

EXPONENTIAL DECAY IN NONSIMPLE THERMOELASTICITY OF TYPE III

ANTONIO MAGAÑA, RAMÓN QUINTANILLA

Dept. Matemática Aplicada 2, UPC
C. Colón 11, 08222 Terrassa, Barcelona, Spain.

Abstract: This paper deals with the model proposed for nonsimple materials with heat conduction of type III. We analyze first the general system of equations, determine the behavior of its solutions with respect to the time and show that the semigroup associated with the system is not analytic. Two limiting cases of the model are studied later.

Keywords: thermoelasticity of type III, exponential decay, semigroup of contractions, analyticity, impossibility of localization.

1. INTRODUCTION AND BASIC EQUATIONS

The response of materials to stimuli relevantly depends on their internal structure. However, the classical theory of elasticity does not consider the inner structure. To overcome this drawback, new mathematical models have been proposed. In particular, there are models to deal with *porous elastic media*, *micropolar elastic solids*, *materials with microstructure* or *nonsimple elastic solids*. Here we consider the last ones. From a mathematical point of view, the models for these materials are characterized by the inclusion of higher order derivatives of the displacement term: the configuration of the points is classified more finely by the values of the higher order gradients at the material points. They were introduced by Green and Rivlin [6], Mindlin [15] or Toupin [22, 23]. Details about these models can also be found in the book of Ciarletta and Ieşan [3] or in the book of Ieşan [7]. Higher order derivatives are introduced because they allow to better clarify the possible configurations of the materials.

One of important questions to be answered for any model is the decay rate of the solutions of the proposed system of equations when certain dissipation mechanisms are taken into account. Without trying to be exhaustive, let us refer to some studies of this kind carried out for porous materials [8, 12, 13, 16, 17, 21], for mixtures of solids [1] or for micropolar materials [14]. Through the paper, to simplify, we speak about *slow decay* or *exponential decay* of the solutions. We say that the decay of the solutions is exponential if they are exponentially stable and, if they are not, we say that the decay of the solutions is slow. The main difference between these two concepts in a thermomechanical context lies in the fact that, if the decay is exponential, then the thermomechanical displacements are very small after a short period of time and can be neglected. However, if the decay is slow, then the solutions weaken in a way that thermomechanical displacements could be appreciated in the system after some time.

It has been shown, for instance, that heat conduction of Fourier's type leads to exponential decay [4], but if hyperbolic heat conduction is considered, then a slow decay is obtained for nonsimple materials. The time decay of solutions for nonsimple elastic materials with memory has been also

studied [18]: if hyperdissipation is considered, then the solutions decay exponentially; otherwise, the decay of the solutions is slow. We think that it is important to know the behavior of the solutions in each particular model.

Green and Nagdhi proposed three thermoelastic theories that they named type I, II and III [5]. The first one coincides with the classical theory in the linear case. The second one is known as thermoelasticity without energy dissipation because the energy is conserved. The third one is the most general, because it contains the former two as a particular case.

The exponential decay of the solutions for the type III thermoelastic solids has been already proved [20]. If the thermal conductivity parameter is localized, then polynomial time decay is obtained [9]. The analyticity of the solutions for the type III thermoelastic plates has been also studied [10].

In this work we consider nonsimple materials with heat conduction of type III, and we prove the exponential decay of the solutions (in contrast to the hyperbolic case). We believe that this kind of results improves the general knowledge about the solutions in thermoelastic problems. In particular, we clarify the relevance of the coupling term in the qualitative behavior of the solutions. In this paper we see how the four order derivative of the displacement term can be controlled by the type III heat conduction by means of suitable coupling terms. However, in contrast to the plate problem, the analyticity of the solutions cannot be obtained.

The system of equations that we want to study is related with the one analyzed for the thermoelasticity of type II (see [19]) but with a new term that turns the system into a dissipative one. To be precise, the system of equations for the one-dimensional case is the following:

$$(1.1) \quad \begin{cases} \rho \ddot{u} = \mu u_{xx} - \gamma u_{xxxx} + \beta \dot{\alpha}_x - \delta \alpha_{xxx}, \\ a \ddot{\alpha} = k \alpha_{xx} + m \dot{\alpha}_{xx} + \beta \dot{u}_x + \delta u_{xxx}. \end{cases}$$

The difference lies in the term $\dot{\alpha}_{xx}$, and therefore, we will assume that $m > 0$.

Here u is the displacement and α stands for the thermal displacement. In this model, the temperature is given by $\dot{\alpha}$. As to the constants, ρ is the mass density, a is the heat capacity, μ and γ are related with the elasticity and hyperelasticity tensors, respectively, β is related with the thermal expansion and m is the thermal conductivity. k and δ are two constitutive constants of the type II and type III theories that have no names in the literature.

For the problem to be properly posed, we impose boundary and initial conditions. Thus, we assume that the solutions satisfy the following boundary conditions

$$(1.2) \quad u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = \alpha_x(0, t) = \alpha_x(\pi, t) = 0,$$

and the following initial conditions

$$(1.3) \quad u(x, 0) = u_0(x), \dot{u}(x, 0) = v_0(x), \alpha(x, 0) = \alpha_0(x), \dot{\alpha}(x, 0) = \theta_0(x).$$

In this paper we use the usual assumptions for the coefficients of system (1.1): $\rho > 0$, $\mu > 0$, $\gamma > 0$, $a > 0$ and $\gamma k > \delta^2$. As we have pointed out, we also assume $m > 0$.

The structure of the paper is as follows. In this first section, we introduce the problem and the system of equations we are going to analyze, with its boundary and initial conditions. In Section 2, we study the decay rate of the solutions and obtain exponential stability. In Section 3, we show that the underlying operator semigroup is not analytic. In Section 4 and 5, we consider only one constitutive constant of coupling. In Section 4, we analyze the case $\beta \neq 0$ and $\delta = 0$, we

obtain again exponential stability and prove the impossibility of localization of the solutions. In Section 5, we analyze the case $\beta = 0$ and $\delta \neq 0$, and note that the solutions decay exponentially and are not given by an analytic semigroup. Finally, we summarize the results in Section 6.

