Nonlinear Micro-Mechanical Analysis of Masonry Periodic Unit Cells

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Highlights

- The orthotropic properties of masonry are derived with very low computational cost
- Several different masonry wall and pillar typologies are investigated
- The model results are compared to experimental data and FE calculations
- A parametric investigation is performed to determine the effect of the mechanical properties of units and mortar to the compressive strength of masonry
- Closed form expressions for the determination of the compressive strength of masonry are proposed

Keywords

Micro-modeling; numerical modeling; masonry; compressive strength; tensile strength

1. Introduction

1.1 State of the Art

Over the last few decades, the interest in the nonlinear behavior of masonry structures has seen a significant increase. The mechanical properties of the materials of which masonry members are comprised, coupled with the mechanical actions to which they are subjected in the composite, make the study of the behavior of masonry...
beyond the elastic limit a necessity. The numerical study of the nonlinear behavior of masonry composites becomes even more relevant considering the difficulty in acquiring masonry samples from existing buildings on which meaningful mechanical tests can be performed.

Micro-modeling techniques for the derivation of the mechanical properties of masonry composites are a powerful tool. They are capable of providing the orthotropic elastic properties of masonry under normal and shear loads as well as its strength domain under uniaxial or complex loading. The information obtained in terms of strength values, the evolution and propagation of damage in each component and the influence of individual material parameters on the response are highly important for the computational study of masonry structures and can prove to be an incentive for guiding experimental studies towards the determination of the critically important mechanical properties of masonry materials. In this context, knowledge- and performance-based design stand to benefit greatly from micro-modeling techniques.

Being composed of (at least) two macroscopically distinguishable material phases with potentially very different mechanical properties, generally arranged in a repeating pattern, masonry structures composed of units bonded with mortar joints are a suitable candidate for analysis using periodic unit cells. The analysis of these cells may be performed using finite element computations [1,2] or analytical expressions [3–5], in order to derive the distribution of stress and strain in the volume of the cell. While the former is capable of providing accurate results, its use is hindered by potentially high computational cost. The latter choice is attractive due to its very low computational cost, but the validity of the assumptions made in the formulation of the analytical expressions needs to be rigorously checked through conceptual reasoning, accurate calculations and comparison with existing experimental data. Comparison with a finite element benchmark may also be required for the determination of the accuracy of analytical models, as experimental data does not often include an exhaustive measurement of all elastic properties of the masonry composite.

It has been demonstrated that numerical modeling of masonry wall structures under in-plane-loading needs to take into account the out-of-plane stresses [6,7]. While this necessity is straightforward in the case of multi-leaf walls or of pillars composed of interlocking masonry units, where the accurate representation of the geometry of the structure demands its full three-dimensional modeling, it is essential in the analysis of single
leaf structures as well, since three-dimensional effects are a governing factor in the behavior of mortar joints in
masonry under compression. While plane stress and plane strain finite element models may present significant
computational cost advantages over full three-dimensional models, they tend to under- or overestimate the
confinement effect on mortar in the joints respectively.

Several models based on computationally inexpensive analytical expressions have been proposed for
various types of masonry. These include early models of stack bond pillars [8] and numerous works on stack
and running bond walls [3,5,9]. Other types of masonry, such as Flemish bond walls, three-leaf walls with infill
and English bond pillars have not been the subject of much investigation. An attractive method of analytically
dealing with masonry periodic unit cells is the micro-mechanical approach introduced by Aboudi [10] for the
analysis of periodically reinforced composites. The masonry cell is suitable for analysis using this method when
seen as a regular arrangement of square or cubic sub-cells with varying mechanical properties and interlocking
patterns.

Detailed micro-modeling requires extensive characterization of the mechanical properties of the masonry
units, mortar, infill and interfaces. Due to the large number of parameters involved, coupled with inherent
difficulties in determining these parameters from samples extracted from existing structures and with the high
scatter that often characterizes them, several of these parameters are routinely given standard values. The study
of the sensitivity of the compressive strength of masonry to some of these parameters is an interesting subject
for investigation, as local or diffuse variation in a material property may be detrimental to the load bearing
capacity of a structural member.

Closed-form expressions for the determination of the compressive strength of masonry have been proposed
based on various analytical formulations [11–14] and are in use in modern design codes [15]. Many of these
expressions have a strong empirical dimension concerning the influence of the material properties of the
constituent materials of masonry. Furthermore, several models have restrictions on their range of application as
defined by the spectrum of elastic properties in which they provide reliable results. A relatively simple closed-
form model based on the principles of detailed micro-modeling which overcomes as much as possible empirical
assumptions and result instability of other closed-form models could be proposed.
1.2 Objectives

A number of objectives is attempted to be tackled through this investigation. A model for the computational modeling of masonry wall and pillar structures based on micro-mechanical modeling techniques and performed through the analysis of periodic unit cells is proposed. The analysis of the masonry cells is carried out using simple analytical expressions based on stress equilibrium, strain conformity and rational assumptions concerning the behavior of masonry geometrical components. By coupling with nonlinear constitutive laws, these models are intended to be used for the calculation of the nonlinear properties of masonry structures.

Several typologies of masonry walls and pillars are treated in this paper. A number of them, such as stack bond pillars, stack bond walls and running bond walls, have garnered the almost complete attention of researchers so far. The present investigation includes analyses on Flemish bond and three-leaf walls, as well as English bond pillars, which have not received the research attention that their abundance in the built environment would warrant.

The models have been employed in the study of the response of masonry under applied normal stress. Special emphasis is placed on compression, which exhibits a rather more complex dependence on material properties due to the effect of triaxial confinement of the mortar in masonry under compression.

In addition, a closed-form expression for the determination of the compressive strength of masonry is proposed based on the micro-mechanical models developed here. A further simplification of this expression is proposed based on empirical data. A number of experimental case studies on the compressive strength of masonry are compared with the results of the analytical model and the closed-form expressions.

Finally, both the micro-mechanical models and the closed form expressions are employed in a parametric investigation, aimed at the study of the influence of several geometrical and material parameters on the compressive strength of masonry.

2. Overview of the Models

2.1 Derivation of the Periodic Unit Cells

Four different masonry wall typologies will be studied in this paper: stack bond, running bond, Flemish bond and three-leaf with a running bond external leaf. The first two typologies are single leaf walls, common as
load-bearing and secondary elements in traditional and new buildings. The latter two present geometrical
complexity and variation along the thickness. Whereas the Flemish bond wall includes header units spanning the
entire thickness of the wall, the three-leaf wall model developed here only accounts for wall typologies with no
units connecting the external leaves: the three leaves interact through a simple continuous interface. Two
different pillar typologies will be examined as well: stack bond and English bond. The first is a simple typology,
consisting of alternating layers of unit and mortar, while the latter features interlocking of the units in two
directions orthogonal to the longitudinal axis of the pillar.

The derivation of the cell to be analyzed for each structural typology depends on the structural complexity
and symmetry conditions. The derivation process for wall structures is graphically presented in Figure 1, while
that of the pillars in Figure 2. Once the basic cell has been derived, it may be discretized into cuboid parts.

2.2 Discretization of the Cells and General Modeling Assumptions

Figure 3 illustrates the discretization of the single leaf wall cells along with the naming scheme for each
component in detail. The discretization of the multi leaf walls is shown in Figure 4 and that of the pillars in
Figure 5. The cuboids are arranged in an orthogonal grid, composed of strips of cuboids, each of which may be
conceived as a set of springs in a linear arrangement. In the cases where cross joints exist, the units are divided
into multiple cuboids, as can be seen in the discretization of all the cells except those of the running bond wall
and pillar.

