Overdetermined partial resolvent kernels for finite networks

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Abstract

In [2], a study of the existence and uniqueness of solution of partial overdetermined boundary value problems for finite networks was performed. These problems involve Schrödinger operators and the novel feature is that no data are prescribed on part of the boundary, whereas both the values of the function and of its normal derivative are given on another part of the boundary. In the present work, we study the resolvent kernels associated with overdetermined partial boundary value problems on finite network and we express them in terms of the well–known Green operator and the Dirichlet–to–Robin map. Moreover, we analyze their main properties and we compute them in the case of a generalized cylinder. The obtained expression involve polynomials that can be seen as a generalization of Chebyshev polynomias, and indeed when the conductances along axes are constant the expressions for the overdetermined partial resolvent kernels are given in terms of second kind Chebyshev polynomials.

Keywords: Overdetermined partial boundary value problems, Dirichlet–to–Neumann map, Inverse problem, resolvent Kernels

1. Preliminaries

A *discrete inverse boundary value problem* consists in recovering the conductances of a network with boundary using only boundary measurements and global equilibrium conditions. In general, inverse problems are exponentially ill–posed, since they are highly sensitive to changes in the boundary data, see [11]. On the other hand, discrete inverse problems appears naturally when discretizing continuous inverse problems, see for instance [5]. Although the discrete inverse problem has been completely characterized in the case of the combinatorial laplacian for planar critical networks, see [7, 9], few works have been done for general networks, where the inverse problem remains open. In [12], an extension of the cited works have been developed for networks embedded in a cylindric surface.

This work is the third in a series on various aspects of the discrete inverse problem. It develops the study corresponding to resolvent kernels associated with overdetermined partial boundary value problems for Schrödinger operators on networks. The appropriate theoretical framework to address the discrete inverse problem is the study of overdetermined partial boundary value problems, while the fundamental tool is the Dirichlet-to-Robin map; which measures the difference of voltages between boundary vertices when electrical currents are applied to them. The theoretical foundations

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of this class of problems have been established in [2]. The results in this framework are of potential application among others, in electrical impedance tomography which is currently one of the non-invasive methods of clinical diagnosis with more development opportunities; see [6]. The data for an inverse problem on a network is the Dirichlet–to–Robin map, since it contains the boundary information, so we worried for their properties, which were analyzed in [3]. The matrix associated with the Dirichlet–to–Robin map is known as the response matrix of the network and it is the Schur complement of a certain submatrix of the Schrödinger matrix. The consideration of Schrödinger operators allowed us to consider a wide class of matrices, not necessarily singular nor weakly diagonally dominant, as response matrices. Therefore, our results represented a generalization of those obtained in [7, 8].

In the study of classical boundary value problems one of the main tools, both for solving as for studying fundamental properties, are the resolvent kernels such as Green, Poisson or Robin kernels. So, once we have established the overdetermined partial boundary value problem, we raise the problem of defining what is a resolvent kernel and what are its main properties. In order to do this, we first obtain an equivalent condition for the existence and uniqueness of solution of these type of problems that can be read directly from a submatrix of the Schrödinger operator. Then, we give expressions of these kernels in terms of the classical Green kernel and the Dirichlet–to–Robin map. In the last section, we obtain the resolvent kernels for a generalized cylinder, which are defined as the product of a path with an arbitrary network. The expressions are given in terms of a generalization of Chebyshev polynomials for higher dimensions, that when the conductances are constant are precisely Chebyshev polynomials of second kind.

Let $\Gamma = (V, c)$ be a finite network, i.e., a finite connected graph without loops nor multiple edges, and with the vertex set V. Let E be the set of edges of the network Γ. Each edge (x, y) is assigned a *conductance* $c(x, y)$, where $c:\ V \times V \longrightarrow [0, +\infty)$. Moreover, $c(x, y) = c(y, x)$ and $c(x, y) = 0$ if $(x, y) \notin E$. Then, $x, y \in V$ are *adjacent*, $x \sim y$, iff $c(x, y) > 0$. We denote by $V(S)$ the *set of neighbours of* $S \subset V$; that is, the set of vertices of $V \setminus S$ adjacent to any vertex $x \in S$.

The set of functions on a subset $F \subseteq V$, denoted by $\mathcal{C}(F)$, and the set of nonnegative functions on F, $\mathcal{C}^+(F)$, are naturally identified with $\mathbb{R}^{|F|}$ and the nonnegative cone of $\mathbb{R}^{|F|}$, respectively. We denote by $\displaystyle\int_F u(x)dx$ the value $\displaystyle\sum_{x\in F} u(x)$. Moreover, if F is a non empty subset of V, its characteristic function is denoted by $\chi_{_F}$. When $F = \{x\}$, its characteristic function will be denoted by ε_x. If $u \in \mathcal{C}(V)$, we define the *support* of u as $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$. Clearly, $\mathcal{C}(F)$ can be identified with the subspace $\{u \in \mathcal{C}(V) : \mathsf{supp}(u) \subset F\}$.

If we consider a proper subset $F \subset V$, then its *boundary* $\delta(F)$ is given by the vertices of $V \setminus F$ that are adjacent to at least one vertex of F. The vertices of $\delta(F)$ are called *boundary vertices* and when a boundary vertex $x \in \delta(F)$ has a unique neighbor in F we call the edge joining them a *boundary spike*. It is easy to prove that $\bar{F} = F \cup \delta(F)$ is connected when F is. Any function $\omega \in \mathcal{C}^+(\bar{F})$ such that $\mathsf{supp}(\omega) = \bar{F}$ and $\displaystyle\int_{\bar{F}} \omega^2(x)\, dx = 1$ is called *weight* on $\bar{F}$. The set of weights is denoted by $\Omega(\bar{F})$. We denote $\kappa_F \in \mathcal{C}(\delta(F))$ as the function $\kappa_F(x) = \sum\limits_{y\in F} c(x, y)$.

We define the *normal derivative of* $u \in \mathcal{C}(\bar{F})$ *on* F as the function in $\mathcal{C}(\delta(F))$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n}_{_F}}\right)(x) = \int_F c(x, y)\,\big(u(x) - u(y)\big)\, dy, \quad \text{for any } x \in \delta(F).$$

If H, F are non–empty subsets of V, any function $K \in \mathcal{C}(H \times F)$ will be called a *kernel*. The *integral operator associated with* K is the endomorphism $\mathscr{K}\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(H)$ that assigns to each $f \in \mathcal{C}(F)$, the function $\mathscr{K}(f)(x) = \displaystyle\int_F K(x,y)\, f(y)\, dy$ for all $x \in H$. Conversely, given an endomorphism $\mathscr{K}\colon \mathcal{C}(F) \longrightarrow \mathcal{C}(H)$, the associated kernel is given by $K(x,y) = \mathscr{K}(\varepsilon_y)(x)$. Clearly, kernels and operators can be identified with matrices, after giving a label on the vertex set. In addition, a function $u \in \mathcal{C}(F)$ can be identified with the kernel, $K \in \mathcal{C}(F \times F)$, defined as $K(x,x) = u(x)$ and $K(x,y) = 0$ otherwise, and hence with a diagonal matrix that will be denoted by D_u. As usual, $K^x = K(x,\cdot)$ and $K^y = K(\cdot,y)$. Along the paper we use the convention that operators and their associated matrices, and functions and their associated vectors, are denoted with the same letter, operators in calligraphic font and matrices and vectors in sans serif font.

Given a matrix M and A, B sets of vertices, $\mathsf{M}(A;B)$ denote the matrix obtained from M with rows indexed by the vertices of A and columns indexed by the vertices of B. Also, given a vector v and a set of vertices A, $\mathsf{v}(A)$ denotes the entries of v indexed by the vertices of A. Moreover, we denoted by C the matrix $\big(c(x,y)\big)_{x,y\in V}$.