2. REGULAR CASE: EXPONENTIAL DECAY

We will prove that the solutions of (1.1) decay exponentially, and for this purpose, we will use the contraction semigroup arguments. In addition to the previous assumptions about the system coefficients, in this case we suppose $\beta \neq 0$ and $\delta \neq 0$.

If we denote $v = \dot{u}$, $\theta = \dot{\alpha}$ and $D = \frac{\partial}{\partial x}$, we can write system (1.1) in the following way:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \frac{1}{\rho} (\mu D^2 u - \gamma D^4 u + \beta D \theta - \delta D^3 \alpha), \\ \dot{\alpha} = \theta, \\ \dot{\theta} = \frac{1}{a} (k D^2 \alpha + m D^2 \theta + \beta D v + \delta D^3 u). \end{cases}$$

Next, we define $L_*^2 = \{f \in L^2 : \int_0^\pi f(x) dx = 0\}$, and let $H_*^1 = H^1 \cap L_*^2$.

To formalize the problem, we assume the evolution to be taking place on the Hilbert space

$$\mathcal{H} = \left\{ U = (u, v, \alpha, \theta) \in (H^2 \cap H_0^1) \times L^2 \times H_*^1 \times L_*^2, \int_0^\pi \alpha(x) dx = \int_0^\pi \theta(x) dx = 0 \right\}$$

driven by the operator

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu D^2 - \gamma D^4}{\rho} & 0 & -\frac{\delta D^3}{\rho} & \frac{\beta D}{\rho} \\ 0 & 0 & 0 & I \\ \frac{\delta D^3}{a} & \frac{\beta D}{a} & \frac{k D^2}{a} & \frac{m D^2}{a} \end{pmatrix},$$

where I denotes the identity operator.

With the notation above, our initial-boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U_0 = (u_0, v_0, \alpha_0, \theta_0).$$

It can be proved that the mild solutions of system (1.1) are given by the semigroup of contractions generated by the operator \mathcal{A} .

We define an inner product in \mathcal{H} . If $U^* = (u^*, v^*, \alpha^*, \theta^*)$, then

$$(2.1) \quad \langle U, U^* \rangle_{\mathcal{H}} = \frac{1}{2} \int_0^\pi (\rho v \bar{v}^* + a \theta \bar{\theta}^* + \mu u_x \bar{u}_x^* + \gamma u_{xx} \bar{u}_{xx}^* + k \alpha_x \bar{\alpha}_x^* + \delta u_{xx} \bar{\alpha}_x^* + \delta \bar{u}_{xx}^* \alpha_x) dx.$$

Here, a superposed bar denotes the complex conjugation. It should be pointed out that this product is equivalent to the usual product in the Hilbert space \mathcal{H} .

The domain of \mathcal{A} is

$$\mathcal{D}(\mathcal{A}) = \{U \in \mathcal{H} : u \in H^4, v \in H_0^1 \cap H^2, D\alpha \in H_0^1 \cap H^2, D\theta \in H_0^1, u_{xx}(0, t) = u_{xx}(\pi, t) = 0\}.$$

Lemma 2.1. *The operator \mathcal{A} defined previously is the infinitesimal generator of a C_0 -semigroup of contractions on \mathcal{H} .*

Proof: First of all, we notice that $\mathcal{D}(\mathcal{A})$ contains a subset that is dense in \mathcal{H} and then $\mathcal{D}(\mathcal{A})$ is also dense in \mathcal{H} (this result comes from the density theorem, see [11], page 9, Theorem 1.4.1). We will show that \mathcal{A} is a dissipative operator and that 0 is in the resolvent set of \mathcal{A} . Using the Lumer-Phillips theorem (see [11], page 3, Theorem 1.2.3), the conclusion will follow.

On the one hand, a direct calculation gives

$$(2.2) \quad \Re \langle \mathcal{A}U, U \rangle = -\frac{1}{2} \int_0^\pi m |D\theta|^2 dx \leq 0,$$

and, therefore, the operator \mathcal{A} is dissipative.

On the other hand, for any $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we will find a unique $U \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{A}U = \mathcal{F}$, or equivalently:

$$(2.3) \quad \left. \begin{aligned} v &= f_1 \\ \frac{1}{\rho} (\mu D^2 u - \gamma D^4 u + \beta D\theta - \delta D^3 \alpha) &= f_2 \\ \theta &= f_3 \\ \frac{1}{a} (k D^2 \alpha + m D^2 \theta + \beta Dv + \delta D^3 u) &= f_4 \end{aligned} \right\}$$

The second and fourth equations can be written in terms of f_1 and f_3 as follows:

$$(2.4) \quad \left. \begin{aligned} \mu D^2 u - \gamma D^4 u - \delta D^3 \alpha &= \rho f_2 - \beta D f_3 \\ k D^2 \alpha + \delta D^3 u &= a f_4 - m D^2 f_3 - \beta D f_1 \end{aligned} \right\}$$

To prove the solvability of this system we develop f_1 , f_2 , f_3 and f_4 in Fourier series. The families of $\sin(nx)$ and $\cos(mx)$ are an orthonormal complete system in the Hilbert space L^2 . In particular, we develop f_1 and f_2 in series of sines and f_3 and f_4 in series of cosines. So, we take $f_1 = \sum_{n=1}^\infty a_n \sin(nx)$, $f_2 = \sum_{n=1}^\infty b_n \sin(nx)$, $f_3 = \sum_{n=1}^\infty c_n \cos(nx)$ and $f_4 = \sum_{n=1}^\infty d_n \cos(nx)$. In view of the fact that the averages of f_3 and f_4 are zero, the developments in the cosines series start from $n = 1$.