The cuboid parts are designated by a set of initials. Throughout all the cases these are: \( u \) for units in general,
\( s \) and \( d \) for stretcher and header units respectively where both are present (such as the Flemish bond case), \( h \) for
head joints, \( c \) for cross joints, \( b \) for bed joints and \( t \) for transversal joints. Infill is designated as \( i \). Dimensions are
designated according to their orientation: \( l \) corresponds to a horizontal length, \( h \) to a vertical height and \( t \) to a
transversal thickness. The dimension symbols are finally suffixed \( u \), \( m \) or \( i \) for units, mortar and infill
respectively, meaning that \( h_u \) is the unit height, \( h_m \) is the thickness of the bed joint, \( l_u \) is the unit length, \( l_m \) is the
thickness of the head joint, \( t_u \) is the width of the unit, \( t_m \) is the thickness of the transversal joint and \( t_i \) is the
transversal thickness of the infill. The \( u \), \( m \) or \( i \) suffixes are also used for designating the material type when the
mechanical properties are defined.
The deformation of the cell faces, and, therefore, the total strain of the cell, depends on the loading applied. Application of normal stress results only in normal global cell stresses and strains. This means that under normal stress the total deformation of all strips, or the average strain of each one, is equal in the three principal directions, or, equivalently, that the external faces of the cell remain plane and parallel. This condition satisfies the periodicity conditions for normal stress loading.

Under the application of normal stress, either only normal or both normal and shear stresses may arise in the cuboid parts. All stresses and strains are assumed constant in the cuboid parts. For linear elastic analysis, the units and the mortar are modeled as three-dimensional isotropic continua. Perfect bond is considered at the unit-mortar interface for the linear elastic computations performed here, so that all deformation of the cell is accounted for in the units, mortar and infill.

Isotropic linear elasticity stress-strain relations in three dimensions apply for every cuboid component under normal and shear stress. This includes the units, the mortar and the infill. These relations are expressed as [16]:

\[ \varepsilon_{ii,n} = \sigma_{ii,n} / E_n - \nu_n (\sigma_{jj,n} + \sigma_{kk,n}) / E_n \]  

(1)

\[ \varepsilon_{ij,n} = \sigma_{ij,n} (1 + \nu_n) / E_n \]

(2)

where the sub-index \( n \) refers to the identifier of the cuboid, \( \sigma_{ii} \) and \( \varepsilon_{ii} \) are the applied normal stress and calculated normal strain along axis \( i \), \( \sigma_{ij} \) and \( \varepsilon_{ij} \) are the shear stress and strain in plane \( ij \), \( E_n \) is the Young’s modulus and \( \nu_n \) is the Poisson’s ratio.

When a cuboid with dimensions \( D_i \) and \( D_j \) in directions \( i \) and \( j \) respectively is subjected to a shear stress \( \sigma_{ij} \) the contribution of the shear deformation of the cuboid \( d_{ij} \) to its normal deformation in the direction \( i \) is taken as being half the displacement at its top due to the shear stress. Therefore, it is defined as

\[ d_{ij} = \varepsilon_{ij,n} D_i / 2 \]

(3)

The three Young’s moduli for each cell configuration are calculated by use of the equation
\[ E_{ij} = \frac{\sigma_{ij}}{\varepsilon_{ii}} \] (4)

The Poisson’s ratios are determined by the equation

\[ \nu_{c,ij} = \frac{-\varepsilon_{ij}}{\varepsilon_{ii}} \] (5)

The \( x \), \( y \) and \( z \) axes in the model correspond to the horizontal, vertical and transversal directions. The transversal direction is normal to the face of the walls and the vertical direction is parallel to the longitudinal axis of the pillars. The total strain in each of the three directions and the three resulting planes depends on the type of cell and can be expressed analytically.

Normal and shear stress equilibrium equations are formed as follows:

\[ \sum_{n} \sigma_{ii,n} A_n - \sigma_{ii} A = 0 \] (6)

\[ \sum_{n} \sigma_{ij,n} A_n - \sigma_{ij} A = 0 \]

where \( A \) is the total cross sectional area of the cell and \( A_n \) is the cross sectional area of cuboid \( n \) in direction \( i \).

All normal and shear stress equilibriums at the faces or cross sections of the cell assure global equilibrium of internal and external stress. External stresses are averaged over the surface of the cell, meaning that there may exist a mismatch between the external average stress and the stress of an individual cuboid.

The resulting systems of equations have been presented in detail in [17]. They are repeated here in Appendix A.

3. **Nonlinear Analysis**

3.1 **Failure Criteria**

Different failure criteria are adopted for the initiation of damage in the cuboids. The choice of criterion depends on the nature of the failure type in question.
For the mortar the Hsieh-Ting-Chen failure curve is adopted [18], while for the units two different failure
criteria have been implemented: a Rankine criterion with a compression and a tension cut-off, equal to the
uniaxial compression and tensile strength of the unit respectively, and a Mohr-Coulomb criterion. In the former
criterion the interaction between tension and compression is disregarded, while in the latter it is taken into
account by reducing the compressive strength of the unit when subjected to laterally applied tensile stress. The
Rankine criterion is also adopted for the infill. All curves are presented in Figure 6a and b.

The mathematical expression of the Hsieh-Ting-Chen curve is [18]:

$$f = C_1 \frac{J_2}{f_c^2} + C_2 \sqrt{J_2} + C_3 \frac{\sigma_1}{f_c} + C_4 \frac{I_1}{f_c} - 1 = 0$$  \hspace{1cm} (7)

where $I_1$ and $J_2$ are the stress invariants, $\sigma_1$ is the maximum principal stress and $f_c$ is the uniaxial
compressive strength. The parameters $C_1$, $C_2$, $C_3$ and $C_4$ are determined by four material tests: uniaxial
compression, uniaxial tension, equibiaxial compression and compression under equibiaxial compression. Figure
6a qualitatively illustrates the effect of out-of-plane stresses in the in-plane strength domain of the mortar.

Similarly, Figure 6b illustrates the effect of lateral tensile stresses on the compressive strength of the unit.

In shear, a Mohr-Coulomb friction failure criterion is adopted, which has the form [19]:

$$|\tau| + \sigma \tan \varphi - c = 0$$  \hspace{1cm} (8)

where $\tau$ is the failure shear stress, $\sigma$ the applied normal stress, $\varphi$ is the friction angle and $c$ the cohesion. The
failure curve is presented in Figure 6c.

3.2 Simulation of Damage

Damage simulation in the cuboids due to compression, tension and shear is accomplished by the use of
integrity variables. These scalar variables are designated $C$, $T$ and $S$ for compression, tension and shear
respectively and range from 1 to 0. They represent the ratio between the actual stress and the effective stress in
the component. A value of 1 signifies that the material is undamaged and a value of 0 signifies that the material
is completely softened. The evolution of the integrity variables is a function of the total uniaxial strain and, in
the case of compressive damage, the lateral compressive stresses. The integrity variables serve an equivalent
purpose to the one served by damage variables in damage mechanics [20], the former being obtained from the
latter by subtracting it from 1.

The determination of these integrity variables requires the description of the hardening and softening
behavior of the materials in tension, compression and shear. For tension an exponential softening law based on
fracture energy is adopted [21]. According to this approach, the integrity variable in tension is:

\[
T(n) = \begin{cases} 
\frac{f_i}{\sigma_{eff}(n)} \exp \left[ -\frac{f_i h}{G_f} \left( \varepsilon(n) - \frac{f_i}{E} \right) \right] & 0 \leq \sigma_{eff}(n) \leq f_i, \\
1 & f_i < \sigma_{eff}(n) 
\end{cases} 
\] (9)

where \( f_i \) is the tensile strength, \( E \) is the Young's modulus, \( G_f \) is the tensile fracture energy, \( h \) is the
characteristic length and \( \varepsilon(n) \) is the total strain. This equation is evaluated in the three principal directions for
each cell component and the lowest integrity variable is chosen to represent the isotropic damage.

The failure due to tension in the interface may be taken into account by adjusting the tensile properties of
the mortar in the chosen direction. For this purpose the tensile strength and fracture energy may be reduced in
the horizontal direction for the head joints, the vertical direction for the bed joints and the transversal direction
for the transversal joints in order for these parameters to correspond to the mechanical properties of the
interface. For the softening of the interface in tension the characteristic length as defined for the mortar joint is
adopted.

