbit with angular frequency ω_1 so it must be $\varpi = \omega_1$. □

2.3 Linear normalization

The main result of this section states that, beyond the adapted coordinates, a new symplectic change (a "canonical Floquet" transformation) can be applied to reduce the *normal variational equations* to a system with constant coefficients.

Remark 2.9. Throughout the text, we shall refer as the "normal directions" those normal to the periodic orbit. Clearly, once an angle and its conjugate action have been introduced to describe the periodic orbit, the normal directions in the phase space will be the ones associated to the rest of the positions and their corresponding conjugate momenta.

Lemma 2.10. *Assuming that the monodromy matrix of the resonant periodic orbit $\mathcal{M}_0$ has the Jordan block structure* (2) *(and hence generiticity of the Krein collision, according to definition 2.3), the Hamiltonian $\widehat{H}(\theta, I, \widetilde{\xi}, \widetilde{\eta})$, obtained after the application of lemma 2.7, can be transformed by means of a symplectic change into*

$$\mathcal{H}(\theta_1, x, I_1, y) = \mathcal{H}_2(x, I_1, y) + \ldots, \tag{10}$$

here (θ_1, x, I_1, y) are the new symplectic coordinates and $\mathcal{H}_2$ is given by,

$$\mathcal{H}_2(x, I_1, y) = \omega_1 I_1 + \omega_2(y_1 x_2 - y_2 x_1) \pm \frac{1}{2}(y_1^2 + y_2^2) \tag{11}$$

where $\omega_2 = \kappa\omega_1$ (see theorem 2.5) and the sign $\pm$ in the above formula is a characteristic of the collision; in particular we also remark that $\mathcal{H}_2$ is free from angular dependence. Furthermore, the canonical transformation is linear in the normal directions $z^* = (x, y)$ and 2π-periodic in the angle $\theta_1 = \theta$ (Floquet canonical reduction). In this sense, we shall say that the transformed Hamiltonian (10) *is "linearly reduced" (with respect to the normal directions).*

Remark 2.11. We note that the normal part of $\mathcal{H}_2$ (i. e., excluding the "rotor" $\omega_1 I_1$) agrees with the quadratic normal form in the classification given in [1] for the non-diagonalizable case.

The present (sub)section is devoted to prove lemma 2.10. With this purpose, we shall first introduce the following (structural) result.

Lemma 2.12. *Let $A \in \mathrm{Sp}(2, \mathbb{R})$, with*

$$\mathrm{Spec}(A) = \{\mathrm{e}^{\pm \mathrm{i}\theta}\}, \ \theta \in (0, \pi) \quad \textit{and} \quad \dim \mathrm{Ker}(A - \mathrm{e}^{\pm \mathrm{i}\theta} I_4) = 1.$$

Then, there exists $C \in \mathrm{Sp}(2, \mathbb{R})$, such that,

$$C^{-1} A C = \left(\begin{array}{c|c} \mathcal{R}_\theta & \epsilon \mathcal{R}_\theta \\ \hline 0 & \mathcal{R}_\theta \end{array} \right), \tag{12}$$

with $\mathcal{R}_\theta = \left(\begin{smallmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{smallmatrix} \right)$, and $\epsilon = \pm 1$, which is a characteristic of the matrix A.

Next, we sketch a proof of lemma 2.12 (see also [5]). A detailed proof can be found in [25].

Proof (of lemma 2.12). We define $\lambda_\pm = \mathrm{e}^{\pm \mathrm{i}\theta}$; and let z and w be the geometric and generalized eigenvector of λ_+ respectively, so

$$Az = \lambda_+ z, \qquad (A - \lambda_+ I_4)\, w = \lambda_+ z. \tag{13}$$

Let us now introduce the 2-form $\omega^2(u,v) = u^* J_2 v$ (i. e., the standard canonical two form in $\mathbb{C}^4$). We define:

$$\Delta := \omega^2(z, \overline{w}),$$

where the bar denotes complex conjugation. As A is a symplectic matrix and taking into account (13), we have that $\Delta \in \mathbb{R} \setminus \{0\}$. So, we introduce ϵ as $\epsilon := \operatorname{sign}(\Delta)$. Then, $\epsilon = \pm 1$, and it is easy to realize that it does not depend on the chosen w, so (as it has been already pointed in the statement of the lemma) it is a sign characteristic of the matrix A. Let us also define

$$\alpha := -\frac{\omega^2(w, \overline{w})}{2\mathrm{i}\Delta} \in \mathbb{R},$$

and introduce the complex basis $\mathfrak{B} = \{u_1, u_2, u_3, u_4\}$, with

$$u_1 = \frac{1}{\sqrt{2|\Delta|}} z, \quad u_2 = \frac{\epsilon}{\sqrt{2|\Delta|}}(w + \mathrm{i}\alpha z), \quad u_3 = \frac{\epsilon}{\sqrt{2|\Delta|}}(\overline{w} - \mathrm{i}\alpha \overline{z}), \quad u_4 = \frac{-1}{\sqrt{2|\Delta|}}\overline{z}.$$

From the previous definitions, $\mathfrak{B}$ is a symplectic basis, and expressed in this basis A takes the form:

$$\widehat{A} = \left(\begin{array}{cc|cc} \lambda_+ & \epsilon\lambda_+ & \multicolumn{2}{c}{0} \\ 0 & \lambda_+ & & \\ \hline \multicolumn{2}{c|}{0} & \lambda_- & 0 \\ & & -\epsilon\lambda_- & \lambda_- \end{array}\right),$$

($\bar{\lambda}_+ = \lambda_-$, by definition). Now, if we consider the (real) canonical basis $\mathfrak{B}' = \{u_1', u_2', u_3', u_4'\}$, given by

$$u_1' = \frac{u_1 - u_4}{\sqrt{2}}, \; u_2' = \frac{u_1 + u_4}{\sqrt{2}\mathrm{i}}, \; u_3' = \frac{u_2 + u_3}{\sqrt{2}}, \; u_4' = \frac{u_2 - u_3}{\sqrt{2}\mathrm{i}},$$

then the matrix A expressed in this basis takes the desired form. □

Lemma 2.12 will be used in the proof of lemma 2.10, which follows below.

Proof of lemma 2.10. Let us denote by $\widehat{H}_2$ the following low order terms in the expansion of the Hamiltonian (8):

$$\widehat{H}_2(\theta, \widetilde{\xi}, I, \widetilde{\eta}) := \widehat{h}_{1,0,0,0,0}(\theta) I + \sum_{|m|_1 + |l|_1 = 2} \widehat{h}_{0,l,m}(\theta) \widetilde{\xi}^l \widetilde{\eta}^m = \omega_1 \left(I + \frac{1}{2} < \widetilde{\zeta}, \Gamma(\theta)\, \widetilde{\zeta} > \right), \tag{14}$$

being $\widetilde{\zeta}^* = (\widetilde{\xi}^*, \widetilde{\eta}^*)$ and $\Gamma(\theta)$ a 4×4 symmetric matrix whose coefficients are real analytic 2π-periodic functions. Moreover, we introduce the norm $|v|_1 = \sum_{i=1}^n |v_i|$, $v \in \mathbb{R}^n$. To achieve the linear normalization, one skips the higher order terms in the Taylor expansion of the Hamiltonian and considers the normal variational equations around the orbit, which are given by the following linear Hamiltonian system,

$$\dot{\theta} = \omega_1, \tag{15a}$$
$$\dot{I} = 0, \tag{15b}$$
$$\dot{\widetilde{\zeta}} = \omega_1 J_2 \Gamma(\theta)\, \widetilde{\zeta}. \tag{15c}$$

Combining (15a), and the *normal* linear system (15c), one obtains

$$\frac{d\widetilde{\zeta}}{d\theta} = J_2 \Gamma(\theta)\, \widetilde{\zeta}. \tag{16}$$

Now, let $X(\theta)$ be a fundamental matrix of the solutions of (16). Then

$$X(\theta + 2\pi) = X(\theta) M_0,$$

where M_0 is a (symplectic) constant nonsingular matrix which, apart from the block $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ of (2), its Jordan form has the same block structure than the monodromy matrix of the resonant periodic orbit $\mathcal{M}_0$ (see (2)). Since (16) is a Hamiltonian system, X_0 is also a canonical matrix. Let us now introduce the following linear substitution,

$$\widetilde{\zeta} = B(\theta) z,$$

$z^* = (x^*, y^*)$, with $B(\theta)$ a canonical and 2π-periodic on θ matrix to be determined later. Therefore, the system (16) transforms into another linear Hamiltonian system:

$$\frac{dz}{d\theta} = J_2 \left(B^* \Gamma B + B^* J_2 B' \right) z. \tag{17}$$

Let us assume the following hypothesis (discussed below):

H1. A constant real symmetric matrix, N_0, exists such that M_0 admits an exponential representation of type,

$$M_0 = \exp(2\pi J_2 N_0).$$

H2. There is a constant canonical matrix D, and a matricial normal form G, to be chosen later, such that the matrix N_0 can be expressed as a product like

$$N_0 = D^* G D.$$

Thus, with hypothesis *H1* and *H2*, we take for $B(\theta)$,

$$B(\theta) = X(\theta) D^{-1} \exp(-\theta J_2 G) \tag{18}$$

and it is easy to check both, the 2π-periodicity of $B(\theta)$ and that the transformed system (17) turns out to be,

$$\frac{dz}{d\theta} = J_2 G z.$$

Next, we come back to the Hamiltonian $\widehat{H}$ and perform the transformation,

$$\theta = \theta_1, \quad I = I_1 + \frac{1}{2} \left\langle z, B^* J_2 B' z \right\rangle, \quad \widetilde{\zeta} = B(\theta_1) z$$

which, in turn, can be checked out to be canonical and 2π-periodic in θ. Direct substitution shows that $\widehat{H}_2$ (defined in (14)) transforms into

$$\begin{aligned} \widetilde{\mathcal{H}}_2(\theta_1, I_1, z) &= \omega_1 I_1 + \frac{\omega_1}{2} \left\langle \boldsymbol{z}, \left(B^* \Gamma B + B^* J_2 B' \right) z \right\rangle \\ &= \omega_1 I_1 + \frac{\omega_1}{2} \left\langle z, G z \right\rangle. \end{aligned} \tag{19}$$

Now we fix the matricial normal form G. If λ_0 and $1/\lambda_0$ are the two double eigenvalues of the monodromy matrix of $\mathcal{M}_0$ let, as in definition 2.2, κ be a real number such that $\lambda_0 = \mathrm{e}^{\mathrm{i} 2\pi\kappa}$; then we take

$$G = \begin{pmatrix} 0 & 0 & 0 & -\kappa \\ 0 & 0 & \kappa & 0 \\ 0 & \kappa & \epsilon/2\pi & 0 \\ -\kappa & 0 & 0 & \epsilon/2\pi \end{pmatrix}, \tag{20}$$

(with $\epsilon = \pm 1$). Hence,

$$2\pi J_2 G = \begin{pmatrix} 0 & 2\pi\kappa & \epsilon & 0 \\ -2\pi\kappa & 0 & 0 & \epsilon \\ 0 & 0 & 0 & 2\pi\kappa \\ 0 & 0 & -2\pi\kappa & 0 \end{pmatrix}$$

and substitution of the above matrix G in (19) gives, explicitly,

$$\widetilde{\mathcal{H}}_2(\theta_1, x, I_1, y) = \omega_1 I_1 + \kappa\omega_1(y_1 x_2 - y_2 x_1) + \epsilon\frac{\omega_1}{4\pi}(y_1^2 + y_2^2)$$

which, bearing in mind that $\omega_2 = \kappa\omega_1$ and $\epsilon = \pm 1$, gives (11) after the trivial symplectic substitution

$$x_i \mapsto \sqrt{\frac{\omega_1}{2\pi}} x_i, \quad y_i \mapsto \sqrt{\frac{2\pi}{\omega_1}} y_i, \quad i = 1, 2.$$

To end up the proof, however, we need to check that –with the choice in (20) for the matrix G–, the two hypothesis *H1* and *H2* are fulfilled. Both items follow immediately applying lemma 2.12, for direct computation shows:

$$\exp(2\pi J_2 G) = \left(\begin{array}{c|c} \mathcal{R}_{2\pi\kappa} & \epsilon\mathcal{R}_{2\pi\kappa} \\ \hline 0 & \mathcal{R}_{2\pi\kappa} \end{array}\right).$$

Thus, a (real) canonic matrix C exists such that $C^{-1} M_0 C = \exp(2\pi J_2 G)$, but this last implies:

$$M_0 = D^{-1}\exp(2\pi J_2 G) D = \exp(2\pi J_2 D^* G D),$$

Hence *H1* and *H2* hold identifying $D = C^{-1}$ (therefore D is a canonical matrix as well) and $N_0 = D^* G D$. Now the proof of lemma 2.10 is completed. □

Remark 2.13. Actually, the matrix G in the proof of lemma 2.10 has been chosen such that the infinitesimal symplectic matrix $2\pi J_2 G$ is in normal form with respect conjugation by elements of $\mathrm{Sp}(2, \mathbb{R})$. For purely imaginary eigenvalues –see [33], and also the reference of [8] quoted therein–, we get in $\mathrm{sp}(2, \mathbb{R})$ the two normal forms,

$$\begin{pmatrix} 0 & 0 & \lambda_1 & 0 \\ 0 & 0 & 0 & \lambda_2 \\ -\lambda_1 & 0 & 0 & 0 \\ 0 & -\lambda_2 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -\alpha & \epsilon & 0 \\ \alpha & 0 & 0 & \epsilon \\ 0 & 0 & 0 & -\alpha \\ 0 & 0 & \alpha & 0 \end{pmatrix},$$

(λ_1, λ_2, α may be positive or negative and $\epsilon = \pm 1$) that correspond to the semi-simple (diagonalizable) and non semi-simple (and also non-nilpotent) cases respectively.

Remark 2.14. After lemma 2.10 we should deal with two different cases which correspond to the plus or minus sign in (11). However, applying first the symplectic change,

$$\begin{aligned} \theta_1 &= \epsilon\theta_1, & x_1 &= \epsilon x_1', & x_2 &= x_2', \\ I_1 &= \epsilon I_1', & y_1 &= \epsilon y_1', & y_2 &= y_2' \end{aligned}$$

to the transformed Hamiltonian $\mathcal{H}$ (see (10)) one may get rid of the $\pm$ in (11) if further the substitution $t \mapsto \epsilon t$ (which reverses the sign of the time when $\epsilon = -1$) is allowed. In the forthcoming we shall assume that both transformations have been made so the $\pm$ sign will no longer appear (i. e., only the case $\epsilon = 1$ is considered). Moreover, the primes will be dropped and the names $\mathcal{H}$ for the linearly reduced Hamiltonian and $\mathcal{H}_2$ for its lower order terms are kept.

In this way, we arrive to a new, linearly reduced (in the sense stated in lemma 2.10) Hamiltonian, whose complete expansion can be written as,

$$\mathcal{H}(\theta_1, x, I_1, y) = \mathcal{H}_2(x, I_1, y) + \sum_{2l+|m|_1+|n|_1 \geq 3} \breve{h}_{l,m,n}(\theta_1) I_1^l x^m y^n \tag{21}$$

($l \in \mathbb{Z}_+, m, n \in \mathbb{Z}_+^2$), where $\mathcal{H}_2$ is now (see remark 2.14 above) given by,

$$\mathcal{H}_2(x, I_1, y) = \omega_1 I_1 + \omega_2(y_1 x_2 - y_2 x_1) + \frac{1}{2}(y_1^2 + y_2^2). \tag{22}$$

2.4 Complexification of the Hamiltonian

Before going on with the nonlinear normalization, and in order to get the homological equations in a simpler form, it is convenient to introduce the following (complex) coordinates,

$$x_1 = \frac{q_1 - p_2}{\sqrt{2}} \quad x_2 = -\frac{q_1 + p_2}{\mathrm{i}\sqrt{2}}, \quad y_1 = \frac{q_2 + p_1}{\sqrt{2}}, \quad y_2 = -\frac{q_2 - p_1}{\mathrm{i}\sqrt{2}}. \tag{23}$$

These last relations define a linear canonical change which transforms the Hamiltonian (21) into

$$H(\theta_1, q, I_1, p) = H_2(q, I_1, p) + \sum_{2l+|m|_1+|n|_1 \geq 3} h_{l,m,n}(\theta_1) I_1^l q^m p^n, \tag{24}$$

where H_2 is (22) expressed in these coordinates (see (25)). As usual, we have put $q^* = (q_1, q_2)$, $p^* = (p_1, p_2)$ and $h_{l,m,n}(\theta_1)$ are analytic 2π-periodic functions.