We will show that it is possible to find $u = \sum_{n=1}^\infty u_n \sin(nx)$ and $\alpha = \sum_{n=1}^\infty \alpha_n \cos(nx)$ such that $\sum_{n=1}^\infty n^4 |u_n|^2 < \infty$ and $\sum_{n=1}^\infty n^2 |\alpha_n|^2 < \infty$. From the assumptions we know that $\sum_{n=1}^\infty n^4 |a_n|^2 < \infty$, $\sum_{n=1}^\infty |b_n|^2 < \infty$, $\sum_{n=1}^\infty n^2 |c_n|^2 < \infty$ and $\sum_{n=1}^\infty |d_n|^2 < \infty$.

Substituting the above expressions in system (2.4) and performing simplifications, we get a linear system for the unknown coefficients u_n and α_n for each n , with the unique solution given by

$$\begin{aligned} u_n &= \frac{-a_n n^2 \beta \delta - b_n k \rho + c_n (n^3 \delta m - n \beta k) + d_n n a \delta}{n^4 (\gamma k - \delta^2) + k \mu n^2}, \\ \alpha_n &= \frac{a_n n (\mu + \sigma n^2) \beta + b_n n \delta \rho - c_n n^2 (m (\mu + \sigma n^2) - \beta \delta) - d_n (\mu + \sigma n^2) a}{n^4 (\gamma k - \delta^2) + k \mu n^2}. \end{aligned}$$

Thus, it is clear that u_n and α_n satisfy the desired conditions. It is not difficult to see that $\|U\|_{\mathcal{H}} \leq C \|\mathcal{F}\|_{\mathcal{H}}$.

Therefore, 0 is in the resolvent set of \mathcal{A} . \square

Theorem 2.2. *The problem given by system (1.1) with boundary conditions (1.2) and initial conditions (1.3) in \mathcal{H} has a unique mild solution.*

Proof: The proof is a direct consequence of the previous Lemma. \square

To show the exponential stability, we use a result due to Gearhart, stated in the book of Liu and Zheng (see [11], page 4, Theorem 1.3.2).

Theorem 2.3. *A semigroup of contractions $\{e^{t\mathcal{A}}\}_{t \geq 0}$ on a Hilbert space \mathcal{H} with norm $\|\cdot\|_{\mathcal{H}}$ is exponentially stable if and only if*

$$(2.5) \quad \{i\lambda, \lambda \text{ is real}\} \text{ is contained in the resolvent of } \mathcal{A},$$

and

$$(2.6) \quad \overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\| < \infty,$$

where \mathcal{I} denotes the identity operator.

We prove the validness of these conditions in the following two lemmas.

Lemma 2.4. *Let \mathcal{A} be the operator from Lemma 2.1. Then condition (2.5) is satisfied.*

Proof: We split the proof into three steps.

- (i) Since 0 is in the resolvent set of \mathcal{A} , by the contraction mapping theorem, for any real λ such that $|\lambda| < \|\mathcal{A}^{-1}\|^{-1}$, the operator $i\lambda\mathcal{I} - \mathcal{A} = \mathcal{A}(i\lambda\mathcal{A}^{-1} - \mathcal{I})$ is invertible. Moreover, $\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}^{-1}\|^{-1}, \|\mathcal{A}^{-1}\|^{-1})$.
- (ii) If $\sup\{\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|, |\lambda| < \|\mathcal{A}^{-1}\|^{-1}\} = M < \infty$, then, using the contraction theorem again, the operator

$$i\lambda\mathcal{I} - \mathcal{A} = (i\lambda_0\mathcal{I} - \mathcal{A})(\mathcal{I} + i(\lambda - \lambda_0)(i\lambda_0\mathcal{I} - \mathcal{A})^{-1}),$$

is invertible for $|\lambda - \lambda_0| < M^{-1}$. Hence, choosing λ_0 close enough to $\|\mathcal{A}^{-1}\|^{-1}$, the set $\{\lambda, |\lambda| < \|\mathcal{A}^{-1}\|^{-1} + M^{-1}\}$ is contained in the resolvent set of \mathcal{A} and $\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $(-\|\mathcal{A}^{-1}\|^{-1} - M^{-1}, \|\mathcal{A}^{-1}\|^{-1} + M^{-1})$.

- (iii) Suppose that the assertion of this lemma is not true. Then, there exists a real number $\sigma \neq 0$ with $\|\mathcal{A}^{-1}\|^{-1} \leq |\sigma| < \infty$ satisfying that the set $\{i\lambda, |\lambda| < |\sigma|\}$ is in the resolvent set of \mathcal{A} and $\sup\{\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|, |\lambda| < |\sigma|\} = \infty$. In this case, we can find a sequence of real numbers, λ_n , with $\lambda_n \rightarrow \sigma$, $|\lambda_n| < |\sigma|$, and a sequence of unit \mathcal{H} -norm vectors in the domain of \mathcal{A} , $U_n = (u_n, v_n, \alpha_n, \theta_n)$, such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \rightarrow 0.$$

Explicitly, this yields

$$(2.7) \quad i\lambda_n u_n - v_n \rightarrow 0 \text{ in } H^2 \text{ as } n \rightarrow \infty,$$

$$(2.8) \quad i\lambda_n v_n - \frac{1}{\rho} (\mu D^2 u_n - \gamma D^4 u_n + \beta D \theta_n - \delta D^3 \alpha_n) \rightarrow 0 \text{ in } L^2 \text{ as } n \rightarrow \infty,$$

$$(2.9) \quad i\lambda_n \alpha_n - \theta_n \rightarrow 0 \text{ in } H^1 \text{ as } n \rightarrow \infty,$$

$$(2.10) \quad i\lambda_n \theta_n - \frac{1}{a} (k D^2 \alpha_n + m D^2 \theta_n + \beta D v_n + \delta D^3 u_n) \rightarrow 0 \text{ in } L^2 \text{ as } n \rightarrow \infty,$$