A parabolic hardening-softening curve based on fracture energy is adopted for compression [21]. The
integrity variables for compression are:
where $f_c$ is the compressive strength, $\varepsilon(n)$ the strain, $\sigma_{eff}(n)$ the effective stress. The strain $\varepsilon_{c/3}$, at which one third of the compressive strength is reached, is expressed as:

$$\varepsilon_{c/3} = -\frac{1}{3} \frac{f_c}{E}$$  \hspace{1cm} (11)

The strain $\varepsilon_c$, at which the maximum compressive strength is reached, is expressed as:

$$\varepsilon_c = -\frac{5}{3} \frac{f_c}{E} = 5 \varepsilon_{c/3}$$  \hspace{1cm} (12)

The ultimate strain $\varepsilon_u$, at which the material has terminated its softening in compression, which is expressed as:

$$\varepsilon_u = \varepsilon_c - \frac{3}{2} \frac{G_f^c}{h f_c}$$  \hspace{1cm} (13)

where, in turn, $G_f^c$ is the compressive fracture energy and $h$ is the characteristic element length. The parabolic curve strain and stress values are scaled due to the application of lateral compressive stress proportionally to the ratio between the uniaxial compressive strength and the compressive strength calculated using the Hsieh-Ting-Chen curve when lateral compression is applied.

The values of the compressive strength and the peak and maximum strain are modified according to the lateral effective stress in order to simulate the pressure dependent behavior of the mortar in certain directions.
This pressure dependence is taken into account for compression in the vertical direction for the bed joints, the horizontal direction for the head joints and the transversal direction for the transversal joints. As in the case for tension, the compressive integrity variable is evaluated in three directions for each component, the lowest value being the eventual isotropic integrity variable.

The expressions of the integrity variables for shear, assuming a Mohr-Coulomb failure criterion and an exponential softening curve based on fracture energy for the cohesion are:

\[
S(n) = \begin{cases} 
1 & \text{for } |\tau_{\text{eff}}(n)| \leq c_0 - \sigma(n) \tan \varphi_0 \\
-\sigma(n) \tan \varphi(n) + c(n) / |\tau_{\text{eff}}(n)| & \text{for } |\tau_{\text{eff}}(n)| > c_0 - \sigma(n) \tan \varphi_0
\end{cases}
\] (14)

where \(c_0\) is the initial cohesion, \(c(n)\) is the cohesion in the load step \(n\), \(\tan \varphi(n)\) the tangent of the friction angle, \(\sigma(n)\) and \(\tau_{\text{eff}}(n)\) the applied normal and effective shear stress. The tangent of the friction angle is assumed to develop according to

\[
\tan \varphi(n) = \tan \varphi_0 + (\tan \varphi_r - \tan \varphi_0) c_0 - c(n) \frac{c_0 - c(n)}{c_0}
\] (15)

where \(\tan \varphi_0\) and \(\tan \varphi_r\) are the initial and residual tangents of the friction angle. The cohesion softens exponentially according to

\[
c(n) = c_0 \exp \left( -\frac{c_0 h}{G_f} \left( \varepsilon(n) - \frac{c_0 - \sigma(n) \tan \varphi_0}{2G} \right) \right)
\] (16)

where \(G\) is the shear modulus, \(G_f^{II}\) the shear fracture energy, \(h\) the characteristic length and \(\varepsilon(n)\) the total strain.

As in the case for tension, shear failure of the bed joint interface may be taken into account by assigning the shear strength and fracture energy of the interface to the mortar joint components. Otherwise, the initial frictional characteristics for each component may be derived from its uniaxial compressive and tensile strength.
All integrity variables increase monotonically, meaning that damage recovery is not possible. Reduction of
the applied strain in a damaged segment results in the integrity variables to stay constant. Therefore, unloading
and reloading take place along the damaged stress-strain path.

The adoption of a micro-mechanical approach and the representation of units, mortar and infill using
cuboids entail a number of differences in the modeling of nonlinearity compared to finite element analysis. The
integrity variables are evaluated in the three principal directions, unlike in finite element plasticity and smeared
 crack models. The geometrical patterns found in masonry result in a limitation of the failure modes capable of
arising for loading along the principal geometric axes, and which can be well approximated by simple micro-
mechanical modeling of the interaction of the masonry composite material phases. The result is a relatively
simple and intuitive modeling of the failure in the masonry components.

The characteristic length required by the laws presented here is calculated differently for each damage type.
In tension the characteristic length for cracking in a given direction is taken as being equal to the dimension of
the cuboid in that direction. The physical interpretation of this assumption is that the width of the crack band is
equal to the length of the cuboid, meaning that a single crack is formed along the length of the cuboid. In
compression and shear it is calculated as the cubic root of the volume of the cuboid, an approach usually
adopted in finite element modeling using solid elements. The physical meaning of the assumption for the
characteristic length is that all of the material in the cuboid yields and is involved in the softening.

Representative hardening and softening curves in compression, tension and shear are presented in Figure 7.

3.3 Solution

For linear elastic analysis the solution from which the elastic moduli of the cell may be derived can be
accomplished in a single analysis step by solving the linear system of equations derived from the stress and
strain conformity expressions, as presented in the previous section. The systems need to be solved for the
unknown normal stresses, shear stresses (where considered), normal strains and shear strains. To each cuboid
correspond three normal stress, three normal strain, one shear stress and one shear strain value for normal stress
loading. In the case of the stack bond and English bond pillars, the shear stresses and strains are disregarded.
Nonlinear elastic analysis requires the modification of these basic systems of equations. Adopting the
damage mechanics concepts of damaged and effective (undamaged) stress this can be achieved by the use of the
integrity variables. The Hooke’s law equations remain unchanged, however, since they describe the relation
between strain and effective stress. The systems are solved for a known applied value of total cell strain and the
total cell stresses are sought.

Overall, the actual damaged stress tensor in load step $n$ for a cuboid can be expressed as:

$$\sigma(n) = T(n)C(n)S(n)\sigma_{eff}(n)$$  \hspace{1cm} (17)

The solution of the system of modified equations is accomplished through the adoption of a multi-variate
Newton-Raphson iterative process. As such, the variables in iteration $i$ of each load step are calculated as
follows [19]:

$$x^{i+1} = x^i - J(x^i)^{-1}F(x^i)$$  \hspace{1cm} (18)

where $x$ is the vector of variables and $J$ is the Jacobian matrix of the vector of equations. The unknown
variables for which the system of equations is solved are the stresses and strains and all the integrity variables
for each cuboid. The additional equations needed to fill out the system of equations are defined as the difference
between the integrity variable assumed at the beginning of the iteration and a trial value which is calculated
based on the stresses and strains of the component in the iteration. The trial values are assigned an initial value
of 1 at the first iteration of the first load step. Due to the simplicity of the expressions used in the systems of
equations to be solved, the Jacobian can be calculated analytically.

4. Results

4.1 General Model Behavior

The results produced by the model will be, as a first step, evaluated qualitatively in terms of the behavior in
vertical compression and horizontal tension. The preliminary numerical analyses were performed using the
material properties and dimensions presented in Table 1. These properties correspond to masonry composed of
strong units, medium strength mortar and low strength, highly deformable infill.
Concerning the numerical parameters of the Hsieh-Ting-Chen failure surface encountered in equation (7), the standard values proposed in [18] were used. These values are obtained by assuming a tensile strength equal to 10% of the compressive strength, a biaxial strength equal to 1.15 times the uniaxial compressive strength and a compressive strength equal to 4.2 times the uniaxial compressive strength under a biaxial confinement equal to 0.8 times the uniaxial strength. According to these assumptions the parameters of equation (7) are: \( C_1 = 2.0108, \)
\( C_2 = 0.9714, C_3 = 9.1412, C_4 = 0.2312. \)

Figure 8 presents the stress-strain curves obtained by the stack bond pillar, stack bond wall, running bond wall, Flemish bond wall and three leaf wall models under horizontal tension and vertical compression. In the horizontal tension case the stack bond wall produced the lowest tensile strength of masonry due to the lack of interlocking between units and mortar and the continuous vertical mortar joint. The running and Flemish bond cases produced nearly equal tensile strength of masonry but the latter exhibited more rapid softening. All stress-strain curves are linear until the peak stress, followed by an exponential softening branch. The softening branch for the three leaf masonry is divided into two separate branches, as different parts of the masonry yield in tension.

