

# Degree in Mathematics

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**Title: Steinhaus Triangles**

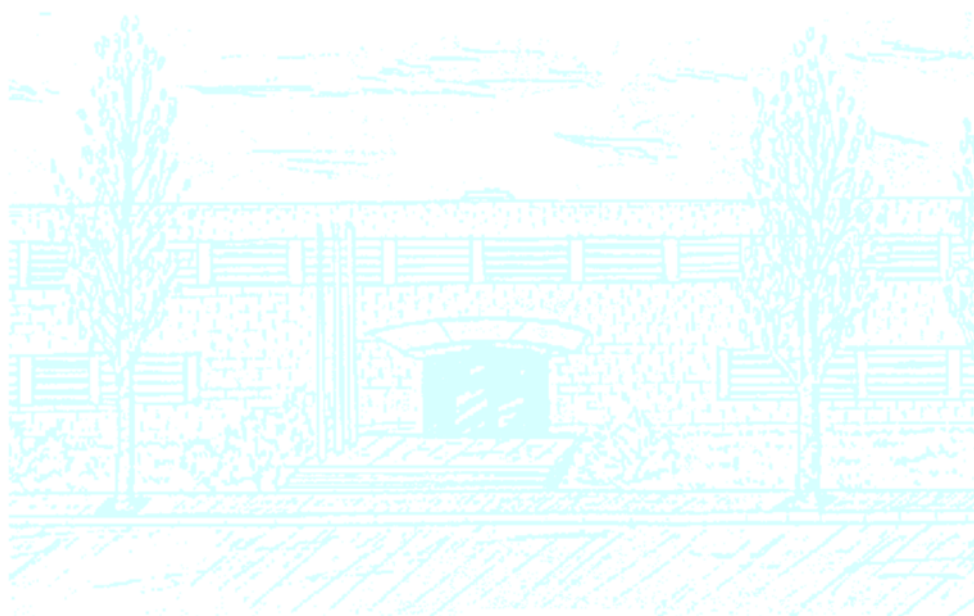
**Author: Núria Pascual Miranda**

**Advisors: Josep M. Brunat Blay**

**Montserrat Maureso Sánchez**

**Department: Matemàtica Aplicada**

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UNIVERSITAT POLITÈCNICA DE CATALUNYA

BACHELOR'S DEGREE THESIS

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# Steinhaus Triangles

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*Author:*

Núria PASCUAL MIRANDA

*Supervisors:*

Dr. Josep M. BRUNAT BLAY  
Dr. Montserrat MAURESÓ SÁNCHEZ

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*To them, they know who they are;  
to the FME, my home*



# Abstract

**Keywords:** Steinhaus's problem, Steinhaus triangle, binary sequence, balanced, zero-sum, strongly-balanced, Molluzzo's problem

**MSC2000:** 05A05, 11B50, 11A07

In this thesis, we study Steinhaus's problem. We begin with the definitions of Steinhaus triangle and balanced binary sequence, and we study sequences with different properties. We then show three different proofs of the existence of balanced binary sequences of any length  $n \equiv 0$  or  $3 \pmod{4}$ .

First, we find strongly-balanced sequences of any valid length. Then, we see that there are balanced symmetric sequences of any length  $n \equiv 0, 3$  or  $7 \pmod{8}$  and that there are balanced antisymmetric binary sequences of any length  $n \equiv 4 \pmod{8}$ . Finally, we show the existence of zero-sum balanced sequences of any length  $n \equiv 0 \pmod{4}$  and we see that this also provides us with the existence of balanced sequences of length  $n \equiv 3 \pmod{4}$ .

In the last chapter, we present Molluzzo's problem, a generalization of Steinhaus's problem. We study arithmetic progressions and antisymmetric sequences and their associated generalized triangles. On the one hand, we show the existence of balanced sequences in  $\mathbb{Z}/3^k\mathbb{Z}$  of any valid length. On the other hand, we see different cases in which Molluzzo's problem is answered negatively.





# Contents

<b>Preface</b>	<b>1</b>
<b>1 Introduction</b>	<b>3</b>
1.1 Introduction . . . . .	3
1.2 The Steinhaus Problem . . . . .	4
1.3 The first proof by Harborth (1972) . . . . .	4
1.4 The derived problems . . . . .	5
1.4.1 The Molluzzo's generalization . . . . .	7
1.5 Notation . . . . .	8
<b>2 Strongly-balanced binary sequences</b>	<b>9</b>
2.1 Introduction . . . . .	9
2.2 Notation and facts . . . . .	9
2.3 The existence . . . . .	12
2.4 The uniqueness . . . . .	16
2.5 Appendix: Figures . . . . .	18
<b>3 Symmetric and Antisymmetric</b>	<b>21</b>
3.1 Introduction . . . . .	21
3.2 The symmetric case . . . . .	22
3.2.1 The valid lengths . . . . .	22
3.2.2 The existence . . . . .	24
3.3 From the symmetric to the antisymmetric case . . . . .	34
3.4 The antisymmetric case . . . . .	35
3.5 Appendix: Figures . . . . .	36
<b>4 Zero-sum balanced binary sequences</b>	<b>43</b>
4.1 Introduction . . . . .	43
4.2 The existence . . . . .	44
4.3 Appendix: Figures . . . . .	49
<b>5 First approach to Molluzzo's problem</b>	<b>55</b>
5.1 Introduction . . . . .	55
5.2 Generalities on balanced sequences . . . . .	56
5.3 Arithmetic progressions . . . . .	57
5.3.1 Balanced arithmetic progressions in $\mathbb{Z}/m\mathbb{Z}$ for $m$ odd . . . . .	59
5.3.2 The antisymmetric case . . . . .	63
5.3.3 Balanced arithmetic progressions in $\mathbb{Z}/m\mathbb{Z}$ for $m$ even . . . . .	67

5.4	Computer simulation . . . . .	70
5.5	Appendix: Figures . . . . .	71
	<b>Bibliography</b>	<b>73</b>

# Preface

Year 1963. Hugo Steinhaus poses a problem without solution that he names *Plus and minus signs*.

He depicts the triangle below, composed of 14 plus signs and 14 minus signs. They are arranged in such a way that under each pair of equal signs there appears a positive sign and under opposite signs a minus sign.

```
+ + - + - + +
+ - - - - +
- + + + -
- + + -
- + -
- -
+
```

Then, he asks the question that nowadays is known as Steinhaus's problem. If the total number of signs is an even number, is it possible to construct a triangle with the same number of plus and minus signs and beginning with a determined number of signs in the highest row?

The mathematics involving the problem could not be easier, just products between ones and minus ones, but the resolution proves to be intricate. The problem might look suitable to think during an afternoon of boredom, but nine years pass until someone solves it. It is Heiko Harborth, and he answers Steinhaus's question positively.

It is possible to construct triangles with the same number of plus and minus signs with the first row of any length that provides an even number of elements of the triangle, as we will see in this thesis.

This type of triangles can be used for detection of errors in binary codes and for the fun of discrete mathematics.



# Chapter 1

## Introduction

### 1.1 Introduction

Let  $n$  be a positive integer, and let  $X = (x_1, x_2, \dots, x_n)$  be a binary sequence of length  $n$ , this is  $x_i \in \{-1, +1\}$  for all  $i$ .

**Definition 1.1.1.** We define the *derived sequence*  $\partial X = (y_1, y_2, \dots, y_{n-1})$  of  $X$  as the binary sequence of length  $(n - 1)$  that fulfills  $y_i = x_i x_{i+1}$  for all  $i$ . In the case  $n = 1$ , we will consider  $\partial X = \emptyset$ .

