# PSI-SERIES, SINGULARITIES OF SOLUTIONS AND INTEGRABILITY OF POLYNOMIAL SYSTEMS

ARNAU MIR AND AMADEU DELSHAMS

**Abstract.** Psi-series (i.e., logarithmic series) of $m$-dimensional polynomial systems are considered. Its existence and convergence is studied, and an algorithm of location of logarithmic singularities is developed. Moreover, the relationship between psi-series and non-integrability is stressed and in particular it is stated that $m$-dimensional polynomial systems with psi-series which do not reduce to Laurent series do not have $m-1$ independent algebraic first integrals.

## 1. Introduction

The integrability of a system of differential equations seems to be very well related with the kind of expansions of its solutions near singularities. Painlevé [2] gave a test to obtain integrable systems, based on looking for systems such that these expansions were simply Laurent series. There are several partial results supporting this test [1], as well as counterexamples [5].

A first generalization of the Laurent series is given by the so called *psi-series*:

$$x(t) = \sum_{n \geq -k} p_n \left( \log \left( \frac{1}{\tau} \right) \right) \tau^n,$$

where $p_n$ are vector polynomials and $\tau = t - t_p$, near a singularity $t_p$ of a solution $x(t)$. To differentiate them from the standard Laurent series, we will call *genuine psi-series* those psi-series such that some $p_n$ is not a constant.

In many cases, these expansions appear in non-integrable systems. However, there are cases of integrable systems with genuine psi-series. In a previous work [4], we have studied all quadratic systems with a center at the origin. It turns out that in all the cases that genuine psi-series appear, the general integral is not an algebraic function. We will see here that this is the general situation, i.e., that the existence of genuine psi-series is an obstruction for algebraic integrals.

In section 2, we give conditions for the existence of psi-series for $m$-dimensional polynomial systems. Section 3 is devoted to the convergence of psi-series and an estimate of the region of convergence is given. The psi-series expansion is an important tool to locate singularities. This tool is explained in section 4.

From the study developed there, one gathers that

– the Painlevé test cannot detect polynomial integrable systems if some of the first integrals are transcendent functions,
– any $m$-dimensional polynomial system with a complete solution in terms of genuine psi-series does *not* have $m-1$ independent algebraic first integrals.

This last result is announced in section 5.

## 2. Existence of formal psi-series

In looking for psi-series of a system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x), \tag{1}$$

one tries first for solutions of the form

$$x(\tau) = \tau^{\alpha} \cdot A + \cdots, \tag{2}$$

where $\tau^{\alpha} \cdot A = (\tau^{\alpha_1} A_1, \ldots, \tau^{\alpha_m} A_m)$, $\tau = t - t_p$, $t_p$ being a singularity, $A$ is a vector of constants, and $\alpha := (\alpha_1, \ldots, \alpha_m) \in \mathbb{Z}^m$ is a vector with integer components. In substituting $\tau^{\alpha}$ in equation (1), one can find $\alpha$ such that $X(x)$ can be separated as $X(x) = F(x) + R(x)$, where $F(x)$ contains the relevant higher order terms and $R(x)$ the lower order terms for $x = \tau^{\alpha} \cdot A$: $R(x) = \mathrm{o}(F(x))$. The following theorem tell us when psi-series appear:

**Theorem 1** *Consider the equation (1), where $X(x) = F(x) + R(x)$, as described above, is an analytic function. Assume*

*H1) There exists a vector $A = (A_1, \ldots, A_m)^{\top}$ with all components non-zero such that $F(A) = \alpha \cdot A$, for some $\alpha \in \mathbb{Z}^m$, where $\alpha \cdot A = (\alpha_1 A_1, \ldots, \alpha_m A_m)$.*
*H2) $r_i \in \{1, 2, \ldots\}$, where $\{-1, r_1, \ldots, r_{m-1}\}$ are different eigenvalues of $DF(A) - \mathrm{diag}(\alpha)$.*

*Then, there exists a (formal) m-parametric family of psi-series*

$$x(t) = \tau^{\alpha} \cdot A + \sum_{n \geq 1} \tau^{\alpha} \cdot p_n \left( \log \left( \frac{1}{\tau} \right) \right) \tau^n. \tag{3}$$

**Sketch of the proof:**

If we differentiate expansion (2) with respect to $t$ and use the Taylor expansion of $F(x) = F(\tau^{\alpha} \cdot A + \cdots)$, we get that $\alpha \cdot A = F(A)$, which due to hypothesis H1) has at least a solution with all components non-zero.

Differentiating twice expansion (2), and taking into account again the Taylor expansion of $F(x) = F(\tau^{\alpha} \cdot A + \cdots)$, we get that $\alpha \cdot A$ is an eigenvector of eigenvalue $-1$ of $DF(A) - \mathrm{diag}(\alpha)$.