The *combinatorial Laplacian* of Γ is the linear operator $\mathcal{L} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function defined for all $x \in V$ as

$$\mathcal{L}(u)(x) = \int_V c(x,y)\,\big(u(x) - u(y)\big)\, dy.$$

Given $q \in \mathcal{C}(V)$ the *Schrödinger operator* on Γ with *potential* q is the linear operator $\mathcal{L}_q : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$. Since q is real, it is well–known that any Schrödinger operator is self–adjoint. For any weight $\sigma \in \Omega(\bar{F})$, the so–called *potential associated with* σ is the function in $\mathcal{C}(\bar{F})$ defined as $q_\sigma = -\sigma^{-1}\mathcal{L}(\sigma)$ on F, $q_\sigma = -\sigma^{-1}\dfrac{\partial \sigma}{\partial \mathsf{n}_F}$ on $\delta(F)$. These authors proved in [3] that $\mathcal{L}_q$ is positive semi–definite on $\mathcal{C}(\bar{F})$ if there exist $\lambda \geq 0$ and $\sigma \in \Omega(\bar{F})$ such that $q = q_\sigma + \lambda\chi_{\delta(F)}$. In this case, it is positive definite iff $\lambda > 0$. So, throughout this paper, we will suppose that the above condition $q = q_\sigma + \lambda\chi_{\delta(F)}$ holds with $\sigma \in \Omega(\bar{F})$ and $\lambda \geq 0$. Therefore, for any $f \in \mathcal{C}(F)$ and $g \in \mathcal{C}(\delta(F))$ the following Dirichlet problem

$$\mathcal{L}_q(u) = f \quad \text{on} \quad F \quad \text{and} \quad u = g \quad \text{on} \quad \delta(F), \tag{1}$$

has a unique solution. In particular, taking $g = 0$ we get that the operator $\mathcal{L}_q$ is invertible on $\mathcal{C}(F)$ and its inverse is called the *Green operator* for F and it is denoted by $\mathscr{G}_q$. On the other hand, the operator that assigns to any $g \in \mathcal{C}(\delta(F))$, the unique solution of Problem (1) when $f = 0$, is called *Poisson operator* for F and denoted by $\mathscr{P}_q$. The relation between the kernels associated with the above operators is given by the following identity that was proved in [4]

$$P_q(x,y) = \varepsilon_x(y) - \frac{\partial G_q^x}{\partial \mathsf{n}_F}(y).$$

The map $\Lambda_q\colon \mathcal{C}(\delta(F)) \longrightarrow \mathcal{C}(\delta(F))$ that assigns to any function $g \in \mathcal{C}(\delta(F))$ the function $\Lambda_q(g) = \dfrac{\partial \mathscr{P}_q(g)}{\partial \mathsf{n}_F} + qg$ is called *Dirichlet–to–Robin map*. In [3], the authors proved that the Dirichlet–to–Robin map, Λ_q, is a self–adjoint and positive semi–definite operator. Moreover, λ is the lowest eigenvalue of Λ_q and its associated eigenfunctions are multiple of σ. In addition, if $N_q \in \mathcal{C}(\delta(F) \times \delta(F))$ is the kernel of Λ_q, its associated matrix $\mathsf{N_q}$ is an irreducible and symmetric M–matrix. Usually $\mathsf{N_q}$ is called the *response matrix* of the network.

2. Overdetermined Partial resolvent kernels

In this section we define the resolvent kernels associated with the overdetermined partial boundary value problems, that where introduced by the authors in [2]. Then, we analyze the main properties of the above mentioned kernels. To begin with we summarize some of the more relevant properties of overdetermined partial boundary value problems that help us to obtain the mentioned results. The proofs of these results can be found in [2].

We fix a proper and connected subset $F \subset V$ and $A, B \subset \delta(F)$ non–empty subsets such that $A \cap B = \emptyset$. Moreover we denote by R the set $R = \delta(F) \setminus (A \cup B)$, so $\delta(F) = A \cup B \cup R$ is a partition of $\delta(F)$. We remark that R can be an empty set. In [2], we introduced a new type of boundary value problems in which the values of the functions and their normal derivatives are known at the same part of the boundary, which represents an overdetermined problem, and there exists another part of the boundary where no data is known. The limit case when $B = R = \emptyset$, the value of the function on the boundary is null and the value of the normal derivative is constant, can be seen as an extension of the so–called discrete Serrin's Problem; see [1].

For any $f \in \mathcal{C}(F)$, $g \in \mathcal{C}(A \cup R)$ and $h \in \mathcal{C}(A)$, the *overdetermined partial Dirichlet–Neumann boundary value problem on F with data f, g, h* consists in finding $u \in \mathcal{C}(\bar{F})$ such that

$$\mathcal{L}_q(u) = f \;\text{ on }\; F, \qquad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = h \;\text{ on }\; A \quad \text{and} \quad u = g \;\text{ on }\; A \cup R. \tag{2}$$

Notice that as the values of u are known in A, the boundary condition $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + q\,u = \hat{h}$ is equivalent to the boundary condition $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = h$, where $h = \hat{h} - qg$ and that Problem (2) is not self–adjoint. So, we consider the *adjoint problem of the overdetermined partial Dirichlet–Neumann boundary value problem (2) on F* given by

$$\mathcal{L}_q(v) = 0 \;\text{ on }\; F, \qquad \frac{\partial v}{\partial \mathsf{n}_\mathsf{F}} = v = 0 \;\text{ on }\; B \quad \text{and} \quad v = 0 \;\text{ on }\; R. \tag{3}$$

Problems (2) and (3) are mutually adjoint since

$$\int_F v(x)\mathcal{L}_q(u)(x)\,dx = \int_F u(x)\mathcal{L}_q(v)(x)\,dx,$$

for any $u, v \in \mathcal{C}(\bar{F})$ such that $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = u = 0$ on A, $\dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}} = v = 0$ on B and $u = v = 0$ on R.

In order to analyze the existence and uniqueness of solution for Problem (2), we consider the *partial Dirichlet–to–Neumann map* as the linear operator $\Lambda_{A,B} : \mathcal{C}(A) \longrightarrow \mathcal{C}(B)$, that assigns to any $v \in \mathcal{C}(A)$ the function

$$\Lambda_{A,B}(v) = \frac{\partial \mathscr{P}_q(v)}{\partial \mathsf{n}_F}\chi_B.$$

In [2] we proved that $\Lambda_{B,A} = \Lambda^*_{A,B}$ and that Problem (2), has a unique solution iff $|A| = |B|$ and $\Lambda_{A,B}$ is non–singular or equivalently iff $\Lambda_{B,A}$ is non–singular. Moreover,

$$\mathsf{N}_q(A;B) = -\mathsf{C}(A;F)\mathsf{G}_q(F;F)\mathsf{C}(F;B). \tag{4}$$

In this study we have obtained an equivalent condition for the existence and uniqueness of solution that can be read directly from a submatrix of the Schrödinger operator. For that, let $\mathscr{K}_q : \mathcal{C}(\bar{F}) \longrightarrow \mathcal{C}(\bar{F})$ the operator defined as

$$\mathscr{K}_q(u) = \begin{cases} \dfrac{\partial u}{\partial \mathsf{n}_F} & \text{on } \delta(F), \\ \mathscr{L}_q(u) & \text{on } F. \end{cases}$$

Theorem 2.1. *The overdetermined partial boundary value problem* (2) *has a unique solution for any data iff the matrix* $\mathsf{K}_q(A \cup F; F \cup B)$ *is invertible. Moreover,*

$$\operatorname{rank} \mathsf{C}(A; V(A)) = \operatorname{rank} \mathsf{C}(B; V(B)) = |A|,$$

which implies that $|A| \leq \min\{|V(A)|, |V(B)|\}$.

Proof. First observe that Problem (2) is equivalent to the overdetermined partial semi–homegeneous boundary value problem

$$\mathcal{L}_q(v) = f - \mathcal{L}_q(g) \ \text{ on } \ F, \quad v = 0 \ \text{ on } \ A \cup R \ \text{ and } \ \frac{\partial v}{\partial \mathsf{n}_F} = h - \frac{\partial g}{\partial \mathsf{n}_F} \ \text{ on } \ A, \tag{5}$$

in the sense that u is the solution of Problem (2) iff $u = v + g$. Therefore, we can restrict ourselves to the study of solution of overdetermined partial semi–homegeneous boundary value problem

$$\mathcal{L}_q(v) = f \ \text{ on } \ F, \quad v = 0 \ \text{ on } \ A \cup R \ \text{ and } \ \frac{\partial v}{\partial \mathsf{n}_F} = h \ \text{ on } \ A. \tag{6}$$

The matrix expression of this problem is

$$\mathsf{K}_q(A \cup F; F \cup B) \begin{bmatrix} \mathsf{u}(F) \\ \mathsf{u}(B) \end{bmatrix} = \begin{bmatrix} -\mathsf{C}(A;F) & 0 \\ \mathsf{L}_q(F;F) & -\mathsf{C}(F;B) \end{bmatrix} \begin{bmatrix} \mathsf{u}(F) \\ \mathsf{u}(B) \end{bmatrix} = \begin{bmatrix} \mathsf{h} \\ \mathsf{f} \end{bmatrix}.$$

Hence, Problem (6) has solution for any data iff $\mathsf{K}_q(A \cup F; F \cup B)$ is invertible. In particular,

$$|A| = \operatorname{rank} \mathsf{C}(A;F) = \operatorname{rank} \mathsf{C}(A;V(A)) \leq |V(A)|.$$

□

From now on we assume that $\mathsf{K}_q(A \cup F; F \cup B)$ is invertible. Recall that this fact is equivalent to de invertibility of $\Lambda_{A,B}$. The fact that the Problem (2) has a unique solution, implies that the value of u on B is determined by the data, as the following result shows; see [2] for the case $f = 0$.