Under vertical compression the stack bond pillar produced the highest strength, followed closely by the stack bond and running bond walls. The softening curve of the running bond wall case was interrupted by an abrupt drop of stress due to tensile damage in the head joint, which has been assigned the properties of the interface. The Flemish bond wall model produced a noticeably lower strength and a steep unloading caused by cracking in the middle of the stretcher unit. The softening curves obtained for the prism pillar, the stack bond wall and the running bond wall are generally characterized by a linear initial part, followed by a parabolic part until after the peak load and followed by a final exponential softening part. Overall, the model is able to produce realistic response curves for both applied tension and compression.

The progression of damage in a Flemish bond wall cell under vertical loading is presented in Figure 9. According to the predictions of the model compressive damage progresses nearly steadily in the cross joint and the bed joint from near the beginning of the loading. The compressive damage in the bed joint increases rapidly near the peak load. Compressive damage is also registered in the head and transversal joint, though not for low
values of vertical stress: due to the lower Young's modulus of the mortar, these joints are less stressed than the neighboring unit cuboids. Tensile damage appears in the last quarter of the loading before the peak and increases rapidly thereafter. This type of damage is registered in the mid part of the stretcher unit in the horizontal direction and in the transversal mortar joints in the transversal direction, the direction in which the interface properties have been assigned. Unlike the running bond wall case, tensile failure in the head joint was not registered.

The progression of normal stresses in the outer and inner masonry leaves will be examined for the three leaf masonry case. The stresses presented here have been averaged across the components comprising the inner and outer leaf respectively in order to present a global comparison of the stress state between the leaves. It should be noted, in any case, that the average stress in the outer leaf closely approximates that of its units since the volume of masonry accounted for by the mortar in the joints is very small. As shown in Figure 10, the outer leaf is in horizontal tension when the wall is subjected to vertical compression and the inner leaf is under horizontal compression. Both leaves are under vertical compression. However, given the difference in their elastic modulus, the vertical compressive stresses in the inner leaf are much lower than the ones in the outer leaf. The stress in the outer leaf reaches nearly the compressive capacity of the running bond wall, which may be seen in Figure 10, but fails to do so before the compressive yielding of the inner leaf when the vertical stress to which it is subjected reaches its compressive strength.

In all the models tested here the dominating failure type for applied horizontal tension is that of the head joint interface, which constitutes the weakest plane of weakness in this direction. In vertical compression, the response is governed by the compressive yielding of the bed joint mortar, which is in a state of triaxial compression for the properties here chosen.

### 4.2 Case Studies

A number of experimental case studies were assembled from the relevant literature on masonry under vertical compression [22–29]. These cases were numerically reproduced using the proposed models in order to predict the vertical compressive strength of masonry. Table 2 summarizes the results. The case studies have been named according to the masonry typology: S stands for stack bond prism, R for running bond wall, F for
Flemish bond wall and P for English bond pillar. An effort was made to include only cases with a sufficient mechanical characterization of the involved materials.

For the sake of comparison, these cases were also simulated using FE models of the full structures. For nonlinear analysis using the finite element models, a combined smeared crack model in tension [30] and a plasticity model in compression based on fracture energy [21]. The latter model is modified for lateral pressure dependence [18] expressed in a three dimensional total strain concept [31] and is employed for both the units and the mortar. The unit-mortar interfaces are modeled using a discrete cracking model for tensile failure. Prior to interfacial failure, the unit mortar bond is considered rigid by assuming a very high elastic stiffness. All finite element calculations were performed using the DIANA general purpose FE program [32]. In addition, a modern masonry design code, the EC6 [15], and two closed-form expressions proposed by Ohler [11] and Hilsdorf [12], were also used for predicting the compressive strength of these case studies. The first expression essentially depends only on the compressive strength of the two material components. The resulting value is the characteristic strength of masonry and is as follows [15]:

$$f_c = K f_{cu}^\alpha f_{cm}^\beta$$  \hspace{1cm} (19)

where $\alpha$ and $\beta$ are constants dependent on the type of mortar and the thickness of the joint, $K$ is a constant dependent on the type of mortar, unit and joint thickness, $f_{cu}$ is the compressive strength of the unit and $f_{cm}$ that of the mortar.

The two latter equations additionally depend on the tensile strength of the unit and the height of the unit and of the mortar joint. The Ohler expression reads [11]:

$$f_c = f_{cm} + \frac{s f_{cu} - f_{cm}}{1 + \frac{t}{m} h_u f_{cu}}$$  \hspace{1cm} (20)

where $s$ and $t$ are parameters describing the failure envelope of the unit and with $m$ being the slope of the failure criterion adopted for the mortar. These parameters are defined according to the specifications presented in Table 3. According to the formulation of the model, three values for the compressive strength are calculated.
and a value is chosen among the three based on its relation to the compressive strength of the unit. Furthermore, it is shown that the model does not account for masonry composites built using mortars with a compressive strength higher than that of the unit.

The Hilsdorf expression reads [12]:

\[
f_c = \frac{f_u + \alpha f_{cm} f_u}{f_u + \alpha f_u U}
\]  

where

\[
\alpha = \frac{h_m}{4.1h_u}
\]

\( f_u \) is the tensile strength of the unit and \( U \) is a safety factor taken as equal to 1.1.

The fracture energies of the materials and the properties of the interface were normally not given in the cited case studies. Instead, a set of standard values was used for the analyses performed. The compressive fracture energy was calculated according to

\[
G_f^c = f_d d
\]

where \( d \) is a ductility index parameter equal to 1 \( mm \). The tensile fracture energy, including that of the unit/mortar interface, was calculated according to the expression

\[
G_f^t = 0.025(2f_t)^{0.7}
\]

in accordance with the recommendations of the Model Code 1990 for concrete with a fine aggregate [33]. The shear fracture energy was calculated using the expression

\[
G_f^s = 0.1c_0
\]

The tensile strength and the cohesion of the interface were taken as 0.2 \( N/mm^2 \) and 0.3 \( N/mm^2 \) respectively, while the friction angle was taken as constant and equal to 45°. For the P1 English bond pillar case the tensile...
strength of the interface was taken as equal to the tensile strength of the mortar, which was lower than the
standard prescribed value.

In Figure 11 a comparison is made between the vertical compressive strength as predicted by the FEM and
micro-mechanical methods and the one obtained in the experimental case studies. The two methods appear to
produce equally good results compared to the experimental values for a wide range of cases.

The comparison of the compressive strength prediction models to the experimental, micro-mechanical,
closed-form and FE results is presented in Table 4. The Ohler model is more accurate than the Hilsdorf model,
while the EC6 model tends to underestimate the compressive strength of masonry for the majority of the cases.
Some of the main differences between the EC6 equation and the micro-mechanical model are related to unusual
cases involving combinations of units and mortar properties and dimensions not encountered in modern
practice. However, such combinations may be encountered in existing or historic masonry.

Finally, the in-plane failure envelope of a running bond periodic unit cell subjected to bilinear stress was
produced. Existing biaxial experimental tests on masonry walls for the derivation of the biaxial strength
envelopes of the composite are scarce and are not accompanied by sufficient material characterization [34]. The
original case study does not include a sufficient characterization of the mechanical properties of the materials.
For this reason, the material properties used in a relevant work simulating the same experimental campaign were
used instead [35]. The obtained failure envelope and its comparison to the biaxial experimental envelope are
shown in Figure 12. The model overall produces an adequate curve, which approximates well the behavior of
the masonry in biaxial tension and vertical compression, but tends to overestimate to a degree the horizontal
compressive strength. This could be the result of the compaction of the head joints not being as good as the one
in the bed joints.

4.3 Parametric Investigation

Overview

A wide parametric investigation was performed, the low computational cost of the method allowing for a
very large number of analyses to be performed in a short time. The objective of the investigation is to determine
the sensitivity of the predicted vertical compressive strength of masonry, as predicted by the micro-mechanical
model, to various material properties and component dimensions. The parameters taken into account for the analyses are summarized in Table 5, and represent a very wide, but not unlikely, range of combinations [36]. In total 62208 combinations of material properties and dimensions were considered and were analyzed using the simple stack bond pillar model.