Therefore, for every  $0 \leq k \leq (n - 1)$  we can define  $\partial^k X$  as the  $k$ th derived sequence of  $X$ , that recursively seen is

$$\partial^k X = \begin{cases} X & \text{if } k = 0; \\ \partial(\partial^{k-1} X) & \text{if } k \geq 1. \end{cases}$$

**Definition 1.1.2.** The *Steinhaus triangle* of  $X$  is the sequence  $\Delta X = (X, \partial X, \dots, \partial^{n-1} X)$ .

**Example 1.1.3.** The sequence  $X = (+1, -1, +1, -1, -1, +1, -1, +1)$  of length 8 (from now on we will write the sequences in a simpler way, for example  $X = + - + - - + - +$ ) has the derived sequence  $\partial X = - - - + - - -$  of length 7, and its Steinhaus triangle is the following.

$$\begin{array}{cccccccc} + & - & + & - & - & + & - & + \\ & - & - & - & + & - & - & - \\ & & + & + & - & - & + & + \\ & & & + & - & + & - & + \\ & & & & - & - & - & - \\ & & & & & + & + & + \\ & & & & & & + & + \\ & & & & & & & + \end{array}$$

**Definition 1.1.4.** We say that a Steinhaus triangle  $\Delta X$  of a binary sequence  $X$  is *balanced* if it has the same number of  $+1$  as  $-1$ .

**Definition 1.1.5.** We say that a binary sequence  $X$  is *balanced* if its Steinhaus triangle  $\Delta X$  is balanced.

**Example 1.1.6.** The Steinhaus triangle shown in example 1.1.3 is a balanced Steinhaus triangle, and therefore its initial row is a balanced binary sequence.

Given a finite collection of numbers  $Y = (y_1, y_2, \dots, y_m)$ , we denote the sum of its elements by  $\sigma(Y) = \sum_{i=1}^m y_i$ . With this notation, we can generalize and denote the sum of the elements of a Steinhaus triangle by  $\sigma(\Delta X) = \sum_{i=0}^n \sigma(\partial^i X)$ . Therefore, clearly, a binary sequence  $X$  will be balanced if and only if  $\sigma(\Delta X) = 0$ .

For a given positive integer  $m$ , we denote by  $[m]$  the set  $\{1, 2, \dots, m\}$ . And we denote by  $\Delta X(i, j)$ , with  $i \in [n]$  and  $j \in [n - i + 1]$ , the  $j$ th element of the  $i$ th row of the triangle, i.e., the  $j$ th element of the  $(i - 1)$ th derived sequence of  $X$ , the sequence  $\partial^{i-1} X$ . Thus, the fundamental relation between elements of a Steinhaus triangle is, if  $i > 1$ ,

$$\Delta X(i, j) = \Delta X(i - 1, j) \Delta X(i - 1, j + 1).$$

## 1.2 The Steinhaus Problem

In the year 1963, Hugo Steinhaus published *One hundred problems in elementary mathematics* [1]. Inside this book, Steinhaus wrote a whole chapter of what were, back then, problems without solution. One of the problems on the list was to determine whether there exists a balanced binary sequence of every length  $n$  or not. Rightaway, himself claimed that certainly it could not be of every  $n$ , since the total number of elements of the Steinhaus triangle  $N = \frac{n(n+1)}{2}$  had to be even.

**Remark 1.2.1.** If a binary sequence  $X = (x_1, x_2, \dots, x_n)$  is balanced, then its length  $n$  is such that  $n \equiv 0$  or  $3 \pmod{4}$ .

In the original book, Steinhaus does not use the term *balanced*, but he suggests the problem as finding the triangles composed on the same number of plus and minus signs. These triangles had to fulfill that under each pair of equal signs it were a positive sign and under opposite signs, a minus sign.

He gave three concrete examples, on length 7, 12 and 20; and the problem was set out.

## 1.3 The first proof by Harborth (1972)

In the year 1972, Heiko Harborth solved the problem for the first time [2]. He proved that, indeed, there is at least four balanced binary sequences of every length  $n$  such that  $n \equiv 0$  or  $3 \pmod{4}$ .

In order to answer Steinhaus's question, he changed the notation. Instead of binary sequences composed of plus and minus signs, one can use the digits 0 and 1, respectively replacing each sign, and transforming the products in the construction of the derived sequences into binary sums. The relation between the elements of a Steinhaus triangle is therefore, if  $i > 1$ ,

$$\Delta X(i, j) \equiv \Delta X(i - 1, j) + \Delta X(i - 1, j + 1) \pmod{2}.$$

Harborth realised an interesting property about these triangles, given the previous property, we have

$$\Delta X(i, j) + \Delta X(i - 1, j) \equiv \Delta X(i - 1, j + 1) \pmod{2} \text{ and}$$

$$\Delta X(i, j) + \Delta X(i - 1, j + 1) \equiv \Delta X(i - 1, j) \pmod{2}.$$

Since these properties, every triangle may be read from three sides, and every side may be read in two directions. This is, except for simmetry cases, if we have a balanced Steinhaus triangle, we actually have 6 of them.

## 1.4 The derived problems

Ever since this first proof was shown, new questions derived from Steinhaus's problem started being proposed. Some of these derived problems consisted on solving the original problem with some new restrictions on the initial binary sequence, i.e., determine if there exists any balanced binary sequences  $X$  of a determined length  $n$  fulfilling a certain property, such as:

- (i) Being symmetric. This is, for all  $i$  from 1 to  $n$ ,  $x_i = x_{n-i+1}$ .

**Example 1.4.1.** The following Steinhaus triangle is balanced and has its first row symmetric (the whole triangle is symmetric as a consequence).

```

- - - + - - -
+ + - - + +
+ - + - +
- - - -
+ + +
+ +
+

```

- (ii) Being antisymmetric. This is, for all  $i$  from 1 to  $n$ ,  $x_i = -x_{n-i+1}$ .

**Example 1.4.2.** The triangle below is balanced and its first row is antisymmetric (the rest of the triangle is therefore symmetric).

```

+ - - - - + + + + -
- + + + + - + + + -
- + + + - - + + + -
- + + - + - + + -
- + - - - - + -
- - + + + - -
+ - + + - +
- - + - -
+ - - +
- + -
- -
+

```

- (iii) Having the same number of  $+1$  as  $-1$ . This is  $\sigma(X) = 0$ , we will denote this property as being *zero-sum*.

**Example 1.4.3.** The next Steinhaus triangle is balanced and zero-sum, since the first row has 4 minus signs and 4 plus signs.

$$\begin{array}{cccccccc}
 - & + & + & - & + & - & + & - \\
 & - & + & - & - & - & - & - \\
 & & - & - & + & + & + & + \\
 & & & + & - & + & + & + \\
 & & & & - & - & + & + \\
 & & & & & + & - & + \\
 & & & & & & - & - \\
 & & & & & & & +
 \end{array}$$

- (iv) Having a balanced binary sequence for initial sequence of each length  $m = n - 4k$ , with  $0 \leq k \leq \lfloor \frac{n}{4} \rfloor$ , we will denote this property as being *strongly-balanced*.

**Example 1.4.4.** The binary sequence in example 1.4.1 is also strongly-balanced, since it is balanced and its initial sequence of length 3 ( $- - -$ ) is balanced too.

Clearly, showing the existence of balanced binary sequences with any of these characteristics provides a new proof for the original Steinhaus's problem.

However, there are other problems related with Steinhaus triangles that do not look for balanced sequences, but for some interesting properties different than being balanced. Some of these problems are:

- (i) The existence of Steinhaus triangles with rotational symmetry, i.e., that their graphical representations are invariant under rotations of 120 and 240 degrees.
- (ii) The existence of Steinhaus triangles with dihedral symmetry. This is, having rotational symmetry and being such that their graphical representation is invariant by axial symmetry with respect to the height of the triangle.

These problems are discussed in [3].