Proposition 2.2. *Let u be the solution of Problem* (2)*, then the values of u on B are determined by the identity*

$$u = \Lambda_{B,A}^{-1}(h) - \Lambda_{B,A}^{-1}\big(\Lambda_{A\cup R,A}(g)\big) - \Lambda_{B,A}^{-1}\left(\frac{\partial \mathscr{G}_q(f)}{\partial \mathsf{n}_\mathsf{F}} \chi_A\right).$$

Proof. If $\psi = u\chi_B$ and $\varphi = \psi + g$, then u is the unique solution of the Dirichlet problem

$$\mathcal{L}_q(u) = f \ \text{ on } \ F \quad \text{ and } \quad u = \varphi \ \text{ on } \ \delta(F).$$

Moreover, from the superposition principle $u = \mathscr{G}_q(f) + \mathscr{P}_q(\psi) + \mathscr{P}_q(g)$ and hence

$$\frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} = \frac{\partial \mathscr{G}_q(f)}{\partial \mathsf{n}_\mathsf{F}} + \frac{\partial \mathscr{P}_q(\psi)}{\partial \mathsf{n}_\mathsf{F}} + \frac{\partial \mathscr{P}_q(g)}{\partial \mathsf{n}_\mathsf{F}}.$$

Therefore,

$$h = \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}}\chi_A = \frac{\partial \mathscr{G}_q(f)}{\partial \mathsf{n}_\mathsf{F}}\chi_A + \Lambda_{B,A}(u) + \Lambda_{A\cup R,A}(g)$$

and the result follows from the invertibility of $\Lambda_{B,A}$. □

Associated with overdetermined partial Dirichlet–Neumann boundary value problems we can define the resolvent operators and their corresponding kernels. For this we consider the following semi–homogeneous overdetermined partial boundary value problems:

$$\mathcal{L}_q(v_f) = f \ \text{ on } \ F, \quad v_f = 0 \ \text{ on } \ A\cup R \quad \text{and} \quad \frac{\partial v_f}{\partial \mathsf{n}_F} = 0 \ \text{ on } \ A, \tag{7}$$

$$\mathcal{L}_q(v_g) = 0 \ \text{ on } \ F, \quad v_g = g \ \text{ on } \ A\cup R \quad \text{and} \quad \frac{\partial v_g}{\partial \mathsf{n}_F} = 0 \ \text{ on } \ A, \tag{8}$$

$$\mathcal{L}_q(v_h) = 0 \ \text{ on } \ F, \quad v_h = 0 \ \text{ on } \ A\cup R \quad \text{and} \quad \frac{\partial v_h}{\partial \mathsf{n}_F} = h \ \text{ on } \ A. \tag{9}$$

As all of them have a unique solution for any data, we define the *partial Green, Poisson and Robin operators* as

$$\widetilde{\mathscr{G}}_{A,B} \colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F\cup B), \qquad \text{where } \widetilde{\mathscr{G}}_{A,B}(f) = v_f \text{ for all } f \in \mathcal{C}(F), \tag{10}$$

$$\widetilde{\mathscr{P}}_{A,B} \colon \mathcal{C}(A\cup R) \longrightarrow \mathcal{C}(\bar{F}), \qquad \text{where } \widetilde{\mathscr{P}}_{A,B}(g) = v_g \text{ for all } g \in \mathcal{C}(A\cup R), \tag{11}$$

$$\widetilde{\mathcal{R}}_{A,B} \colon \mathcal{C}(A) \longrightarrow \mathcal{C}(F\cup B), \qquad \text{where } \widetilde{\mathcal{R}}_{A,B}(h) = v_h \text{ for all } h \in \mathcal{C}(A), \tag{12}$$

respectively. With these definitions the unique solution of the overdetermined partial Dirichlet–Neumann boundary value problem (2) can be written as

$$u = \widetilde{\mathscr{G}}_{A,B}(f) + \widetilde{\mathscr{P}}_{A,B}(g) + \widetilde{\mathcal{R}}_{A,B}(h).$$

We can define analogously the resolvent kernels for the adjoint problem

$$\widetilde{\mathscr{G}}_{B,A} \colon \mathcal{C}(F) \longrightarrow \mathcal{C}(F\cup A), \qquad \text{where } \widetilde{\mathscr{G}}_{B,A}(f) = v_f \text{ for all } f \in \mathcal{C}(F) \tag{13}$$

$$\widetilde{\mathscr{P}}_{B,A} \colon \mathcal{C}(B\cup R) \longrightarrow \mathcal{C}(\bar{F}), \qquad \text{where } \widetilde{\mathscr{P}}_{B,A}(g) = v_g \text{ for all } g \in \mathcal{C}(B\cup R) \tag{14}$$

$$\widetilde{\mathcal{R}}_{B,A} \colon \mathcal{C}(B) \longrightarrow \mathcal{C}(F\cup A), \qquad \text{where } \widetilde{\mathcal{R}}_{B,A}(h) = v_h \text{ for all } h \in \mathcal{C}(B). \tag{15}$$

The following relations between the overdetermined partial operators are a straightforward consequence of the second Green's identity, see [4].

Proposition 2.3. *The overdetermined partial Green and Poisson operators, satisfy:*

$$\int_F \widetilde{\mathscr{G}}_{A,B}(f)g = \int_F \widetilde{\mathscr{G}}_{B,A}(g)f, \qquad f,g \in \mathcal{C}(F),$$

$$\int_{B\cup R} \frac{\partial \widetilde{\mathscr{P}}_{A,B}(f)}{\partial \mathsf{n}_F} g = \int_{A\cup R} \frac{\partial \widetilde{\mathscr{P}}_{B,A}(g)}{\partial \mathsf{n}_F} f, \qquad f \in \mathcal{C}(A\cup R), \quad g \in \mathcal{C}(B\cup R),$$

$$\int_B \widetilde{\mathcal{R}}_{A,B}(f)g = \int_A \widetilde{\mathcal{R}}_{B,A}(g)f, \qquad f \in \mathcal{C}(A), \quad g \in \mathcal{C}(B).$$

Our next aim is to express the kernels of the above operators in terms of the Green kernel $\mathscr{G}_q$ and the Dirichlet–to–Robin map Λ_q, for whom we already know important attributes.

Let $\widetilde{G}_{A,B}\colon (F\cup B)\times F \longrightarrow \mathbb{R}$, $\widetilde{P}_{A,B}\colon \bar{F}\times (A\cup R) \longrightarrow \mathbb{R}$ and $\widetilde{R}_{A,B}\colon (F\cup B)\times A \longrightarrow \mathbb{R}$ be the *partial Green, Poisson* and *Robin kernels*, respectively. We denote by $\widetilde{\mathsf{G}}_{A,B}\in \mathcal{M}_{|F\cup B|\times|F|}(\mathbb{R})$, $\widetilde{\mathsf{P}}_{A,B}\in \mathcal{M}_{|\bar{F}|\times|A\cup R|}(\mathbb{R})$ and $\widetilde{\mathsf{R}}_{A,B}\in \mathcal{M}_{|F\cup B|\times|A|}(\mathbb{R})$ their associated matrices.

It will be useful to define the following bilinear form $\mathscr{B}_{B,A}:\mathcal{C}(B)\times\mathcal{C}(A)\longrightarrow\mathbb{R}$, given by

$$\mathscr{B}_{B,A}(f,g)=\langle f,\Lambda^{-1}_{B,A}(g)\rangle,\quad f\in\mathcal{C}(A),\quad g\in\mathcal{C}(B).$$

Proposition 2.4. *The partial Green kernel $\widetilde{G}_{A,B}$ can be expressed in terms of the Green kernel and its normal derivatives and the partial Dirichlet–to–Robin map as*

$$\widetilde{G}_{A,B}(x,y)=G_q(x,y)-\Lambda^{-1}_{B,A}\left(\frac{\partial G^y_q}{\partial \mathsf{n}_F}\chi_A\right)(x)+\mathscr{B}_{B,A}\Big(\frac{\partial G^x_q}{\partial \mathsf{n}_F}\chi_B,\frac{\partial G^y_q}{\partial \mathsf{n}_F}\chi_A\Big).$$