Taking the inner product between $(i\lambda_n\mathcal{I} - \mathcal{A})U_n$ and U_n in \mathcal{H} , using (2.2) and taking its real part, we obtain $D\theta_n \rightarrow 0$ and then $\theta_n \rightarrow 0$. Thus, from (2.9) $\alpha_n \rightarrow 0$ and $D\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

Multiplying (2.10) by Du_n , which is bounded, we obtain

$$\beta \langle Dv_n, Du_n \rangle + \delta \langle D^3 u_n, Du_n \rangle \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Integrating by parts and using (2.7), we get

$$-i\lambda_n \beta \langle Du_n, Du_n \rangle - \delta |D^2 u_n|^2 \rightarrow 0 \text{ as } n \rightarrow \infty.$$

This proves that $D^2u_n \rightarrow 0$ and $Du_n \rightarrow 0$ (therefore, $u_n \rightarrow 0$ and $v_n \rightarrow 0$, too).

This argument shows that U_n cannot be of unit \mathcal{H} -norm, which finishes the proof of this lemma. \square

Lemma 2.5. *Let \mathcal{A} be the operator defined above. Then condition (2.6) holds true.*

Proof: Suppose that the claim of the lemma is not true. Then, there is a sequence $(\lambda_n)_n$ with $|\lambda_n| \rightarrow \infty$ and a sequence of unit \mathcal{H} -norm vectors in the domain of \mathcal{A} , $U_n = (u_n, v_n, \alpha_n, \theta_n)$, such that conditions (2.7)–(2.10) hold. Again $D\theta_n \rightarrow 0$ and thus $\theta_n \rightarrow 0$ as $n \rightarrow \infty$ by the virtue of the second Poincaré's inequality. Hence, again from (2.9), $\alpha_n \rightarrow 0$ and $D\alpha_n \rightarrow 0$.

From (2.7), it is obvious that $(\lambda_n u_n)_n$ is bounded (because so is $(v_n)_n$). If we multiply (2.7) by $D\theta_n$, we get that $\langle \lambda_n u_n, D\theta_n \rangle \rightarrow 0$.

Now, we multiply (2.10) by Du_n , which is bounded, and obtain:

$$\langle i\lambda_n \theta_n, Du_n \rangle - \frac{k}{a} \langle D^2 \alpha_n, Du_n \rangle - \frac{m}{a} \langle D^2 \theta_n, Du_n \rangle - \frac{\beta}{a} \langle Dv_n, Du_n \rangle - \frac{\delta}{a} \langle D^3 u_n, Du_n \rangle \rightarrow 0.$$

Integrating by parts, we get that the first, the second and the third terms of the above expression tend to zero. And we also obtain

$$-\beta \langle Dv_n, Du_n \rangle + \delta \langle D^2 u_n, D^2 u_n \rangle \rightarrow 0,$$

which is equivalent to

$$-\beta i \langle \lambda_n Du_n, Du_n \rangle + \delta |D^2 u_n|^2 \rightarrow 0.$$

Therefore, $D^2 u_n \rightarrow 0$ and $Du_n \rightarrow 0$.

It remains to show that $v_n \rightarrow 0$. To this end, we multiply (2.8) by v_n , which is bounded, and remove the terms that we already know to tend to zero:

$$\langle i\lambda_n v_n, v_n \rangle + \frac{\gamma}{\rho} \langle D^4 u_n, v_n \rangle + \frac{\delta}{\rho} \langle D^3 \alpha_n, v_n \rangle \rightarrow 0.$$

We divide by λ_n and substitute v_n with $i\lambda_n u_n$ in the second and third terms of the above expression:

$$i \langle v_n, v_n \rangle + \frac{\gamma}{\rho} \langle D^4 u_n, iu_n \rangle + \frac{\delta}{\rho} \langle D^3 \alpha_n, iu_n \rangle \rightarrow 0.$$

Finally, integrating by parts we obtain

$$i |v_n|^2 + \frac{\gamma}{\rho} \langle D^2 u_n, iD^2 u_n \rangle + \frac{\delta}{\rho} \langle D\alpha_n, iD^2 u_n \rangle \rightarrow 0,$$

which proves that $v_n \rightarrow 0$. Hence U_n cannot be of unit \mathcal{H} -norm. \square

Theorem 2.6. *Let (u, α) be a mild solution of the problem determined by (1.1), with boundary conditions (1.2) and initial conditions (1.3) in \mathcal{H} . Then, (u, α) decays exponentially to zero when the time tends to infinity.*

Proof: The proof is a direct consequence of Lemmas 2.4 and 2.5. \square

3. REGULAR CASE: LACK OF ANALYTICITY

The aim of this section is to show that the semigroup associated with system (1.1) is not analytic in general. To this end, we use the following characterization of analytic semigroups.

Theorem 3.1. *A semigroup of contractions $\{e^{t\mathcal{A}}\}_{t \geq 0}$ on a Hilbert space \mathcal{H} with norm $\|\cdot\|_{\mathcal{H}}$ is of analytic type if and only if*

$$(3.1) \quad \{i\lambda, \lambda \text{ is real}\} \text{ is contained in the resolvent of } \mathcal{A},$$

and

$$(3.2) \quad \overline{\lim}_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \|\lambda(i\lambda\mathcal{I} - \mathcal{A})^{-1}\| < \infty,$$

where \mathcal{I} denotes the identity matrix.

This theorem can be found in the book by Liu and Zheng (see [11], page 5, Theorem 1.3.3).