Concerning these material properties, the $E/u_{cu}$ ratio of 300, $f_{tu}/f_{cu}$ of 0.10, $E/m_{cm}$ of 700 are considered "standard values", as are $u$ of 0.15, $m$ of 0.2, $u$ of 52mm and $m$ of 10mm.

In addition to the material properties, the influence of the adopted failure surfaces was investigated as well. Two cases were considered for the failure criterion of the units and the mortar alike. For the units the two failure curves presented in Figure 6b were used. In the Rankine criterion no interaction is assumed to take place between the applied compressive stress on the unit and the lateral tensile stresses that arise due to unit-mortar interaction. When the Mohr-Coulomb criterion is adopted, the developed tensile stresses reduce the capacity of the unit in compression. The two criteria as implemented in the model are identical in the tension-tension and compression-compression range.

The influence of the biaxial and triaxial behavior of the mortar was also investigated. This has been accomplished by altering the numerical parameters of the Hsieh-Ting-Chen failure curve. Initially, the same parameters were used as proposed in [18]. A second case with a biaxial strength equal to 1.10 times the uniaxial strength was considered while maintaining the same behavior under triaxial stress. The numerical parameters have not changed substantially by altering the biaxial strength alone. They now read: $C_1 = 1.9862$, $C_2 = 0.8575$, $C_3 = 9.2816$, $C_4 = 0.1571$.

Finally, a case with a compressive strength of 2.6 times the uniaxial strength under a bilateral compression equal to 0.8 times the uniaxial compressive strength was considered. The behavior of masonry under compression is particularly sensitive to the triaxial compressive behavior of the mortar and the behavior of the units under tension. Therefore it was considered desirable to investigate the influence of the adopted material laws on the predicted compressive strength. The numerical parameters are now as follows: $C_1 = 9.8064$, $C_2 = 2.8545$, $C_3 = 10.7004$, $C_4 = 0.6208$. Adjusting the triaxial strength of the material has greatly changed the numerical parameters, especially the parameter linked to the $J2$ stress invariant and its square root.
Results

The large number of results produced by the parametric investigation makes their complete presentation unwieldy. Certain aspects of the parametric investigation results will be presented in a general fashion, while others will be discussed more in-depth.

Assuming the Mohr-Coulomb criterion for the unit failure criterion moderately influenced the results. On average the predicted value for the Mohr-Coulomb curve was 93% that of the Rankine curve, the minimum value being 53% and the maximum being 100%, or equal to the original value.

Decreasing the biaxial strength of the mortar produces results identical to those of the standard case. Given the fact that the failure surface is three-dimensional, this observation is rational.

The change in the triaxial strength results in very noticeable differences in the predicted compressive strength. On average, for a lower triaxial strength the predicted compressive strength of masonry was 60% that of the standard value, with the minimum value being 17% and the maximum being 100%. The influence of the triaxial behavior of the material is therefore shown to be the decisive factor for compression perpendicular to the bed joint as far as the modeling of the behavior of the mortar is concerned.

The results of the model appear to be insensitive to the biaxial strength of the mortar and heavily dependent on its behavior under triaxial compressive strength. This observation raises questions concerning the adequacy of simple biaxial tests on mortars for the full characterization in their behavior in masonry, given its trivial role in the compressive strength of the composite. Triaxial tests appear to be far more relevant.

The comparison of the micro-mechanical model results to the EC6, Ohler and Hilsdorf models is presented in Figure 13 in the form of a cumulative frequency distribution graph, the number of individual points being too numerous to illustrate. The graph depicts the distribution of ratios between the numerical model result and the closed-form expressions. All three closed-form expressions may give results as much as ten times higher than the micro-mechanical result. In turn, the micro-mechanical model can produce values almost twice as much as the Ohler and Hilsdorf models and more than four times the value given by the EC6 equation. Overall, for the parameters and dimensions considered in this study, the micro-mechanical model of the stack bond prism produces results higher than the closed-form expressions more often than it produces lower.
The study of the influence of the material properties of the materials warrants a closer examination. While the number of parameters investigated is prohibitive of a complete overview of the results in the context of this work, the sensitivity of the result to a number of individual parameters may be easily studied. This sensitivity is investigated by altering a single parameter and keeping the remaining parameters fixed to the already mentioned standard values. Figure 14 graphically illustrates the influence of three material properties on the predicted compressive strength of masonry. According to the obtained results, an increase in the Poisson's ratio of mortar can dramatically increase the compressive strength of masonry composed of weak mortar while it may decrease it in the case of strong mortars. An increase of the Young's modulus of the units always results in an increase in the predicted masonry compressive strength. Finally, there appears to be little difference in the predicted compressive strength for units with a tensile higher than 10% their compressive strength, while a moderate decrease is observed for units with a tensile strength equal to 5% their tensile strength.

While understanding the influence of the tensile strength of the units on the compressive strength of masonry is straightforward, a few clarifying comments on the effect of the elastic properties of the materials on the compressive strength of masonry may be necessary, especially given that the closed-form expressions used for comparison do not take them into account. The influence of the Poisson's ratio of the mortar on the compressive strength of masonry is logical in the sense that a higher ratio results in an increased tendency of the mortar to laterally expand when subjected to vertical loads. The increased tendency results, in turn, to a higher lateral confinement by the unit. Excessive expansion, however, may lead to premature cracking of the brick and result in a decrease in the compressive strength of masonry, especially in the case of weak units. The effect of the Young's modulus of the units can be explained along the same lines: a higher value results in smaller lateral deformation of the units and, therefore, an increase in the lateral confinement on the mortar.

The apparent bias of the micro-mechanical model to produce results higher than the Ohler or the EC6 model for the majority of the cases may be explained in light of the influence of the Poisson's ratio of the mortar on the result. For the parametric investigation the values assigned to this parameter generally moved above the average noted for masonry mortars. While mortars may indeed be characterized by lower Poisson's ratios in the linear elastic range, much higher and rapidly increasing values are registered early in the nonlinear range or even
before the onset of any noticeable softening in the response of the mortar in compression [22,37]. Therefore, it was considered appropriate to investigate the effect of this material parameter above the usual range.

5. **Closed-Form Prediction of the Compressive Strength**

*Derivation of the Expression*

A new closed-form expression taking into account geometrical dimensions, elastic and inelastic properties of the constituent materials is proposed based on the micro-mechanical models examined and the material constitutive laws adopted for analysis. The simple stress distribution obtained from the stack bond pillar model is used for the analysis and a linear response of the materials is assumed until failure. The Mohr-Coulomb failure surface is adopted for the units and the Hsieh-Ting-Chen failure surface is adopted for the mortar. The closed-form expression is developed for masonry under compression perpendicular to the bed joint and does not include the influence of head or transversal joints.

The closed-form expression serves to create a model for the prediction of the compressive strength of masonry taking into account the interaction of the failure modes most common in masonry under compression but without requiring any computational resources. This expression, while based on simple algebraic equations, takes into account the geometrical and elastic properties of the masonry components and quantifies the unit-mortar interaction in masonry under compression. The range of activated failure modes in the components depends on this quantification.

Four types of failure are identified: failure of the unit in compression (UC), failure of the unit in combined compression/tension (UT), failure of the mortar in multiaxial compression (MC) and failure of the mortar in combined compression/tension (MT).

The mortar fails under multiaxial compression (mode MC) or combined tension-compression when the Hsieh-Ting-Chen criterion, as presented in equation (7), is satisfied.