Also, by direct substitution of (23) in the Hamiltonian (21), it can be seen that the quadratic part in (24) is,

$$H_2 = \omega_1 I_1 + \mathrm{i}\omega_2(q_1 p_1 + q_2 p_2) + q_2 p_1. \tag{25}$$

This will be the lowest-order term in our normal form. Note that, in the change (23) x and y will be real provided that,

$$\bar{q}_1 = -p_2, \quad \bar{q}_2 = p_1. \tag{26}$$

Remark 2.15. If the above relations are assumed to hold and, as the complex Hamiltonian H is the transformed of a real Hamiltonian $\mathcal{H}$, it must be $\overline{H} = H$. More precisely, if we expand H in Poisson (Taylor-Fourier) series,

$$H(\theta_1, q, I_1, p) = \sum_{k,l,m,n} h_{k,l,m,n} I_1^l \, q^m p^n \exp(\mathrm{i}k\theta_1),$$

with $k \in \mathbb{Z}$, then it is readily checked that the inverse change of (23) transforms H back to the Poisson series of a real function if and only if the relations:

$$\bar{h}_{k,l,m_1,m_2,n_1,n_2} = (-1)^{m_1+n_2} h_{-k,l,n_2,n_1,m_2,m_1} \tag{27}$$

hold between the coefficients of the expansion of H.

2.5 Nonlinear normalization

Here, we shall apply a normal form process to remove the nonresonant higher degree terms of the Hamiltonian (24). We notice that if this normalization is carried out up to any order, it leads to a generically divergent system (due to the small divisors involved). So, the computation of the complete normal form, as described in the forthcoming, has to be regarded as a formal process. If we want to work with a convergent Hamiltonian, we have to stop after a finite number of steps of the normalizing process.

A very natural way to compute the normalizing transformation is to look for it as a composition of a sequence of canonical transformations, in such a way that the normal form is computed, degree-by-degree, by choosing the s^{th}-transformation to remove the nonresonant terms of degree $s+2$ from the Hamiltonian obtained after the previous step, for $s \geq 1$. However, if one is interested in further applications of the normal form (e. g., a quantitative analysis or a numerical implementation), it is advisable to use some "closed" transformation algorithm which computes the normalizing transformation from a *single* canonical change. More precisely, throughout this work we shall use the Giorgilli-Galgani algorithm (see [11, 12, 10, 32]) applied to formal Taylor-Fourier series (see definition 2.16 in (sub)section 2.5.1 below).

2.5.1 Some notation and definitions

Prior to the introduction of the Giorgilli-Galgani algorithm, we place here some definitions and introduce the appropriate (sub)spaces to work with.

Definition 2.16. We shall denote, by $\mathfrak{E}$ the space of formal Taylor-Fourier series of type

$$\mathfrak{E} := \{f = f(\theta_1, q, I_1, p) : f = \sum_{k,l,m,n} f_{k,l,m,n} I_1^l q^m p^n \exp(\mathrm{i}k\theta_1)\},$$

with $k \in \mathbb{Z}$, $l \in \mathbb{Z}_+$ and $m, n \in \mathbb{Z}_+^2$.

Definition 2.17 (adapted degree). Given a monomial $f_{k,l,m,n} I_1^l q^m p^n \exp(\mathrm{i}k\theta_1) \in \mathfrak{E}$, its *adapted degree* is defined as,

$$\deg\left(I_1^l q^m p^n \exp(\mathrm{i}k\theta_1)\right) := 2l + |m|_1 + |n|_1,$$

where $|\cdot|_1$ is the 1-norm in $\mathbb{R}^n$ i. e.: $|x|_1 := \sum_{i=1}^n |x_i|, \quad x \in \mathbb{R}^n$.

Remark 2.18. Hence, according to definition 2.17 the degree of the action variable I_1 is counted twice with respect to the degree of the normal coordinates (q, p). As we shall see, this will be convenient for the application of the Giorgilli-Galgani algorithm. Moreover, from now on and up to the end of section 2 the term *degree* will mean *adapted degree.*

Definition 2.19. Given $s \in \mathbb{N}$, $s \geq 3$; $\mathfrak{E}_s$ will denote the subspace of $\mathfrak{E}$ containing the homogeneous polynomials of degree s in (q, I_1, p) and with 2π-periodic coefficients in θ_1.

Definition 2.20. Let $\mathcal{S} : \mathfrak{E} \to \mathfrak{E}$ be the (linear) operator defined by,

$$f \mapsto \mathcal{S}(f) := \sum_{k,l,m,n} (-1)^{m_1+n_2} \bar{f}_{-k,l,n_2,n_1,m_2,m_1} I_1^l q^m p^n \exp(\mathrm{i}k\theta_1),$$

for all $f = \sum_{k,l,m,n} f_{k,l,m,n} I_1^l q^m p^n \exp(\mathrm{i}k\theta_1) \in \mathfrak{E}$.

Definition 2.21. If $f \in \mathfrak{E}$ (or $f \in \mathfrak{E}_s$) is such that $\mathcal{S}(f) = f$, then it is said that f *satisfies the* $\mathcal{S}$*-symmetries.* Further, the following subspaces

$$\mathfrak{E}^{\mathcal{S}} := \{f \in \mathfrak{E} : \mathcal{S}(f) = f\}$$

of $\mathfrak{E}$ and

$$\mathfrak{E}_s^{\mathcal{S}} := \{f \in \mathfrak{E}_s : \mathcal{S}(f) = f\}$$

of $\mathfrak{E}_s$ $(s \geq 3)$, will be considered.

Remark 2.22. In particular, if $f \in \mathfrak{E}^{\mathcal{S}}$ (or $\mathfrak{E}_s^{\mathcal{S}}$), their coefficients must satisfy relations (27).

Definition 2.23 (Poisson bracket). Given $f, g \in \mathfrak{E}$, one can write their Poisson bracket through,

$$\{f,g\} = \frac{\partial f}{\partial \theta_1}\left(\frac{\partial g}{\partial I_1}\right)^* - \frac{\partial f}{\partial I_1}\left(\frac{\partial g}{\partial \theta_1}\right)^* + \frac{\partial f}{\partial z} J_2 \left(\frac{\partial g}{\partial z}\right)^*.$$

Remark 2.24. Two properties of the Poisson bracket will be worth for our purposes. On the one hand, if $f \in \mathfrak{E}_s$ and $g \in \mathfrak{E}_p$, then $\{f,g\} \in \mathfrak{E}_{s+p-2}$; this will be the key to extend the Giorgilli-Galgani formulas (see proposition 2.26 below) to functions in the space $\mathfrak{E}$. On the other hand, it is just a straightforward verification to realize that the Poisson bracket preserves the $\mathcal{S}$-symmetries. More precisely: $\mathcal{S}(\{f,g\}) = \{f,g\}$ whenever $f, g \in \mathfrak{E}^S$.

Definition 2.25 (Lie operator). To each $u \in \mathfrak{E}$ we associate the (linear) operator

$$\begin{array}{rcl} L_u : \mathfrak{E} & \longrightarrow & \mathfrak{E} \\ f & \mapsto & L_u f = \{f,u\} \end{array}$$

(briefly $L_u := \{\cdot, u\}$), where $\{\cdot,\cdot\}$ is the Poisson bracket introduced in definition 2.23 above. The operator L_u is often referred as the *Lie operator* associated to u.

With the above definitions of the spaces $\mathfrak{E}$, $\mathfrak{E}_s$ and that of the Lie operator, we can now go on giving a formulation of the Giorgilli-Galgani algorithm.

Proposition 2.26 (The Giorgilli-Galgani algorithm). *Let $G = \sum_{s\geq 3} G_s \in \mathfrak{E}$ with $G_s \in \mathfrak{E}_s$; we define the map $T_G : \mathfrak{E} \to \mathfrak{E}$ in the following way: if $f = \sum_{l\geq 1} f_l \in \mathfrak{E}$, with $f_l \in \mathfrak{E}_l$, then*

$$T_G f = \sum_{s\geq 1} F_s,$$

where

$$F_s = \sum_{l=1}^{s} f_{l,s-l},$$

and the terms $f_{l,s}$ can be computed recursively by the formulas,

$$f_{l,0} = f_l, \quad f_{l,s} = \sum_{j=1}^{s} \frac{j}{s} L_{G_{2+j}} f_{l,s-j}. \tag{28}$$

Usually, the sum G is known as the generating function of the transformation.

The important property of transformation T_G is that, for $f \in \mathfrak{E}$ and assuming the convergence of f and G: $T_G f = f \circ \phi_1^{\breve{G}}$, where $\phi_1^{\breve{G}}$ is the time one flow of the Hamiltonian $\breve{G} = \sum_{s\geq 3} sG_s$. So, the coordinate transformation given by $\theta_1 = T_G\theta_1', I_1 = T_G I_1', q_i = T_G q_i'$ and $p_i = T_G p_i'$ $(i = 1,2)$ is canonical and 2π-periodic in θ_1'. The inverse transformation is obtained just considering T_{-G}. For an account of these properties, together with their corresponding proofs, see [11].

Remark 2.27. Nevertheless, there are also two other essential points to stress here. First, taking into account the behaviour of the Poisson bracket with respect to the (adapted) degree (see remark 2.24), it can be checked by induction, that the sums $f_{l,s}$ defined by (28) belong to $\mathfrak{E}_{l+s}$ and hence $F_s \in \mathfrak{E}_s$. This will allow us to set up (and solve) the homological equations *degreewise* in proposition 2.29. Second, if $G \in \mathfrak{E}^S$, then T_G preserves the $\mathcal{S}$-symmetries, i. e.: if $G \in \mathfrak{E}^{\mathcal{S}}$, then $\mathcal{S}(T_G f) = T_G f$ for all $f \in \mathfrak{E}^{\mathcal{S}}$. This last property is due to the preservation of the $\mathcal{S}$-symmetries under the Poisson bracket (in the sense stated in remark 2.24) and also to the structure of the operator T_G which consists, basically, on linear combinations of nested Poisson brackets with rational coefficients.

Remark 2.28 (notation). We will not use new names neither for the transformed functions, nor for the new coordinates so, to simplify the notation, the primes will be omitted.

The idea is thus to take $f = H$, the Hamiltonian function, to construct an *ad hoc* generating function of the form G, and employ the algorithm 2.26 to cast it into its (formal) normal form. This reduction process can be done recursively, and the following algorithm can be given to determine both G and the normal form.

Proposition 2.29. *Consider* $H = \sum_{s\geq 2} H_s \in \mathfrak{E}$*, with* $H_s \in \mathfrak{E}_s$*, and the generating function* $G = \sum_{s\geq 3} G_s$*, with* $G_s \in \mathfrak{E}_s$*. If we write* $T_G H = \sum_{s\geq 2} Z_s$*, the following relations are satisfied:*

$$\begin{aligned} Z_2 &= H_2 \\ L_{H_2} G_s + Z_s &= F_s, \quad s \geq 3, \end{aligned} \tag{29}$$

where,

$$\begin{aligned} F_3 &= H_3, \\ F_s &= \sum_{j=1}^{s-3} \frac{j}{s-2} L_{G_{2+j}} Z_{s-j} + \sum_{j=1}^{s-2} \frac{j}{s-2} H_{2+j,s-j-2}, \quad s \geq 4. \end{aligned}$$

Here, the quantities $H_{l,k}$ *may be computed recursively from the formulas* (28) *of the Giorgilli-Galgani algorithm.*

The proof of proposition 2.29 is formally identical to the proof of the corresponding classical one in [10] and in [32], so the reader is referred there.

From the relation (29), it is clear that the important point for us is to investigate the solvability, in terms of G_s and Z_s, of the *homological equation*

$$L_{H_2} G_s + Z_s = F_s, \tag{30}$$

for a given $F_s \in \mathfrak{E}_s$, in such a way Z_s takes the simplest possible form. This constitutes the subject of the next section.

2.6 Algebraic properties of the homological equations

Given $F_s \in \mathfrak{E}_s^{\mathcal{S}}$, our target is to look for the simplest expression for $Z_s \in \mathfrak{E}_s^{\mathcal{S}}$ (the normal form) in such a way there exists $G_s \in \mathfrak{E}_s^{\mathcal{S}}$ verifying identically equation (30).

First of all, to simplify notations, we assume the degree s fixed and we skip the subscript s of F_s and Z_s. At this stage, we consider a generic $F_s \in \mathfrak{E}_s$, whereas the rôle of the symmetries will be considered later on.

Now, what we shall do is to set $Z = 0$ and to investigate the solvability of the equations

$$L_{H_2} G = F. \tag{31}$$

In particular, we shall find out the possible resonant monomials in F, which will determine the form of Z in (30). The way in which the solvability of (31) is discussed also gives a constructive process to obtain G and Z from F. The proof is based on how the operator L_{H_2} acts on a monomial $g = I_1^l q_1^{m_1} q_2^{m_2} p_1^{n_1} p_2^{n_2} \exp(ik\theta_1) \in \mathfrak{E}_s$. It can be seen, by direct computation from the definition of L_{H_2}, that

$$L_{H_2} g = \{g, H_2\} = \left(\Omega + m_1 \frac{q_2}{q_1} - n_2 \frac{p_1}{p_2}\right) g, \tag{32}$$

where Ω is introduced as,

$$\Omega \equiv \Omega_{k,|m|_1,|n|_1} = \mathrm{i}\omega_1 k + \mathrm{i}\omega_2 (|m|_1 - |n|_1). \tag{33}$$

On the other hand, the quotient q_2/q_1 does not appear if the monomial g has $m_1 = 0$. Similarly for the quotient p_1/p_2. With this remark, the expression (32) is fully justified. Immediately from there, it follows that L_{H_2} preserves:

(i) The value of $k \in \mathbb{Z}$ of the monomial g.

(ii) The degree of the action, l.

(iii) The sums of the degrees $|m|_1$ of q, and $|n|_1$ of p, respectively.

(iv) As a consequence of the last two points, the global degree s is also preserved. This is consistent with our definition of the adapted degree.

In a natural way, we can consider the subspaces $\mathfrak{E}_{k,l,M,N}$, given by Poisson series $F \in \mathfrak{E}_s$ of the form

$$F = \sum_{\substack{0\le m\le M\\ 0\le n\le N}} f_{k,l,m,M-m,N-n,n} I_1^l\, q_1^m q_2^{M-m} p_1^{N-n} p_2^n \exp(\mathrm{i}k\theta_1), \tag{34}$$

with $2l + M + N = s$. Therefore,

$$\mathfrak{E}_s = \bigoplus_{\substack{k\in\mathbb{Z}\\ 2l+M+N=s}} \mathfrak{E}_{k,l,M,N}.$$

Hence, since every one of the subspaces $\mathfrak{E}_{k,l,M,N}$ is invariant under L_{H_2}, we can split the homological equations for the degree s into "boxes" with fixed k and fixed l, M, N verifying $2l + M + N = s$.

Another point to note is that the coefficients of a $F \in \mathfrak{E}_{k,l,M,N}$ are readily determined by just a pair of indices because, in view of (34), it is advisable to denote: $f_{m,n} = f_{k,l,m,M-m,N-n,n}$ (we skip both, k and l since they are held fixed).

With this notation, the homological equations (31) restricted to the different finite-dimensional subspaces, $\mathfrak{E}_{k,l,M,N}$, can be translated into an algebraic system of linear equations for the (complex) coefficients. Explicitly,

$$\Omega_{k,M,N}\, g_{m,n} + (m+1)\, \widetilde{\delta}_{m,M}\, g_{m+1,n} - (n+1)\, \widetilde{\delta}_{n,N}\, g_{m,n+1} = f_{m,n}, \tag{35}$$

where $\widetilde{\delta}_{i,j} = 1 - \delta_{i,j}$, being $\delta_{i,j}$ the Kronecker's delta and $g_{m,n}, f_{m,n}$ (for $0 \le m \le M$, $0 \le n \le N$) are the coefficients of the projections (on $\mathfrak{E}_{k,l,M,N}$) of the generating function G_s and the right hand side term F_s of (31), respectively.

