On the perturbation of bimodal systems

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Abstract

Given a bimodal system defined by the equations

$$\left\{ \begin{array}{lll} \dot{x}(t) = & A_1x(t) + Bu(t) & \text{if } c^t x(t) \leq 0 \\ \dot{x}(t) = & A_2x(t) + Bu(t) & \text{if } c^t x(t) \geq 0 \end{array} \right. \tag{1}$$

where $B \in \mathcal{M}_{n,m}$ and $A_i \in \mathcal{M}_n$, $i = 1, 2$, are such that A_1, A_2 coincide on the hyperplane $\mathcal{V}$ =Kerc^t. We consider in the set of matrices defining the above systems the simultaneous feedback equivalence defined by $([A_1, B], [A_2, B]) \sim ([A'_1, B'], [A'_2, B'])$ if

$$[A'_i\, B'] = S^{-1}[A_i\, B] \begin{bmatrix} S & 0 \\ R & T \end{bmatrix} \quad i = 1, 2 \text{ with } S(\mathcal{V}) = \mathcal{V}$$

This equivalent relation corresponds to the action of a Lie group. Under this action we obtain, in the case $m \leq 1$, the semiuniversal deformation, following Arnold's technique. Then the problem of structural stability is studied.

1 Preliminaries

(1.1) From now on, $\mathcal{M}_{n,m}$ denotes the set of $n \times m$ complex matrices. We write $\mathcal{M}_{n,n} = \mathcal{M}_n$. If $A \in \mathcal{M}_{n,m}$, A^* (resp. A^t) denotes *conjugate transpose* of A, (resp. *transpose* of A) and *tr* A the *trace* of A.
(1.2) In [3] the following reduced form is obtained under the above equivalent relation: Let J be a $h \times h$ complex Jordan matrix and N be the $l \times l$ standard nilpotent matrix. Then any pair $((A_1\, b), (A_2\, b))$ with $A_i \in \mathcal{M}_n, b \in \mathcal{M}_{n,1}$ and $A_1|\mathcal{V} = A_2|\mathcal{V}$, $\mathcal{V} =$ Ker $(0, ..., 0, 1)^t$, is equivalent to a pair $((A_{10}, b_0), (A_{20}, b_0))$ where

$$A_{10} = \begin{pmatrix} J & 0 & \alpha^1 \\ 0 & N & 0 \\ 0 & \alpha_1 & 0 \end{pmatrix}, with\, \alpha_1 = (0, ..., 0, 1),\ \alpha^1 = (\alpha^1_1, ..., \alpha^1_h)^t$$

$$A_{20} = \begin{pmatrix} J & 0 & \beta^1_1 \\ 0 & N & \beta^1_2 \\ 0 & \alpha_1 & \beta \end{pmatrix},\ b_0 = \begin{pmatrix} 0 \\ p \\ \epsilon_0 \end{pmatrix}$$

with

$$\beta^1_1 = (\beta^1_{11}, ..., \beta^1_{1h})^t,\ \beta^1_2 = (\beta^1_{21}, ..., \beta^1_{2l})^t,\ p = (0, ...0, 1)^t.$$

We shall say that this pair is in *Kronecker reduced form.*
(1.3) Let $\mathcal{M} = \{((A_1\, b), (A_2\, b)); A_1|\mathcal{V} = A_2|\mathcal{V}\}$ and

$$\mathcal{G} = \begin{pmatrix} S & O \\ f & t \end{pmatrix}; S \in Gl(n), S(\mathcal{V}) = \mathcal{V}, t \neq 0$$

Notice that S has the form $S = \begin{pmatrix} S_{11} & s^1 \\ 0 & s \end{pmatrix}$, so that $\mathcal{G}$ can be identified with an open set of $\mathbb{C}^{n^2+2}$.