According to the stack bond pillar model, the stresses in the components can be analyzed into a vertical stress, equal in the unit and the mortar, the horizontal stress in the mortar and the horizontal stress in the unit.
The transversal stresses are equal to the horizontal stresses in either component. The ratio of vertical to horizontal stress in the mortar may be expressed as

\[ s_{b1} = \frac{\sigma_{m,x}}{\sigma_y} = \frac{\sigma_{m,z}}{\sigma_y} = \frac{h_u (E_m v_u - E_m v_m)}{E_u h_u (v_m - 1) + E_m h_m (v_u - 1)} \]  

while in the units as

\[ s_{u1} = \frac{\sigma_{u,x}}{\sigma_y} = \frac{\sigma_{u,z}}{\sigma_y} = \frac{h_m (E_u v_m - E_m v_u)}{E_u h_u (v_m - 1) + E_m h_m (v_u - 1)} = -s_{b1} \frac{h_u}{h_m} \]

These two parameters depend on the elastic and geometric characteristics of the two materials. When \( s_{b1} \) is negative, meaning that simultaneously \( s_{u1} \) is positive, the mortar is under triaxial compression and the unit is under vertical compression and horizontal and transversal tension when the masonry is subjected to vertical compression. The higher the value is from 1 the more significant the mismatch of the elastic properties and, therefore, the confining compressive stress on the mortar. In the case of the existence of confining stresses in the mortar, substituting \( s_{b1} \) in the Hsieh-Ting-Chen equation a rational function of \( \sigma_y \) is obtained whose root is the vertical compression stress causing MC failure. Further dividing the function by the uniaxial compressive strength of mortar allows for the failure criterion to be expressed in a dimensionless manner.

For the standard values of the Hsieh-Ting-Chen criterion the root of the rational function normalized by division with the compressive strength of mortar is plotted in Figure 15. For \( s_{b1} \) equal to 0 the horizontal and transversal stresses are 0 and the compressive strength of the failure mode is equal to the compressive strength of mortar: the ordinate of the graph is equal to -1. The part of the function with positive abscissae may be approximated very satisfactorily through nonlinear regression analysis by the third order polynomial

\[ \frac{f_{c,MC}}{f_{cm}} = -162.004 \times s_{b1}^3 - 2.768 \times s_{b1}^2 - 10.035 \times s_{b1} - 1 \]

thus allowing for a closed-form expression of the failure stress of mortar under confinement, which here is a negative value, to be obtained. The part with negative abscissae, for which \( MT \) failure arises, can be similarly approximated by the polynomial
The numerical parameters of the polynomials depend on the numerical parameters of the Hsieh-Ting-Chen criterion and have been here calculated based on the standard values already given: C1 = 2.0108, C2 = 0.9714, C3 = 9.1412, C4 = 0.2312.

Failure mode UC is obtained when the vertical compressive stress equals the compressive strength of the unit and arises when the unit is in triaxial compression:

\[ f_{c,UC} = -f_{cu} \]  

(30)

Failure mode UT is obtained in the unit under combined tension and compression when

\[ -\frac{f_{c,UT}}{f_{cu}} + \frac{\sigma_{n,k}}{f_{mu}} = 1 \]  

(31)

Substituting \( s_{bl} \) in this equation and solving for the vertical failure stress one obtains

\[ f_{c,UT} = \frac{-1}{1 + \frac{s_{bl} h_{m}}{f_{cu} h_{u}}} \]  

(32)

Summing up the failure modes, it may be stated that the (negative) compressive strength of masonry is equal to

\[ f_c = \begin{cases} \max \left( f_{c,MT}, -f_{cu} \right) & \text{for} \: s_{bl} \leq 0 \\ \max \left( f_{c,MC}, f_{c,UT} \right) & \text{for} \: s_{bl} \geq 0 \end{cases} \]  

(33)

In summary, the model quantifies the mismatch of elastic properties between the units and the mortar through the calculation of \( s_{bl} \). The failure mode activated depends on whether this value is positive. For negative values, the unit is in triaxial confinement. It is assumed that the mortar in the joint is not affected by lateral tension. For \( s_{bl} = 0 \) (no lateral stresses arise in the mortar and the unit) the two branches of the piecewise equation converge and failure is governed by the lowest component uniaxial strength. For \( s_{bl} < 0 \) the masonry fails due to uniaxial failure of the units or multiaxial failure of the mortar. For \( s_{bl} > 0 \) the mortar is in triaxial

\[ \frac{f_{c,MT}}{f_{cm}} = -12.622 \times s_{bl}^3 - 14.077 \times s_{bl}^2 - 5.5407 \times s_{bl} - 1 \]  

(29)
compression and the unit in uniaxial compression and bilateral tension. In this case the masonry will fail either
due to crushing of the mortar or combined crushing and cracking of the unit.

**Derivation of a Semi-Empirical Expression**

The closed-form analytical expression may be converted to a simpler semi-empirical expression, in which
the only parameters considered are the compressive strength of the units and the mortar and the height of the
unit and mortar layers. In order to reach this semi-empirical expression the tensile strength of the units and the
elastic properties of the units and the mortar are given representative values. These values have been estimated
as the average values across a large number of case studies [36] and are presented in Table 6.

**Results**

The results obtained from the closed-form expression and the semi-empirical model are shown in absolute
values Table 4. No general tendency to under- or overestimate the compressive strength was noted throughout
the series of experimental results. The results of both the closed-form analytical and the semi-empirical model
are illustrated in Figure 16a, as compared to the experimental results. A comparison of the proposed closed-form
and semi-empirical expression to the predictions of the micro-mechanical model is presented in Figure 17, again
in the form of a cumulative frequency distribution graph.

Figure 16b presents a contour plot of the results obtained from the semi-empirical expression for a wide
range of combinations of units and mortars of varying strengths. The compressive strength iso-lines illustrate the
interaction of the failure modes in masonry under compression. The complexity of the contours does not allow
for a simple regression model to approximate well the obtained curve. Direct application of the equation (33) is
preferable.

The correlation between the closed-form and the semi-empirical model is good, as is their correlation with
the experimental results here investigated. The derived semi-empirical model, while slightly less accurate
compared to similar proposed models, such as that of Ohler [11], is more inclusive in terms of application scope
and is more straightforward in its application.
Neither of the two proposed models produces diverging values, regardless of the combination of material properties. The adequacy of these models may also be evaluated through a comparison with the micro-mechanical model results using the parametric result database as a benchmark. Comparing, again the results obtained using cumulative curves, it is shown that the closed-form expression is able to produce good results, with large over- and underestimations accounting for a small percentage of the results. The semi-empirical expression is less accurate in that regard, but it is comparable to the Ohler model.

6. **Conclusions**

A series of masonry micro-mechanical models based on the analysis of periodic unit cells has been developed and used for the prediction of the tensile and compressive strength of masonry. The cells are analyzed according to closed-form expressions for the determination of the stress distribution in the cell coupled with an iterative nonlinear solution method for the implementation of damage laws in the analysis.

The models are capable of making good predictions of the compressive strength of various types of masonry pillar and wall structures. Several of the typologies investigated have received very little attention in the existing literature. The evolution of damage levels for increasing applied loads of stress in different parts of the masonry has been studied and commented upon. Differences in the response of different masonry typologies under the same loading conditions are also presented. The formulation of the models using analytical expressions serves to critically reduce the computational effort required for analysis. Being computationally advantageous to, for example, finite element computations, the models of the cells are appropriate for two-scale analyses of large structures, thus bridging the gap between the accuracy of detailed micro-modeling techniques and the need to investigate the structural behavior of large masonry assemblages.

The proposed models serve as a useful tool for performing quick but accurate calculations for the derivation of the nonlinear properties of masonry structures. In addition, they have been here employed in a deep and wide parametric investigation, which, taking advantage of their low computational cost, provides insights on the sensitivity of the compressive strength of masonry to several material and geometrical parameters over a wide spectrum of values. The parametric study has revealed a dependence of the compressive strength of masonry to material parameters usually not taken into account in structural design and often ignored in the mechanical characterization of masonry composites and materials, such as the Poisson's ratio of the mortar. Furthermore,
the study of the three-dimensional pressure-dependent behavior of the mortar, as opposed to merely its biaxial strength, appears to be crucial for its use in computational modeling of masonry.

Two closed-form expressions for the prediction of the compressive strength of masonry are proposed, based on the aforementioned micro-mechanical models: a closed-form adaptation of a micro-mechanical model fully based on detailed micro-modeling and a semi-empirical adaptation relying on the compressive strength of the units and the mortar and their height in the composite. Both approaches produce adequate results.