To investigate the existence of solutions of (35), we first write them in a convenient matrix form of type,

$$\Lambda\, g = f. \tag{36}$$

Here $\Lambda \equiv \Lambda_{M,N,k}$ is a square matrix of dimension $(M+1) \times (N+1)$, $f \equiv f_{l,M,N,k}$ and $g \equiv g_{l,M,N,k}$ are the arrays holding $g_{m,n}$ and $f_{m,n}$ respectively, for $0 \le m \le M$ and $0 \le n \le N$ ($M + N = s - 2l$, fixed). Suppose that these coefficients have been ordered through the following claim: $g_{\mu,\beta} \prec g_{\alpha,\sigma}$ ($g_{\mu,\beta}$ precedes $g_{\alpha,\sigma}$) if $\mu > \alpha$ or, when $\mu = \alpha$, if $\beta > \sigma$ (the same for $f_{m,n}$). Therefore, g and f will take the form,

$$g^* = (g_M^*, g_{M-1}^*, \ldots, g_0^*), \quad f^* = (f_M^*, f_{M-1}^*, \ldots, f_0^*),$$

still with $g_J^* = (g_{J,N}, g_{J,N-1}, g_{J,N-2}, \ldots, g_{J,1}, g_{J,0})$, for $J = 0, \ldots, M$ (and identically for f). It is then straightforward to check that the system (36) is written by blocks as,

$$\begin{pmatrix} D_N &&&&& \\ E_M & D_N &&&& \\ & E_{M-1} & D_N &&& \\ && \ddots & \ddots && \\ &&& E_2 & D_N & \\ &&&& E_1 & D_N \end{pmatrix} \begin{pmatrix} g_M \\ g_{M-1} \\ g_{M-2} \\ \vdots \\ g_1 \\ g_0 \end{pmatrix} = \begin{pmatrix} f_M \\ f_{M-1} \\ f_{M-2} \\ \vdots \\ f_1 \\ f_0 \end{pmatrix}. \tag{37}$$

The different blocks stand for $E_j = j \cdot I_{N+1}$ (i. e., the product of the integer j with the identity $(N+1)\times(N+1)$ matrix, for $j = 1, 2, \ldots, M$), whereas $D_N = \Omega \cdot I_{N+1} - P_N$, with $\Omega = \Omega_{k,M,N}$ and being P_N the $(N+1)\times(N+1)$ nilpotent matrix,

$$P_N = \begin{pmatrix} 0 & & & & & \\ N & 0 & & & & \\ & N-1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & & 1 & 0 \end{pmatrix}.$$

It follows from this description, that the matrix Λ in (36) –now identified with the one of the linear system (37)–, is a band matrix, lower triangular, where all the entries different from zero are placed at the main diagonal and on two bands (sub-diagonals) below the main diagonal. Moreover, the elements of the diagonal are all them equal to Ω. Hence, if $\Omega \neq 0$, this specific part of the homological equations (we mean their components in the subspace $\mathfrak{E}_{k,l,M,N}$), has a unique solution which can be easily obtained. Then, the next lemma is readily deduced.

Lemma 2.30. *If we suppose $\omega_1/\omega_2 \notin \mathbb{Q}$ (i. e., assuming irrational collision, see definition 2.2); then, given $F \in \mathfrak{E}_{k,l,M,N}$, with $k \neq 0$ or $M \neq N$, there exists a unique $G \in \mathfrak{E}_{k,l,M,N}$ such that $L_{H_2}G = F$. Furthermore, the solution G satisfies the $\mathcal{S}$-symmetries if F does.*

Proof. From definition (33), it is clear that, if $\omega_1/\omega_2 \notin \mathbb{Q}$, then $\Omega_{k,M,N} = 0$ if and only if $k = 0$ and $M = N$. Thus, the first part follows from the lower band structure of the homological equations (restricted to $\mathfrak{E}_{k,l,M,N}$) shown above. On the other hand, if the vector g obtained from the coefficients of G is a solution of (36), then the vector defined by exchanging the coefficient g_{k,l,m_1,m_2,n_1,n_2} by $(-1)^{m_1+n_2}\bar{g}_{-k,l,n_2,n_1,m_2,m_1}$ (see relation (27) that defines the $\mathcal{S}$-symmetries) is also a solution of the same system. Thus, the last assertion of the lemma follows from the uniqueness of solutions. ☐

This result motivates the following definition.

Definition 2.31. A monomial $f = f_{k,l,m,n} I_1^l q^m p^n \exp(\mathrm{i}k\theta_1) \in \mathfrak{E}_s$ is said to be of $\mathcal{M}$-type if $k = 0$ and $|m|_1 = |n|_1$.

Then, returning to equation (31), we can state the following result referring to the "partial" resolution of this equation, which arises from lemma 2.30.

Proposition 2.32. *If $F \in \mathfrak{E}_s$ is free of $\mathcal{M}$-type monomials and $\omega_1/\omega_2 \notin \mathbb{Q}$, then, there is a solution for $G \in \mathfrak{E}_s$ in equation (30), which is unique if we look for it also free of $\mathcal{M}$-type monomials. Furthermore, the solution G satisfies the $\mathcal{S}$-symmetries if F does.*

Remark 2.33. We point out that Diophantine conditions on ω_1 and ω_2 are not required to ensure convergence of G if F does. This is because, for a fixed s, in the divisors $\mathrm{i}\omega_1 k + \mathrm{i}\omega_2(M-N)$ appearing in the homological equation we have that $M - N$ ranges between $-s$ and s, and therefore, for all $k \in \mathbb{Z}$, these divisors are bounded from below whenever ω_1/ω_2 is irrational.

2.7 Study of the homological equations for $\mathcal{M}$-type monomials

Now we are going to discuss the solvability of the homological equations when $\mathcal{M}$-type monomials are taken into account. Thus, we shall restrict the linear operator L_{H_2} to the space $\mathfrak{E}_{0,l,M,M} \subset \mathfrak{E}_s$, $(s = 2l + 2M)$. More precise definitions are introduced below.

Definition 2.34. Let $\mathcal{P}_M$ be the space of homogeneous polynomials of degree $2M$ in q_1, q_2, p_1, p_2, with complex coefficients, such that:

$$\widehat{F} \in \mathcal{P}_M \Leftrightarrow \widehat{F} = \sum_{j,\nu=0}^{M} \widehat{F}_{j,\nu}\, q_1^j q_2^{M-j} p_1^{M-\nu} p_2^{\nu}. \tag{38}$$

Next, the subspace $\mathcal{P}_M^{\mathcal{S}}$ is defined by those polynomials of $\mathcal{P}_M$ satisfying the $\mathcal{S}$- symmetries, i. e., for $\widehat{F} \in \mathcal{P}_M$, written as in (38),

$$\widehat{F} \in \mathcal{P}_M^{\mathcal{S}} \Leftrightarrow \overline{\widehat{F}}_{j,\nu} = (-1)^{j+\nu} \widehat{F}_{\nu,j}. \tag{39}$$

Therefore, $\mathfrak{E}_{0,l,M,M} = \{I^l \widehat{F},\ \text{with}\ \widehat{F} \in \mathcal{P}_M\}$, and we put symbolically,

$$\mathfrak{E}_{0,l,M,M} = I^l \mathcal{P}_M.$$

Remark 2.35. One can check that $\mathcal{P}_M^{\mathcal{S}}$ is a real linear subspace of $\mathcal{P}_M$ with $\dim_{\mathbb{R}} \mathcal{P}_M^{\mathcal{S}} = (M+1)^2$, the number of independent real coefficients, taking into account the symmetries –see the proof of lemma 2.43–.

Definition 2.36. Let $\widehat{L}$ be the linear operator,

$$\begin{array}{rcl} \widehat{L} : \mathcal{P}_M & \rightarrow & \mathcal{P}_M \\ \widehat{F} & \mapsto & \widehat{L}\widehat{F} = \{\widehat{F}, H_2\}, \end{array}$$

with H_2 given in (25) and being now

$$\{\widehat{F}, H_2\} = \sum_{i=1}^{2} \left(\frac{\partial \widehat{F}}{\partial q_i} \frac{\partial H_2}{\partial p_i} - \frac{\partial \widehat{F}}{\partial p_i} \frac{\partial H_2}{\partial q_i} \right).$$

Hence, $\widehat{L}$ can be thought of as the restriction of L_{H_2} on $\mathcal{P}_M$. It has then full sense, to consider the homological equations (30) restricted to the space $\mathcal{P}_M$:

$$\widehat{L}\widehat{G} + \widehat{Z} = \widehat{F}. \tag{40}$$

These will be the *reduced* homological equations. So, if $\widehat{G}$, $\widehat{Z} \in \mathcal{P}_M$ satisfy the above equation, then $I^l\widehat{G}$, $I^l\widehat{Z}$ will be a solution of (40).

What is important for us is to investigate Range $\widehat{L}$ and to find a complementary of this space.

Definition 2.37. Given two polynomials

$$F = \sum_{m,n} F_{m,n} q^m p^n, \qquad G = \sum_{m',n'} G_{m',n'} q^{m'} p^{n'}$$

we define their bracket (see [9]) by,

$$\langle F|G \rangle = \sum_{m_1,m_2,n_1,n_2} m_1!\, m_2!\, n_1!\, n_2!\, F_{m_1,m_2,n_1,n_2} \overline{G}_{m_1,m_2,n_1,n_2}.$$

It is straightforward to check that this bracket satisfies all the properties of a (complex) Hermitian product: $\langle F|G \rangle = \overline{\langle G|F \rangle}$, and so on. In particular, for two polynomials in $\mathcal{P}_M$, say

$$\widehat{F} = \sum_{j,\nu=0}^{M} \widehat{F}_{j,\nu}\, q_1^j q_2^{M-j} p_1^{M-\nu} p_2^{\nu}, \qquad \widehat{G} = \sum_{j',\nu'=0}^{M} \widehat{G}_{j',\nu'}\, q_1^{j'} q_2^{M-j'} p_1^{M-\nu'} p_2^{\nu'},$$

the bracket $\langle \widehat{F}|\widehat{G} \rangle$ results,

$$\langle \widehat{F}|\widehat{G} \rangle = \sum_{j,\nu=0}^{M} j!\,(M-j)!\,(M-\nu)!\,\nu!\, \widehat{F}_{j,\nu} \overline{\widehat{G}}_{j,\nu}. \tag{41}$$

Remark 2.38. On the other hand, the Hermitian product defined by (41) restricted to $\mathcal{P}_M^{\mathcal{S}}$ is a real inner product. To see this, take $\widehat{F}, \widehat{G} \in \mathcal{P}_M^{\mathcal{S}}$ and realize that,

$$\langle \widehat{F} | \widehat{G} \rangle = \ldots + j!\,(M-j)!\,(M-\nu)!\,\nu!\,\widehat{F}_{j,\nu}\overline{\widehat{G}}_{j,\nu} + \cdots + \nu!\,(M-\nu)!\,(M-j)!\,j!\,\widehat{F}_{\nu,j}\overline{\widehat{G}}_{\nu,j} + \ldots,$$

and application of (39) transforms the term on right hand side in:

$$\ldots + 2(-1)^{j+\nu} j!\,(M-j)!\,(M-\nu)!\,\nu!\,\mathrm{Re}(\widehat{F}_{j,\nu}\widehat{G}_{j,\nu}) + \ldots,$$

which is a real number (recall that, when $j = \nu$, then the coefficients $\widehat{F}_{j,\nu}$ and $\widehat{G}_{j,\nu}$ are real). Thus, $\langle \widehat{F} | \widehat{G} \rangle = \langle \widehat{G} | \widehat{F} \rangle$ and the rest of the properties for the real inner product follow.

Now, the following decomposition works,

$$\mathcal{P}_M = \mathrm{Range}\ \widehat{L} \oplus \mathrm{Ker}\ \widehat{L}^\dagger \tag{42}$$

where $\widehat{L}^\dagger$ is the adjoint operator of $\widehat{L}$. In fact, Ker $\widehat{L}^\dagger$ = (Range $\widehat{L})^\perp$ (the kernel of the adjoint operator of $\widehat{L}$ is the orthogonal complement of its image). Thus, it is a quite natural choice to take the complementary terms in this space and, as they will give rise to the compatibility terms Z_s in the homological equations (30), we shall refer to them (and further to the Z_s themselves) as the *resonant* terms. The next lemma determines the operator $\widehat{L}^\dagger$.

Lemma 2.39. *The operator* $\widehat{L}^\dagger = \{\cdot, H_2^\dagger\}$, *with* $H_2^\dagger = q_1 p_2$, *is the adjoint operator of* $\widehat{L}$ *with respect to the Hermitian product defined by* (41).

Proof. We have to see that if $\widehat{F}, \widehat{G} \in \mathcal{P}_M$, then $\langle \widehat{L}\widehat{F} | \widehat{G} \rangle = \langle \widehat{F} | \widehat{L}^\dagger \widehat{G} \rangle$. However, it is enough to check this for monomials: $\widehat{F} = q_1^j\, q_2^{M-j}\, p_1^{M-\nu}\, p_2^\nu$ and $\widehat{G} = q_1^{j'}\, q_2^{M-j'}\, p_1^{M-\nu'}\, p_2^{\nu'}$. Direct computation shows,

$$\widehat{L}\widehat{F} = \{\widehat{F}, H_2\} = j q_1^{j-1}\, q_2^{M-j+1}\, p_1^{M-\nu}\, p_2^\nu - \nu q_1^j\, q_2^{M-j}\, p_1^{M-\nu+1}\, p_2^{\nu-1},$$

$$\langle \widehat{L}\widehat{F} | \widehat{G} \rangle = \delta_{j',j-1}\,\delta_{\nu',\nu}\, j!\,(M-j+1)!\,(M-\nu)!\,\nu! - \delta_{j',j}\,\delta_{\nu',\nu-1}\, j!\,(M-j)!\,(M-\nu+1)!\,\nu!,$$

and, in the same way,

$$\widehat{L}^\dagger\widehat{G} = \{\widehat{G}, H_2^\dagger\} = -(M-\nu')q_1^{j'}\, q_2^{M-j'}\, p_1^{M-\nu'-1}\, p_2^{\nu'+1} + (M-j')q_1^{j'+1}\, q_2^{M-j'-1}\, p_1^{M-\nu'}\, p_2^{\nu'},$$

$$\langle \widehat{F} | \widehat{L}^\dagger \widehat{G} \rangle = -\delta_{j',j}\,\delta_{\nu,\nu'+1}\, j!\,(M-j)!\,(M-\nu+1)!\,\nu! + \delta_{j'+1,j}\,\delta_{\nu',\nu}\, j!\,(M-j+1)!\,(M-\nu)!\,\nu!.$$

Then, the proof of the lemma is completed. □

Remark 2.40. If should $\widehat{L}_{H_2}$ (and hence $\widehat{L}$) be self adjoint, giving rise to diagonal homological equations, then Ker $\widehat{L}$ = Ker $\widehat{L}^\dagger$. But in view of the system (37), this is clearly not the case.

Now, we can compute Ker $\widehat{L}^\dagger$, which gives a complement of Range $\widehat{L}$.

Lemma 2.41. *Let us define*

$$\xi_1 = \frac{\mathrm{i}}{2}\left(q_1 p_1 + q_2 p_2\right), \qquad \xi_2 = q_1 p_2. \tag{43}$$

Then, $\widehat{F} \in \mathcal{P}_M$ *belongs to* Ker $\widehat{L}^\dagger$ *if and only if*

$$\widehat{F} = \sum_{j=0}^{M} a_j \xi_1^{M-j} \xi_2^j.$$

Proof. Let $\widehat{F}$ be a polynomial in $\mathcal{P}_M$, then $\widehat{F} \in \mathrm{Ker}\,\widehat{L}^\dagger$ if and only if $\widehat{L}^\dagger \widehat{F} = \{\widehat{F}, H_2^\dagger\} = 0$. Thus, $\widehat{F}$ must be a solution of the partial differential equation (see [30]),

$$-p_2 \frac{\partial u}{\partial p_1} + q_1 \frac{\partial u}{\partial q_2} = 0. \tag{44}$$

A function u is a solution of a first order linear equation like this, if and only if it is a first integral of the associated characteristic system (see [2] Chap. 2, §7.B, or eventually, any textbook on differential equations).