We consider in $\mathcal{M}$ the hermitian product defined by

$$< ((A_1, b), (A_2, b)), ((A_1', b'), (A_2', b')) >= tr((A_1, b), (A_2, b)) \begin{pmatrix} A_1'^* \\ b'* \\ A_2'* \\ b'^* \end{pmatrix}$$

and the action of $\mathcal{G}$ on $\mathcal{M}$ defined by

$$\begin{pmatrix} S & O \\ f & t \end{pmatrix} * ((A_1, b), (A_2, b)) = (S(A_1, b) \begin{pmatrix} S^{-1} & O \\ f & t \end{pmatrix}, S(A_2, b) \begin{pmatrix} S^{-1} & O \\ f & t \end{pmatrix}).$$

We fix a pair $((A_{10}, b_0), (A_{20}, b_0)) \in \mathcal{M}$ and let $\phi : \mathcal{G} \to \mathcal{M}$ be the map defined by

$$\phi(\mathcal{S}) = \mathcal{S} * ((A_{10}, b_0), (A_{20}, b_0))$$

with $\mathcal{S} = \begin{pmatrix} S & O \\ f & t \end{pmatrix}$.

Let $\mathcal{A}_0 = ((A_{10},\, b_0), (A_{20},\, b_0))$ and denote $\mathcal{O}_0 = \{\mathcal{S} * \mathcal{A}_0; \mathcal{S} \in \mathcal{G}\}$. We know that the orbit $\mathcal{O}_0$ is a locally closed submanifold of $\mathcal{M}$ (see for example [2]). Then if we denote $\mathfrak{B} = (T_{\mathcal{A}_0}\mathcal{O}_0)^\perp$ and $\mathcal{I}$ the unit element in $\mathcal{G}$, we have the following theorem due to Arnold ([1]; see also [4]).

Theorem 1 *The linear variety $\mathcal{A}_0 + \mathfrak{B}$ has the following universal property. Let $\psi : \mathfrak{B} \to \mathcal{M}$ defined by $\psi(\chi) = \mathcal{A}_0 + \chi$. Then for any differentiable map $\varphi : \mathbb{C}^N \to \mathcal{M}$ such that $\varphi(0) = \mathcal{A}_0$, there exist a neighborhood U of 0 in $\mathbb{C}^N$ a differentiable map $\eta : U \to \mathfrak{B}$ such that $\eta(0) = 0$ and a differentiable map $\xi : U \to \mathcal{G}$ with $\chi(0) = \mathcal{I}$ such that $\varphi(\mu) = \xi(\mu) \star \psi(\eta(\mu))$.*

The linear variety $\mathcal{A}_0 + \mathfrak{B}$ has the minimum dimension having this universal property. It is called a *miniversal deformation* of $\mathcal{A}_0$.

Finally we recall that $\mathcal{A}_0$ is said to be *structural stable* if it is an interior point of its orbit. Equivalently, if $\mathfrak{B} = 0$.

2 Construction of a miniversal deformation

As we have said, in order to obtain a miniversal deformation of $\mathcal{A}_0$ we have to compute $(T_{\mathcal{A}_0}\mathcal{O}_0)^\perp$.

Let $\mathcal{I}$ be the unit element in $\mathcal{G}$ and $\mathcal{P} = \begin{pmatrix} P & O \\ p_1 & q \end{pmatrix} \in T_{\mathcal{I}}\mathcal{G}$ with $P = \begin{pmatrix} P_{11} & p^1 \\ 0 & p \end{pmatrix}$. Then we have the following lemma.

Lemma 1

$$d\phi_{\mathcal{I}}(\mathcal{P}) = (([P,\, A_{10}] + b_0 p_1,\, b_0 q + P b_0),\, ([P,\, A_{20}] + b_0 p_1, b_0 q + P b_0)).$$

Since $T_{\mathcal{A}_0}\mathcal{O}_0 = Imd\phi_{\mathcal{I}}$ one has that $(A_1,\, b), (A_2,\, b)) \in (T_{\mathcal{A}_0}\mathcal{O}_0)^{\perp}$ if and only if

$$< (([P,\, A_{10}] + b_0 p_1,\, b_0 q + P b_0),\, ([P,\, A_{20}] + b_0 p_1, b_0 q + P b_0)),\, ((A_1,\, b), (A_2,\, b)) >= 0$$

for every $\mathcal{P} \in T_{\mathcal{I}}\mathcal{G}$.