Acknowledgments

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Notation

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**Stress and Strain**

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**Integrity Variables**

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Closed-Form Expressions

1. $s_{bl}$  Ratio of lateral to vertical stress in mortar

2. $s_{ul}$  Ratio of lateral to vertical stress in unit

3. $f_{c,UC}$  Unit compression mode strength

4. $f_{c,UCT}$  Unit tension mode strength

5. $f_{c,MC}$  Mortar compression mode strength

Material Subscripts

6. $u$  Unit

7. $m$  Mortar

8. $i$  Infill

9. $if$  Interface

10. $c$  Masonry composite

Cuboid Subscripts

11. $u$  Unit (general)

12. $d$  Header unit

13. $s$  Stretcher unit

14. $b$  Bed joint

15. $c$  Cross joint

16. $h$  Head joint

17. $t$  Transversal joint
References


Appendix

1. **Stack Bond Wall**

Due to the simple geometrical layout of the cell, which does not include complex geometrical interlocking of mortar joints and units, shear stress components are disregarded for applied normal stress.

Normal strain conformity is assumed as follows:

\[
\varepsilon_{zz,h} = \varepsilon_{zz,u} = \varepsilon_{zz,c} = \varepsilon_{zz,b}
\]

\[
\varepsilon_{xx,h} = \varepsilon_{xx,c}
\]

\[
\varepsilon_{xx,u} = \varepsilon_{xx,b} \quad (34)
\]

\[
\varepsilon_{yy,h} = \varepsilon_{yy,u}
\]

\[
\varepsilon_{yy,c} = \varepsilon_{yy,b}
\]
The above strain conformity relations assume equal out-of-plane strains for all cuboids and equal in-plane strains for parallel cuboids in a given direction.

Normal stress equilibriums are considered in the horizontal direction at the left and right face of the cell, in the vertical direction at the top and bottom faces of the cell and in the transversal direction at the front face.

2. **Running Bond Wall**

The geometrical interlocking of the units and the mortar requires that the in plane shear stresses be taken into account for normal stress loading conditions. Therefore, apart from the three components of normal stress for each cuboid component, the $xy$ shear stress and strains also have been included for the formulation of the system of equations. In turn, the compatibility conditions take into account the shear deformation of the cuboids, primarily those of the bed and cross joints, for applied loads in the horizontal direction. The resulting shear stresses are negligible for loading in the vertical and transversal directions.

Normal stress equilibriums are considered in the horizontal direction at the external faces and at a cross section across the middle of the cell and in the vertical and transversal directions at the external faces of the cell. Conversely, shear stress equilibriums are taken at the top and left external faces and at vertical cross section across the middle of the cell.

The assumptions made concerning the normal strains of the cuboids are as follows:

\[
\varepsilon_{zz,h} = \varepsilon_{zz,a1} = \varepsilon_{zz,a2} = \varepsilon_{zz,c} = \varepsilon_{zz,b} \\
\varepsilon_{yy,h} = \varepsilon_{yy,a1} \\
\varepsilon_{yy,c} = \varepsilon_{yy,b} \\
\varepsilon_{xx,h} = \varepsilon_{xx,c} \\
\varepsilon_{xx,a1} = \varepsilon_{xx,b}
\]

The following assumptions are made about the normal and shear stress distribution in the cell:
3. **Flemish Bond Wall**

As in the running bond cell model, the in-plane shear deformation of the bed joint is taken into account for normal stress loading. The transfer of stress across the transversal joint is also taken into account without, however, considering the effects of shear stress in other planes. Also, as in the case of the running bond cell, shear stresses are negligible for vertical and transversal applied normal stress.

Normal stress equilibriums are considered in the horizontal direction at the external faces and at a cross section traversing the s1 and b2 cuboids, in the vertical direction at the top face and at a cross section across the middle of the cell and in the transversal direction at the front and back faces of the cell.

Shear stress equilibriums are considered at the top face, the left face, at a horizontal cross section across the middle of the cell and at a vertical cross section traversing the h1, c1, s2, h2, c3 and t2 cuboids.

The following assumptions are made concerning the normal strains in the cuboids:

\[
\varepsilon_{x,b1} = \varepsilon_{x,c2} = \varepsilon_{x,d1}
\]

\[\varepsilon_{x,c3} = \varepsilon_{x,t3}\]

\[\varepsilon_{x,c1} = \varepsilon_{x,c3} = \varepsilon_{x,h1} = \varepsilon_{x,h2}\]

\[\varepsilon_{x,t2} = \varepsilon_{x,t2}\]

\[\varepsilon_{x,t1} = \varepsilon_{x,t1} = \varepsilon_{x,b2} = \varepsilon_{x,c4}\]

\[\varepsilon_{y,d1} = \varepsilon_{y,h1} = \varepsilon_{y,s1} = \varepsilon_{y,s2} = \varepsilon_{y,s3} = \varepsilon_{y,d2} = \varepsilon_{y,h2} = \varepsilon_{y,t1} = \varepsilon_{y,t2} = \varepsilon_{y,t3}\]

\[\varepsilon_{y,b1} = \varepsilon_{y,c1} = \varepsilon_{y,h2} = \varepsilon_{y,c2} = \varepsilon_{y,c3} = \varepsilon_{y,c4}\]
$\varepsilon_{zz,1} = \varepsilon_{zz,2} = \varepsilon_{zz,3}$

$\varepsilon_{zz,d1} = \varepsilon_{zz,h1} = \varepsilon_{zz,d2} = \varepsilon_{zz,h2}$

$\varepsilon_{xy,r1} = \varepsilon_{xy,r1}$

$\varepsilon_{xy,r2} = \varepsilon_{xy,r2}$

$\varepsilon_{xy,r3} = \varepsilon_{xy,r3}$

$\varepsilon_{xy,b1} = \varepsilon_{xy,c2}$

$\varepsilon_{xy,c1} = \varepsilon_{xy,c3}$

The following assumptions are made about the normal stress distribution in the cell:

$\sigma_{zz,c1} = \sigma_{zz,c3}$

$\sigma_{zz,b2} = \sigma_{zz,c4}$

$\sigma_{zz,d1} = \sigma_{zz,d2}$

(38)

$\sigma_{zz,r1} = \sigma_{zz,r1}$

$\sigma_{zz,s2} = \sigma_{zz,s2}$

$\sigma_{zz,s3} = \sigma_{zz,s3}$

The above set of equations assumes that the transversal normal stress is equal for the cuboids of the two leaves of masonry, be they unit or mortar. For the shear stresses it is assumed that

$\sigma_{xy,s1} = \sigma_{xy,s3} = \sigma_{xy,b1}$

(39)
\[ \sigma_{xy,d1} = \sigma_{xy,b1} \]

4. Three Leaf Wall

The three leaf-model retains the set of assumptions made for the running bond model, which still apply to the outer masonry leaf. A number of further assumptions are made for the infill and the stress equilibrium conditions are modified to accommodate the new geometrical entities. Finally, some adjustments need to be made for the definition of the cell total strain.

An additional normal stress equilibrium condition is considered in the transversal direction at the back face of the cell. Constant normal stress is assumed in the infill, so that

\[ \sigma_{xx,i1} = \sigma_{xx,i2} = \sigma_{xx,i3} = \sigma_{xx,i4} = \sigma_{xx,i5} \]

\[ \sigma_{yy,i1} = \sigma_{yy,i2} = \sigma_{yy,i3} = \sigma_{yy,i4} = \sigma_{yy,i5} \]  \hspace{1cm} (40)

\[ \sigma_{zz,i1} = \sigma_{zz,i2} = \sigma_{zz,i3} = \sigma_{zz,i4} = \sigma_{zz,i5} \]

Concerning the shear strains in the cuboids the following assumptions are made:

\[ \varepsilon_{xy,i1} = \varepsilon_{xy,i2} = \varepsilon_{xy,i3} = \varepsilon_{xy,a1} \]  \hspace{1cm} (41)

\[ \varepsilon_{xy,i4} = \varepsilon_{xy,i5} \]

5. Stack Bond Pillar

For the stack bond pillar model the system of equations developed by Haller for vertical normal stress is used [8].