Now it is very convenient to perform a change of time $s = -\log \tau$, or $\tau = \mathrm{e}^{-s}$, where $\tau = t - t_p$, and a change of variables

$$x = \mathrm{e}^{-s\alpha} \cdot A + \mathrm{e}^{-s\alpha} \cdot (Bw), \tag{4}$$

where $\mathrm{e}^{-s\alpha} \cdot v$ denotes $(\mathrm{e}^{-s\alpha_1} v_1, \ldots, \mathrm{e}^{-s\alpha_m} v_m)$, for $v = (v_1, \ldots, v_m) \in \mathbb{C}^m$, $w$ is the new variable and $B$ is the matrix formed with eigenvectors of $DF(A) - \mathrm{diag}(\alpha)$.

Denoting $' = \mathrm{d}/\mathrm{d}s$, then $\mathrm{d}/\mathrm{d}t = -\mathrm{e}^{s}\,'$, and it follows that the equation for the new variables $w$ is

$$w' + \mathrm{diag}(-1, r_1, \ldots, r_{m-1})w = \text{ (lower order terms)}. \tag{5}$$

Introducing $w_n(s) = B^{-1}p_n$, the psi-series (3) takes the form

$$x(s) = \mathrm{e}^{-s\alpha} \cdot A + \sum_{n\geq 1} \left(\mathrm{e}^{-s\alpha} \cdot p_n\right) \mathrm{e}^{-ns} = \mathrm{e}^{-s\alpha} \cdot A + \sum_{n\geq 1} \left(\mathrm{e}^{-s\alpha} \cdot (Bw_n)\right) \mathrm{e}^{-ns}, \tag{6}$$

and hence, by change (4), we look for an expansion of $w$ of the following form

$$w(s) = \sum_{n\geq 1} w_n(s)\, \mathrm{e}^{-ns},$$

$w_n$ being polynomials in the variable $s$.

Taking into account that $w' = \sum_{n\geq 1}(w_n' - nw_n)\,\mathrm{e}^{-ns}$, we get the differential equation for $w_n$

$$w_n' + \Lambda_n w_n = c_n, \tag{7}$$

where $\Lambda_n = \mathrm{diag}(-1, r_1, \ldots, r_{m-1}) - n\mathrm{Id}$ and $c_n$ is a polynomial function that depends only on $w_k, k = 1, \ldots, n-1$.

The next step is to obtain the polynomial vector $w_n$.

If $n < \min\{r_1, \ldots, r_{m-1}\}$, by induction, we get that the only polynomial solution of the equation (7) is the constant vector $w_n = -{\Lambda_n}^{-1}c_n$, since $\Lambda_n$ is an invertible matrix. So, $p_n = Bw_n$ is a constant vector too.

If $n = r_i,\ i = 1, \ldots, m-1$, the matrix $\Lambda_n$ is singular and it has eigenvalues $n+1$, $n - r_1, \ldots, 0^{i)}, \ldots, n - r_{m-1}$. In this case, we have that the $i$-component of the differential equation (7) is

$$w_{r_i,i}' = c_{r_i,i}.$$

So, the solution of the above equation is constant unless $c_{r_i,i} \neq 0$. In the case where $c_{r_i,i} = 0, \forall i = 1, \ldots, m-1$, the polynomial solution of the equation (7) is constant for all $n$ because $\Lambda_n$ is a non singular matrix for $n \neq r_i,\ i = 1, \ldots, m-1$. Under these circumstances, the expansion of $x(t)$ are Laurent series. According to the Painlevé property, system (1) is candidate to be an integrable system.

On the other hand, if there exists $i \in \{1, \ldots, m-1\}$ such that $c_{r_i,i} \neq 0$, the solutions of the equation (7) are non constants polynomials vectors and genuine psi-series appear.□

## 3. Convergence of the psi-series

From now on, we shall suppose that the vector field $X(x)$ is a polynomial vector. In this case, the vector $c_n$ of the formula (7) can be written as:

$$c_n = \sum_{j=2}^{k} \sum_{i_1+\cdots+i_j=n} R_j\left[Bw_{i_1}, \ldots, Bw_{i_j}\right], \tag{8}$$

where $R_j$ is a $j$-linear form.