**Example 1.4.5.** The triangle below has dihedral symmetry, since it is invariant by axial symmet and rotational symmetry of 120 and 240 degrees.

$$\begin{array}{cccccc}
 + & - & - & - & + \\
 & - & + & + & - \\
 & & - & + & - \\
 & & & - & - \\
 & & & & +
 \end{array}$$

- (iii) The weights distribution problem.

With the notation used by Harborth, in which the signs  $+$  and  $-$  are replaced by 0 and 1 respectively, a binary sequence  $X = (x_1, \dots, x_n)$ , with  $x_i \in \{0, 1\}$  for all  $i \in [n]$ , is balanced if  $\sigma(\Delta X) = \frac{N}{2}$ , where  $N = \frac{n(n+1)}{2}$  is the number of elements of  $\Delta X$ . Harborth himself realised that the minimum weight that a not-all-zero Steinhaus triangle could have was  $\sigma(\Delta X_{min}) = n$ .

This weight is achieved in any length for the sequence  $(1, \dots, 1)$ .



**Example 1.4.6.** In the case  $n = 4$ , the sequence  $(1, 1, 1, 1)$  fulfills  $\sigma(\Delta(1, 1, 1, 1)) = 4$ .

$$\begin{array}{cccc} 1 & 1 & 1 & 1 \\ & 0 & 0 & 0 \\ & & 0 & 0 \\ & & & 0 \end{array}$$

On the other extrem, Harborth established that the greatest number of digits 1 being possible was

$$\sigma(\Delta X_{max}) = \left\lfloor \frac{2N + 1}{3} \right\rfloor.$$

**Example 1.4.7.** In the case  $n = 5$ , the sequence  $(1, 1, 0, 1, 1)$  fulfills

$$\sigma(\Delta(1, 1, 0, 1, 1)) = \left\lfloor \frac{2 \cdot 15 + 1}{3} \right\rfloor = 10,$$

as we can see below.

$$\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ & 0 & 1 & 1 & 0 \\ & & 1 & 0 & 1 \\ & & & 1 & 1 \\ & & & & 0 \end{array}$$

The weights distribution problem asks which are the extrem and intermediate weights that a Steinhaus triangle can have and how are they distributed.

### 1.4.1 The Molluzzo's generalization

One of the most interesting derived problems was posed by John C. Molluzzo in 1976 [4]. He took Harborth's notation of 0s and 1s and sums modulo 2 instead of the usual notation and he generalized Steinhaus's problem into larger modulus  $m$ , with  $m \geq 3$ .

If  $n$  and  $m$  are positive integers, is it possible to find a sequence  $X = (x_1, x_2, \dots, x_n)$  with  $x_i \in \mathbb{Z}/m\mathbb{Z}$  for all  $i$ , such that its generalized Steinhaus triangle, i.e. operating modulo  $m$  and not modulo 2, has the same number of elements in each equivalence class in  $\mathbb{Z}/m\mathbb{Z}$ ?

**Remark 1.4.8.** As in the original problem, one can clearly see that it is not possible finding this sequence  $X$  of any length  $n$ , since the total number of elements in the triangle  $N = \frac{n(n+1)}{2}$  has to be a multiple of  $m$ . In other words, a necessary condition for finding this sequence  $X$  is that  $n \equiv 0$  or  $(2m - 1) \pmod{2m}$ .

**Example 1.4.9.** This triangle is of length 4 and has exactly 2 elements of each equivalence class in  $\mathbb{Z}/5\mathbb{Z}$

$$\begin{array}{cccc} 2 & 2 & 3 & 3 \\ & 4 & 0 & 1 \\ & & 4 & 1 \\ & & & 0. \end{array}$$

## 1.5 Notation

Given  $n, m \in \mathbb{N}$ , let  $X = (x_1, x_2, \dots, x_n)$  and  $Y = (y_1, y_2, \dots, y_m)$  two binary sequences, we establish the following notation:

- The initial segment of length  $i$  of  $X$ , for all  $i \in [n]$ , is denoted by

$$X[i] = (x_1, \dots, x_i).$$

- The repetition of  $X$   $k$  times,  $k \in \mathbb{N}$ , is denoted by

$$X^k = (x_1, \dots, x_n, x_1, \dots, x_n, \dots, x_1, \dots, x_n).$$

- The infinite periodic sequence with period  $X$  is denoted by

$$X^\infty = (x_1, \dots, x_n, x_1, \dots, x_n, \dots).$$

- The concatenation of  $X$  and  $Y$  is denoted by

$$XY = (x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m).$$

- The *eventually periodic* sequence with initial segment  $X$  and period  $Y$  is denoted by

$$XY^\infty = (x_1, \dots, x_n, y_1, \dots, y_m, y_1, \dots, y_m, \dots).$$

## Chapter 2

# Strongly-balanced binary sequences

### 2.1 Introduction

In this chapter, we study the existence of balanced binary sequences  $X$  of a given length  $n \in \mathbb{N}$  whose initial sequences  $X[n-4]$ ,  $X[n-8]$ ,  $X[n-12], \dots$  are also balanced. This derived question of Steinhaus's problem was discussed and solved in [5].

**Definition 2.1.1.** Let  $X = (x_1, \dots, x_n)$  be a binary sequence of length  $n$ . We say that  $X$  is *strongly-balanced* if its initial sequence  $X[n-4k] = (x_1, \dots, x_{n-4k})$  is balanced for every  $0 \leq k \leq \frac{n}{4}$ .

**Example 2.1.2.** As seen in example 1.4.4, the sequence  $X = ----+----$  is strongly-balanced since it is balanced and so is its initial sequence of length 3,  $X[3] = ---$ .

**Remark 2.1.3.** The property of being strongly-balanced can be also defined recursively. If the binary sequence  $X = (x_1, \dots, x_n)$  is balanced and  $n = 3$  or  $4$ , then  $X$  is strongly-balanced. If  $n > 4$ , the binary sequence  $X = (x_1, \dots, x_n)$  is strongly-balanced if it is balanced and the initial sequence  $X[n-4] = (x_1, \dots, x_{n-4})$  is strongly-balanced.

### 2.2 Notation and facts

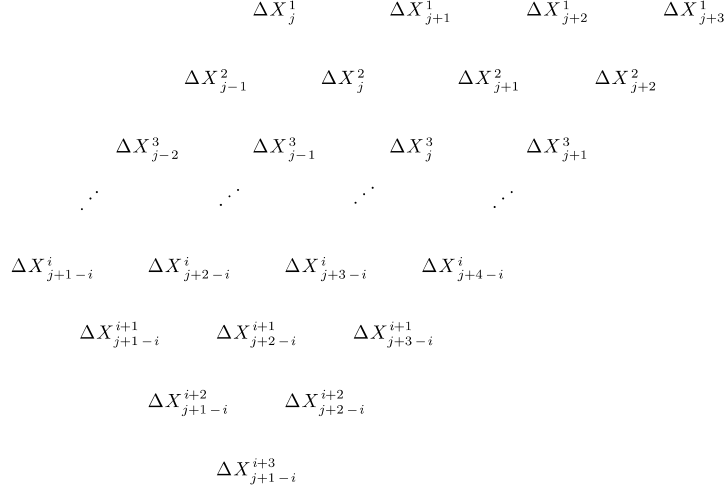
In order to prove the existence of strongly-balanced binary sequences of every length  $n \equiv 0$  or  $3 \pmod{4}$ , we need to introduce some notation that will lead us to some interesting remarks.

If  $X = (x_1, \dots, x_n)$  is a binary sequence and  $\Delta X$  its Steinhaus triangle, then:

- As we introduced in the previous chapter, we denote the  $j$ th element of the  $i$ th row of the triangle  $\Delta X$  by  $\Delta X(i, j)$ , with  $i \in [n]$  and  $j \in [n-i+1]$ .
- $d_k$  denotes the  $k$ th NE/SW diagonal of  $\Delta X$ , i.e.,

$$d_k = (\Delta X(1, k), \Delta X(2, k-1), \dots, \Delta X(k-1, 2), \Delta X(k, 1)).$$

- For  $j \in [n-3]$  and  $i \in [j]$ , then  $Z_j^i$  denotes the trapezoid of figure 2.1.