According to Haller, the horizontal and transversal deformation equality of the two components reads:

\[ \varepsilon_{xx,u} = \varepsilon_{xx,b} \text{ and } \varepsilon_{zz,u} = \varepsilon_{zz,b} \]  \hspace{1cm} (42)

The horizontal and transversal stress equilibrium reads:
\[ \sigma_{xx}(h_u + h_a) = \sigma_{xx,a} h_u + \sigma_{xx,b} h_a \quad \text{and} \quad \sigma_{zz}(h_u + h_a) = \sigma_{zz,a} h_u + \sigma_{zz,b} h_a \]  \quad (43)

The vertical stress equilibrium demands that both components develop vertical stress equal to the external load according to

\[ \sigma_{yy,a} = \sigma_{yy,b} = \sigma_{yy} \]  \quad (44)

6. **English Bond Pillar**

Shear stresses in the components are disregarded for applied normal stress. Normal strain conformity in the cell is achieved by assuming

\[ \epsilon_{xx,a1} = \epsilon_{xx,b1} = \epsilon_{xx,a3} = \epsilon_{xx,b2} = \epsilon_{xx,a4} \quad \text{and} \quad \epsilon_{xx,a2} = \epsilon_{xx,b1} = \epsilon_{xx,c1} = \epsilon_{xx,b4} = \epsilon_{xx,c3} = \epsilon_{xx,c5} \]

\[ \epsilon_{yy,a3} = \epsilon_{yy,b1} = \epsilon_{yy,a4} = \epsilon_{yy,b5} = \epsilon_{yy,c1} = \epsilon_{yy,c2} = \epsilon_{yy,c3} \quad \text{and} \quad \epsilon_{yy,a2} = \epsilon_{yy,c4} = \epsilon_{yy,a1} = \epsilon_{yy,b2} \]  \quad (45)

\[ \epsilon_{zz,a2} = \epsilon_{zz,c1} = \epsilon_{zz,b1} = \epsilon_{zz,a1} = \epsilon_{zz,b1} = \epsilon_{zz,a3} \quad \text{and} \quad \epsilon_{zz,c4} = \epsilon_{zz,c3} = \epsilon_{zz,b5} = \epsilon_{zz,b2} = \epsilon_{zz,c2} = \epsilon_{zz,a4} \]

Normal stress equilibrium in the horizontal, vertical and transversal directions are taken at all the external faces of the cell.

**Figure Captions**

Figure 1 Derivation of wall cells: full wall, repeating pattern and cell derived from geometrical symmetry of repeating pattern. (a) Stack bond wall, (b) running bond wall, (c) Flemish bond wall and (d) three-leaf wall with running bond outer leaf.

Figure 2 Derivation of pillar cells: full wall, repeating pattern and cell derived from geometrical symmetry of repeating pattern. (a) stack bond pillar and, (b) English bond pillar.

Figure 3 Discretization and component designation of single leaf wall periodic unit cells: (a) stack bond and (b) running bond.

Figure 4 Discretization and component designation of multi leaf wall periodic unit cells: (a) Flemish bond and (b) three-leaf wall with running bond outer leaf.

Figure 5 Discretization and component designation of pillar periodic unit cells: (a) stack bond and (b) English bond.
Figure 6  Failure curves used for (a) mortar for various levels of out-of-plane stress, (b) units and (c) interfaces.

Figure 7  Hardening-softening curves for (a) compression: basic and scaled curve for an applied confining stress, (b) tension and (c) shear under varying levels of applied normal stress.

Figure 8  Stress strain curves: (a) horizontal tension and (b) vertical compression.

Figure 9  Damage progression curves for Flemish bond wall model under vertical compression. (a) Compressive damage: b1 stands for a bed joint, c1 for a cross joint, h1 for a head joint and t1 for a transversal joint cuboid. (b) Tensile damage: s2 stands for a stretcher unit, and t1 and t2 for two separate transversal joint cuboids.

Figure 10  Progression of stresses in components of three leaf wall model under vertical compression: (a) horizontal stress and (b) vertical stress.

Figure 11  Comparison of vertical compressive strength according to (a) FEM and micro-mechanical results and (b) experimental and micro-mechanical results. Dashed lines indicate 10% difference.

Figure 12  In-plane failure envelope for masonry under biaxial stress: comparison of periodic unit cell numerical results with biaxial experimental envelope.

Figure 13  Cumulative frequency distribution of micro-mechanical to analytical model result ratio. Comparison of micro-mechanical results with design code and existing closed form expressions.

Figure 14  Influence of material properties on the compressive strength of masonry: (a) Poisson’s ratio of mortar for $f_{tu} = 5\% f_{cu}$ (5) Poisson’s ratio of mortar for $f_{tu} = 10\% f_{cu}$ (c) Young’s modulus of units and (d) tensile strength of units.

Figure 15  Relation between $s_{b1}$ parameter and normalized strength for failure mode MC.

Figure 16  (a) Comparison of closed-form analytical and semi-empirical models for the prediction of the compressive strength of masonry with experimental results. (b) Contour plot of semi-empirical expression results as a function of the compressive strength of the units and the mortar.

Figure 17  Cumulative frequency distribution of micro-mechanical to analytical model result ratio. Comparison of micro-mechanical results with proposed closed-form expression and semi-empirical model results.
Table Captions

Table 1  Material properties and component dimensions for preliminary numerical analysis.

Table 2  Case studies of masonry under vertical compression: material and geometrical properties and comparison of experimental to micro-mechanical model results. Assumed values in curly brackets.

Table 3  Determination of the $s$, $t$ and $m$ numerical parameters for Ohler's model.

Table 4  Case studies of masonry under vertical compression: comparison between experimental, finite element, empirical and micro-mechanical model results.

Table 5  Material properties and dimensions for nonlinear elastic parametric analysis. "Standard" values in bold.

Table 6  General material parameters adopted for the semi-empirical model for the prediction of the compressive strength of masonry.
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Figure 3

(a)

(b)
Figure 4

(a)

(b)
Figure 5

(a)

(b)
\[ \sigma_z = 0 \]
\[ \sigma_z = -\frac{f_t}{10} \]
\[ \sigma_z = \frac{f_t}{10} \]

Figure 6

(a) (b) (c)
Figure 7

(a) Basic
Scaled

(b) Compressive Stress
No Stress
Tensile Stress

(c) $G_f/h$

$\varepsilon_u$, $\varepsilon_c$, $\varepsilon_{e/3}$, $f_t$, $f_c/3$, $\varepsilon_{el,\text{max}}$, $\varepsilon_c$, $\varepsilon_{c/3}$
Figure 8

(a) Stress [N/mm²] vs. Strain [-] for different wall types:
- Stack Wall
- Running Wall
- Flemish Wall
- Three Leaf Wall

(b) Stress [N/mm²] vs. Strain [-] for different wall types:
- Stack Prism
- Stack Wall
- Running Wall
- Flemish Wall
- Three Leaf Wall
Figure 9

(a) (b)
Figure 11

(a) Comparison of micro-mechanical and FEM calculated strengths ($f_c$) with experimental strengths ($f_{c,\text{exp}}$) for different samples (S1 to S16).

(b) Comparison of micro-mechanical and experimental strengths ($f_c$) with FEM calculated strengths ($f_{c,\text{FEM}}$) for different samples (S1 to S17, R1, F1 to F3).
Figure 12

Experimental Envelope

Micro-Mechanical
Figure 13

Cumulative Percentage of All Cases

f_c Micro Mechanical/f_c Analytical
Figure 14

(a) (b)

(c) (d)
Figure 15
Figure 16

(a) Comparison of model predictions with experimental data for concrete strength. The dots represent model predictions, while the triangles represent semi-empirical data. The diagonal line indicates perfect agreement.

(b) Contour plot showing the relationship between concrete compressive strength ($f_{cm}$) and cube strength ($f_{cu}$). The color gradient represents the range of concrete strengths, with darker shades indicating higher strength.
Figure 17

Cumulative Percentage of All Cases

Closed Form

Semi Empirical

\( f_c \) Micro Mechanical/\( f_c \) Analytical