Proof. Let $y\in F$ and let $v=\widetilde{\mathscr{G}}_{A,B}(\varepsilon_y)\in\mathcal{C}(F\cup B)$ be the unique solution of Problem (7) for $f=\varepsilon_y$ and consider $v_B=v\chi_B\in\mathcal{C}(B)$. Then, v is the unique solution of

$$\mathcal{L}_q(v)=\varepsilon_y \ \text{ on } \ F \quad\text{and}\quad v=v_B \ \text{ on } \ \delta(F) \tag{16}$$

and it satisfies the additional condition $\dfrac{\partial v}{\partial \mathsf{n}_F}=0$ on A. Thus, $v=\mathscr{G}_q(\varepsilon_y)+\mathscr{P}_q(v_B)$ on $\bar{F}$ and from Proposition 2.2 we obtain that

$$v_B=-\Lambda^{-1}_{B,A}\left(\frac{\partial G^y_q}{\partial \mathsf{n}_F}\chi_A\right).$$

On the other hand, for all $x\in\bar{F}$,

$$\widetilde{G}_{A,B}(x,y)=G_q(x,y)-\Lambda^{-1}_{B,A}\left(\frac{\partial G^y_q}{\partial \mathsf{n}_F}\chi_A\right)(x)+\mathscr{B}_{B,A}\Big(\frac{\partial G^x_q}{\partial \mathsf{n}_F}\chi_B,\frac{\partial G^y_q}{\partial \mathsf{n}_F}\chi_A\Big).$$

□

Corollary 2.5. *The blocks of the overdetermined partial Green matrix $\widetilde{\mathsf{G}}_{A,B}$ can be expressed in terms of the conductances, the Green and the partial Dirichlet–to–Robin matrices as*

$$\widetilde{\mathsf{G}}_{A,B}(F;F)=\mathsf{G}_q(F;F)+\mathsf{G}_q(F;F)\cdot\mathsf{C}(F;B)\cdot\mathsf{N}_q(A;B)^{-1}\cdot\mathsf{C}(A;F)\cdot\mathsf{G}_q(F;F),$$

$$\widetilde{\mathsf{G}}_{A,B}(B;F)=\mathsf{N}_q(A;B)^{-1}\cdot\mathsf{C}(A;F)\cdot\mathsf{G}_q(F;F).$$

Moreover, $\widetilde{\mathsf{G}}_{B,A}(F;F)=\widetilde{\mathsf{G}}_{A,B}(F;F)^\top$.

The next propositions show analogous results for the overdetermined partial Poisson and Robin operators. The proofs are analogous to the last one, so we let them to the reader.

Proposition 2.6. *The overdetermined partial Poisson kernel $\widetilde{P}_{A,B}$ is expressed as*

$$\widetilde{P}_{A,B}(x,y)=P_q(x,y)-\mathscr{B}_{B,A}\Big(P^x_q\chi_B,N^y_q(A;A\cup R)\Big).$$

Corollary 2.7. *The overdetermined partial Poisson matrix* $\widetilde{\mathsf{P}}_{A,B}$ *is expressed in blocks as*

$$\widetilde{\mathsf{P}}_{A,B}(F;A\cup R)=\mathsf{G}_q(F;F)\cdot\mathsf{C}(F;A\cup R)-\mathsf{G}_q(F;F)\cdot\mathsf{C}(F;B)\cdot\mathsf{N}_q(A;B)^{-1}\cdot\mathsf{N}_q(A;A\cup R)$$
$$\widetilde{\mathsf{P}}_{A,B}(B;A\cup R)=-\mathsf{N}_q(A;B)^{-1}\cdot\mathsf{N}_q(A;A\cup R)$$
$$\widetilde{\mathsf{P}}_{A,B}(A\cup R;A\cup R)=\mathsf{I}(A\cup R;A\cup R)$$

in terms of the conductances, the Green and the partial Dirichlet–to–Robin matrices.

Proposition 2.8. *The overdetermined partial Robin kernel* $\widetilde{R}_{A,B}$ *is given by*

$$\widetilde{R}_{A,B}(x,y)=\mathscr{B}_{B,A}\Big(P_q^x\chi_B,\varepsilon_y\Big).$$

Corollary 2.9. *The overdetermined partial Robin matrix* $\widetilde{\mathsf{R}}_{A,B}$ *is given by the blocks*

$$\widetilde{\mathsf{R}}_{A,B}(F;A)=\mathsf{G}_q(F;F)\cdot\mathsf{C}(F;B)\cdot\mathsf{N}_q(A;B)^{-1}$$
$$\widetilde{\mathsf{R}}_{A,B}(B;A)=\mathsf{N}_q(A;B)^{-1}$$

all in terms of the conductances, the Green and the partial Dirichlet–to–Robin matrices.

Hence, the last results provide the matrix expression of the solution of the partial Dirichlet–Neumann boundary value problem (3) in terms of the classical Green and the Dirichlet–to–Robin matrices.

Corollary 2.10. *The unique solution* $u\in\mathcal{C}(\bar{F})$ *of the overdetermined partial Dirichlet–Neumann boundary value problem* (3) *is given by the matrix equations*

$$\mathsf{u}(A\cup R)=\mathsf{g}(A\cup R)$$
$$\mathsf{u}(B)=\mathsf{N}_q(A;B)^{-1}\cdot\Big(\mathsf{C}(A;F)\cdot\mathsf{G}_q(F;F)\cdot\mathsf{f}(F)-\mathsf{N}_q(A;A\cup R)\cdot\mathsf{g}(A\cup R)+\mathsf{h}(A)\Big),$$
$$\mathsf{u}(F)=\mathsf{G}_q(F;F)\cdot\Big(\mathsf{f}(F)+\mathsf{C}(F;B)\cdot\mathsf{u}(B)+\mathsf{C}(F;A\cup R)\cdot\mathsf{g}(A\cup R)\Big).$$

The next result displays the values of the first blocks of the overdetermined partial resolvent kernels under when a geometrical hypothesis fulfills.

Corollary 2.11. *If* $|A|=|V(A)|$, *then* $\mathsf{C}(A,V(A))$ *is invertible and*

$$\widetilde{\mathsf{G}}_{A,B}(V(A);F)=\mathsf{0},$$
$$\widetilde{\mathsf{P}}_{A,B}(V(A);R)=\mathsf{0},$$
$$\widetilde{\mathsf{P}}_{A,B}(V(A);A)=\mathsf{C}(A,V(A))^{-1}\mathsf{D}_{k_F},$$
$$\widetilde{\mathsf{R}}_{A,B}(V(A);A)=-\mathsf{C}(A,V(A))^{-1}.$$

Proof. Under the assumption $|A|=|V(A)|$, from Theorem 2.1, the matrix $\mathsf{C}(A;V(A))$ is invertible. Suppose that $t\in F$ and denote by $v_t=\widetilde{G}_{A,B}(\cdot,t)$. Then,

$$0 = \chi_A \frac{\partial v_t}{\partial \mathsf{n}_F} = -\mathsf{C}(A, V(A))\mathsf{v}_t(V(A))$$

and hence the result follows. The proof of the second equality is analogue. Moreover, if $z \in A$ and we denote by $u_z = \widetilde{P}_{A,B}(\cdot, z)$ and $v_z = \widetilde{R}_{A,B}(\cdot, z)$, then

$$0 = \chi_A \frac{\partial u_z}{\partial \mathsf{n}_F} = k_F(z)\varepsilon_z - \mathsf{C}(A, V(A))\mathsf{u}_z(V(A)) \quad \text{and} \quad \varepsilon_z = \chi_A \frac{\partial v_z}{\partial \mathsf{n}_F} = -\mathsf{C}(A, V(A))\mathsf{v}_z(V(A))$$

and hence the result follows. □

Now we can obtain the inverse of the matrix $\mathsf{K}_q(A \cup F; F \cup B)$ in terms of the overdetermined partial resolvent kernels.