Let $\mathcal{P}$ be as above and introduce the following notation:

$$A_{10} = \begin{pmatrix} A_{11} & \alpha^1 \\ \alpha_1 & \alpha \end{pmatrix}, A_{20} = \begin{pmatrix} A_{11} & \beta^1 \\ \alpha_1 & \beta \end{pmatrix}, b_0 = \begin{pmatrix} b_0^1 \\ \epsilon_0 \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} B_{11} & \delta^1 \\ \delta_1 & \delta \end{pmatrix}, A_2 = \begin{pmatrix} B_{11} & \gamma^1 \\ \delta_1 & \gamma \end{pmatrix}, b = \begin{pmatrix} b^1 \\ \epsilon \end{pmatrix}.$$

Then, we have the following result.

Theorem 2 *A miniversal deformation of* $\mathcal{A}_0$ *is given by the linear variety* $\mathcal{A}_0 + ((A_1,\, b), (A_2,\, b))$*, where* A_1, A_2 *and* b *are any solution of the following system:*

(i) $2[A_{11},\, B_{11}^*] + \alpha^1\delta^{1*} - 2\delta_1^*\alpha_1 + \beta^1\gamma^{1*} + b_0^1 b^{1*} = 0$

(ii) $2\alpha_1 B_{11}^* + \alpha\delta^{1*} - (\delta^{1*} + \gamma^{1*})A_{11} - \bar{\delta}\alpha_1 + \beta\gamma^{1*} - \bar{\gamma}\alpha_1 + \epsilon_0 b^{1*} = 0$

(iii) $2\alpha_1\delta_1^* - \delta^{1*}\alpha^1 - \gamma^{1*}\beta^1 + \epsilon_0\bar{\epsilon} = 0$

(iv) $B_{11}^* b_0^1 + \delta_1^*\epsilon_0 = 0$

(v) $(\delta^{1*} + \gamma^{1*})b_0^1 + (\bar{\delta} + \bar{\gamma})\epsilon_0 = 0$

(vi) $tr(b_0^1 b^{1*}) + \epsilon_0\bar{\epsilon} = 0$

Since the number of unknowns is $n^2 + 2n$ and the number of equations is $n^2 - n + 4$, we have the following result.

Proposition 1 There is no pair structural stable in $\mathcal{M}$.

Remark 1 If A_{10}, A_{20} and b_0 are real matrices, we can substitute the symbol $\star$ for the symbol t, corresponding to the *transpose matrix.*

If the pair $((A_1,\, b), (A_2,\, b))$ is in Kronecker reduced form, the above equations take a simplified form allowing in many cases the obtention of an explicit solution of a miniuniversal deformation. In fact, we have in this case,

$$A_{10} = \begin{pmatrix} J & 0 & \alpha^1 \\ 0 & N & 0 \\ 0 & \alpha_1 & 0 \end{pmatrix},\ \alpha_1 = (0, ..., 0, 1),\ \alpha^1 = (\alpha_1^1, ..., \alpha_h^1)^t (= d(\gamma)^t)$$

$$A_{20} = \begin{pmatrix} J & 0 & \beta_1^1 \\ 0 & N & \beta_2^1 \\ 0 & \alpha_1 & \beta \end{pmatrix}, \; b_0 = \begin{pmatrix} 0 \\ p \\ \epsilon_0 \end{pmatrix}, \; p = (0, ...0, 1)^t.$$

Then if accordingly with the above notation we write

$$A_1 = \begin{pmatrix} B_{11} & B_{12} & \delta_1^1 \\ B_{21} & B_{22} & \delta_2^1 \\ \delta_{11} & \delta_{12} & \delta \end{pmatrix}, \; A_2 = \begin{pmatrix} B_{11} & B_{12} & \gamma_1^1 \\ B_{21} & B_{22} & \gamma_2^1 \\ \delta_{11} & \delta_{12} & \gamma \end{pmatrix}, \; b = \begin{pmatrix} b_1^1 \\ b_2^1 \\ \epsilon \end{pmatrix}$$

the following proposition follows.