For (44) the mentioned characteristic equations (i. e., the Hamiltonian equations of $H_2^\dagger$) are,

$$\dot{q_1} = 0, \qquad \dot{q_2} = q_1, \qquad \dot{p_1} = -p_2, \qquad \dot{p_2} = 0, \tag{45}$$

then, the functions, q_1, p_2, $q_1p_1 + q_2p_2$ are first integrals of the system (45), and u has to be a combination of them. However, if we want solutions in $\mathcal{P}_M$ we have to consider polynomials in ξ_1 and ξ_2. □

Remark 2.42. We note that ξ_1, ξ_2 are real under the symmetries (26) introduced by the complexification.

Furthermore, as the $\mathcal{S}$-symmetries are preserved under L_{H_2}, and hence under $\widehat{L}$, we can consider the restriction of $\widehat{L}$ to $\mathcal{P}_M^{\mathcal{S}}$ (say, $\widehat{L}^{\mathcal{S}} = \widehat{L}|_{\mathcal{P}_M^{\mathcal{S}}}$) and apply there the same decomposition (42) to have

$$\mathcal{P}_M^{\mathcal{S}} = \mathrm{Range}\,\widehat{L}^{\mathcal{S}} \oplus \mathrm{Ker}\,(\widehat{L}^{\mathcal{S}})^\dagger.$$

Now, we are in conditions to discuss the solvability of the reduced homological equations (40). The basic result is given by lemma 2.43, in which we characterize the resonant terms $\widehat{Z}$, and we give a constructive algorithm to determine $\widehat{G}$ from $\widehat{F}$. In order to state this lemma, first of all we introduce the following expressions:

$$\xi_3 = \frac{1}{2}(q_1p_1 - q_2p_2), \qquad \xi_4 = q_2p_1. \tag{46}$$

We remark that both, ξ_3 and ξ_4, verify the $\mathcal{S}$-symmetries (see definition 2.21) and so correspond to real expressions according to the complexification (23). They will be used, combined with ξ_1 and ξ_2 (see (43)), to describe the action of $\widehat{L}^{\mathcal{S}}$.

Lemma 2.43. *Given $\widehat{F} \in \mathcal{P}_M^{\mathcal{S}}$, there is a unique decomposition $\widehat{F} = \widehat{R} + \widehat{N}$ with $\widehat{R} \in \mathrm{Ker}(\widehat{L}^{\mathcal{S}})^\dagger$ and $\widehat{N} \in \mathrm{Range}\,\widehat{L}^{\mathcal{S}}$. More precisely, if we set*

$$\mathfrak{Q} = \{\xi_1^{M-j}\xi_2^j\}_{j=0,\dots,M}, \qquad \mathfrak{T} = \{\xi_1^\nu \xi_2^m \xi_3^n, \xi_1^\nu \xi_3^m \xi_4^n\}_{\nu+m+n=M, n\neq 0},$$

then $\mathrm{Ker}(\widehat{L}^{\mathcal{S}})^\dagger = \mathrm{Span}_{\mathbb{R}}\,\mathfrak{Q}$ *and* $\mathrm{Range}\,\widehat{L}^{\mathcal{S}} = \mathrm{Span}_{\mathbb{R}}\,\mathfrak{T}$. *Thus, there exists $\widehat{G} \in \mathcal{P}_M^{\mathcal{S}}$ such that $\widehat{L}^{\mathcal{S}}\widehat{G} = \widehat{N}$, and is uniquely determined except by the addition of terms in* $\mathrm{Ker}(\widehat{L}^{\mathcal{S}})^\dagger$*, which is given by* $\mathrm{Ker}(\widehat{L}^{\mathcal{S}})^\dagger = \mathrm{Span}_{\mathbb{R}}\{\xi_1^{M-j}\xi_4^j\}_{j=0,\dots,M}$.

Remark 2.44. So, given $\widehat{F} \in \mathcal{P}_M^{\mathcal{S}}$, equation $\widehat{L}\widehat{G} + \widehat{Z} = \widehat{F}$ is solved in $\mathcal{P}_M^{\mathcal{S}}$ by taking $\widehat{Z} = \widehat{R}$ (*the normal form*) and $\widehat{G}$ a solution of $\widehat{L}\widehat{G} = \widehat{N}$. The (constructive) way in which $\widehat{G}$ and $\widehat{Z}$ can be obtained from $\widehat{F}$ is described in the proof.

Proof. Let $\widehat{F} \in \mathcal{P}_M^{\mathcal{S}}$ (see definition 2.34). As any monomial $q_1^j q_2^{M-j} p_1^{M-\nu} p_2^\nu$, can be re-arranged into the form,

$$q_1^j q_2^{M-j} p_1^{M-\nu} p_2^\nu = \begin{cases} (q_1p_1)^j (q_2p_2)^\nu (q_2p_1)^{M-j-\nu} & \text{if } j+\nu \le M, \\ (q_1p_1)^{M-\nu} (q_2p_2)^{M-j} (q_1p_2)^{j+\nu-M} & \text{if } j+\nu > M, \end{cases}$$

$\widehat{F}$ may be written as,

$$\widehat{F} = \sum_{\nu=1}^{M} \eta_2^{\nu} \sum_{\nu+m+n=M} \widetilde{f}_{\nu,m,n}\, \eta_1^m \eta_3^n + \sum_{\nu=0}^{M} \eta_4^{\nu} \sum_{\nu+m+n=M} \widetilde{g}_{\nu,m,n} \eta_1^m \eta_3^n,$$

where,

$$\eta_1 = q_1 p_1, \quad \eta_2 = q_1 p_2, \quad \eta_3 = q_2 p_2, \quad \eta_4 = q_2 p_1,$$

and with the complexification symmetries,

$$\overline{\widetilde{f}}_{l,m,n} = (-1)^{m+n} \widetilde{f}_{l,n,m}, \qquad \text{and} \qquad \overline{\widetilde{g}}_{l,m,n} = (-1)^{m+n} \widetilde{g}_{l,n,m}. \tag{47}$$

Then, it is clear that,

$$\begin{aligned} \mathfrak{M} \;\; = \;\; & \{\eta_2^{\nu} \eta_1^m \eta_3^n : 1 \le \nu \le M \text{ and } \nu + m + n = M\} \\ & \cup \{\eta_4^{\nu} \eta_1^m \eta_3^n : 0 \le \nu \le M \text{ and } \nu + m + n = M\}, \end{aligned}$$

is a basis of $\mathcal{P}_M$ (with complex coefficients). Hence, the real dimension of $\mathcal{P}_M^{\mathcal{S}}$ is given by the number of real independent coefficients according to the symmetries (47). It is not difficult to check that, for a fixed ν in the sums above, the quantity of real and imaginary parts –not related by the symmetries (47)– necessary to form all the complex coefficients $\widetilde{f}_{\nu,m,n}$ is $M - \nu + 1$ (and the same for $\widetilde{g}_{\nu,m,n}$). Thus, summation over ν gives,

$$\dim_{\mathbb{R}} \mathcal{P}_M^{\mathcal{S}} = \sum_{\nu=1}^{M} (M - \nu + 1) + \sum_{\nu=0}^{M} (M - \nu + 1) = (M+1)^2.$$

Now, we consider the set of $(M+1)^2$ vectors of $\mathcal{P}_M$ defined by $\mathfrak{T} \cup \mathfrak{Q}$. It turns out that they are linearly independent in $\mathcal{P}_M$, because the map $(\xi_1, \xi_3) \mapsto (\eta_1, \eta_3)$ is linear and invertible (see formulas (43) and (46)); so $\mathfrak{T} \cup \mathfrak{Q}$ constitutes a *real* basis of $\mathcal{P}_M^{\mathcal{S}}$ (in the sense that all the elements of $\mathcal{P}_M^{\mathcal{S}}$ can be put as linear combinations of elements of $\mathfrak{T} \cup \mathfrak{Q}$ with *real* coefficients). We know, see Lemma 2.41, that $\mathfrak{Q}$ constitutes a real basis of $\mathrm{Ker}(\widehat{L}^{\mathcal{S}})^{\dagger}$, and so, we can decompose $\widehat{F} = \widehat{R} + \widehat{N}$ with $\widehat{R} \in \mathrm{Ker}(\widehat{L}^{\mathcal{S}})^{\dagger}$ and $\widehat{N} \in \mathrm{Span}_{\mathbb{R}}(\mathfrak{T})$. What we have to prove is that there is $\widehat{G} \in \mathcal{P}_M^{\mathcal{S}}$ such that $\widehat{L}\widehat{G} = \widehat{N}$. To show this, we take $\widehat{N}$ and $\widehat{G}$ and write them down in the basis $\mathfrak{T}$ and $\mathfrak{T} \cup \mathfrak{Q}$, respectively,

$$\begin{aligned} \widehat{N} &= \sum_{\substack{\nu+m+n=M \\ (\text{with } n \neq 0)}} \widehat{f}_{\nu,m,n}\, \xi_2^{\nu} \xi_1^m \xi_3^n + \sum_{\substack{\nu+m+n=M \\ (\text{with } \nu \neq 0)}} \widehat{g}_{\nu,m,n}\, \xi_4^{\nu} \xi_1^m \xi_3^n \\ \widehat{G} &= \sum_{\substack{\nu+m+n=M \\ (\text{with } \nu \neq 0)}} f_{\nu,m,n}\, \xi_2^{\nu} \xi_1^m \xi_3^n + \sum_{\nu+m+n=M} g_{\nu,m,n}\, \xi_4^{\nu} \xi_1^m \xi_3^n. \end{aligned}$$

Remark 2.45. We stress that the sums defining $\widehat{N}$ and $\widehat{G}$ are arranged in a different way (note specially the ranges of the index ν). This will be useful to compute explicitly the coefficients of $\widehat{G}$ (see below).

On the other hand, the operator $\widehat{L}^{\mathcal{S}}$ acting on the elements of the basis $\mathfrak{T} \cup \mathfrak{Q}$ yields,

$$\begin{aligned} \widehat{L}^{\mathcal{S}} (\xi_2^{\nu} \xi_1^m \xi_3^n) &= \frac{\partial}{\partial \xi_2} (\xi_2^{\nu} \xi_1^m \xi_3^n) \{\xi_2, H_2\} + \frac{\partial}{\partial \xi_1} (\xi_2^{\nu} \xi_1^m \xi_3^n) \{\xi_1, H_2\} + \frac{\partial}{\partial \xi_3} (\xi_2^{\nu} \xi_1^m \xi_3^n) \{\xi_3, H_2\}, \\ \widehat{L}^{\mathcal{S}} (\xi_4^{\nu} \xi_1^m \xi_3^n) &= \frac{\partial}{\partial \xi_4} (\xi_4^{\nu} \xi_1^m \xi_3^n) \{\xi_4, H_2\} + \frac{\partial}{\partial \xi_1} (\xi_4^{\nu} \xi_1^m \xi_3^n) \{\xi_1, H_2\} + \frac{\partial}{\partial \xi_3} (\xi_4^{\nu} \xi_1^m \xi_3^n) \{\xi_3, H_2\}, \end{aligned}$$

but,

$$\{\xi_1, H_2\} = 0, \quad \{\xi_2, H_2\} = -2\xi_3, \quad \{\xi_3, H_2\} = \xi_4, \quad \{\xi_4, H_2\} = 0.$$

Therefore,

$$\widehat{L}^{\mathcal{S}}\left(\xi_2^{\nu}\xi_1^{m}\xi_3^{n}\right) = -2\nu\xi_2^{\nu-1}\xi_1^{m}\xi_3^{n+1} + n\xi_2^{\nu}\xi_1^{m}\xi_3^{n-1}\xi_4,$$

and when $1 \le \nu \le M$, we can arrange the second term on the right using that $\xi_2\xi_4 = -\xi_1^2 - \xi_3^2$, so $n\xi_2^{\nu-1}\xi_1^{m}\xi_3^{n-1}\xi_2\xi_4 = -n\xi_2^{\nu-1}\xi_1^{m+2}\xi_3^{n-1} - n\xi_2^{\nu-1}\xi_1^{m}\xi_3^{n+1}$, which when it is joined to the first term gives

$$\widehat{L}^{\mathcal{S}}\left(\xi_2^{\nu}\xi_1^{m}\xi_3^{n}\right) = -(2\nu+n)\xi_2^{\nu-1}\xi_1^{m}\xi_3^{n+1} - n\xi_2^{\nu-1}\xi_1^{m+2}\xi_3^{n-1}, \tag{48}$$

with $1 \le \nu \le M$ and $\nu + m + n = M$. Similarly,

$$\widehat{L}^{\mathcal{S}}\left(\xi_4^{\nu}\xi_1^{m}\xi_3^{n}\right) = n\xi_4^{\nu+1}\xi_1^{m}\xi_3^{n-1} \tag{49}$$

for $0 \le \nu \le M$, $\nu + m + n = M$. With (48) and (49) we can compute the action of $\widehat{L}^{\mathcal{S}}$ on $\widehat{G}$ and write down explicitly the equation $\widehat{L}^{\mathcal{S}}\widehat{G} = \widehat{N}$ in the unknown real coefficients $f_{\nu,m,n}$ ($1 \le \nu \le M$, $\nu+m+n = M$), and $g_{\nu,m,n}$ ($0 \le \nu \le M$, $\nu+m+n = M$). In this way, one gets,

$$\begin{aligned}
&-\sum_{\nu=1}^{M}\xi_2^{\nu-1}\sum_{\nu+m+n=M}\left((2\nu+n)f_{\nu,m,n}\,\xi_1^{m}\xi_3^{n+1} + nf_{\nu,m,n}\,\xi_1^{m+2}\xi_3^{n-1}\right)\\
&\quad+\sum_{\nu=0}^{M}\xi_4^{\nu+1}\sum_{\nu+m+n=M} ng_{\nu,m,n}\,\xi_1^{m}\xi_3^{n-1}\\
&=\sum_{\nu=0}^{M-1}\xi_2^{\nu}\sum_{\substack{\nu+m+n=M\\(n\neq 0)}}\widehat{f}_{\nu,m,n}\,\xi_1^{m}\xi_3^{n} + \sum_{\nu=1}^{M}\xi_4^{\nu}\sum_{\nu+m+n=M}\widehat{g}_{\nu,m,n}\,\xi_1^{m}\xi_3^{n}.
\end{aligned} \tag{50}$$

By comparison of coefficients in the second sums of both sides, we arrive to the relations:

$$g_{\nu-1,m,n+1} = \frac{\widehat{g}_{\nu,m,n}}{n+1}, \tag{51}$$

with $1 \le \nu \le M$ ($\nu + m + n = M$).

Remark 2.46. Equation (51) does not determine $g_{\nu,m,0}$ ($\nu + m = M$), so these coefficients can be chosen arbitrarily (for they play no rôle in the homological equations). In particular we shall set them to zero, i. e., we take: $g_{\nu,m,0} = 0$ ($\nu + m = M$). It will be clear in a moment that the corresponding monomials constitute a basis of $\mathrm{Ker}(\widehat{L}^{\mathcal{S}})^{\dagger}$.