Proposition 2.12. *The inverse of* $\mathsf{K}_q(A \cup F; F \cup B)$ *is*

$$\mathsf{K}_q(A \cup F; F \cup B)^{-1} = \begin{bmatrix} \widetilde{\mathsf{R}}_{A,B}(F; A) & \widetilde{\mathsf{G}}_{A,B}(F; F) \\ \widetilde{\mathsf{R}}_{A,B}(B; A) & \widetilde{\mathsf{G}}_{A,B}(B; F) \end{bmatrix}.$$

Proof. Let

$$\mathsf{M} = \begin{bmatrix} \widetilde{\mathsf{R}}_{A,B}(F; A) & \widetilde{\mathsf{G}}_{A,B}(F; F) \\ \widetilde{\mathsf{R}}_{A,B}(B; A) & \widetilde{\mathsf{G}}_{A,B}(B; F) \end{bmatrix} \in \mathcal{M}_{|A \cup F|}(\mathbb{R}),$$

then the product $\mathsf{K}_q(A \cup F; F \cup B) \cdot \mathsf{M}$ is given by the matrix

$$\begin{bmatrix} -\mathsf{C}(A; F) \cdot \widetilde{\mathsf{R}}_{A,B}(F; A) & -\mathsf{C}(A; F) \cdot \widetilde{\mathsf{G}}_{A,B}(F; F) \\ -\mathsf{C}(F; B) \cdot \widetilde{\mathsf{R}}_{A,B}(B; A) + \mathsf{L}_q(F; F) \cdot \widetilde{\mathsf{R}}_{A,B}(F; A) & -\mathsf{C}(F; B) \cdot \widetilde{\mathsf{G}}_{A,B}(B; F) + \mathsf{L}_q(F; F) \cdot \widetilde{\mathsf{G}}_{A,B}(F; F) \end{bmatrix}.$$

From Equation (4) and Corollaries 2.5 and 2.9 we get that

$$\begin{aligned} -\mathsf{C}(A; F) \cdot \widetilde{\mathsf{R}}_{A,B}(F; A) &= \mathsf{I}(A; A), \\ -\mathsf{C}(F; B) \cdot \widetilde{\mathsf{R}}_{A,B}(B; A) + \mathsf{L}_q(F; F) \cdot \widetilde{\mathsf{R}}_{A,B}(F; A) &= 0, \\ -\mathsf{C}(A; F) \cdot \widetilde{\mathsf{G}}_{A,B}(F; F) &= 0, \\ -\mathsf{C}(F; B) \cdot \widetilde{\mathsf{G}}_{\mathsf{A,B}}(\mathsf{B}; \mathsf{F}) + \mathsf{L}_q(F; F) \cdot \widetilde{\mathsf{G}}_{A,B}(F; F) &= \mathsf{I}(F; F), \end{aligned}$$

and hence M is the inverse of $\mathsf{K}_q(A \cup F; B \cup F)$. □

3. Generalized cylinders

In this section we give an explicit expression for the overdetermined partial resolvent kernels on generalized cylinder. To begin with and starting from a network, we can define a *network with boundary* as $\Gamma = (H \cup \delta(H), c_H)$ where H is a proper subset and $c_H = c \cdot \mathbf{1}_{(\bar{H} \times \bar{H}) \setminus (\delta(H) \times \delta(H))}$. From now on we will work with networks with boundary and for sake of simplicity we denote $c_H = c$.

Given a network with boundary $\Gamma = (H \cup \delta(H), c)$ with a weight $\sigma \in \Omega(H \cup \delta(H))$ and a path P with vertex set $\{x_0, \ldots, x_{\ell+1}\}$ and conductance $c_j = c(x_{j-1}, x_j) > 0$, for all $j = 1, \ldots, \ell + 1$. We define the *generalized cylinder, with base* Γ as the network with boundary whose vertex set is

$$\big(\{x_j\}_{j=0}^{\ell+1} \times H\big) \cup \big(\{x_j\}_{j=1}^{\ell} \times \delta(H)\big)$$

and whose conductance is given by

$$\begin{aligned} c\big((x_j,y),(x_j,z)\big) &= c(y,z), && j=1,\dots,\ell, && y,z\in H\cup\delta(H),\\ c\big((x_{i-1},y),(x_i,y)\big) &= c_i, && i=1,\dots,\ell+1, && y\in H,\\ c\big((x_i,y),(x_j,z)\big) &= 0, && \text{otherwise.} \end{aligned}$$

Moreover, we consider the weight $\omega(x_i,y)=(\ell+2||\sigma_{_H}||^2)^{-\frac{1}{2}}\sigma(y)$.

If $F=\{x_j\}_{j=1}^{\ell}\times H$, then $\delta(F)=A\cup B\cup R$, where the sets that provide this partition are defined as $A=\{x_0\}\times H$, $B=\{x_{\ell+1}\}\times H$ and $R=\{x_j\}_{j=1}^{\ell}\times\delta(H)$. On account of simplicity, we also define the sets $A_k=\{x_k\}\times H$, $k=0,\dots,\ell+1$ and $R_k=\{x_k\}\times\delta(H)$, $k=1,\dots,\ell$. In particular, $A_0=A$, $A_1=V(A)$, $A_\ell=V(B)$ and $A_{\ell+1}=B$, see Figure 1 for an illustration of a generalized cylinder. From now on, whenever an ordering in $F\cup B$ is needed we consider $\{A_1;\dots;A_\ell;B\}$.

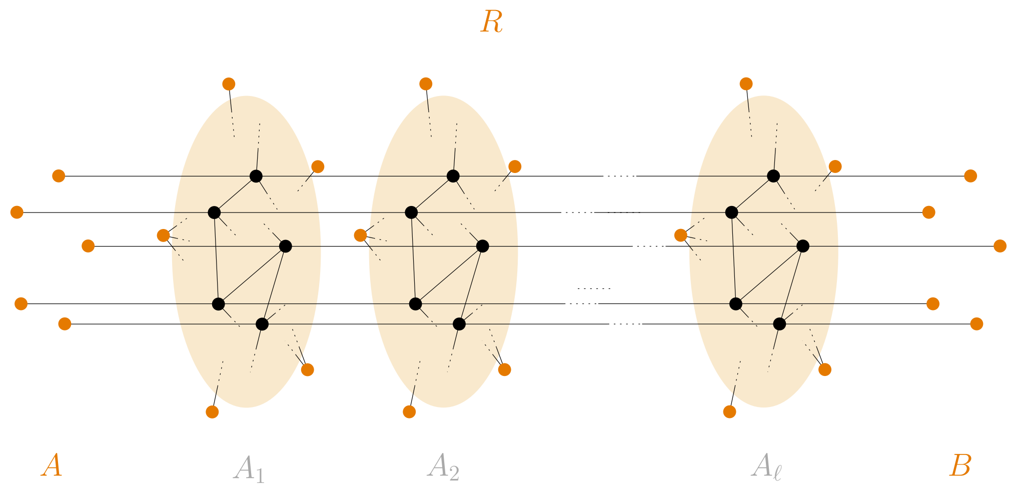

Figure 1: Graphical representation of a generalized cylinder.

In what follows we consider $q=q_\omega+\lambda\chi_{_{\delta(F)}}$, where $\lambda\geq 0$.

Proposition 3.1. *The Schrödinger operator on F with respect ω and λ for a generalized cylinder is given by*

$$\mathcal{L}_{q_\omega}=\mathcal{L}^{\Gamma}_{q_\sigma}+\mathcal{L}^{P},$$

where $\mathcal{L}^{\Gamma}_{q_\sigma}$ is the Schrödinger operator with respect σ and λ for the network Γ and $\mathcal{L}^{P}$ is the combinatorial laplacian for the path.

Proof. Observe that $\mathcal{L}=\mathcal{L}^{\Gamma}+\mathcal{L}^{P}$, where $\mathcal{L}^{P}u(x_i,y)=\mathcal{L}^{P}u^y(x_i)=c_i(u^y(x_i)-u^y(x_{i-1}))+c_{i+1}(u^y(x_i)-u^y(x_{i+1}))$ and $\mathcal{L}^{\Gamma}u(x_i,y)=\mathcal{L}^{\Gamma}u^{x_i}(y)$, for any $i=1,\dots,\ell$ and $y\in H$. Therefore,

$$q_\omega(x_i,y)=-\frac{\mathcal{L}\omega}{\omega}(x_i,y)=-\frac{\mathcal{L}^{\Gamma}\sigma(y)}{\sigma(y)}-\frac{\mathcal{L}^{P}\omega^y(x_i)}{\omega^y(x_i)}=q_\sigma(y)$$

and hence

$$\mathcal{L}_{q_\omega}=\mathcal{L}^{\Gamma}_{q_\sigma}+\mathcal{L}^{P}.$$

□

Corollary 3.2. *The submatrix* $\mathsf{K}_q(A\cup F;F\cup B)$ *has the following block structure*

$$\mathsf{K}_q(A\cup F;F\cup B)=\begin{pmatrix} -c_1\mathsf{I} & 0 & 0 & \cdots & 0 & 0\\ \mathsf{Q}_1 & -c_2\mathsf{I} & 0 & \cdots & 0 & 0\\ -c_2\mathsf{I} & \mathsf{Q}_2 & -c_3\mathsf{I} & \cdots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & \cdots & -c_{\ell-1}\mathsf{I} & \mathsf{Q}_{\ell-1} & -c_\ell\mathsf{I} & 0\\ 0 & \cdots & \cdots & -c_\ell\mathsf{I} & \mathsf{Q}_\ell & -c_{\ell+1}\mathsf{I}\end{pmatrix},$$

where $\mathsf{Q}_i=\mathsf{L}^{\Gamma}_{q_\sigma}(H;H)+(c_i+c_{i+1})\mathsf{I}$. *Therefore,* $\Lambda_{A,B}$ *is an isomorphism.*

Now we study the overdetermined partial resolvent kernels associated with the overdetermined partial boundary value problem.