Theorem 3.2. *The semigroup of contractions associated with system (1.1) is not analytic.*

Proof: Taking into account the previous theorem, to prove our statement, it suffices to show that there exist a sequence $(\lambda_n)_n$ of real numbers and a bounded sequence $(F_n)_n$ in \mathcal{H} with $\lim_{n \rightarrow \infty} \lambda_n = \infty$ such that

$$\lim_{n \rightarrow \infty} \|\lambda_n(i\lambda_n\mathcal{I} - \mathcal{A})^{-1}\| = \infty.$$

For each $n \in \mathbb{N}$, we consider $F_n = (0, \sin(nx), 0, 0)$, which is bounded in \mathcal{H} . Let $U_n = (u_n, v_n, \alpha_n, \theta_n) \in D(\mathcal{A})$ be the unique solution of the resolvent equation $(i\lambda_n\mathcal{I} - \mathcal{A})U_n = F_n$. This means that

$$(3.3) \quad i\lambda_n u_n - v_n = 0,$$

$$(3.4) \quad i\lambda_n v_n - \frac{1}{\rho} (\mu D^2 u_n - \gamma D^4 u_n + \beta D \theta_n - \delta D^3 \alpha_n) = \sin(nx),$$

$$(3.5) \quad i\lambda_n \alpha_n - \theta_n = 0,$$

$$(3.6) \quad i\lambda_n \theta_n - \frac{1}{a} (k D^2 \alpha_n + m D^2 \theta_n + \beta D v_n + \delta D^3 u_n) = 0.$$

Due to the boundary conditions (1.2), the solution of the above system is of the form

$$u_n = A_n \sin(nx), \quad \alpha_n = B_n \cos(nx).$$

Then, the two following equations must be satisfied:

$$(3.7) \quad A_n (\gamma n^4 + \mu n^2 - \lambda_n^2 \rho) + B_n (i\beta \lambda_n n + \delta n^3) = \rho,$$

$$(3.8) \quad A_n (\delta n^3 - i\beta \lambda_n n) + B_n (k n^2 + i\lambda_n m n^2 - a \lambda_n^2) = 0.$$

We let $\lambda_n = \frac{n}{\sqrt{\rho}} \sqrt{\gamma n^2 + \mu}$, which tends to infinity as $n \rightarrow \infty$. Therefore, from the above system, we get

$$A_n = \frac{a\rho(\mu + \gamma n^2) - k\rho^2 - imn\rho^{3/2}\sqrt{\mu + \gamma n^2}}{\delta^2 n^4 \rho + \beta^2 n^2 (\mu + \gamma n^2)}, \quad B_n = \frac{\delta n \rho^2 - i\beta \rho^{3/2} \sqrt{\mu + \gamma n^2}}{\delta^2 n^4 \rho + \beta^2 n^2 (\mu + \gamma n^2)}.$$

A direct computation yields

$$\|U_n\|_{\mathcal{H}}^2 = \frac{\pi n^6 \rho^2 (2a^2 \gamma^3 + 2\gamma^2 m^2 \rho) + R(n)}{4(\delta^2 n^3 \rho + \beta^2 n (\mu + \gamma n^2))^2} = \frac{\pi n^6 \rho^2 (2a^2 \gamma^3 + 2\gamma^2 m^2 \rho) + R(n)}{4n^6 (\beta^4 \gamma^2 + 2\beta^2 \gamma \delta^2 \rho + \delta^4 \rho^2) + S(n)}, \quad \forall n \in \mathbb{N},$$

where $R(n)$ and $S(n)$ are fourth degree polynomials in n . To be precise,

$$\begin{aligned} R(n) = & \pi n^4 \rho^2 (6a^2 \gamma^2 \mu + a\beta^2 \gamma^2 + 3a\gamma \delta^2 \rho - 4a\gamma^2 k\rho + 4\gamma \mu m^2 \rho + 2\beta\gamma \delta m\rho) \\ & + \pi n^2 \rho^2 (6a^2 \gamma \mu^2 + 2a\beta^2 \gamma \mu + 3a\delta^2 \mu\rho - 8a\gamma k\mu\rho + 2\gamma k^2 \rho^2 + \beta^2 \gamma k\rho - \delta^2 k\rho^2 + 2\mu^2 m^2 \rho + 2\beta\delta \mu m\rho) \\ & + \pi \rho^2 (2a^2 \mu^3 + a\beta^2 \mu^2 - 4ak\mu^2 \rho + 2k^2 \mu \rho^2 + \beta^2 k\mu\rho) \end{aligned}$$

and

$$S(n) = 4n^4 (2\beta^4 \gamma \mu + 2\beta^2 \delta^2 \mu\rho) + 4\beta^4 \mu^2 n^2.$$

Therefore, $\lim_{n \rightarrow \infty} \|\lambda_n U_n\|_{\mathcal{H}} = \infty$, which completes the proof of the theorem. \square

4. LIMITING CASE: $\beta \neq 0$ AND $\delta = 0$

If we assume that $\delta = 0$, system 1.1 reduces to

$$(4.1) \quad \begin{cases} \rho \ddot{u} = \mu u_{xx} - \gamma u_{xxxx} + \beta \dot{\alpha}_x, \\ a \ddot{\alpha} = k \alpha_{xx} + m \dot{\alpha}_{xx} + \beta \dot{u}_x. \end{cases}$$

We use the same boundary and initial conditions as in the regular case. Unfortunately, the exponential stability of the solutions that we have proved for the regular case is not an obvious fact now. It deserves special attention.

4.1. Exponential decay. The proof follows, mutatis mutandis, the same scheme we used in Section 2. We omit obvious details and write only the relevant facts. To some extent, we abuse the notation as we use the same symbols we have used in the previous sections to denote the operators. In this case, the operator is given by

$$\mathcal{A} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{\mu D^2 - \gamma D^4}{\rho} & 0 & 0 & \frac{\beta D}{\rho} \\ 0 & 0 & 0 & I \\ 0 & \frac{\beta D}{a} & \frac{k D^2}{a} & \frac{m D^2}{a} \end{pmatrix}.$$

The mild solutions of system (4.1) are given by the semigroup of contractions generated by the operator \mathcal{A} .

We use the same inner product (2.1) used before but take $\delta = 0$.