Similarly, comparison of coefficients in the first sums on the left and on the right hand side of (50), with $0 \le \nu < M$ held fixed, leads to the linear system,

$$\widehat{f}_{\nu,0,M-\nu} = -(M+\nu+1)f_{\nu+1,0,M-\nu-1}, \tag{52a}$$

$$\widehat{f}_{\nu,1,M-\nu-1} = -(M+\nu)f_{\nu+1,1,M-\nu-2}, \tag{52b}$$

$$\begin{aligned}\widehat{f}_{\nu,m,M-\nu-m} &= -(M+\nu-m+1)f_{\nu+1,m,M-\nu-m-1}\\ &\quad-(M-\nu-m+1)f_{\nu+1,m-2,M-\nu-m+1}\end{aligned} \tag{52c}$$

with $2 \le m \le M-\nu-1$ (and $\nu \le M-3$) in the last equation. If we introduce the vectors $f, \widehat{f} \in \mathbb{R}^{M-\nu}$ by,

$$f = \begin{pmatrix} f_{\nu+1,0,M-\nu-1}\\ f_{\nu+1,1,M-\nu-2}\\ f_{\nu+1,2,M-\nu-3}\\ \vdots\\ f_{\nu+1,M-\nu-2,1}\\ f_{\nu+1,M-\nu-1,0}\end{pmatrix}, \qquad \widehat{f} = \begin{pmatrix} \widehat{f}_{\nu,0,M-\nu}\\ \widehat{f}_{\nu,1,M-\nu-1}\\ \widehat{f}_{\nu,2,M-\nu-2}\\ \vdots\\ \widehat{f}_{\nu,M-\nu-2,2}\\ \widehat{f}_{\nu,M-\nu-1,1}\end{pmatrix},$$

for $\nu = 0, \dots, M-1$. Then, the equations (52a)–(52c) can be expressed in vector notation as,

$$\Lambda' f = \widehat{f} \tag{53}$$

with the $(M-\nu)\times(M-\nu)$ dimensional matrix $\Lambda' \equiv \Lambda'_\nu$ given by $\Lambda'_{j,j} = -(M+\nu+2-j)$, for $j = 1, \cdots, M-\nu$; $\Lambda'_{j,j-2} = -(M-\nu+2-j)$, for $j = 3, \cdots, M-\nu$, and with the rest of the coefficients equal to zero. Hence, it is a nonsingular, lower-triangular, matrix for every $0 \le \nu < M$, and so it has a unique solution. We conclude then that $\mathrm{Range}(\widehat{L}^{\mathcal{S}}) = \mathrm{Span}_{\mathbb{R}}(\mathfrak{T})$. These considerations close the proof of lemma 2.43. □

Remark 2.47. Observe that in the proof of the last lemma, we have set up the homological equations for the $\mathcal{M}$-type monomials, but to solve them and compute the corresponding terms of the generating function, we need to write the elements $F \in \mathcal{P}_M^{\mathcal{S}}$ from its natural form (38)

$$F = \sum_{j,\alpha=0}^{M} F_{j,\alpha}\, q_1^j q_2^{M-j} p_1^{M-\alpha} p_2^\alpha$$

to a polynomial in ξ_1, ξ_2, ξ_3 and ξ_4 of type

$$F = \sum_{\nu=0}^{M} \xi_2^\nu \sum_{\nu+\vartheta+j=M} f_{\nu,\vartheta,j}\, \xi_1^\vartheta \xi_3^j + \sum_{\nu=1}^{M} \xi_4^\nu \sum_{\nu+\vartheta+j=M} g_{\nu,\vartheta,j}\, \xi_1^\vartheta \xi_3^j,$$

which allows to identify easily its components $\widehat{R}$ and $\widehat{N}$ (see lemma 2.43 above). By direct substitution it can be seen that the relation between the coefficients of the two expressions for F are,

$$f_{\nu,\vartheta,j} = \mathrm{i}^\vartheta \sum_{m+n=M-\nu} (-1)^n C(m,n,\vartheta) F_{M-n,M-m} \qquad \text{if } 0 \le \nu \le M, \tag{54}$$

$$g_{\nu,\vartheta,j} = \mathrm{i}^\vartheta \sum_{m+n=M-\nu} (-1)^n C(m,n,\vartheta) F_{m,n} \qquad \text{if } 1 \le \nu \le M, \tag{55}$$

with the form factors $C(m,n,\vartheta)$ defined by,

$$C(m,n,\vartheta) = \sum_{\beta=\max(0,\vartheta-m)}^{\min(\vartheta,n)} \binom{m}{\vartheta-\beta}\binom{n}{\beta}(-1)^{\vartheta-\beta}$$

and satisfying in addition the symmetries,

$$C(n,m,\vartheta) = (-1)^\vartheta C(m,n,\vartheta).$$

Finally we recall that, as the coefficients (54) and (55) are real: they expand F, an element of the real vector space $F \in \mathcal{P}_M^{\mathcal{S}}$, with respect to the real basis $\mathfrak{T} \cup \mathfrak{Q}$.

Finally, lemmas 2.30 and 2.43, considered together, answer the question of the (formal) solvability of the homological equations (30), and allow us to state the following proposition.

Proposition 2.48. *The homological equations* (30)*, with $F_s \in \mathfrak{C}_s^{\mathcal{S}}$, $s \ge 3$, are identically fulfilled with $G_s, Z_s \in \mathfrak{C}_s^{\mathcal{S}}$, where $Z_s = 0$ when s is odd or, when s is even, Z_s can be written as an homogeneous polynomial, with real coefficients, of degree $s/2$ in $q_1p_2, I_1, \mathrm{i}(q_1p_1+q_2p_2)$.*

Hence, theorem 2.5 follows immediately from lemma 2.10, proposition 2.29 and finally from proposition 2.48, since applying the inverse of the change (23), one gets

$$q_1p_2 = -\frac{1}{2}(x_1^2+x_2^2), \quad \mathrm{i}(q_1p_1+q_2p_2) = y_1x_2 - y_2x_1$$

and therefore, for $s \ge 3$ and even, Z_s depends on the real coordinates I_1, x, y in the form stated in the theorem. Note, however, that due to the minus sign in the first of the formulas above, the coefficients of Z_s may differ in a sign when expressed in real or in complex coordinates. Also, we remark that an additional reversion in the time t, $t \mapsto -t$, is necessary if $\epsilon = -1$ after the linear reduction (see lemma 2.10 and remark 2.14). This closes the section.

3 Dynamics of the normal form

In this section, the normal form $Z^{(r)}$ is analyzed. Then, after the setting of the Hamiltonian equations corresponding to $Z^{(r)}$ and the discussion of their first integrals (section 3.1) we derive, in section 3.2, a parametrization of the family of periodic orbits and discuss their stability in terms of such parametrization (lemma 3.4). Next, in section 3.3 we show that –under certain generic conditions which depend intrinsically on the low order terms of the normal form– there unfolds, "surrounding" the periodic orbits, a two-parameter family of two dimensional invariant tori. Furthermore, a study of the normal behaviour of such bifurcating tori is done, and the results are summarized in proposition 3.7. The global picture resembles the classical Andronov-Hopf bifurcation, in the sense that unfolded *stable* objects (2D-invariant tori in our case) appear around lower dimensional *unstable* ones –here, the periodic orbits of the family–, whereas conversely, *unstable* 2D-invariant tori may unfold around *stable* periodic orbits. Whether the former or the latter phenomenon takes place, depends again upon the nature of the low-order normal form. In the literature –see [33]–, this kind of bifurcation is known as the *Hamiltonian Andronov-Hopf bifurcation*. Next, in sections 3.4 and 3.5, we give parametrizations of the invariant manifolds of the hyperbolic periodic orbits of the family and of the bifurcated (hyperbolic) 2D-invariant tori. Section 3.6 deals with the 3D-invariant tori branching off the 2D-elliptic tori of proposition 3.7.

However, it is worth to realize that if only a qualitative description is needed, all these forementioned objects (periodic orbits, invariant tori and manifolds) and the dynamics generated from them, can be detected using the normal form (56) up to an order as low as four (hence $r = 2$). This is carried out at the end, in appendix A. Nevertheless, if one looks for accurate parametrizations of those backbone dynamical invariants, then they must be computed from a normal form of higher order and therefore, our approach describes the (local) dynamics around the non semi-simple resonant periodic orbit not only qualitatively, but quantitatively as well, in the sense that allows all these computations effectively and up to any arbitrary order. On the other hand, though, some close related problems are left open and are not treated here; mainly, the derivation –as a function of the distance to the critical periodic orbit– of the "optimal" order of the normal form (i. e., such that it minimizes the size of the remainder in given neighbourhood of the resonant periodic orbit) and the preservation of the bifurcated invariant tori when the complete (transformed) Hamiltonian is considered. These are more tricky matters and, in particular, to tackle the latter, one cannot avoid getting involved with KAM schemes (see [25]).

3.1 Hamiltonian equations of the truncated normal form

From now on we shall concentrate on the study of the normal form $Z^{(r)}$, skipping the remainder $\mathfrak{R}^{(r)}$ off and working only with *real* coordinates throughout. Hence, in view of theorem 2.5, $Z^{(r)}$ can be put into the form:

$$Z^{(r)}(x, I_1, y) = \omega_1 I_1 + \omega_2 y \times x + \tfrac{1}{2}|y|_2^2 + \mathcal{Z}_r\left(\tfrac{1}{2}|x|_2^2, I_1, y \times x\right), \tag{56}$$

with the notation,

$$|x|_2 = (x_1^2 + x_2^2)^{1/2}, \quad |y|_2 = (y_1^2 + y_2^2)^{1/2}, \quad y \times x = x_2 y_1 - x_1 y_2$$

and $\mathcal{Z}_r(u, v, w)$ being a polynomial of degree $\lfloor r/2 \rfloor$ (we use $\lfloor x \rfloor$ to denote the integer part of $x \in \mathbb{R}$), beginning with quadratic terms. We shall express it as

$$\mathcal{Z}_r(u, v, w) = \frac{1}{2}(au^2 + bv^2 + cw^2) + duv + euw + fvw + \mathcal{F}_r(u, v, w), \tag{57}$$

with

$$\mathcal{F}_r(u, v, w) = \sum_{3 \le j+m+n \le \lfloor r/2 \rfloor} f_{j,m,n} u^j v^m w^n, \tag{58}$$

if $r \geq 6$ or zero otherwise.

Remark 3.1. Actually, the coefficients of the term of degree two are those which will play an essential rôle in the dynamics of $Z^{(r)}$. It becomes clear throughout the main results of this section: lemma 3.4, theorem 3.5, proposition 3.7 and also in the appendix A.

Now, if we define

$$\eta(x, I_1, y) := \left(\tfrac{1}{2}|x|_2^2, I_1, y \times x\right),$$

the corresponding Hamiltonian equations can be written in the form

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 + \partial_2 \mathcal{Z}_r \circ \eta, \\
\dot{I}_1 &= 0, \\
\dot{x}_1 &= \omega_2 x_2 + y_1 + x_2 \partial_3 \mathcal{Z}_r \circ \eta, \\
\dot{x}_2 &= -\omega_2 x_1 + y_2 - x_1 \partial_3 \mathcal{Z}_r \circ \eta, \\
\dot{y}_1 &= \omega_2 y_2 - x_1 \partial_1 \mathcal{Z}_r \circ \eta + y_2 \partial_3 \mathcal{Z}_r \circ \eta, \\
\dot{y}_2 &= -\omega_2 y_1 - x_2 \partial_1 \mathcal{Z}_r \circ \eta - y_1 \partial_3 \mathcal{Z}_r \circ \eta.
\end{aligned} \tag{59}$$

Moreover the system above is integrable, since it can be shown that the three functions

$$\mathcal{I}_1 = I_1, \quad \mathcal{I}_2 = y \times x \quad \text{and} \quad \mathcal{I}_3 = \tfrac{1}{2}|y|_2^2 + \mathcal{Z}_r\left(\tfrac{1}{2}|x|_2^2, I_1, y \times x\right) \tag{60}$$

are, outside the zero measure set defined by

$$y_1 = 0, \qquad y_2 = 0, \qquad \partial_1 \mathcal{Z}_r = 0,$$

three functionally independent integrals in involution of the system (59).

3.2 Parametrization of the family of periodic orbits

It is straightforward to check that these Hamiltonian equations have a one-parameter family of periodic orbits given by

$$\mathcal{M}_\sigma : \begin{cases} \theta_1 = (\omega_1 + \partial_2 \mathcal{Z}_r(0, I_1, 0))t + \theta_1^0, \\ I_1 = \sigma, \\ x_1 = x_2 = y_1 = y_2 = 0, \end{cases} \tag{61}$$

This implies that the action I_1 is a good parameter for the (local) description of the initial family of periodic orbits. So we can identify $\sigma \equiv I_1$ as the parameter and, in the forthcoming, denote the family by $\{\mathcal{M}_{I_1}\}_{I_1 \in \mathbb{R}}$.

Remark 3.2. With the parametrization (61), the "twist" condition (see [31]) $\omega'(0) \neq 0$, asking the angular frequency to move with the parameter of the family, can be expressed as $\partial_{2,2}\mathcal{Z}_r(0,0,0) = b \neq 0$.

Remark 3.3. One may wonder if such parametrization is preserved when the remainder is added to the normal form and the complete transformed Hamiltonian is considered. In fact, the only monomials in $\mathfrak{R}^{(r)}$ which could destroy the given parametrization are:

$$x_m^j I_1^l \left\{ \begin{smallmatrix} \sin k\theta_1 \\ \cos k\theta_1 \end{smallmatrix} \right\} \quad \text{and} \quad y_m^j I_1^l \left\{ \begin{smallmatrix} \sin k\theta_1 \\ \cos k\theta_1 \end{smallmatrix} \right\}$$

or –when complex remainder is considered–,

$$q_m^j I_1^l \exp(\mathrm{i}k\theta_1) \quad \text{and} \quad p_m^j I_1^l \exp(\mathrm{i}k\theta_1);$$

with $m = 1, 2$, $k \in \mathbb{Z}$, $j = 0, 1$ and $l \in \mathbb{N}$ such that $j + 2l > r$. It can be readily seen then, that the denominators associated to these monomials are $(k \pm j\kappa)\omega_1$ with $\kappa = \omega_2/\omega_1 \notin \mathbb{Q}$ (see theorem 2.5) and k, j in the same range than before. So, small divisors do not appear here and a (semi) normal form additional transformation –constructed as the limit of the successive canonical changes removing those monomials–, will be convergent in the appropriate domain.

It turns out that, if the coefficient d in (57) is different from zero (this is a generic condition that will be assumed in the sequel), the stability of the family $\{\mathcal{M}_{I_1}\}_{I_1\in\mathbb{R}}$ depends, for $|I_1|$ small, on the product dI_1. This is stated in the next lemma.

Lemma 3.4. *If the coefficient d of the polynomial $\mathcal{Z}_r$ given by* (57) *is $d \neq 0$, then for $|I_1|$ small enough, the periodic orbits in $\{\mathcal{M}_{I_1}\}_{I_1\in\mathbb{R}}$ are complex-unstable when $dI_1 < 0$ or (linearly) stable when $dI_1 > 0$.*

Proof. To compute the characteristic exponents of the periodic orbits, we write down the variational equations of (59) around $\mathcal{M}_{I_1}$. Using the parametrization (61), one may check that in the normal directions (x, y) these equations are given by the linear system:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_2 & 1 & 0 \\ -\sigma_2 & 0 & 0 & 1 \\ -\sigma_1 & 0 & 0 & \sigma_2 \\ 0 & -\sigma_1 & -\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix},$$

with σ_1, σ_2 defined by,

$$\begin{aligned} \sigma_1 &:= \partial_1 \mathcal{Z}_r(0, I_1, 0) = dI_1 + \mathrm{O}(I_1^2), \\ \sigma_2 &:= \omega_2 + \partial_3 \mathcal{Z}_r(0, I_1, 0) = \omega_2 + fI_1 + \mathrm{O}(I_1^2) \end{aligned} \qquad (62)$$

and then the characteristic exponents of the periodic orbits are,

$$\begin{aligned} \alpha_{I_1}^{\pm} &= \mathrm{i}\sigma_2 \pm \sqrt{-\sigma_1} \\ &= \mathrm{i}(\omega_2 + fI_1 + \mathrm{O}(I_1^2)) \pm \sqrt{-dI_1 + \mathrm{O}(I_1^2)}, \\ \beta_{I_1}^{\pm} &= -\mathrm{i}\sigma_2 \pm \sqrt{-\sigma_1} \\ &= -\mathrm{i}(\omega_2 + fI_1 + \mathrm{O}(I_1^2)) \pm \sqrt{-dI_1 + \mathrm{O}(I_1^2)} \end{aligned}$$

Thus, if $|I_1|$ is sufficiently, the sign of the terms inside the square roots at the expansions for $\alpha_{I_1}^{\pm}$ and $\beta_{I_1}^{\pm}$ in the above formulas, depends on the sign of dI_1 in the way described by the lemma. ☐

Figure 2 sketches the evolution of the characteristic multipliers as I_1 moves. Therefore, according to lemma 3.4, and for increasing values of I_1, the orbits in the family change from complex-unstable to stable for $d > 0$ (figure 2(a)) or from stable to complex-unstable for $d < 0$ (figure 2(b)).