Figure 2.1: Trapezoid  $Z_j^i$ , where  $\Delta X_j^i$  denotes  $\Delta X(i, j)$ 

- For  $j \in [n - 3]$ , then  $S_j$  denotes  $Z_j^j$ , a *strip* made of the four NE/SW diagonals  $d_j, d_{j+1}, d_{j+2}$  and  $d_{j+3}$ .
- For  $j \in [n - 3]$ ,  $i \in [j]$  and  $k \in [j]$ , then  $P_j^i(k)$  denotes the parallelogram of width 4 and length  $k$  depicted in figure 2.2.

With this notation established, we can realize a few remarks that will be very useful on the proof of the existence of balanced binary sequences with different properties, in this case, being strongly-balanced.

Since the basic property relating the elements of the triangle is

$$\Delta X(i, j) = \Delta X(i - 1, j) \Delta X(i - 1, j + 1),$$

we can see that the trapezoid  $Z_j^i$  is completely determined by its first row, i.e.,

$$\Delta X(1, j), \Delta X(1, j + 1), \Delta X(1, j + 2) \text{ and } \Delta X(1, j + 3);$$

and its left side, which is

$$\Delta X(1, j), \Delta X(2, j - 1), \dots, \Delta X(i, j + 1 - i).$$

In turn, if  $i, j > 4$ , this left side of  $Z_j^i$  is determined by its first element,  $\Delta X(1, j)$ , and the right-side of the adjacent trapezoid  $Z_{j-4}^{i-4}$ .

Therefore, the trapezoid  $Z_j^i$  is completely determined by its top row and the right side of  $Z_{j-4}^{i-4}$ . We can generalize this and obtain the following proposition.

**Proposition 2.2.1.** *Any trapezoid  $Z_j^i$  is completely determined by:*

- If  $i = 1$ , its top row. In this case, the trapezoid is actually a triangle.*
- Otherwise, its top row and the diagonal just above its left side,*

$$\Delta X(1, j - 1), \Delta X(2, j - 2), \dots, \Delta X(i - 1, j + 1 - i).$$

$$\begin{array}{ccccccc}
& & & & \Delta X_j^i & & \\
& & & & & & \\
& & & & \Delta X_{j-1}^{i+1} & & \Delta X_j^{i+1} \\
& & & & & & \\
& & & & \Delta X_{j-2}^{i+2} & & \Delta X_{j-1}^{i+2} & & \Delta X_j^{i+2} \\
& & & & & & \\
& & & & \Delta X_{j-3}^{i+3} & & \Delta X_{j-2}^{i+3} & & \Delta X_{j-1}^{i+3} & & \Delta X_j^{i+3} \\
& & & & & & \\
& & & & \Delta X_{j-4}^{i+4} & & \Delta X_{j-3}^{i+4} & & \Delta X_{j-2}^{i+4} & & \Delta X_{j-1}^{i+4} \\
& & & & \vdots & & \vdots & & \vdots & & \vdots \\
& & & & \Delta X_{j-(k-1)}^{i+(k-1)} & & \Delta X_{j-(k-2)}^{i+(k-1)} & & \Delta X_{j-(k-3)}^{i+(k-1)} & & \Delta X_{j-(k-4)}^{i+(k-1)} \\
& & & & & & \\
& & & & \Delta X_{j-(k-1)}^{i+k} & & \Delta X_{j-(k-2)}^{i+k} & & \Delta X_{j-(k-3)}^{i+k} \\
& & & & & & \\
& & & & \Delta X_{j-(k-1)}^{i+k+1} & & \Delta X_{j-(k-2)}^{i+k+1} \\
& & & & & & \\
& & & & \Delta X_{j-(k-1)}^{i+k+2}
\end{array}$$

Figure 2.2: Parallelogram  $P_j^i(k)$ , where  $\Delta X_j^i$  denotes  $\Delta X(i, j)$ 

In the same way, the parallelogram  $P_j^i(k)$  is completely determined by its upper side,

$$\Delta X(i, j), \Delta X(i+1, j), \Delta X(i+2, j) \text{ and } \Delta X(i+3, j);$$

and its left side

$$\Delta X(i, j), \Delta X(i+1, j-1), \dots, \Delta X(i+(k-1), j-(k-1)).$$

In fact, its upper side is determined by the bottom of the quadrilater (either a trapezoid or another parallelogram) above it, while, if  $i > 4$ , the left side of  $P_j^i(k)$  is determined by the right side of  $P_j^{i-4}(k)$ .

Once again, we can generalize this and obtain the following proposition:

**Proposition 2.2.2.** *Any parallelogram  $P_j^i(k)$  is completely determined by:*

a) *If  $i > 1$ , the diagonal just above its left side,*

$$\Delta X(i-1, j), \Delta X(i, j-1), \dots, \Delta X(i+(k-2), j-(k-1));$$

*and the antidiagonal just above its right side,*

$$\Delta X(i-1, j+1), \Delta X(i, j+1), \Delta X(i+1, j+1), \Delta X(i+2, j+1).$$

b) *If  $i = 1$ , its upper element,  $\Delta X(1, j)$ , the diagonal next to its left side*

$$\Delta X(1, j-1), \dots, \Delta X(k-1, j-(k-1))$$

*and the antidiagonal next to its right side,*

$$\Delta X(1, j+1), \Delta X(2, j+1), \Delta X(3, j+1).$$

These two propositions will be very useful whenever we find a periodic structure on the initial sequence of the triangle in order to justify that the triangle itself has a certain periodic structure.

## 2.3 The existence

In this section we prove the existence of strongly-balanced binary sequences of every length  $n \equiv 0$  or  $3 \pmod{4}$ . Therefore, we will find a new proof of Steinhaus's problem.

**Theorem 2.3.1.** *For every positive integer  $n \equiv 0$  or  $3 \pmod{4}$ , there exists a binary sequence  $X$  of length  $n$  which is strongly-balanced.*

*Proof.* There are two separated cases:

### Case $n \equiv 0 \pmod{4}$

Let  $X = +-++(+ + - + + - + - - - ++)^{\infty}$ . We will show by induction that every initial segment  $X[n]$  with  $n \equiv 0 \pmod{4}$  is strongly-balanced. By the recursive definition of being strongly-balanced, if we know that  $X[n-4]$  is strongly-balanced, we just need to check that  $X[n]$  is balanced.

We denote by  $T_n$  the triangle  $\Delta X[n]$ . We start checking the base cases. In figure 2.3, we can see  $T_{16}$ , but we can also observe  $T_4$ ,  $T_8$  and  $T_{12}$ . These triangles fulfill