The following theorem give us a real region depending of two parameters: $C$ and $K$, where $C$ appears in the following lemma:

**Lemma 2** *Let $j$ be an integer number. So, there exists a constant $C$ such that:*

$$a_n := \sum_{i_1+\cdots+i_j=n} \frac{1}{(i_1+1)\ldots(i_j+1)} \leq \frac{C}{(n+1)^{\beta}},$$

*with $0 < \beta < 1$.*

**Theorem 3** *Let $n_0$ be the integer such that the following inequality is true for $n \geq n_0$:*

$$\frac{C \sum_{j=2}^{k} \|R_j\| \, \|B\|^j}{(n+1)^{\beta}} \cdot \frac{3}{n} < \frac{1}{n+1},$$

*where $0 < \beta < 1$ is the exponent that appears in lemma 2.*

*Let $K$ be a real positive number that satisfies the following finite number of inequalities:*

$$\|w_n\| \leq (n+1)^{-1}(2K + Ks)^{\frac{n}{r}}, \ \ n = 0, 1, \ldots, \max\{n_0, 4r\},$$

*and $r = \max\{r_1, \ldots, r_{m-1}\}$.*

*Then the psi-series $\sum w_n(s)\, \mathrm{e}^{-ns}$ and $\sum p_n(s)\, \mathrm{e}^{-ns}$ are convergent for real $s > s_0$, where $s_0$ is the positive root of the equation:*

$$K(2+s) = \mathrm{e}^{rs}.$$

**Remark 4** The study of convergence for a complex value $s_c = s - i\theta$, $s \in \mathbb{R}$ and $\theta \in [0, 2\pi)$, can be reduced to the real case [4].

## 4. How to find singularities using psi-series

Our aim is to find the singularity $t_p$ and the others free constants $c_1, \ldots, c_{m-1}$ of system (1) of the solution with initial conditions $x(t_0) = x_0$.

To solve the problem, we suppose that we know $x(t)$, where $t$ belongs to the region of convergence of the psi-series expansion. We can compute $x(t)$ by using a standard numerical integration method like Runge-Kutta 7-8.

Next, we consider the following application: (we identify $\mathbb{C} \equiv \mathbb{R}^2$)

$$\begin{aligned} F_{\Delta t}: \ & \mathbb{R}^{2m} \longrightarrow \mathbb{R}^{2m} \\ & (t_p, c_1, \ldots, c_{m-1}) \longrightarrow x(t) = \left( \tau^{\alpha} \cdot A + \sum_{n \geq 1} \tau^{\alpha} \cdot p_n \left( \log \tfrac{1}{\tau} \right) \tau^n \right), \end{aligned} \tag{9}$$

where $\Delta t = t - t_p$ small.

So, we have to solve numerically the following numerical equation in $\mathbb{R}^{2m}$:

$$F_{\Delta t}(t_p, c_1, \ldots, c_{m-1}) = (x(t)). \tag{10}$$

We can solve it by using, for example, the Newton-Raphson method. For a more details about this section, see [4].

## 5. Relation between psi-series and General Integrals

In [4] we have studied all the integrable quadratic systems with a center at the origin. We have proved by using the results of Lunkevich and Sibirskii [3] that if the General Integral of the system is algebraic, then the only possible psi-series expansions reduce to Laurent series.

We have here an analogous result.

**Theorem 5** *Let*

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X(x), \tag{11}$$

*be a differential system of dimension $m$ where $X(x)$ is a polynomial vector field.*

*Assume that a solution $x(t)$ of (11) and a singularity $t_p$ of $x(t)$, has a psi-series expansion as a function of $\tau = t - t_p$*

$$x(t) = \tau^{\alpha} \cdot A + \sum_{n \geq 1} \left( \tau^{\alpha} \cdot p_n \left( \log \frac{1}{\tau} \right) \right) \tau^n .$$

*Suppose that (11) is an integrable system and the $m-1$ general integrals are algebraic integrals:*

$$\begin{array}{ll} Q_1(x, H) & = 0, \\ \vdots & \vdots \\ Q_{m-1}(x, H) & = 0, \end{array} \tag{12}$$

*where $Q_i$ are polynomials in the variables $x$ and $H = (H_1, \ldots, H_{m-1})^{\top}$ (free constants of integration).*

*Then this psi-series is in fact a Laurent series (all the polynomial $p_n$ are simply constants).*

In few words, polynomial systems with genuine psi-series cannot have a complete set of algebraic integrals.

## Acknowledgements

Research by Amadeu Delshams is partially supported by the spanish grant DGICYT PB94–0215, the EC grant ERBCHRXCT940460 and the catalan grant CIRIT GRQ93–1135.

## References

1. L. Gavrilov. Non-integrability of the equations of heavy gyrostat. *Compositio Mathematica*, 82:275–291, 1992.
2. E.L. Hille. *Ordinary Differential Equations in the Complex Domain.* Wiley-Interscience, New York, 1976.
3. V. A. Lunkevich and K. S. Sibirskii. Integrals of a general quadratic differential systems in cases of a center. *Differential Equations*, 18(5):563–568, 1982.
4. A. Mir and A. Delshams. Psi-series of quadratic systems on the plane. Preprint, 1995.
5. A. Ramani, B. Grammaticos, and T. Bountis. The Painlevé property and singularity analysis of integrable and non-integrable systems. *Phys. Reports*, 180(3):159–245, 1989.