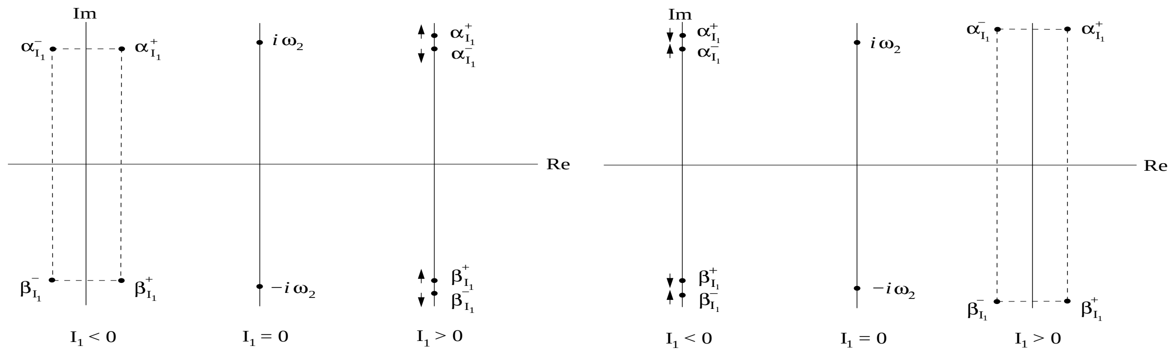
3.3 An unfolding of a two-parameter family of 2D-invariant tori

Generically, the collision of characteristic multipliers (of a family of periodic orbits) shown in figure 2 carries on quasiperiodic bifurcation phenomena. These may be described using the r-order normal form $Z^{(r)}$.

Before, to simplify the identification of the requested solutions, it is convenient to introduce new coordinates through the change:

$$\begin{aligned} x_1 &= \sqrt{2q}\cos\theta_2, \quad & y_1 &= -\frac{I_2}{\sqrt{2q}}\sin\theta_2 + p\sqrt{2q}\cos\theta_2, \\ x_2 &= -\sqrt{2q}\sin\theta_2, \quad & y_2 &= -\frac{I_2}{\sqrt{2q}}\cos\theta_2 - p\sqrt{2q}\sin\theta_2, \end{aligned} \qquad (63)$$

with $q > 0$. (63) is canonical, for one immediately verifies: $d\theta_1 \wedge dI_1 + dx \wedge dy = d\theta \wedge dI + dq \wedge dp$ and is properly defined and regular except in the set $x_1 = x_2 = 0$. It introduces a second action I_2, together



(a) $d > 0$. (b) $d < 0$.

Figure 2: Evolution of the characteristic multipliers. We note that when $I_1 = 0$, then $\alpha_0^- = \alpha_0^+ = \mathrm{i}\omega_2$ and $\beta_0^- = \beta_0^+ = -\mathrm{i}\omega_2$ (collision on the imaginary axis).

with its conjugate angle θ_2 while q and p are the new normal position and its conjugate momentum respectively; in these coordinates, the Hamiltonian $Z^{(r)}$ takes the form,

$$Z^{(r)}(\theta_1, \theta_2, q, I_1, I_2, p) = \omega_1 I_1 + \omega_2 I_2 + qp^2 + \frac{I_2^2}{4q} + \mathcal{Z}_r(q, I_1, I_2), \tag{64}$$

(we keep the same name for the transformed Hamiltonian). Assuming, as in lemma 3.4, $d \neq 0$, the Hamiltonian system corresponding to (64)

$$\begin{aligned}
\dot{\theta}_1 &= \omega_1 + \partial_2 \mathcal{Z}_r(q, I_1, I_2), \\
\dot{\theta}_2 &= \omega_2 + \frac{I_2}{2q} + \partial_3 \mathcal{Z}_r(q, I_1, I_2), \\
\dot{q} &= 2qp, \\
\dot{I}_1 &= 0, \\
\dot{I}_2 &= 0, \\
\dot{p} &= -p^2 + \frac{I_2^2}{4q^2} - \partial_1 \mathcal{Z}_r(q, I_1, I_2),
\end{aligned} \tag{65}$$

has a two-parameter family of bifurcated 2D invariant tori. Theorem 3.5 sets a precise formulation of this assertion.

Theorem 3.5. *If the coefficient of d of $\mathcal{Z}_r$ (see equation (57)) is $d \neq 0$ there exists a real analytic function $\mathcal{I} : \mathfrak{D} \subset \mathbb{R}^2 \to \mathbb{R}$, $\mathfrak{D}$ a neighbourhood of $(0,0)$, defined implicitly by the equation*

$$\eta^2 = \partial_1 \mathcal{Z}_r\left(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta\right),$$

with $\mathcal{I}(0,0) = 0$ and such that, for $(\xi, \eta) \in \mathfrak{D}$, the two-dimensional torus

$$\mathcal{T}_{\xi,\eta} = \{(\theta, q, I, p) \in \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} : q = \xi,\ I_1 = \mathcal{I}(\xi, \eta),\ I_2 = 2\xi\eta,\ p = 0\} \tag{66}$$

is invariant under the flow of (65) with parallel dynamics determined by the vector $\Omega^* = (\Omega_1, \Omega_2)$ of intrinsic frequencies:

$$\begin{aligned}
\Omega_1(\xi, \eta) &= \omega_1 + \partial_2 \mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta), \\
\Omega_2(\xi, \eta) &= \omega_2 + \eta + \partial_3 \mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta);
\end{aligned} \tag{67}$$

moreover, the corresponding invariant tori of the Hamiltonian (56) are real whenever $\xi > 0$.

Proof. It follows directly by substitution in (65) equations, whereas the last point about the real character of the invariant tori follows from (66) and the change (63). Here we only stress that the condition $d \neq 0$ (the non-degeneracy of the transition) is the necessary hypothesis for the implicit analytic function $\mathcal{I}$ to exist in a neighbourhood of $(0,0)$, since $\partial^2_{1,2}\mathcal{Z}_r(0,0,0) = d$. Finally, transformation (63) shows the real character of the (corresponding) tori of (56) for $\xi > 0$ (see also equations (68) below). □

Then $\{\mathcal{T}_{\xi,\eta}\}_{(\xi,\eta)\in\mathfrak{D}}$ with $\xi > 0$ constitutes a two-parameter family of real invariant tori filled up with quasiperiodic solutions of the system (65). Whence, changing back to rectangular (with respect to the normal directions) coordinates by means of (63), one obtains a family of two-parameter quasiperiodic solutions winding 2D real invariant tori of (56). Explicitly:

$$\begin{array}{rclrcl} \theta_1 &=& \Omega_1(\xi,\eta)t + \theta_1^0, & I_1 &=& \mathcal{I}(\xi,\eta), \\ x_1 &=& \sqrt{2\xi}\cos(\Omega_2(\xi,\eta)t + \theta_2^0), & y_1 &=& -\sqrt{2\xi}\,\eta\sin(\Omega_2(\xi,\eta)t + \theta_2^0), \\ x_2 &=& -\sqrt{2\xi}\sin(\Omega_2(\xi,\eta) + \theta_2^0), & y_2 &=& -\sqrt{2\xi}\,\eta\cos(\Omega_2(\xi,\eta)t + \theta_2^0). \end{array} \tag{68}$$

Using the expressions for Z_r given by (57) and (58), a formal expansion of the implicit function $\mathcal{I}$ can easily be derived. Up to second order in ξ, η one gets:

$$\mathcal{I}(\xi,\eta) = -\frac{a}{d}\xi - \frac{1}{d}\left(3f_{3,0,0} - \frac{2af_{2,1,0}}{d} + \frac{a^2 f_{1,2,0}}{d^2}\right)\xi^2 - \frac{2e}{d}\xi\eta + \frac{1}{d}\eta^2 + \mathrm{O}_3(\xi,\eta), \tag{69}$$

and then substitution in (67) yields, for the frequencies Ω_1, Ω_2,

$$\begin{aligned} \Omega_1(\xi,\eta) &= \omega_1 + \left(d - \frac{ab}{d}\right)\xi \\ &+ \left(-\frac{3b}{d}f_{3,0,0} - \frac{a^2 b}{d^3}f_{1,2,0} + \frac{2ab}{d^2}f_{2,1,0} + f_{2,1,0} - \frac{2a}{d}f_{1,2,0} + \frac{3a^2}{d^2}f_{0,3,0}\right)\xi^2 \\ &+ \left(-\frac{2eb}{d} + 2f\right)\xi\eta + \frac{b}{d}\eta^2 + \mathrm{O}_3(\xi,\eta) \end{aligned} \tag{70a}$$

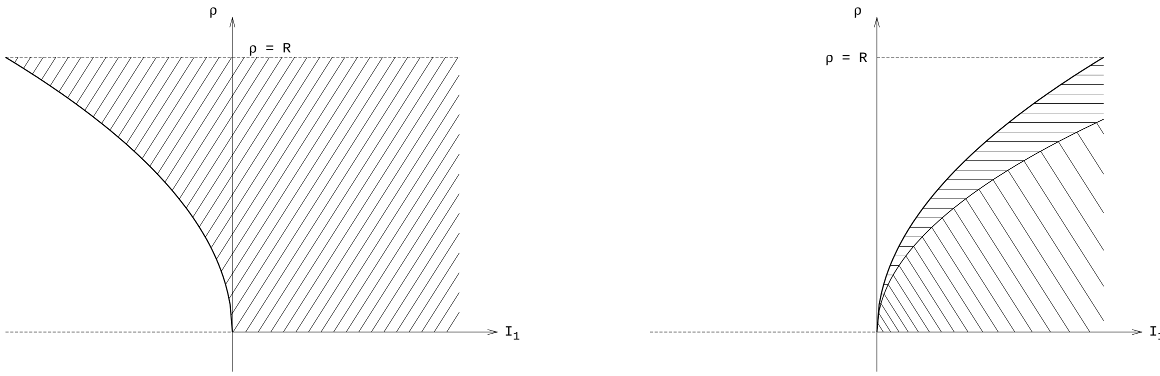
$$\begin{aligned} \Omega_2(\xi,\eta) &= \omega_2 + \left(e - \frac{af}{d}\right)\xi + \eta \\ &+ \left(-\frac{3f}{d}f_{3,0,0} - \frac{a^2 f}{d^3}f_{1,2,0} + \frac{2af}{d^2}f_{2,1,0} + f_{2,0,1} - \frac{a}{d}f_{1,1,1} + \frac{a^2}{d^2}f_{0,2,1}\right)\xi^2 \\ &+ \left(2c - \frac{2ef}{d}\right)\xi\eta + \frac{f}{d}\eta^2 + \mathrm{O}_3(\xi,\eta). \end{aligned} \tag{70b}$$

Similarly as in the lemma 3.4 for the stability of periodic orbits, the normal character (elliptic, hyperbolic) of the unfolded tori has to do with the sign of one of the coefficients of the polynomial $\mathcal{Z}_r$ (see proposition below).

Remark 3.6. In view of the expansions (70a) and (70b) one easily computes $\det D_\zeta \Omega = d - ab/d + \mathrm{O}_1(\xi,\eta)$ –with $\zeta^* = (\xi,\eta)$, $\Omega^* = (\Omega_1,\Omega_2)$–, so the family of invariant tori will be nondegenerated (in the Kolmogorov's sense) under the condition $d^2 \neq ab$.

Proposition 3.7. *With the assumptions of theorem 3.5 –including the reality condition $\xi > 0$– and if the coefficient a of the polynomial $\mathcal{Z}_r$ in (57) is $a \neq 0$; the type of the bifurcation is determined by the sign of the coefficient a. More precisely:*

Case 1. If $a > 0$; besides the elliptic tori around stable periodic orbits –which correspond to excitations in their normal elliptic directions– there appear elliptic tori around complex-unstable periodic orbits.



(a) $a > 0$ (direct bifurcation)

(b) $a < 0$ (inverse bifurcation)

Figure 3: Qualitative sketch of the size of the 2D bifurcated tori and their stability in the case $d > 0$ (the figures for $d < 0$ follow straightforward). In these plots complex-unstable periodic orbits lie on the negative horizontal semiaxis (in dashed lines) and the stable ones in the positive horizontal semiaxis. In figure (a) the shaded area corresponds to the domain of existence of elliptic invariant tori. In figure (b) the regions shaded obliquely and horizontally are the domains of the elliptic and hyperbolic tori respectively whilst the separating curve holds parabolic tori. Here ρ, $0 < \rho < R$, is defined by $\rho := (x_1^2 + x_2^2)^{1/2}$ (so $\rho = (2\xi)^{1/2}$, according to (63)). It can be thought of as the radius of the invariant torus $\mathcal{T}_{\xi,\eta}$ in the normal directions x_1, x_2. R is the "maximum allowed radius" and is determined by the domain $\mathfrak{D}$ of $\mathcal{I}(\xi,\eta)$. If η is allowed to range in a neighbourhood of $\eta = 0$, the regions shaded in the figures may be derived from (69) setting $\xi = \rho^2/2$, i. e. from: $dI_1 = -a\rho^2/2 + \eta^2 + \mathrm{O}_3(\rho,\eta)$ and the normal character of the tori follows from the characteristic exponents (71).

Case 2. If $a < 0$, then, hyperbolic invariant tori unfold around stable periodic orbits. In this case, the family described in theorem 3.5 contains also elliptic tori (of the same nature than those in the previous case) and parabolic tori.

By analogy with the classical Andronov-Hopf bifurcation, the former and the latter cases in the proposition are often referred as the "direct" and the "inverse" bifurcation respectively. In figure 3 the bifurcation pattern is sketched in both contexts.

Proof (of proposition 3.7). Consider the system (65) and the family of invariant tori $\{\mathcal{T}_{\xi,\eta}\}_{(\xi,\eta)}$ of theorem 3.5. Around one of these tori, the first variational equations in the normal directions are given by the linear system,

$$\begin{aligned} \dot{X} &= 2\xi Y, \\ \dot{Y} &= -\left(2\frac{\eta^2}{\xi} + a + \partial^2_{1,1}\mathcal{F}_r(\xi,\mathcal{I}(\xi,\eta),2\xi\eta)\right) X \end{aligned}$$

whose eigenvalues (the characteristic exponents of the torus) are:

$$\mu_{\pm}(\xi,\eta) = \pm\sqrt{-4\eta^2 - 2a\xi - 2\xi\partial^2_{1,1}\mathcal{F}_r(\xi,\mathcal{I}(\xi,\eta),2\xi\eta)}. \tag{71}$$

As $\xi\partial^2_{1,1}\mathcal{F}_r(\xi,\mathcal{I}(\xi,\eta),2\xi\eta)$ is $\xi\mathrm{O}_1(\xi,\eta)$, this implies that –at least in a small neighbourhood of $(\xi,\eta) = (0,0)$–, the normal behaviour of the tori is determined by the sign of the first two terms inside the square root $-4\eta^2 - 2a\xi$. In particular, if the coefficient a is positive (case 1) then the family only holds the elliptic invariant tori, whilst for negative values of a (case 2), elliptic and hyperbolic tori will be present, but parabolic tori will appear as well. Indeed, if one considers the equation: $2\eta^2 + a\xi + \xi\partial^2_{1,1}\mathcal{F}_r(\xi,\mathcal{I}(\xi,\eta),2\xi\eta) = 0$, just the implicit function theorem applied at $(\xi,\eta) = (0,0)$ shows the existence –in the space of parameters (ξ,η)–, of a path $\xi = g(\eta)$ giving rise to a one-parameter family of parabolic tori. Of course, the same can be done when $a < 0$ but then ξ, as a