Theorem 3.3. *Let* $\widetilde{\mathsf{G}}_{ij}=\widetilde{G}_{A,B}(A_i;A_j)$*, for all* $i=1,\dots,\ell+1$ *and* $j=1,\dots,\ell$*. Then, for any* $j=1,\dots,\ell$*, it is satisfied that*

$$\begin{aligned}\widetilde{\mathsf{G}}_{ij}&=0, && 1\le i\le j,\\ \widetilde{\mathsf{G}}_{j+1j}&=-\frac{1}{c_{j+1}}\mathsf{I},\\ c_{i+2}\widetilde{\mathsf{G}}_{i+2j}-\mathsf{Q}_{i+1}\widetilde{\mathsf{G}}_{i+1j}+c_{i+1}\widetilde{\mathsf{G}}_{ij}&=0, && j\le i\le \ell-1.\end{aligned}$$

Proof. Since $\mathsf{L}_q(F;F\cup B)\widetilde{\mathsf{G}}_{A,B}(F\cup B;F)=\mathsf{I}$, it is satisfied

$$\begin{aligned}\mathsf{Q}_1\widetilde{\mathsf{G}}_{1j}-c_2\widetilde{\mathsf{G}}_{2j}&=\varepsilon_j(1)\mathsf{I}, \quad j=1,\dots,\ell,\\ c_i\widetilde{\mathsf{G}}_{i-1j}-\mathsf{Q}_i\widetilde{\mathsf{G}}_{ij}+c_{i+1}\widetilde{\mathsf{G}}_{i+1j}&=-\varepsilon_j(i)\mathsf{I}, \quad i=2,\dots,\ell,\ \ j=1,\dots,\ell.\end{aligned}$$

Let us prove by induction that for any $j=1,\dots,\ell$, $1\le i\le j$, $\widetilde{\mathsf{G}}_{ij}=0$ and $\widetilde{\mathsf{G}}_{j+1j}=-\frac{1}{c_{j+1}}\mathsf{I}$. The case $i=1$ follows from Corollary 2.11 and hence from the first equation of the above system we have that $\widetilde{\mathsf{G}}_{2j}=0$ for any $j=2,\dots,\ell$ and when $j=1$, $\widetilde{\mathsf{G}}_{21}=-\frac{1}{c_2}\mathsf{I}$. Suppose that $\widetilde{\mathsf{G}}_{kj}=0$ for any $1\le k\le i$. From the above system of equations, we have that

$$c_i\widetilde{\mathsf{G}}_{i-1j}-\mathsf{Q}_i\widetilde{\mathsf{G}}_{ij}+c_{i+1}\widetilde{\mathsf{G}}_{i+1j}=-\varepsilon_j(i)\mathsf{I}$$

and hence for $i+1\le j$, $\widetilde{\mathsf{G}}_{i+1j}=0$ and for $i=j$, $\widetilde{\mathsf{G}}_{j+1j}=-\frac{1}{c_{j+1}}\mathsf{I}$. □

Similar techniques allow us to obtain the overdetermined partial Poisson and Robin kernels associated with the overdetermined partial boundary value problem.

Theorem 3.4. *For all* $i=1\dots,\ell+1$ *and* $j=1,\dots,\ell$ *let* $\widetilde{\mathsf{P}}_{ij+1}=\widetilde{P}_{A,B}(A_i;R_j)$ *and* $\widetilde{\mathsf{P}}_{i1}=\widetilde{P}_{A,B}(A_i;A)$ *be the blocks of* $\widetilde{P}_{A,B}$*. Then, defining* $\widetilde{\mathsf{P}}_{01}=\mathsf{I}$ *and* $\widetilde{\mathsf{P}}_{0j+1}=0$*, for any* $j=1,\dots,\ell$*, it is satisfied that*

$$\begin{aligned}\widetilde{\mathsf{P}}_{ij+1}&=0, && 1\le i\le j,\\ \widetilde{\mathsf{P}}_{11}=\mathsf{I},\widetilde{\mathsf{P}}_{j+1j+1}&=-\frac{1}{c_{j+1}}\mathsf{C},\\ c_{i+1}\widetilde{\mathsf{P}}_{i+1j}-\mathsf{Q}_i\widetilde{\mathsf{P}}_{ij}+c_i\widetilde{\mathsf{P}}_{i-1j}&=0, && j\le i\le \ell.\end{aligned}$$

Proof. Since $\mathsf{K}_q(F;F\cup B)\widetilde{\mathsf{P}}_{A,B}(F\cup B;A\cup R)=-\mathsf{K}_q(F;A\cup R)$, it is satisfied

$$\begin{aligned} c_2\widetilde{\mathsf{P}}_{21}-\mathsf{Q}_1\widetilde{\mathsf{P}}_{11}&=-c_1\mathsf{I},\\ c_2\widetilde{\mathsf{P}}_{2j+1}-\mathsf{Q}_1\widetilde{\mathsf{P}}_{1j+1}&=-\varepsilon_{j+1}(2)\mathsf{C}, \qquad j=1,\dots,\ell,\\ c_{i+1}\widetilde{\mathsf{P}}_{i+1j+1}-\mathsf{Q}_i\widetilde{\mathsf{P}}_{ij+1}+c_i\widetilde{\mathsf{P}}_{i-1j+1}&=-\varepsilon_{j+1}(i+1)\mathsf{C}, \quad i=2,\dots,\ell, \quad j=0,\dots,\ell. \end{aligned}$$

Therefore, defining $\widetilde{\mathsf{P}}_{01}=\mathsf{I}$ and $\widetilde{\mathsf{P}}_{0j+1}=\mathsf{0}$, for all $j=1,\dots,\ell$, the above system becomes

$$c_{i+1}\widetilde{\mathsf{P}}_{i+1j+1}-\mathsf{Q}_i\widetilde{\mathsf{P}}_{ij+1}+c_i\widetilde{\mathsf{P}}_{i-1j+1}=-\varepsilon_{j+1}(i+1)\mathsf{C}, \quad i=1,\dots,\ell, \quad j=0,\dots,\ell.$$

Let us prove by induction that for any $i=1,\dots,\ell$ and $1\le i\le j$, $\widetilde{\mathsf{P}}_{ij+1}=\mathsf{0}$ and $\widetilde{\mathsf{P}}_{j+1j+1}=-\dfrac{1}{c_{j+1}}\mathsf{C}$. From Corollary 2.11 we know that $\widetilde{\mathsf{P}}_{1j+1}=\mathsf{0}$ and that $\widetilde{\mathsf{P}}_{11}=\mathsf{I}$. Moreover,

$$c_2\widetilde{\mathsf{P}}_{2j+1}-\mathsf{Q}_1\widetilde{\mathsf{P}}_{1j+1}+c_1\widetilde{\mathsf{P}}_{0j+1}=-\varepsilon_{j+1}(2)\mathsf{C}, \quad j=0,\dots,\ell$$

and hence for $j=1$, $\widetilde{\mathsf{P}}_{22}=-\dfrac{1}{c_2}\mathsf{C}$ and for $j=2,\dots,\ell$, $\widetilde{\mathsf{P}}_{2j+1}=\mathsf{0}$. Suppose that $\widetilde{\mathsf{P}}_{kj+1}=\mathsf{0}$ for any $1\le k\le i\le j-1$. From the above system of equations, we have that

$$c_{i+1}\widetilde{\mathsf{P}}_{i+1j+1}-\mathsf{Q}_i\widetilde{\mathsf{P}}_{ij+1}+c_i\widetilde{\mathsf{P}}_{i-1j+1}=-\varepsilon_{j+1}(i+1)\mathsf{C}$$

and hence for $i+1\le j$, $\widetilde{\mathsf{P}}_{i+1j+1}=\mathsf{0}$ and for $i=j$, $\widetilde{\mathsf{P}}_{j+1j+1}=-\dfrac{1}{c_{j+1}}\mathsf{C}$. □

Theorem 3.5. *For all $i=1,\dots,\ell+1$, let $\widetilde{\mathsf{R}}_i=\widetilde{R}_{A,B}(A_i;A)$ be the blocks of $\widetilde{R}_{A,B}$. Then, it is satisfied that*

$$c_{i+1}\widetilde{\mathsf{R}}_{i+1}-\mathsf{Q}_i\widetilde{\mathsf{R}}_i+c_i\widetilde{\mathsf{R}}_{i-1}=\mathsf{0}, \quad 1\le i\le\ell, \qquad \widetilde{\mathsf{R}}_0=\mathsf{0}, \quad \widetilde{\mathsf{R}}_1=-\frac{1}{c_1}\mathsf{I}.$$

Proof. Since $\mathsf{K}_q(F;F\cup B)\widetilde{\mathsf{R}}_{A,B}(F\cup B;A)=\mathsf{0}$, it is satisfied

$$\begin{aligned} c_2\widetilde{\mathsf{R}}_2-\mathsf{Q}_1\widetilde{\mathsf{R}}_1&=\mathsf{0},\\ c_{i+1}\widetilde{\mathsf{R}}_{i+1}-\mathsf{Q}_i\widetilde{\mathsf{R}}_i+c_i\widetilde{\mathsf{R}}_{i-1}&=\mathsf{0}, \quad i=2,\dots,\ell. \end{aligned}$$

Moreover, from Corollary 2.11 we get that $\widetilde{\mathsf{R}}_1=-\dfrac{1}{c_1}\mathsf{I}$ and hence, defining $\widetilde{\mathsf{R}}_0=\mathsf{0}$, the result follows. □

Note that from the above theorem the blocks of the partial kernels that are not yet determined can be obtained from the solution of a initial value problem for a linear second order difference equation over the matrix ring. This class of equations was studied in a more general context by some of the authors in [10]. For the sake of completeness we introduce here the basic notation to give the solution of the initial value problem and to prove directly the claim.