$$\sigma(T_4) = \sigma(T_8) = \sigma(T_{12}) = \sigma(T_{16}) = 0.$$

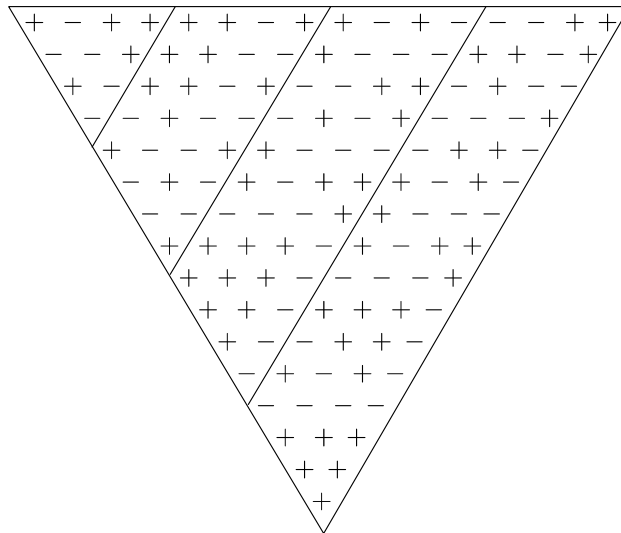
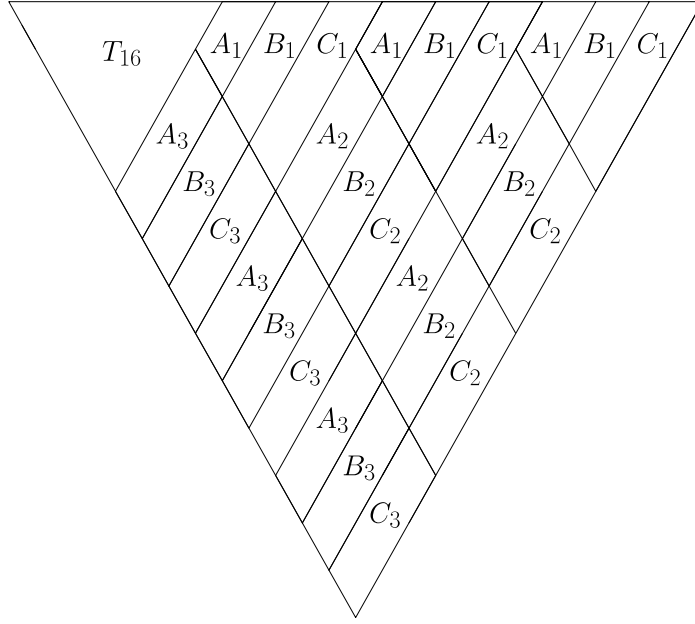


Figure 2.3: The triangles  $T_4$ ,  $T_8$ ,  $T_{12}$  and  $T_{16}$ .

Therefore we have seen that, for  $n \equiv 0 \pmod{4}$  and  $n \leq 16$ , the binary sequence  $X[n]$  is strongly-balanced.

Now, we want to prove that, for  $n > 16$ , the triangle  $T_n$  follows the structure depicted in figure 2.4.

First, suppose  $n = 20$ . We want to see that  $X[20]$  is strongly-balanced. Since we already know that  $X[16]$  is strongly-balanced, we just need to check that  $X[20]$  is balanced. In order to see that, we start by observing that  $T_{20}$  is actually  $T_{16}$  with a glued strip on the right side,  $S_{17}$ . Since we know that  $T_{16}$  is balanced, we just need to see that  $\sigma(S_{17}) = 0$ . We can divide

Figure 2.4: Structure of  $T_n = \Delta X[n]$ 

$S_{17}$  in two different blocks, the trapezoid  $Z_{17}^5$  and the parallelogram  $P_{12}^6(12)$ , named  $A_1$  and  $A_3$  respectively. As we can see in figure 2.6, these quadrilaterals fulfill

$$\sigma(A_1) = 0, \quad \sigma(A_3) = 0.$$

Therefore,  $\sigma(T_{20}) = 0$  and  $X[20]$  is strongly-balanced.

A very similar argument can be used if  $n = 24$ . The strip  $S_{21}$ , composed of  $Z_{21}^9$  and  $P_{12}^{10}(12)$ , named  $B_1$  and  $B_3$  and both depicted in figure 2.7, is glued to the right side of  $T_{20}$  in order to obtain  $T_{24}$ . Since

$$\sigma(B_1) = -4, \quad \sigma(B_3) = 4,$$

we can conclude that  $\sigma(T_{24}) = 0$  and  $X[24]$  is strongly-balanced.

In the same way, we can proceed with the case  $n = 28$ . The triangle  $T_{28}$  is composed of the triangle  $T_{24}$  and the strip  $S_{25}$ , who is composed by  $C_1$  and  $C_3$ , where  $C_1 = Z_{25}^{13}$  and  $C_3 = P_{12}^{14}(12)$ , both depicted in figure 2.8. Since

$$\sigma(C_1) = -4, \quad \sigma(C_3) = 4,$$

we have that  $T_{28}$  is balanced and thus,  $X[28]$  is strongly-balanced.

Now, let us suppose  $n = 32$ . By construction of  $X[32]$ , we have that

$$\Delta X(1, 16 + \ell) = X(1, 28 + \ell), \quad \text{for all } \ell \in [4].$$

Moreover, the first four elements of the right side of  $T_{16}$  are equal to the first four elements of the right side of  $C_1$ . In other words, the first four elements of  $d_{16}$  are equal to the first four elements of  $d_{28}$ . Then, using proposition 2.2.1, we know that the strip  $S_{29}$  starts with a  $A_1$ . Then, we define the parallelogram  $A_2 = P_{24}^6(12)$ , which is depicted on figure 2.6. We can observe that  $\sigma(A_2) = 0$ , and that the bottom of  $A_2$  is equal to the bottom of  $A_1$ . This fact, together

with the fact that the last 12 elements of  $d_{16}$  are equal to the elements of the right side of  $C_3$ , implies that the strip  $S_{29}$  ends with a parallelogram  $A_3$ , using proposition 2.2.2. Therefore, we can conclude that the triangle  $T_{32}$  is formed by a triangle  $T_{28}$ , a trapezoid  $A_1$ , a parallelogram  $A_2$  and a parallelogram  $A_3$ . Since each one of these blocks has entry sum 0, we have that  $\sigma(T_{32}) = 0$  and thus, the sequence  $X[32]$  is strongly-balanced.

The case  $n = 36$  follows a similar argument. By construction of  $X[36]$ , we have that

$$\Delta X(1, 20 + \ell) = X(1, 32 + \ell), \text{ for all } \ell \in [4].$$

Moreover, the strip  $S_{29}$  starts with  $A_1$ , just as the strip  $S_{17}$  does. These two facts imply that the strip  $S_{33}$  starts with  $B_1$ , just as  $S_{21}$ . We define  $B_2 = P_{24}^{10}(12)$ , that is depicted in figure 2.7 and fulfills  $\sigma(B_2) = 0$ . Then, since the bottom of  $B_2$  is equal to the bottom of  $B_1$  and the strip  $S_{29}$  ends with  $A_3$  just as the strip  $S_{17}$ , the strip  $S_{33}$  ends with  $B_3$ . Therefore, the triangle  $T_{36}$  has the following entry sum,

$$\sigma(T_{36}) = \sigma(T_{32}) + \sigma(B_1) + \sigma(B_2) + \sigma(B_3) = 0 + (-4) + 0 + 4 = 0.$$

Thus  $T_{36}$  is balanced and  $X[36]$  is strongly-balanced.

Analogously,  $T_{40}$  is balanced, since  $T_{36}$  is balanced and the strip  $S_{37}$  is composed of a trapezoid  $C_1$ , a parallelogram  $P_{24}^{14}(12)$ , named  $C_2$  and depicted in figure 2.8 fulfilling  $\sigma(C_2) = 0$ , and a parallelogram  $C_3$ . This gives us

$$\sigma(T_{40}) = \sigma(T_{36}) + \sigma(C_1) + \sigma(C_2) + \sigma(C_3) = 0 + (-4) + 0 + 4 = 0.$$

Thus,  $X[40]$  is strongly-balanced.

At this point, we know now all the constructing blocks of the structure shown in figure 2.4. We need to note that the elements on the right side of  $C_2$  coincide with the last twelve elements of the right side of  $C_1$ . Therefore, applying proposition 2.2.2, between a  $A_2$  and a  $C_2$  we find another  $A_2$ . This, together with the periodic structure of  $X[n]$ , guarantees the structure of  $T_n$ , for  $n > 40$ . Now that we know that the structure is valid, we need to see that this structure provides balanced triangles for  $n > 40$ .