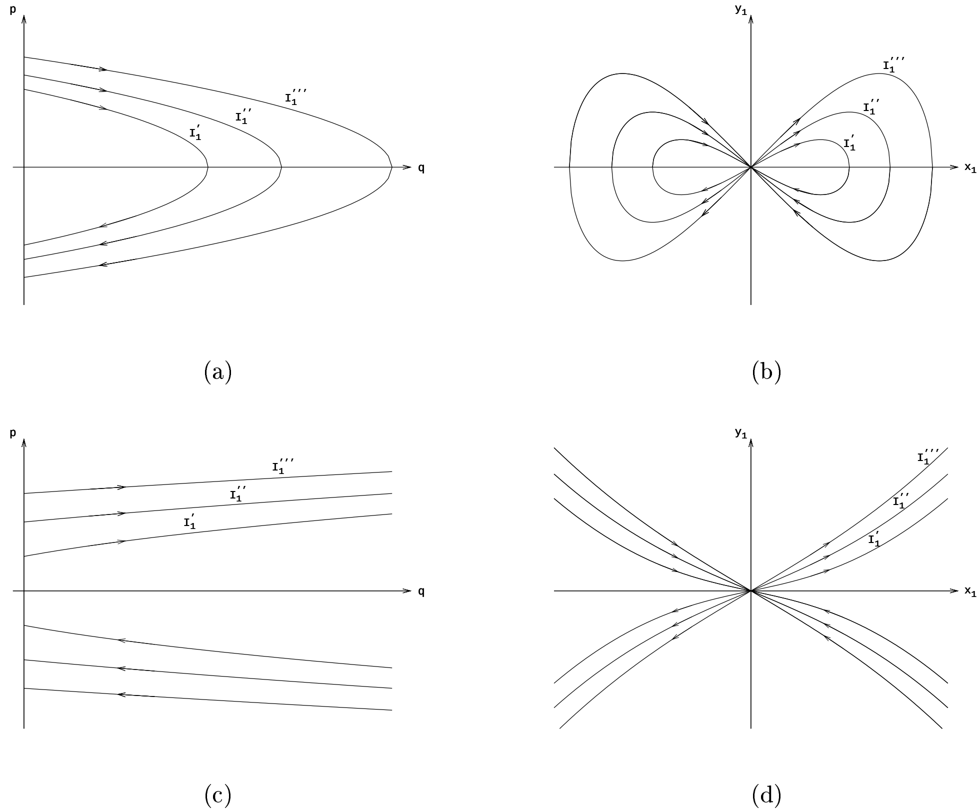


Figure 4: Invariant manifolds of the hyperbolic periodic orbits. Figures (a) and (b), corresponding to $a > 0$, are the projections of the invariant manifolds on the planes (q, p) and (x_1, y_1) respectively. The same, but for $a < 0$ is plotted in figures (c) and (d). In each drawn, these are represented for three different values of I_1: $I_1''' < I_1'' < I_1' < 0$. In (a) and (c) stable and unstable invariant manifolds are on $p < 0$ and on $p > 0$ respectively, whilst in (b), (d) the stable manifolds have $x_1 y_1 < 0$ and $x_1 y_1 > 0$ the unstable ones.

function of η, will take locally (i. e., in a small neighbourhood of the origin) only negative values, against the condition for real invariant tori.

To discuss the position –relative to the family of periodic orbits– of the bifurcated invariant tori, one looks at (69) and realizes that for (ξ, η) given, the action I_1 of the corresponding invariant torus can be expressed as

$$I_1 = -\frac{a}{d}\xi + \frac{1}{d}\eta^2 + \xi \mathrm{O}_1(\xi, \eta) + \mathrm{O}_3(\xi, \eta);$$

hence, the sign of dI_1 is determined locally by the first two terms. In particular, for $a > 0$ (case 1) dI_1 can take positive or negative values so elliptic bifurcated tori of the first case unfold "around" both stable and unstable periodic orbits (see figure 3(a)). On the contrary, for $a < 0$ the sign of dI_1 is (locally) always positive and therefore, in the second case, bifurcated hyperbolic, elliptic and parabolic tori appear around –in the sense just stated– stable periodic orbits of the family (figure 3(b)). This ends the proof. □

3.4 Parametrization of the invariant manifolds of the hyperbolic periodic orbits

We recall that when $\sigma_1 < 0$ in (62) (i.e., when $dI_1 < 0$ with I_1 small) then the orbit $\mathcal{M}_{I_1}$ given by the parametrization (61) is a hyperbolic periodic orbit of the Hamiltonian equations of the normal form $Z^{(r)}$. So, one may use $Z^{(r)}$ to get parametrizations of the stable and unstable invariant manifolds of this orbits. If we consider a fixed I_1 such that $\sigma_1 < 0$, then the corresponding stable and unstable manifolds of $\mathcal{M}_{I_1}$ are three dimensional and can be obtained by setting the values of the first integrals in (60) to the ones of $\mathcal{M}_{I_1}$. If we write $\mathcal{I}_2$ and $\mathcal{I}_3$ in the coordinates (63) we have,

$$\mathcal{I}_2 = I_2, \qquad \mathcal{I}_3 = qp^2 + \frac{I_2^2}{4q} + \mathcal{Z}_r(q, I_1, I_2).$$

So, the invariant manifolds of $\mathcal{M}_{I_1}$ are defined by

$$I_2 = 0, \qquad qp^2 + \mathcal{Z}_r(q, I_1, 0) = \mathcal{Z}_r(0, I_1, 0),$$

obtaining:

$$p = \pm\sqrt{\frac{1}{q}(\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0))} = \pm\sqrt{-\frac{a}{2}q - dI_1 + \mathrm{O}_2(q, I_1)}.$$

The choice $+$ corresponds to the unstable manifold and the choice $-$ to the stable one. Going back to the rectangular coordinates through (63), we obtain the following parametrization of the manifolds:

$$x_1 = \sqrt{2q}\cos\theta_2, \qquad y_1 = \pm\sqrt{2(\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0))}\cos\theta_2,$$
$$x_2 = -\sqrt{2q}\sin\theta_2, \qquad y_2 = \mp\sqrt{2(\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0))}\sin\theta_2.$$

Alternatively, the invariant manifolds can be given as graphs:

$$\begin{aligned} y_i &= \pm x_i\sqrt{\frac{2}{x_1^2 + x_2^2}\left(\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r\left(\frac{1}{2}\left(x_1^2 + x_2^2\right), I_1, 0\right)\right)} \\ &= \pm x_i\sqrt{-\frac{a}{4}(x_1^2 + x_2^2) - dI_1 + \Gamma}, \end{aligned}$$

where Γ stands for the terms of (adapted) degree at least 3. These parametrizations are represented in figures 4(a)–(d) for three different (negative) values of the action I_1 and according to the sign of a (see the details in the caption).

However, the range of available parameters (q, I_1) is restricted by the condition that the expressions inside the square roots must be positive. By defining $F(q, I_1) = (\mathcal{Z}_r(0, I_1, 0) - \mathcal{Z}_r(q, I_1, 0))/q$, we need $F(q, I_1) \geq 0$. We notice that $F(0, 0) = 0$, $\partial_1 F(0, 0) = -a/2 \neq 0$ and $\partial_2 F(0, 0) = -d \neq 0$. Thus, the boundary of this domain can be (locally) expressed as function of I_1 or q.

3.5 Parametrization of the invariant manifolds of the hyperbolic 2D-invariant tori

In the inverse case (when $a < 0$) we have shown that for certain range of the parameters (ξ, η) the 2-dimensional bifurcated torus $\mathcal{T}_{\xi,\eta}$ given in (66) is normally hyperbolic. More precisely, it happens for the values of $\xi > 0$ and η such that $4\eta^2 + 2\xi\partial_{1,1}^2\mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta) < 0$ (see theorem 3.5 and proposition 3.7). For these tori we can also compute its stable and unstable manifold which, for any given torus, have dimension three. Again, they are implicitly defined by fixing the values of the first integrals in (60):

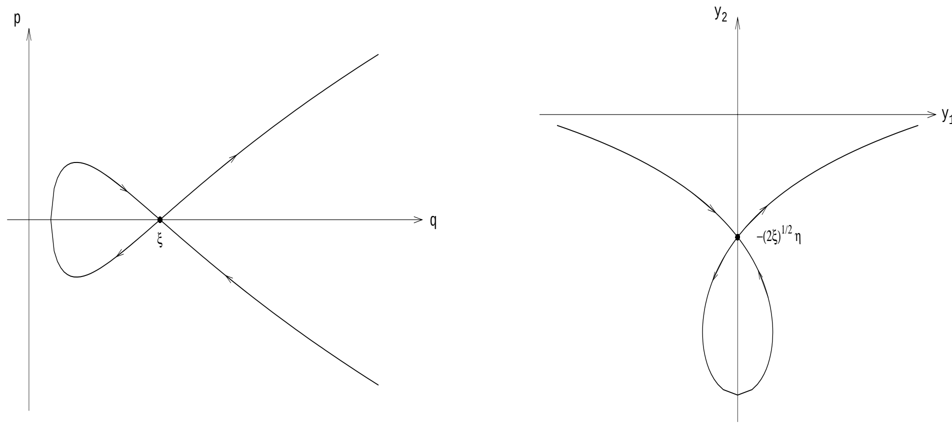
$$\mathcal{I}_1 = \mathcal{I}(\xi, \eta), \qquad \mathcal{I}_2 = 2\xi\eta, \qquad \mathcal{I}_3 = \xi\eta^2 + \mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta).$$

Using the coordinates (63) we obtain the following expression for p:

$$p = \pm\sqrt{F(q, \xi, \eta)} = \pm\sqrt{\frac{1}{q}\left\{\mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta) - \mathcal{Z}_r(q, \mathcal{I}(\xi, \eta), 2\xi\eta) + \xi^2\eta^2\left(\frac{1}{\xi} - \frac{1}{q}\right)\right\}}, \tag{72}$$

and the corresponding manifold will be stable if $p(q-\xi) < 0$ or unstable if $p(q-\xi) > 0$ (see figure 5(a)). Of course, using the expressions (63) we can go back to the original normal form coordinates. As for the hyperbolic periodic orbits, we can also see these manifolds as graphs, so that

$$y_1 = 2\xi\eta\frac{x_2}{x_1^2 + x_2^2} \pm \sqrt{F\left(\frac{x_1^2 + x_2^2}{2}, \xi, \eta\right)}x_1, \qquad y_2 = -2\xi\eta\frac{x_1}{x_1^2 + x_2^2} \pm \sqrt{F\left(\frac{x_1^2 + x_2^2}{2}, \xi, \eta\right)}x_2.$$



(a) (b)

Figure 5: Invariant manifolds of the hyperbolic invariant tori corresponding to (ξ, η). In figure (b) we set $x_2 = 0$ and $\eta > 0$ is assumed. The position of the hyperbolic invariant tori is $(q, p) = (\xi, 0)$ in (a) and $(y_1, y_2) = (0, -\eta\sqrt{2\xi})$ in (b). In both cases is marked with a dot.

Moreover, if we use the equation defining $\mathcal{I}(\xi, \eta)$ (see (69)) we can make more clear the expression of $F(q, \xi, \eta)$. Indeed, we can expand it in powers of $q - \xi$, obtaining:

$$\begin{aligned}\mathcal{Z}_r(q, \mathcal{I}(\xi,\eta), 2\xi\eta) - \mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta) &= \eta^2(q-\xi) + \frac{1}{2}\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta)(q-\xi)^2 \\ &\quad + G(q,\xi,\eta)(q-\xi)^3,\end{aligned}$$

where $G(q, \xi, \eta)(q - \xi)^3$ stands for the complementary term in the Taylor expansion, and thus

$$F = -\frac{(q-\xi)^2}{4q^2}\left\{4\eta^2 + 2\xi\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta) + 2(q-\xi)\left[\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta) + 2qG(q,\xi,\eta)\right]\right\}.$$

Using this last expression of F the condition on the torus $\mathcal{T}_{\xi,\eta}$ to be hyperbolic appears in a natural form, as in equation (72) we need $F(q, \xi, \eta) \geq 0$. Thus, if $4\eta^2 + 2\xi\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi, \eta), 2\xi\eta) < 0$, this condition is fulfilled provided $|q - \xi|$ is small. In figure 5 we plot, for some given values of the parameters (ξ, η), the invariant manifolds of the hyperbolic tori. The projections are displayed on the plane (q, p) –in figure 5(a)– and on the plane (y_1, y_2) –in figure 5(b)–.

Moreover, if we want to characterize the range of available parameters in (72), we have to study the solutions of $g(q, \xi, \eta) = 0$, with

$$g = 4\eta^2 + 2\xi\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta) + 2(q-\xi)\left[\partial^2_{1,1}\mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta) + 2qG(q,\xi,\eta)\right].$$

As $g(0, 0, 0) = 0$ and $\partial_1 g(0, 0, 0) = 2a \neq 0$, we can give the solutions of $g(q, \xi, \eta) = 0$ by writing $q = f(\xi, \eta)$, so that the boundary of values of q in the parameter space of F is given (locally around $(\xi, \eta) = (0, 0)$) as $q \geq f(\xi, \eta)$.

3.6 Computation of 3D-invariant tori

$3D$-invariant tori of the normal form (56) can be obtained from periodic orbits of the 1-degree of freedom Hamiltonian system given by:

$$H(q, p; I_1, I_2) = qp^2 + \frac{I_2^2}{4q} + \mathcal{Z}_r(q, I_1, I_2), \tag{73}$$

where I_1 and I_2 have to be treated as parameters (see (64)). Thus, given a couple of values I_1 and I_2 (fixed), let $(\widetilde{q}(\theta_3), \widetilde{p}(\theta_3))$ be a 2π-periodic parametrization of a periodic orbit of (73) such that $\dot{\theta}_3 = \widetilde{\omega}_3$. Then, the dynamics of the corresponding $3D$-invariant torus of (64) can be obtained by direct integration of the expressions,

$$\dot{\theta}_1 = \omega_1 + \partial_2 \mathcal{Z}_r(\widetilde{q}(\widetilde{\omega}_3 t), I_1, I_2),$$
$$\dot{\theta}_2 = \omega_2 + \frac{I_2}{2\widetilde{q}(\widetilde{\omega}_3 t)} + \partial_3 \mathcal{Z}_r(\widetilde{q}(\widetilde{\omega}_3 t), I_1, I_2),$$

being the vector of intrinsic frequencies of this torus $\widetilde{\omega} = (\widetilde{\omega}_1, \widetilde{\omega}_2, \widetilde{\omega}_3)$, with $\widetilde{\omega}_1$ and $\widetilde{\omega}_2$ defined by

$$\widetilde{\omega}_1 = \omega_1 + \langle \partial_2 \mathcal{Z}_r(\widetilde{q}(\theta_3), I_1, I_2) \rangle,$$
$$\widetilde{\omega}_2 = \omega_2 + \left\langle \frac{I_2}{2\widetilde{q}(\theta_3)} \right\rangle + \langle \partial_3 \mathcal{Z}_r(\widetilde{q}(\theta_3), I_1, I_2) \rangle,$$

where $\langle \cdot \rangle$ denotes the average with respect to the angle θ_3 (of course, we need $\widetilde{\omega}_1$, $\widetilde{\omega}_2$ and $\widetilde{\omega}_3$ to be independent frequencies in order to have a legitimate $3D$-torus).

If we want to discuss the range of parameters for which we obtain periodic orbits of (73), we point out that these orbits can be obtained implicitly as energy levels of the system, $\{H(q, p; I_1, I_2) = h\}$, for suitable values of h. The extremal values for the interval of allowed values of h (for any given I_1 and I_2), correspond to the ones of the critical points of H. The discussion of these critical points for the system (73) can be easily derived from the discussion of the fourth order normal form done in appendix A.

Appendix A. Study of the low order normal form

We shall consider the normal form (64) with $r = 2$ (fourth order Hamiltonian). Actually this truncated Hamiltonian contains the relevant dynamics and phenomena described so far.

The aim of this appendix is mainly descriptive, so the straightforward computations are omitted. However, the dynamics is here explored fixing the value of the energy and deriving the corresponding phase portrait in the normal directions. This does constitute a different sight that complements the description in section 3.