A *binary multi–index of length* $p\in\mathbb{N}^*$ is a p–tupla $\alpha=(\alpha_1,\dots,\alpha_p)\in\{0,1\}^p$ and its *strength* is $|\alpha|=\sum\limits_{j=1}^{p}\alpha_j$. The set of multi–indices of length p is denoted by ℓ_p and we have that $\#\ell_p=2^p$, for any $p \in \mathbb{N}^*$. Given $p \in \mathbb{N}^*$ and $\alpha \in \ell_p$, then $0 \le |\alpha| \le p$. Moreover, if $|\alpha| = m \ge 1$, we denote by $i_1, \dots, i_m$ the indices such that $1 \le i_1 < \cdots < i_m \le p$ and $\alpha_{i_j} = 1$, $j = 1, \dots, m$.

For any $p \in \mathbb{N}^*$, we are interested in considering only some binary multi–index of length p whose strength is $\lfloor \frac{p}{2} \rfloor$ at most. Next, we define the following sets:

(i) For $p \in \mathbb{N}^*$, $\ell_p^0 = \Big\{\alpha \in \ell_p : |\alpha| = 0\Big\} = \Big\{(0, \dots, 0)\Big\}$.

(ii) For $p \ge 2$, $\ell_p^1 = \Big\{\alpha \in \ell_p : \alpha_p = 0 \text{ and } |\alpha| = 1\Big\}$.

(iii) For $p \ge 4$ and $m = 2, \dots, \lfloor \frac{p}{2} \rfloor$,

$$\ell_p^m = \Big\{\alpha \in \ell_p : \alpha_p = 0,\ \ |\alpha| = m \text{ and } i_{j+1} - i_j \ge 2,\ \ j = 1, \dots, m-1\Big\}.$$

Clearly, $\#\ell_p^0 = 1$ for any $p \in \mathbb{N}^*$ and $\#\ell_p^1 = p - 1$ for any $p \ge 2$. On the other hand, if $p \ge 4$, since choosing m locations for the ones in $\alpha \in \ell_p^m$, $m = 2, \dots, \lfloor \frac{p}{2} \rfloor$, implies fixing other $m - 1$ locations with zeroes between i_1 an i_m, we can choose m locations among $p - 1 - (m - 1)$ available, which implies that $\#\ell_p^m = \binom{p-m}{m}$. Moreover, this formula also works for $\#\ell_p^0$, $p \in \mathbb{N}^*$, and for $\#\ell_p^1$, $p \ge 2$.

Given $p \in \mathbb{N}^*$ and $m = 0, \dots, \lfloor \frac{p}{2} \rfloor$, for any $\alpha \in \ell_p^m$, its *complementary* is $\bar{\alpha} = (\bar{\alpha}_1, \dots, \bar{\alpha}_p) \in \ell_p$, the binary multi–index of length p defined as

$$\bar{\alpha}_{i_j} = \bar{\alpha}_{i_j+1} = 0, \quad j = 1, \dots, m \text{ and } \bar{\alpha}_i = 1, \quad i = 1, \dots, p, \ \ i \ne i_j, i_j + 1, \quad j = 1, \dots, m.$$

It is clear that $|\bar{\alpha}| = p - 2m$. In particular, if $\alpha \in \ell_p^0$, $p \in \mathbb{N}^*$, then $\bar{\alpha} = (1, \dots, 1)$.

Given $r, s \in \mathbb{N}^*$, $a_0, \dots, a_r \in \mathbb{R}^*$ and $\mathsf{B}_1, \dots, \mathsf{B}_r \in \mathcal{M}_s(\mathbb{R})$, where $\mathcal{M}_s(\mathbb{R})$ is the space of square matrices of order s and real coefficients, we define the sequence of matrices in $\mathcal{M}_s(\mathbb{R})$

$$\begin{aligned}
&\mathsf{K}_0(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r) = \mathsf{I},\\
&\mathsf{K}_1(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r) = \mathsf{0},\\
&\mathsf{K}_2(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r) = -\frac{a_0}{a_1}\mathsf{I},\\
&\mathsf{K}_n(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r) = -a_0\Big(\prod_{i=1}^{n-1} a_i\Big)^{-1} \sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^m \sum_{\alpha \in \ell_{n-2}^m} a_2^{2\alpha_1} \cdots a_{n-1}^{2\alpha_{n-2}} \mathsf{B}_{n-1}^{\bar{\alpha}_{n-2}} \cdots \mathsf{B}_2^{\bar{\alpha}_1},
\end{aligned} \tag{17}$$

for any $n = 3, \dots, r + 1$, and the sequence of matrices in $\mathcal{M}_s(\mathbb{R})$

$$\begin{aligned}
&\mathsf{M}_0(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r) = \mathsf{0},\\
&\mathsf{M}_1(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r) = \mathsf{I},\\
&\mathsf{M}_n(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r) = \Big(\prod_{i=1}^{n-1} a_i\Big)^{-1} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^m \sum_{\alpha \in \ell_{n-1}^m} a_1^{2\alpha_1} \cdots a_{n-1}^{2\alpha_{n-1}} \mathsf{B}_{n-1}^{\bar{\alpha}_{n-1}} \cdots \mathsf{B}_1^{\bar{\alpha}_1},
\end{aligned} \tag{18}$$

for any $n = 2, \dots, r + 1$. We omit the reference to $a_0, \dots, a_r$ and to $\mathsf{B}_1, \dots, \mathsf{B}_r$ in the above expressions when it does not lead to confussion.

The following result displays an explicit formula for the solution of initial value problems for second order difference equations, see [10] for its proof.

Proposition 3.6. *Given* $r, s \in \mathbb{N}^*$, $a_0, \dots, a_r \in \mathbb{R}^*$ *and* $\mathsf{B}_1, \dots, \mathsf{B}_r \in \mathcal{M}_s(\mathbb{R})$, *then for any* $\mathsf{X}_0, \mathsf{X}_1 \in \mathcal{M}_s(\mathbb{R})$, *the unique solution of the initial value problem*

$$a_n \mathsf{Z}_{n+1} - \mathsf{B}_n \mathsf{Z}_n + a_{n-1} \mathsf{Z}_{n-1} = \mathsf{0}, \quad n = 1, \dots, r, \quad \mathsf{Z}_0 = \mathsf{X}_0, \ \mathsf{Z}_1 = \mathsf{X}_1$$

is

$$\mathsf{Z}_n = \mathsf{K}_n(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r)\mathsf{X}_0 + \mathsf{M}_n(a_0, \dots, a_r; \mathsf{B}_1, \dots, \mathsf{B}_r)\mathsf{X}_1, \quad n = 0, \dots, r+1.$$

The following result refers to the case in which the coefficients of the difference equation are constant, where we use the Chebyshev polynomials of second kind that are defined as

$$U_{-2}(x) = -1, \quad U_{-1}(x) = -0 \ \text{ and } \ U_n(x) = \sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^m \binom{n-m}{m} (2x)^{n-2m}, \quad n \in \mathbb{N},$$

see [13].