Lemma 4.1. *The operator \mathcal{A} is the infinitesimal generator of a C_0 -semigroup of contractions on \mathcal{H} .*

Proof: The proof is analogous to that of Lemma 2.1. Thus, we only focus on the differences.

The operator \mathcal{A} is dissipative.

For any $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we can find a unique $U \in \mathcal{H}$ such that $\mathcal{A}U = \mathcal{F}$, or equivalently:

$$(4.2) \quad \left. \begin{aligned} v &= f_1 \\ \frac{1}{\rho} (\mu D^2 u - \gamma D^4 u + \beta D \theta) &= f_2 \\ \theta &= f_3 \\ \frac{1}{a} (k D^2 \alpha + m D^2 \theta + \beta D v) &= f_4 \end{aligned} \right\}$$

The second and fourth equations can be written in terms of f_1 and f_3 as follows:

$$(4.3) \quad \left. \begin{aligned} \mu D^2 u - \gamma D^4 u &= \rho f_2 - \beta D f_3 \\ k D^2 \alpha &= a f_4 - m D^2 f_3 - \beta D f_1 \end{aligned} \right\}$$

The unique solvability of this system is furnished by the usual elliptic arguments. Notice that, in this particular case, from the second equation a solution for α can be obtained by integration. And, with respect to u , the Lax-Milgram lemma can be applied.

As a consequence, 0 is in the resolvent set of \mathcal{A} . \square

Theorem 4.2. *The problem defined from Equation (4.1) with boundary conditions (1.2) and initial conditions (1.3) in \mathcal{H} has a unique mild solution.*

Proof: The proof is a direct consequence of the previous Lemma. \square

We prove now the exponential stability using again conditions (2.5) and (2.6).

Lemma 4.3. *The operator \mathcal{A} satisfies condition (2.5).*

Proof: We omit the two first steps of the proof and we write only the third one:

- (iii) Suppose that the claim of this lemma is not true. Then, there exists a real number $\sigma \neq 0$ satisfying $\|\mathcal{A}^{-1}\|^{-1} \leq |\sigma| < \infty$ and having a property that the set $\{i\lambda, |\lambda| < |\sigma|\}$ is in the resolvent set of \mathcal{A} and $\sup\{\|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|, |\lambda| < |\sigma|\} = \infty$. In this case, we can find a sequence of real numbers, $(\lambda_n)_n$, with $\lambda_n \rightarrow \sigma$, $|\lambda_n| < |\sigma|$, and a sequence of unit norm vectors in the domain of \mathcal{A} , $U_n = (u_n, v_n, \alpha_n, \theta_n)$, such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \rightarrow 0.$$

Writing this condition term by term we get

$$(4.4) \quad i\lambda_n u_n - v_n \rightarrow 0 \text{ in } H^2 \text{ as } n \rightarrow \infty,$$

$$(4.5) \quad i\lambda_n v_n - \frac{1}{\rho} (\mu D^2 u_n - \gamma D^4 u_n + \beta D \theta_n) \rightarrow 0 \text{ in } L^2 \text{ as } n \rightarrow \infty,$$

$$(4.6) \quad i\lambda_n \alpha_n - \theta_n \rightarrow 0 \text{ in } H^1 \text{ as } n \rightarrow \infty,$$

$$(4.7) \quad i\lambda_n \theta_n - \frac{1}{a} (k D^2 \alpha_n + m D^2 \theta_n + \beta D v_n) \rightarrow 0 \text{ in } L^2 \text{ as } n \rightarrow \infty,$$

Taking the inner product of $(i\lambda_n\mathcal{I} - \mathcal{A})U_n$ with U_n in \mathcal{H} , using (2.2) and taking its real part we obtain $D\theta_n \rightarrow 0$ and then $\theta_n \rightarrow 0$. Thus, from (4.6) $\alpha_n \rightarrow 0$ and $D\alpha_n \rightarrow 0$.

Multiplying (4.7) by Du_n , which is bounded, we obtain

$$\langle Dv_n, Du_n \rangle \rightarrow 0.$$

But from (4.4),

$$\langle Dv_n, Du_n \rangle = -i\lambda_n \langle Du_n, Du_n \rangle \rightarrow 0.$$

This proves that $Du_n \rightarrow 0$ (and, therefore, $u_n \rightarrow 0$ and $v_n \rightarrow 0$).

Using these facts and multiplying (4.5) with u_n , we obtain that $D^2 u_n \rightarrow 0$ as $n \rightarrow \infty$.

This argument shows that U_n cannot be of unit \mathcal{H} -norm, which finishes the proof of this lemma. \square

Lemma 4.4. *The operator \mathcal{A} satisfies condition (2.6).*

Proof: Suppose that the claim of the lemma is not true. Then, there is a sequence $(\lambda_n)_n$ with $|\lambda_n| \rightarrow \infty$ and a sequence of unit \mathcal{H} -norm vectors in the domain of \mathcal{A} , $U_n = (u_n, v_n, \alpha_n, \theta_n)$, such that conditions (4.4)–(4.7) hold. Again $D\theta_n \rightarrow 0$ as $n \rightarrow \infty$ and then $\theta_n \rightarrow 0$. Thus, again from (4.6) $\alpha_n \rightarrow 0$ and $D\alpha_n \rightarrow 0$ as $n \rightarrow \infty$.