Proposition 2 $((A_1, b), (A_2, b)) \in (T_{\mathcal{A}_0}\mathcal{O}_0)^\perp$ if and only if

(i) $2[J, B_{11}] + \alpha^1\delta_1^{1*} + \beta_1^1\gamma_1^{1*} = 0$

(ii) $2(JB_{21}^* - B_{21}^*J) + \alpha^1\delta_2^{1*} - 2\delta_{11}^*\alpha_1 + \beta_1^1\gamma_2^{1*} = 0$

(iii) $2(NB_{12}^* - B_{12}^*J) + \beta_2^1\gamma_1^{1*} + pb_1^{1*} = 0$

(iv) $2[N, B_{22}^*] - 2\delta_{12}^*\alpha_1 + \beta_2^1\gamma_2^{1*} + pb_2^{1*} = 0$

(v) $-(\delta_1^{1*} + \gamma_1^{1*})J + \beta\gamma_1^{1*} + \epsilon_0 b_1^{1*} + 2\alpha_1 B_{12}^* = 0$

(vi) $2\alpha_1 B_{22}^* - (\delta_2^{1*} + \gamma_2^{1*})N - \delta\alpha_1 + \beta\gamma_2^{1*} - \gamma\alpha_1 + \epsilon_0 b_2^{1*} = 0$

(vii) $2\alpha_1\delta_{12}^* - \delta_1^{1*}\alpha^1 - \gamma_1^{1*}\beta_1^1 - \gamma_2^{1*}\beta_2^1 + \epsilon_0\epsilon = 0$

(viii) $B_{21}^*p + \delta_{11}^*\epsilon_0 = 0$

(ix) $B_{22}^*p + \delta_{12}^*\epsilon_0 = 0$

(x) $(\delta_2^{1*} + \gamma_2^{1*})p + (\delta + \gamma)\epsilon_0 = 0$

(xi) $tr \begin{pmatrix} 0 & 0 \\ pb_1^{1*} & pb_2^{1*} \end{pmatrix} + \epsilon_0\epsilon = 0$

Notice that $l = 0$ implies $\epsilon_0 = 1$, $\alpha_1 = 0$ and $l > 0$ implies $\epsilon_0 = 0$, so that we have

Corollary 1 *The above equations reduced to:*

(1) If $l = 0$*:*

(i) $2[J, B_{11}^*] + \alpha^1\delta_1^{1*} + \beta_1^1\gamma_1^{1*} = 0$

(ii) $-(\delta_1^{1*} + \gamma_1^{1*})J + \beta\gamma_1^{1*} + b_1^{1*} = 0$

(iii) $-\delta_1^{1*}\alpha^1 - \gamma_1^{1*}\beta_1^1 = 0$

(iv) $\delta_{11}^* = 0$

(v) $\delta + \gamma = 0$

(2) If $h = 0$*:*

(i) $2[N, B^*_{22}] - 2\delta^*_{12}\alpha_1 + \beta^1_2\gamma^{1*}_2 + pb^{1*}_2 = 0$

(ii) $2\alpha_1 B^*_{22} - (\delta^{1*}_2 + \gamma^{1*}_2)N - \delta\alpha_1 + \beta\gamma^{1*}_2 - \gamma\alpha_1 = 0$

(iii) $2\alpha_1\delta^*_{12} - \gamma^{1*}_2\beta^1_2 = 0$

(iv) $B^*_{22}p = 0$

(v) $(\delta^{1*}_2 + \gamma^{1*}_2)p = 0$

(vi) $trpb^{1*}_2 = 0$

3 The case n = 3

If $n = 3$ (and of course if $n = 2$) the above equations can be solved easily. We limit ourselves to give in the following three examples the dimension of the corresponding orbit. We denote this orbit by $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$, respectively.

(1) $J = (\lambda), N = (0)$, so that $\alpha_1 = 1$, $\alpha^1 = 1$, $p = 1$, $\epsilon_0 = 0$. Then $dim\mathcal{O}_1 = 9$.

(2) $J = (0), N = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, so that $\alpha_1 = (0, 1)$, $\alpha^1 = 0, p = (0, 1)^t, \epsilon_0 = 0$. Then $dim\mathcal{O}_2 = 11$.

(3) $J = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$ so that $\alpha_1 = (0, 0), \alpha^1 = (1, 0)^t, \alpha = 0, \epsilon_0 = 1$. Then $dim\mathcal{O}_3 = 11$.

Notice that, according Proposition 1, any of these pairs is structurally stable.

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