For the induction step, suppose that  $X[n-4]$  is strongly-balanced. Then, the triangle  $T_n$  can be obtained gluing the strip  $S_{n-3}$  to the right side of the triangle  $T_{n-4}$ . This strip is the same that  $S_{n-15}$ , the strip glued to  $T_{n-16}$  in order to obtain  $T_{n-12}$ , with an additional parallelogram. This parallelogram is  $A_2$  if  $n \equiv 8 \pmod{12}$ ,  $B_2$  if  $n \equiv 0 \pmod{12}$  or  $C_2$  if  $n \equiv 4 \pmod{12}$ , in either case it has entry sum 0. Now, since  $X[n-4]$  is strongly-balanced, we know that both  $T_{n-16}$  and  $T_{n-12}$  are balanced. Thus, the strip  $S_{n-15}$  is such that  $\sigma(S_{n-15}) = 0$  and therefore, so is  $S_{n-3}$ . This implies that  $T_n$  is balanced and then,  $X[n]$  is strongly balanced for all  $n$ .

### Case $n \equiv 3 \pmod{4}$

Let  $\tilde{X} = + - + (+ + + - + - + + + - +)^\infty$ . We want to see that, for all  $n \equiv 3 \pmod{4}$ , the initial sequence  $\tilde{X}[n]$  is balanced. Therefore,  $\tilde{X}[n]$  will be strongly-balanced.

The proof of the fact that  $\tilde{X}[n]$  is balanced for all  $n \equiv 3 \pmod{4}$  is analogous to the proof done in the previous case in order to see that  $X[n]$  is balanced for all  $n \equiv 0 \pmod{4}$ .

Let  $m \equiv 3 \pmod{4}$ . Then,  $(m+1) \equiv 0 \pmod{4}$ . There are only two main differences between the case  $m$  and the case  $(m+1)$ .

1. The structure of  $\tilde{T}_m$  corresponds to the structure of  $T_{m+1}$  shown in figure 2.4, but in this case, there are only 3 different parallelograms, since  $\tilde{A}_2 = \tilde{A}_3$ ,  $\tilde{B}_2 = \tilde{B}_3$  and  $\tilde{C}_2 = \tilde{C}_3$ .



Besides that, we have an initial triangle  $\tilde{T}_{15}$  instead of  $T_{16}$ . We can see in figure 2.5 that  $\tilde{T}_{15}$  is strongly-balanced since

$$\sigma(\tilde{T}_3) = \sigma(\tilde{T}_7) = \sigma(\tilde{T}_{11}) = \sigma(\tilde{T}_{15}) = 0.$$

The six blocks of the periodic structure are depicted on figures 2.9, 2.10 and 2.11, and as can be observed,

$$\sigma(\tilde{A}_1) = \sigma(\tilde{A}_2) = \sigma(\tilde{B}_1) = \sigma(\tilde{B}_2) = \sigma(\tilde{C}_1) = \sigma(\tilde{C}_2) = 0.$$

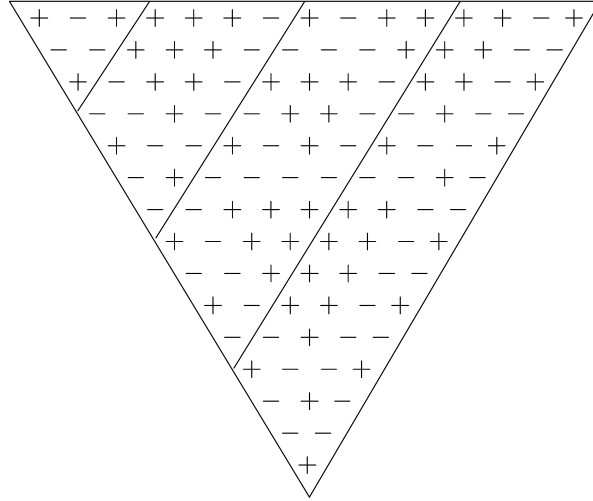


Figure 2.5: The triangles  $\tilde{T}_3$ ,  $\tilde{T}_7$ ,  $\tilde{T}_{11}$  and  $\tilde{T}_{15}$ .

2. The bottom of  $\tilde{A}_2$ , i.e.  $++--$ , is not equal to  $\tilde{A}_1$ , which is  $-+--$ , their first element is different. In the same way, the first element of the right side of  $\tilde{C}_2$ ,  $+1$ , is opposed to the fourth element of the right side of  $\tilde{C}_1$ ,  $-1$ , which is the first of the last twelve elements of this side. These two sequences are the ones that determine the parallelogram under  $\tilde{A}_1$ . However, these two differences actually cancel each other, since

$$(+1) \cdot (+1) = (-1) \cdot (-1) = +1,$$

and the periodic structure is also valid in this case.

With these two differences, the proof is analogous and the triangles  $\tilde{T}_n$  are balanced for all  $n \equiv 3 \pmod{4}$ , so the sequences  $\tilde{X}[n]$  are strongly-balanced for all  $n \equiv 3 \pmod{4}$ .

▽

**Theorem 2.3.2.** *For every positive integer  $n \equiv 0$  or  $3 \pmod{4}$ , there exists a balanced binary sequence  $X$  of length  $n$ .*

*Proof.* Trivial since theorem 2.3.1 and a strongly-balanced sequence is always balanced.

▽

And hence we have found an alternative proof of Steinhaus's problem.

## 2.4 The uniqueness

The property of being strongly-balanced is strong, and very few binary sequences fulfill it. In this section, we present the behaviour of the number of strongly-balanced sequences and the unique sequences that are strongly-balanced for a large  $n$ .

By a computer simulation, one can observe that the number of strongly-balanced sequences of a given length  $n \equiv 0 \pmod{4}$  starts strictly increasing. When  $n = 36$ , we found a maximum, and it decreases until  $n = 92$ . Then, the number of strongly-balanced sequences stabilizes to 4. This is, for all  $n \geq 92$  such that  $n \equiv 0 \pmod{4}$ , the only four strongly-balanced sequences of length  $n$  that exist are the initial segments of length  $n$  of the following infinite binary sequences:

$$\begin{aligned} X_1 &= + - + + ( + + - + + - + - - - + + )^\infty, \\ X_2 &= ( + - + + - + + + + - - - )^\infty, \\ X_3 &= + - + - ( + - - + + + + - - + + + )^\infty, \\ X_4 &= + - + - ( - + - + + + - + - + + + )^\infty. \end{aligned}$$

The proof that these four sequences are the unique strongly-balanced sequences with  $n \equiv 0 \pmod{4}$  and  $n > 92$ , was realized by Eliahou and Hachez in [5]. They proved the result by induction, showing that the only balanced extension of length  $n$  of the strongly-balanced sequence  $X_i[n-4]$  is  $X_i[n]$ , for all  $i \in [4]$ .

Table 2.1, under these lines, shows the number of strongly-balanced binary sequences,  $SB(n)$ , for  $n \leq 96$ , with  $n \equiv 0 \pmod{4}$ :

$n$	$SB(n)$	$n$	$SB(n)$	$n$	$SB(n)$
4	6	36	124	68	32
8	18	40	106	72	20
12	30	44	92	76	20
16	52	48	92	80	8
20	80	52	90	84	8
24	88	56	64	88	6
28	106	60	44	92	4
32	116	64	38	96	4

Table 2.1: Behaviour of  $SB(n)$  with  $n$  small and  $n \equiv 0 \pmod{4}$

The case  $n \equiv 3 \pmod{4}$  shows a similar behaviour, but more complicated. Table 2.2 shows the number of strongly-balanced binary sequences,  $SB(n)$ , for  $n \leq 127$ , with  $n \equiv 3 \pmod{4}$ . Once the length is larger than 127,  $SB(n)$  stabilizes. However, it does not stabilize to a value, but to a certain period. If  $n \equiv 3$  or  $7 \pmod{12}$ , then  $SB(n) = 14$ , but if  $n \equiv 11 \pmod{12}$ , then  $SB(n) = 12$ .