From (4.4), it is clear that $(\lambda_n u_n)_n$ is bounded due to the boundedness of $(v_n)_n$. Multiplying (4.4) by $D\theta_n$, we get that $\langle \lambda_n u_n, D\theta_n \rangle \rightarrow 0$. Now, we multiply (4.7) by Du_n , which is bounded, and obtain:

$$\langle i\lambda_n \theta_n, Du_n \rangle - \frac{k}{a} \langle D^2 \alpha_n, Du_n \rangle - \frac{m}{a} \langle D^2 \theta_n, Du_n \rangle - \frac{\beta}{a} \langle Dv_n, Du_n \rangle \rightarrow 0.$$

Integrating by parts, we get that the first term of the above expression tends to zero and

$$k \langle D\alpha_n, D^2 u_n \rangle + m \langle D\theta_n, D^2 u_n \rangle - \beta \langle Dv_n, Du_n \rangle \rightarrow 0.$$

Since $(D^2 u_n)_n$ is bounded and $\beta \neq 0$, this expression is equivalent to

$$\langle Dv_n, Du_n \rangle \rightarrow 0,$$

and, hence,

$$-i\lambda_n \|Du_n\|^2 \rightarrow 0.$$

As a consequence, $Du_n \rightarrow 0$ and $u_n \rightarrow 0$ as $n \rightarrow \infty$.

It remains to show that $(v_n)_n$ and $(D^2 u_n)_n$ tend to zero. In order to do so, we introduce the following notation: let $\phi_n = \int_0^x v_n(y) dy$ and $\xi_n = \int_0^x \theta_n(y) dy$. Notice that ϕ_n and ξ_n are both bounded in L^2 since the antiderivative is a continuous map.

We multiply (4.7) by ϕ_n and we get

$$\langle i\lambda_n \theta_n, \phi_n \rangle - \frac{k}{a} \langle D^2 \alpha_n, \phi_n \rangle - \frac{m}{a} \langle D^2 \theta_n, \phi_n \rangle - \frac{\beta}{a} \langle Dv_n, \phi_n \rangle \rightarrow 0.$$

In the following argumentation we use repeatedly integration by parts and we take into account the boundary conditions and the fact that $\int_0^\pi \theta_n(x) dx = 0$.

Using the properties of the inner product, we know that

$$\langle i\lambda_n \theta_n, \phi_n \rangle = \langle \theta_n, -i\lambda_n \phi_n \rangle.$$

And, performing an integration by parts,

$$\langle \theta_n, -i\lambda_n \phi_n \rangle = \int_0^\pi \frac{d}{dx} (\xi_n i\lambda_n \phi_n) dx - \int_0^\pi \xi_n i\lambda_n v_n dx.$$

The first term of the right-hand side of the above equality is zero. And, in the second term, we substitute $i\lambda_n v_n$ with $\frac{1}{\rho} (\mu D^2 u_n - \gamma D^4 u_n)$ (from (2.8)). Thus, we have

$$\langle \xi_n, i\lambda_n v_n \rangle = \langle \xi_n, \frac{1}{\rho} (\mu D^2 u_n - \gamma D^4 u_n) \rangle = \langle \xi_n, \frac{\mu}{\rho} D^2 u_n \rangle - \langle \xi_n, \frac{\gamma}{\rho} D^4 u_n \rangle = -\langle \theta_n, \frac{\mu}{\rho} Du_n \rangle - \langle D\theta_n, \frac{\gamma}{\rho} D^2 u_n \rangle.$$

Since $(D\theta_n)_n$ and $(\theta_n)_n$ tend to zero as $n \rightarrow \infty$ and $(Du_n)_n$ and $(D^2 u_n)_n$ are bounded, we conclude that

$$\langle i\lambda_n \theta_n, \phi_n \rangle \rightarrow 0,$$

and, therefore,

$$-\frac{k}{a} \langle D^2 \alpha_n, \phi_n \rangle - \frac{m}{a} \langle D^2 \theta_n, \phi_n \rangle - \frac{\beta}{a} \langle Dv_n, \phi_n \rangle \rightarrow 0.$$

But, integrating by parts, the above expression is equal to

$$k \langle D\alpha_n, v_n \rangle + m \langle D\theta_n, v_n \rangle + \beta \langle v_n, v_n \rangle \rightarrow 0.$$

This implies $v_n \rightarrow 0$ as $n \rightarrow \infty$.

We now multiply (4.5) with u_n (we discard the last term of (4.5) because it tends to 0):

$$\langle i\lambda_n v_n, u_n \rangle - \frac{1}{\rho} \langle \mu D^2 u_n, u_n \rangle + \frac{\gamma}{\rho} \langle D^4 u_n, u_n \rangle \rightarrow 0$$

Since $(\lambda_n u_n)_n$ is bounded and $(Du_n)_n$ tends to zero, this last expression gives $D^2 u_n \rightarrow 0$ as $n \rightarrow \infty$. And this proves that U_n cannot be of unit \mathcal{H} -norm. \square

Theorem 4.5. *Let (u, α) be a mild solution of the problem determined by (4.1), with boundary conditions (1.2) and initial conditions (1.3) in \mathcal{H} . Then, (u, α) decays exponentially to zero as the time tends to infinity.*

Proof: The proof is a direct consequence of Lemmas 4.3 and 4.4. \square

4.2. Lack of analyticity and impossibility of localization.

Theorem 4.6. *The semigroup of contractions associated with system (4.1) is not analytic.*

Proof: The proof is, essentially, the same as the one of Theorem 3.2, after substituting $\delta = 0$ at each of its occurrences. \square

In this subsection, we investigate also the impossibility for the solutions to localize in time. This means that the only solution that vanishes after a finite period of time is the null solution. The main idea is to show the uniqueness of solutions for the backward in time problem. Therefore, we consider the following system:

$$(4.8) \quad \begin{cases} \rho \ddot{u} = \mu u_{xx} - \gamma u_{xxxx} - \beta \theta_x. \\ a \ddot{\alpha} = k \alpha_{xx} - m \theta_{xx} - \beta v_x. \end{cases}$$

We use the same boundary conditions (1.2) and initial conditions (1.3) used throughout the paper.