Previously, though, it is convenient to introduce more suitable coordinates by means of the canonical change,

$$Q = \sqrt{2q}, \qquad P = p\sqrt{2q},$$

In these new coordinates, the Hamiltonian $Z^{(2)}$ will take the form,

$$Z^{(2)}(Q, I_1, I_2, P) = H_0(I_1, I_2) + H_1(Q, I_1, I_2, P), \tag{74}$$

with H_0 and H_1 given respectively by,

$$H_0(I_1, I_2) = \omega_1 I_1 + \omega_2 I_2 + \frac{1}{2} b I_1^2 + \frac{1}{2} c I_2^2 + f I_1 I_2,$$

and

$$H_1(Q, I_1, I_2, P) = \frac{1}{2} P^2 + \frac{1}{2} \left\{ \frac{I_2^2}{Q^2} + (dI_1 + eI_2) Q^2 \right\} + \frac{a}{8} Q^4, \tag{75}$$

The actions I_1, I_2 are constants of the motion, and so is H_0. Hence it is enough to investigate the energy levels of H_1, which is an (integrable) Hamiltonian of type

$$H_1(Q, I_1, I_2, P) = \tfrac{1}{2} P^2 + V(Q, I_1, I_2),$$

where the potential $V(Q, I_1, I_2)$ must be

$$V(Q, I_1, I_2) = \frac{1}{2} \left\{ \frac{I_2^2}{Q^2} + (dI_1 + eI_2) Q^2 \right\} + \frac{a}{8} Q^4.$$

Remark A.1. In particular, if in (75) one fixes the value of the first integrals I_1 and I_2, the equilibrium points and the periodic orbits of the resulting one-degree of freedom Hamiltonian will correspond to 2D and 3D-invariant tori respectively of the whole phase space.

A systematic account of the dynamics of (74) can be obtained if one considers three independent first integrals of the system, fixes the value of two of them, and shows in the (Q, P)-plane all the possible motions according to the values of the third one. Thus, what we will do is to consider fixed values of α and h, and to restrict I_1 and I_2 to the linear manifold $dI_1 + eI_2 = \alpha$ and to the energy level $\{H_1(Q, I_1, I_2, P) = h\}$. Then, the allowed motions can be easily derived using I_2 as a parameter (see also [14]). These results are accounted in the proposition below.

Proposition A.2. *Consider the Hamiltonian* (75) *and assume that the coefficients a and d are both different from* 0. *Given fixed values of α and h, we introduce the energy level sets*

$$\mathcal{E}_{I_2} := \{(Q,P) \in \mathbb{R}^2,\ Q > 0 :\ H_1(Q,P,I_1,I_2) = h,\ dI_1 + eI_2 = \alpha\}, \qquad I_2 \in \mathbb{R}.$$

The phase portrait of these sets can be described, according to the sign of a, by the following two cases:

Case 1. $a > 0$; depending on the sign of α, we have the following lower bounds for h: $h > h_0 = -a^{-1}\alpha^2/2$ if $\alpha < 0$; $h > 0$ if $\alpha \geq 0$ (for smaller values of h the motion is not allowed); in any case, we have an elliptic point surrounded by invariant closed curves. All the flow is confined by different objects associated to $\mathcal{E}_0$. If $\alpha > 0$ these confiners are: a closed curve for $h_0 < h < 0$; the (matching) stable and unstable manifolds of the point $(Q,P) = (0,0)$ (which is an equilibrium point when $I_2 = 0$) for $h = 0$; and an open curve joining the points $(0,(2h)^{1/2})$, $(0,-(2h)^{1/2})$ for $h > 0$ (recall that $Q > 0$). If $\alpha \geq 0$, the confiner is always the open curve.

Case 2. $a < 0$; in this case, any value of h is allowed. If $\alpha < 0$, we have that for any value of h the motion is unbounded: no equilibrium points take place and for all the trajectories Q goes to the positive infinity when t does. The same happens for $\alpha > 0$, if $h < 0$ or $h > 4h_0/3 = -2a^{-1}\alpha^2/3$. However, in the case $\alpha > 0$ we have the following additional possibilities: if $0 < h < h_0 = -a^{-1}\alpha^2/2$, an elliptic fixed point surrounded by closed invariant curves appear. The region of the invariant tori is confined by a curve joining the points $(0,(2h)^{1/2})$, $(0,-(2h)^{1/2})$. Outside this separatrix, at a distance

$$\sqrt{\frac{2}{a}\left(-\alpha - \sqrt{\alpha^2 + 2ah}\right)}$$

of the origin $(Q,P) = (0,0)$, escape trajectories evolve. When h is increased and $h_0 < h < 4h_0/3$, an hyperbolic equilibrium point merges and the elliptic point with its accompanying family of closed invariant curves are contained inside the loop formed by the connecting branches of invariant manifolds of the hyperbolic point. With the exception of the flow lying on the branch of the stable manifold not belonging to the loop (that on the right of the hyperbolic point), escape trajectories take place outside. When $h > 4h_0/3$, both fixed points disappear through a parabolic collision for $h = 4h_0/3$.

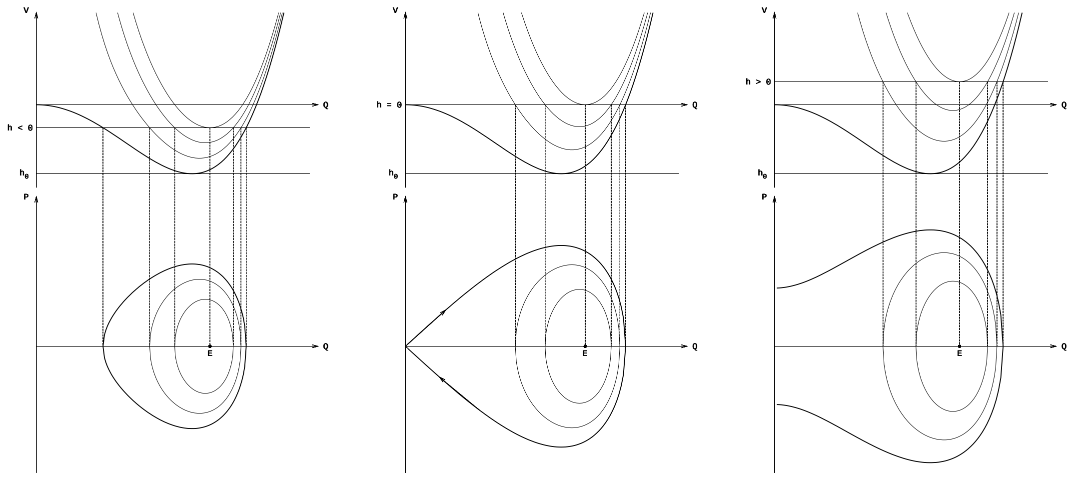
The first case, $a > 0$, is represented in figures 6(a)-(c) only for $\alpha < 0$ ($\alpha > 0$ leads to a plot similar to the figure 6(c), but with $V(Q;0)$ –see equation (76) in the proof below– being a monotonic increasing positive function. The case 2, $a < 0$, is outlined, for $\alpha > 0$, in figures 7(a), (b).

Proof (of proposition A.2). Setting $dI_1 + eI_2 = \alpha$ in (75) one gets a family of one-degree of freedom Hamiltonians

$$H(Q,P;I_2) = \frac{1}{2}P^2 + V(Q;I_2),$$

with

$$V(Q;I_2) = \frac{1}{2}\left(\frac{I_2^2}{Q^2} + \alpha Q^2\right) + \frac{a}{8}Q^4, \tag{76}$$



(a) $h_0 = -\frac{\alpha^2}{2a} < h < 0$. (b) $h = 0$ (c) $h > 0$

Figure 6: Phase portrait for $a > 0$ and $\alpha < 0$. h_0 is defined as the value of $V(Q; 0)$ (see (76)) at its minimum point. The lowermost potential (i. e., the only graph not growing to $+\infty$ when $Q \to 0^+$) and the outermost curves in the (Q, P)-plane below –both drawn with thicker lines–, correspond to $I_2 = 0$. For higher (absolute) values of I_2, more precisely, for $0 < |I_2| < |I_2^E|$ closed invariant curves appear for the fixed energy level h (see the proof of proposition A.2 for the details). As pointed in the text, for $h = 0$, figure (b), this curve identifies the stable and unstable invariant manifolds of the origin, which is an equilibrium point for $I_2 = 0$ and $h = 0$. In particular, the situation described in (b) corresponds with the direct bifurcation introduced in proposition 3.7 if one identifies the invariant manifolds of the origin with the invariant manifolds of the (complex-unstable) periodic orbit of the family with $I_1 = \alpha/d$. Here, though, the scene is envisaged on a section of constant energy.

where I_2 is regarded now as the parameter of the family. For each I_2 given, they are the sum of the kinetic term $P^2/2$ plus the corresponding potential $V(Q; I_2)$; hence their phase portraits are straightforward constructed from the shapes of $V(Q; I_2)$.

In the first case, $a > 0$, it is checked immediately that the inequalities: $-a^{-1}\alpha^2/2 \le V(Q;0) < V(Q; I_2)$ if $\alpha < 0$ and $0 < V(Q; 0) < V(Q; I_2)$ if $\alpha \ge 0$, hold for $Q > 0$ and all $I_2 \ne 0$. This leads to the lower bound for h. Then, given a value h of the energy, to find the equilibrium points one looks for (Q, I_2) satisfying simultaneously the equations

$$V(Q; I_2) = h, \qquad \partial_1 V(Q; I_2) = 0, \tag{77}$$

with $Q > 0$. Hence, for this value of the parameter I_2, Q is a critical point of the potential lying on that energy level.

When $a > 0$, (77) has only one solution, say (E, I_2^E), corresponding to an elliptic equilibrium point. The surrounding closed invariant curves can be parametrized by

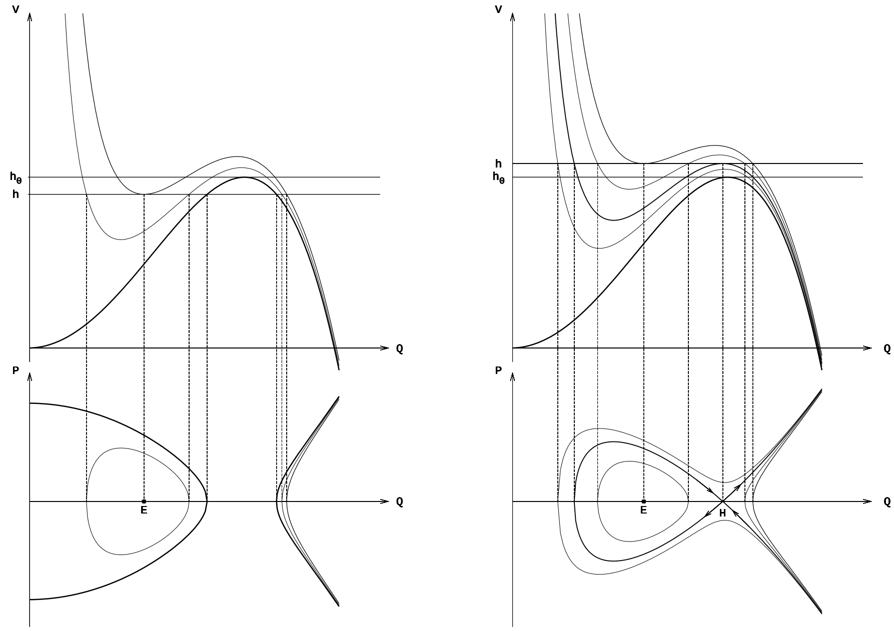
$$Q = s, \qquad P = \pm\sqrt{2h - V(s; I_2)} \tag{78}$$

with $0 < |I_2| < |I_2^E|$, $E' < s < E''$; being E', E'' the two positive solutions of the equation $V(s; I_2) = h$. The same parametrization works for the curves confining the phase flow, but setting $I_2 = 0$ and $0 < s < E'''$ where E''' is the (unique) positive solution of $V(s; 0) = h$.

For $a < 0$ (second case), one realizes that $V(Q; I_2)$ decreases monotonically for any value of the parameter I_2 when $\alpha \le 0$. Therefore no equilibrium points may appear, so all the solutions will escape when $t \to \infty$. On the other hand, if $\alpha > 0$ it is straightforward to check that the system (77) has

(i) no solutions for $h < 0$ or $h > -2a^{-1}\alpha^2/3$,

(ii) two solutions $(E, \pm I_2^E)$, for $0 \le h < -a^{-1}\alpha^2/2$ and



(a) $0 < h < h_0 = -\frac{\alpha^2}{2a}$ (b) $h_0 < h < \frac{4}{3}h_0$

Figure 7: $a < 0$, $\alpha > 0$. Construction of the phase portrait of the Hamiltonian (75) from the graph of the potential $V(Q; I_2)$ (see the proof of proposition A.2 for the details). (a) $0 < h < h_0$. The graphs of the potential correspond to three values of the action $I_2 = 0$ (thicker line), I_2, I_2^E with $0 < |I_2| < |I_2^E|$ and hence: $V(Q;0) < V(Q;I_2) < V(Q;I_2^E)$ for all $Q > 0$. In the phase portrait below, it can be seen that the flow is whether confined to wind closed curves around the elliptic equilibrium point (marked with E), or escape to infinity following one of the hyperbola-like curves on the right. (b) $h_0 < h < 4h_0/3$. As in (a), the potential is represented for several values of the actions: $I_2 = 0$ (in thicker line) and I_2', I_2^H, I_2, I_2^E with $0 < |I_2'| < |I_2^H| < |I_2| < |I_2^E|$, which yields $V(Q;0) < V(Q;I_2') < V(Q;I_2^H) < V(Q;I_2) < V(Q;I_2^E)$ for all $Q > 0$. Again, an elliptic equilibrium point (marked with E below) appears but this time together with an hyperbolic one (marked with H). In the (Q,P)-plane there are represented the closed invariant curves around the elliptic point, the invariant manifolds of the hyperbolic point (in thicker line) and the escape trajectories corresponding to the values I_2' (the outermost curve, wrapping the invariant manifolds), I_2^E and I_2 –this latter two are the hyperbola-like curves close to the rightward, not connected, branches of the invariant manifolds–. The portrait drawn in (b) may be thought of as the projection, in the plane of the normal directions and for a fixed energy h, of the inverse bifurcation of proposition 3.7.

(iii) four solutions when $-a^{-1}\alpha^2/2 \leq h \leq -2a^{-1}\alpha^2/3$. We denote them by: $(E, \pm I_2^E)$, $(H, \pm I_2^H)$, with $0 < |I_2^H| < |I_2^E|$. Moreover, if I_2 is such that: $|I_2^H| < |I_2| < |I_2^E|$ then the relations

$$V(Q, I_2^H) < V(Q, I_2) < V(Q, I_2^E), \tag{79}$$

are satisfied for all $Q > 0$.

Therefore in (i) there are no equilibrium points in the phase plane (Q,P) and consequently all the trajectories escape as $t \to +\infty$. In item (ii), an elliptic equilibrium point is present and it is encircled by a family of (closed) invariant curves with $0 < |I_2| < |I_2^E|$. In item (iii) an additional equilibrium point, hyperbolic, appears. Furthermore, there exists H', $0 < H' < E < H$ such that $V(H', I_2^H) = V(E, I_2^E) = V(H, I_2^H) = h$. Thus the stable and unstable manifolds of the hyperbolic point form a loop which embraces the elliptic point and –in view of (79)– is filled up with closed invariant curves having I_2 in the range $|I_2^H| < |I_2| < |I_2^E|$. We stress here that, when $h > -a^{-1}\alpha^2/3$ then $(E, \pm I_2^E) = (H, \pm I_2^H)$, so the elliptic and the hyperbolic points collide on this energy section giving rise to a parabolic equilibrium point (Note: a parabolic torus in the six-dimensional phase space). The different trajectories on the (Q,P)-plane can also be parametrized by the equations (78), setting there the corresponding values of h, I_2 and letting the parameter s to range between the appropriate limits: e. g., for an invariant closed curve I_2 must satisfy $0 < |I_2| < |I_2^E|$, (if $0 < h < -a^{-1}\alpha^2/2$)

or $|I_2^H| < |I_2| < |I_2^E|$ (if $-a^{-1}\alpha^2/2 \leq h \leq -a^{-1}\alpha^2/3$) and s should take values in the interval $E' < s < E''$, now with E', E'' the two smaller positive solutions of $V(s, I_2) = h$. Recall that in this case, the last equation has still a larger solution $E''' > E''$. Then, for $s > E'''$, (78) parametrizes an escape curve.

So long we have just only given a few hints for the construction of the phase portrait of the solutions of (75). Further details can be appreciated in the figures 6, 7. □

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