Then, if we denote by

$$\begin{aligned} Y_1 &= + + + - ( + - + + + + - + + - )^\infty, \\ Y_2 &= + + - - - - + ( + + - - + - + - + - - + )^\infty, \\ Y_3 &= + - + ( + + + - + - + + + + - + )^\infty, \\ Y_4 &= + - + + + + - ( + - + + - + - - - - + + + + - + - + - - - - )^\infty, \\ Y_5 &= + - + + + + - ( - + + - + + + + - + + - )^\infty, \\ Y_6 &= + - + - + - - - ( + - + - + - - + + + - - )^\infty, \end{aligned}$$

$$\begin{aligned}
 Y_7 &= + - + - + - - - (+ - + - - - - - - + - + - - + - + + - - - + -)^\infty, \\
 Y_8 &= + - + (- + - - - + - - + + - +)^\infty, \\
 Y_9 &= - + + (+ + - + + + + - + - + +)^\infty, \\
 Y_{10} &= - + + + + - + (- - + + + - - + - + - +)^\infty, \\
 Y_{11} &= - - - - - + - (+ - - + + + - - + - + -)^\infty, \\
 Y_{12} &= - - - (- - + - - + + + + - - -)^\infty,
 \end{aligned}$$

the unique strongly-balanced binary sequences of a given length  $n \geq 127$ , with  $n \equiv 3 \pmod{4}$ , are the following.

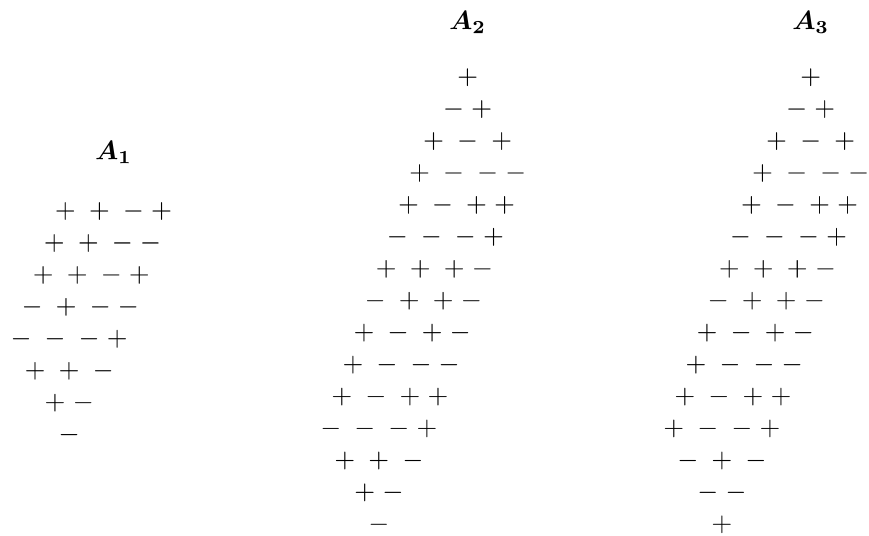
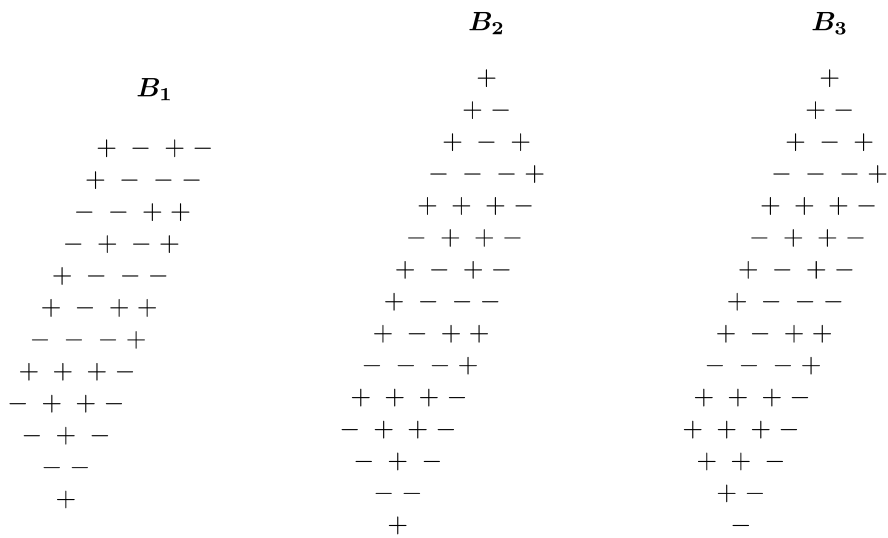
- If  $n \equiv 3 \pmod{12}$ , the initial segments  $Y_i[n]$ , for all  $i \in [12]$ , and the binary sequences  $Y_5[n - 4] + - + -$  and  $Y_8[n - 4] + - + +$ .
- If  $n \equiv 7 \pmod{12}$ , the initial segments  $Y_i[n]$ , for all  $i \in [12]$ , and the binary sequences  $Y_8[n - 8] + - + + - + + +$  and,
  - if  $n \equiv 7 \pmod{24}$ , the sequence  $Y_5[n - 8] + - + - - - -$ .
  - if  $n \equiv 19 \pmod{24}$ , the sequence  $Y_5[n - 8] + - + - - + - +$ .
- If  $n \equiv 11 \pmod{12}$ , just the initial segments  $Y_i[n]$ , for all  $i \in [12]$ .

With the same method of the case  $n \equiv 0$ , Eliahou and Hachez proved that these are the unique strongly-balanced sequences of length  $n$ , with  $n \equiv 3 \pmod{4}$  and  $n > 127$ .

$n$	$SB(n)$	$n$	$SB(n)$	$n$	$SB(n)$	$n$	$SB(n)$
3	4	35	88	67	48	99	14
7	8	39	108	71	28	103	14
11	16	43	114	75	26	107	14
15	26	47	90	79	26	111	16
19	36	51	88	83	20	115	14
23	48	55	104	87	16	119	14
27	48	59	92	91	18	123	16
31	66	63	60	95	14	127	14

Table 2.2: Behaviour of  $SB(n)$  with  $n$  small and  $n \equiv 3 \pmod{4}$

## 2.5 Appendix: Figures

Figure 2.6: The blocks forming the strips  $S_k$  with  $k \equiv 5 \pmod{12}$ Figure 2.7: The blocks forming the strips  $S_k$  with  $k \equiv 9 \pmod{12}$