In view of (2.1), we define

$$E_1(t) = \frac{1}{2} \int_0^\pi (\rho |v|^2 + a |\theta|^2 + \mu |u_x|^2 + \gamma |u_{xx}|^2 + k |\alpha_x|^2) dx$$

and

$$E_2(t) = \frac{1}{2} \int_0^\pi (\rho |v|^2 - a |\theta|^2 + \mu |u_x|^2 + \gamma |u_{xx}|^2 - k |\alpha_x|^2) dx.$$

Direct calculations give

$$\dot{E}_1(t) = \int_0^\pi m |\theta_x|^2 dx$$

and

$$\dot{E}_2(t) = - \int_0^\pi 2\beta v \theta_x dx - \int_0^\pi m |\theta_x|^2 dx.$$

Using the Lagrange identity method (see [2]), we get

$$\begin{aligned} \frac{\partial}{\partial s} (\rho \dot{u}(s) \dot{u}(2t-s)) &= \rho \ddot{u}(s) \dot{u}(2t-s) - \rho \dot{u}(s) \ddot{u}(2t-s) \\ \frac{\partial}{\partial s} (a \dot{\alpha}(s) \dot{\alpha}(2t-s)) &= a \ddot{\alpha}(s) \dot{\alpha}(2t-s) - J \dot{\alpha}(s) \ddot{\alpha}(2t-s) \end{aligned}$$

Therefore, it could be seen that

$$\int_0^\pi (\rho|v|^2 + k|\alpha_x|^2) dx = \int_0^\pi (a|\theta|^2 + \mu|u_x|^2 + \gamma|u_{xx}|^2) dx.$$

And, therefore,

$$E_2(t) = \int_0^\pi (\mu|u_x|^2 + \gamma|u_{xx}|^2 - k|\alpha_x|^2) dx.$$

Letting

$$E_3(t) = \int_0^\pi \left(\frac{m}{2} |\alpha_x|^2 - a\alpha\theta \right) dx,$$

we obtain

$$\dot{E}_3(t) = \int_0^\pi (k|\alpha_x|^2 - a|\theta|^2 - \beta v\alpha_x) dx.$$

Finally, we define

$$E_0(t) = \epsilon E_1(t) + E_2(t) + \lambda E_3(t),$$

where ϵ is a sufficiently small positive real number and λ is an appropriately large positive real number. From the definition of E_0 , it is clear that

$$E_0(t) = \int_0^\pi \left(\frac{\epsilon\rho}{2} |v|^2 + \frac{\epsilon a}{2} |\theta|^2 + \frac{\epsilon + 2}{2} (\mu|u_x|^2 + \gamma|u_{xx}|^2) + \frac{1}{2} ((\epsilon - 2)k + m\lambda) |\alpha_x|^2 - \lambda a\alpha\theta \right) dx.$$

We take now $\mathcal{E}(t) = \int_0^t E_0(s) ds$. Notice that $\dot{\mathcal{E}}(t) = \int_0^t \frac{dE_0(s)}{ds} ds$. And, hence

$$\dot{\mathcal{E}}(t) = \int_0^t \left(\int_0^\pi \left((\epsilon - 1)m|\theta_x|^2 - 2\beta v\theta_x + \lambda k|\alpha_x|^2 - \lambda a|\theta|^2 - \lambda\beta v\alpha_x \right) dx \right) ds.$$

Applying the arithmetic–geometric inequality, there exist two positive constants M_1 and M_2 such that

$$\dot{\mathcal{E}}(t) \leq \int_0^t M_1 \left(\int_0^\pi |v|^2 dx \right) ds + \int_0^t \lambda k M_2 \left(\int_0^\pi |\alpha_x|^2 dx \right) ds.$$

And, therefore, another constant M_3 exists such that

$$\dot{\mathcal{E}}(t) \leq M_3 \mathcal{E}(t).$$

This implies $\mathcal{E}(t) \leq \mathcal{E}(0) \exp(M_3 t)$. And, since $\mathcal{E}(0) = 0$, we get $\mathcal{E}(t) \equiv 0$ for all $t \in (0, t_0)$ where t_0 is a sufficiently small positive number. \square

We can summarize this result in the following statement.

Theorem 4.7. *Let (u, α) be a classical solution of the problem defined by system (4.1), the boundary conditions (1.2) and the initial conditions (1.3) such that $u = \alpha \equiv 0$ after a finite time $t_0 > 0$. Then, $u = \alpha \equiv 0$ for every $t \geq 0$.*

5. LIMITING CASE: $\beta = 0$ AND $\delta \neq 0$

If we assume that $\beta = 0$, the system (1.1) reduces to

$$(5.1) \quad \begin{cases} \rho \ddot{u} = \mu u_{xx} - \gamma u_{xxxx} - \delta \alpha_{xxx}, \\ a \ddot{\alpha} = k \alpha_{xx} + m \dot{\alpha}_{xx} + \delta u_{xxx}. \end{cases}$$

We use the same boundary and initial conditions as in the regular case.

5.1. Exponential decay.

Theorem 5.1. *Let (u, α) be a mild solution of the problem determined by (1.1), with boundary conditions (1.2) and initial conditions (1.3) in \mathcal{H} . Then, (u, α) decays exponentially to zero as the time tends to infinity.*

Proof: The proof is a direct consequence of Lemmas 2.4 and 2.5, when writing $\beta = 0$ wherever it appears. The same conclusions are obtained. \square

5.2. Lack of analyticity.

Theorem 5.2. *The semigroup of contractions associated with system (5.1) is not analytic.*

Proof: The proof is, essentially, the same as the proof of Theorem 3.2, when writing $\beta = 0$ wherever it appears. \square

6. CONCLUSIONS

We analyzed a linear model proposed for nonsimple materials with heat conduction of type III. First, we proved the existence of solutions for the underlying system of equations. Then, we showed that the solutions are exponentially stable with respect to the time variable. However, we showed that the semigroup associated with the system is not analytic. Later, we focused on two limiting cases of the model, and for each one of them we also proved the exponential decay of the solutions and the lack of analyticity. Moreover, in the first case, we proved that the only solution that vanishes after a finite period of time is the null solution. The impossibility of localization for the solutions in the regular case is still an open question.

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