Corollary 3.7. *Given* $r, s \in \mathbb{N}^*$, $a \neq 0$ *and* $\mathsf{B} \in \mathcal{M}_s(\mathbb{R})$ *defining* $\mathsf{P} = \dfrac{1}{2a}\mathsf{B}$ *then, for any* $\mathsf{X}_0, \mathsf{X}_1 \in \mathcal{M}_s(\mathbb{R})$, *the unique solution of the initial value problem*

$$a\mathsf{Z}_{n+1} - \mathsf{B}\mathsf{Z}_n + a\mathsf{Z}_{n-1} = \mathsf{0}, \quad n = 1, \dots, r, \quad \mathsf{Z}_0 = \mathsf{X}_0, \ \mathsf{Z}_1 = \mathsf{X}_1$$

is

$$\mathsf{Z}_n = U_{n-1}(\mathsf{P})\mathsf{X}_1 - U_{n-2}(\mathsf{P})\mathsf{X}_0, \quad n = 0, \dots, r+1.$$

Proof. From the above Proposition it suffices to prove that for any $n = 0, \dots, r+1$ we have that

$$\mathsf{K}_n(a, \dots, a; \mathsf{B}, \dots, \mathsf{B}) = -U_{n-2}(\mathsf{P}) \ \text{ and } \ \mathsf{M}_n(a, \dots, a; \mathsf{B}, \dots, \mathsf{B}) = U_{n-1}(\mathsf{P}).$$

From identities (17) and (18) we have that $\mathsf{K}_0 = \mathsf{I} = -U_{-2}(\mathsf{P})$, $\mathsf{K}_1 = \mathsf{M}_0 = \mathsf{0} = U_{-1}(\mathsf{P})$ and $\mathsf{K}_2 = -\mathsf{I} = -U_0(\mathsf{P})$.

On the other hand, since $\gamma_n = a^{-n}$ and $\#\ell_n^m = \binom{n-m}{m}$, then for many $n = 3, \dots, r+1$ we have

$$\begin{aligned} \mathsf{K}_n &= -a a^{-(n-1)} \sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^m \binom{n-2-m}{m} a^{2m} \mathsf{B}^{n-2-2m} \\ &= -\sum_{m=0}^{\lfloor \frac{n-2}{2} \rfloor} (-1)^m \binom{n-2-m}{m} (2\mathsf{P})^{n-2-2m} = -U_{n-2}(\mathsf{P}), \end{aligned}$$

and moreover, for any $n = 2, \dots, r+1$,

$$\begin{aligned} \mathsf{M}_n &= a^{-(n-1)} \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^m \binom{n-1-m}{m} a^{2m} \mathsf{B}^{n-1-2m} \\ &= \sum_{m=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^m \binom{n-1-m}{m} (2\mathsf{P})^{n-1-2m} = U_{n-1}(\mathsf{P}). \quad \square \end{aligned}$$

Theorem 3.8. *The non–null blocks for the overdetermined partial resolvent kernels are given by the following expressions.*

(i) For all $j = 1, \dots, \ell - 1$ and $n = 0, \dots, \ell + 1 - j$, it is satisfied that

$$\widetilde{\mathsf{G}}_{j+n\,j} = -\frac{1}{c_{j+1}} \mathsf{M}_n(c_{j+1}, \dots, c_{\ell+1}; \mathsf{Q}_{j+1}, \dots, \mathsf{Q}_\ell)$$

and moreover, $\widetilde{\mathsf{G}}_{\ell\,\ell} = 0$ and $\widetilde{\mathsf{G}}_{\ell+1\,\ell} = -\dfrac{1}{c_{\ell+1}}\mathsf{I}$.

(ii) For all $n = 0, \dots, \ell + 1$, it is satisfied that

$$\widetilde{\mathsf{P}}_{n\,1} = \mathsf{K}_n(c_1, \dots, c_{\ell+1}; \mathsf{Q}_1, \dots, \mathsf{Q}_\ell) + \mathsf{M}_n(c_1, \dots, c_{\ell+1}; \mathsf{Q}_1, \dots, \mathsf{Q}_\ell)$$

whereas for all $j = 2, \dots, \ell$ and $n = 0, \dots, \ell + 2 - j$, it holds

$$\widetilde{\mathsf{P}}_{j-1+n\,j} = -\frac{1}{c_j} \mathsf{M}_n(c_j, \dots, c_{\ell+1}; \mathsf{Q}_j, \dots, \mathsf{Q}_\ell)\mathsf{C},$$

and moreover $\widetilde{\mathsf{P}}_{\ell\,\ell+1} = 0$ and $\widetilde{\mathsf{P}}_{\ell+1\,\ell+1} = -\dfrac{1}{c_{\ell+1}}\mathsf{C}$.

(iii) For all $n = 1, \dots, \ell + 1$, it is satisfied that

$$\widetilde{\mathsf{R}}_n = -\frac{1}{c_1} \mathsf{M}_n(c_1, \dots, c_{\ell+1}; \mathsf{Q}_1, \dots, \mathsf{Q}_\ell), \quad n = 1, \dots, \ell + 1.$$

Proof. In all cases the claimed expression for the corresponding resolvent kernel follows applying Proposition 3.6 to a suitable initial value problem that we describe explicitly below.

(i) Fixed $j = 1, \dots, \ell - 1$, we consider $a_k = c_{j+1+k}$, $\mathsf{Z}_k = \widetilde{\mathsf{G}}_{j+k,j}$ and $\mathsf{B}_k = \mathsf{Q}_{j+k}$, $k = 0, \dots, \ell - j$, then $\{\mathsf{Z}_k\}_{k=0}^{\ell+1-j}$ is the unique solution of the initial value problem

$$a_k \mathsf{Z}_{k+1} - \mathsf{B}_k \mathsf{Z}_k + a_{k-1} \mathsf{Z}_{k-1} = 0, \quad k = 1, \dots, \ell - j, \quad \mathsf{Z}_0 = 0, \ \mathsf{Z}_1 = -\frac{1}{c_{j+1}}\mathsf{I}.$$

(ii) Fixed $j = 2, \dots, \ell$, we consider $a_k = c_{j+k}$, $\mathsf{Z}_k = \widetilde{\mathsf{P}}_{j-1+k,j}$ and $\mathsf{B}_k = \mathsf{Q}_{j+k-1}$ $k = 1, \dots, \ell + 1 - j$, then $\{\mathsf{Z}_k\}_{k=0}^{\ell+2-j}$ is the unique solution of the initial value problem

$$a_k \mathsf{Z}_{k+1} - \mathsf{B}_k \mathsf{Z}_k + a_{k-1} \mathsf{Z}_{k-1} = 0, \quad k = 1, \dots, \ell + 1 - j, \quad \mathsf{Z}_0 = 0, \ \mathsf{Z}_1 = -\frac{1}{c_j}\mathsf{C}.$$

(iii) We consider $a_k = c_{k+1}$, $k = 0, \dots, \ell$, $\mathsf{Z}_0 = \widetilde{\mathsf{R}}_0$, $\mathsf{Z}_k = \widetilde{\mathsf{R}}_k$ and $\mathsf{B}_k = \mathsf{Q}_k$, $k = 1, \dots, \ell$, then $\{\mathsf{Z}_k\}_{k=0}^{\ell}$ is the unique solution of the initial value problem

$$a_k \mathsf{Z}_{k+1} - \mathsf{B}_k \mathsf{Z}_k + a_{k-1} \mathsf{Z}_{k-1} = 0, \quad k = 1, \dots, \ell, \quad \mathsf{Z}_0 = 0, \ \mathsf{Z}_1 = -\frac{1}{c_1}\mathsf{I}. \quad \square$$

Corollary 3.9. *When the path has constant conductances; i.e., $c_i = a > 0$ for all $i = 1, \dots, \ell + 1$, then $\mathsf{Q}_i = \mathsf{L}_{q_\omega}(H; H) + 2a\mathsf{I}$, $i = 1, \dots, \ell$. Moreover, defining $\mathsf{P} = \dfrac{1}{2a}\mathsf{L}_{q_\omega}(H; H) + \mathsf{I}$ it is satisfied that* *(i) For all* $j = 1, \dots, \ell$ *and* $n = 0, \dots, \ell + 1 - j$,

$$\widetilde{\mathsf{G}}_{j+n\,j} = -\frac{1}{a} U_{n-1}(\mathsf{P}).$$

(ii) For all $j = 1, \dots, \ell + 1$ *and* $n = 0, \dots, \ell + 2 - j$,

$$\text{when } j = 1,\ \widetilde{\mathsf{P}}_{n\,1} = U_{n-1}(\mathsf{P}) - U_{n-2}(\mathsf{P}) \ \text{ and when } j \geq 2,\ \widetilde{\mathsf{P}}_{j-1+n\,j} = -\frac{1}{a} U_{n-1}(\mathsf{P})\mathsf{C}.$$

(iii) For all $n = 1, \dots, \ell + 1$,

$$\widetilde{\mathsf{R}}_n = -\frac{1}{a} U_{n-1}(\mathsf{P}).$$

Acknowledgement This work has been partly supported by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología,) under project MTM2011-28800-C02-02 and MTM2014-60450-R.

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