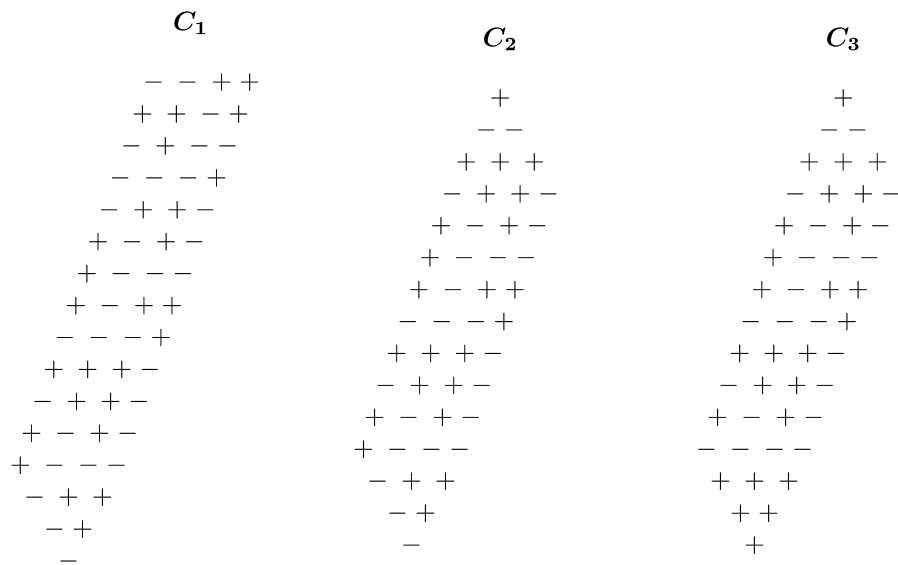


Figure 2.8: The blocks forming the strips  $S_k$  with  $k \equiv 1 \pmod{12}$

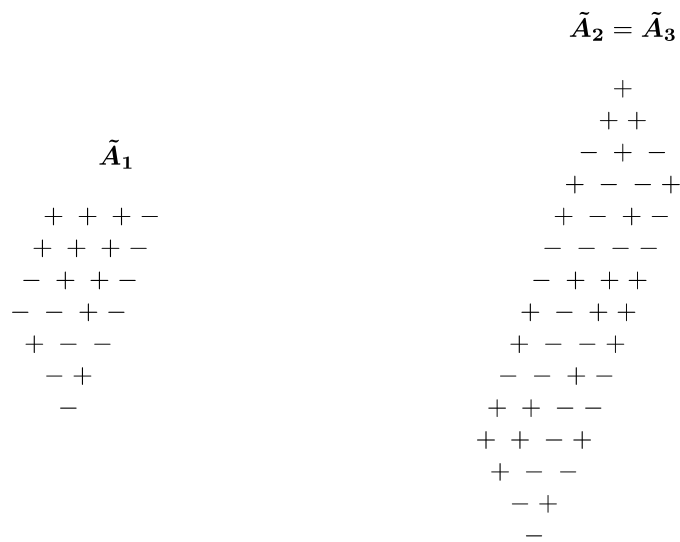
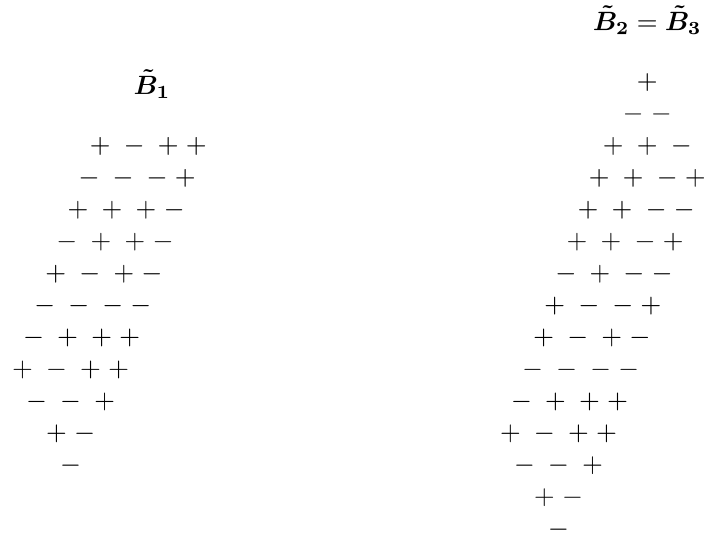
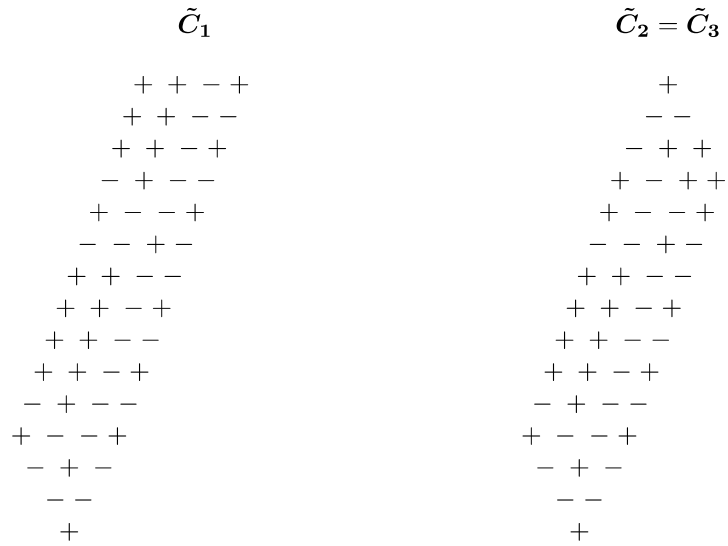


Figure 2.9: The blocks forming the strips  $S_k$  with  $k \equiv 4 \pmod{12}$

Figure 2.10: The blocks forming the strips  $S_k$  with  $k \equiv 8 \pmod{12}$ Figure 2.11: The blocks forming the strips  $S_k$  with  $k \equiv 0 \pmod{12}$

## Chapter 3

# Symmetric and Antisymmetric balanced binary sequences

### 3.1 Introduction

In this chapter, we answer if there exist any symmetric or antisymmetric balanced binary sequences of a given length  $n \in \mathbb{N}$ . This derived question of Steinhaus's problem was discussed and solved in [6].

**Definition 3.1.1.** Let  $X = (x_1, x_2, \dots, x_n)$  be a finite binary sequence. We say that the *reversed sequence* of  $X$  is  $\bar{X} = (x_n, \dots, x_2, x_1)$ .

**Definition 3.1.2.** We say that a finite binary sequence  $X = (x_1, x_2, \dots, x_n)$  is *symmetric* if  $\bar{X} = X$ . This is, for all  $i \in [n]$ , it fulfills  $x_i = x_{n-i+1}$ .

**Definition 3.1.3.** We say that a finite binary sequence  $X = (x_1, x_2, \dots, x_n)$  is *antisymmetric* if  $\bar{X} = -X$ . This is, for all  $i \in [n]$ , it fulfills  $x_i = -x_{n-i+1}$ .

**Remark 3.1.4.** If  $X = (x_1, \dots, x_n)$  is an antisymmetric binary sequence of length  $n$ , then  $n$  is even. This is clear since if  $n$  were odd, there would be a middle term in the sequence  $X$ , that would be a fixed point in the reversed sequence  $\bar{X}$ , therefore it would have to be equal to its opposite, which is impossible because  $x_i \in \{-1, +1\}$  for all  $i$ .

Furthermore, from this remark follows straightforwardly the next proposition.

**Proposition 3.1.5.** *If  $X = (x_1, \dots, x_n)$  is a binary sequence of length  $n$  such that  $X$  is both balanced and antisymmetric, then  $n \equiv 0 \pmod{4}$ .*

*Proof.* For  $X$  to be balanced, it is necessary that  $n \equiv 0$  or  $3 \pmod{4}$ , and for  $X$  to be antisymmetric, its length has to be even, therefore,  $n \equiv 0 \pmod{4}$ . ▽

**Remark 3.1.6.** If  $X = (x_1, \dots, x_n)$  is an antisymmetric binary sequence, it is zero-sum, as introduced in example 1.4.3, it fulfills  $\sigma(X) = 0$ .

The existence of zero-sum balanced binary sequences is studied in Chapter 4, but here we will provide a different proof of such existence.

## 3.2 The symmetric case

### 3.2.1 The valid lengths

We start seeing that does not exist any balanced binary sequence  $X = (x_1, \dots, x_n)$  of length  $n$  that is both symmetric and balanced if  $n \equiv 4 \pmod{8}$ , i.e., there can only be symmetric balanced binary sequences of length  $n$  if the length is such that  $n \equiv 0, 3$  or  $7 \pmod{8}$ . To do so, we use the following results.

**Lemma 3.2.1.** *Let  $X = (x_1, \dots, x_n)$  be a symmetric binary sequence of length  $n$ , with  $n$  even. Then,  $\sigma(X) \equiv n \pmod{4}$ .*

*Proof.* Let  $k = \frac{n}{2}$ . Then, since  $X$  is symmetric by hypothesis,

$$X = (x_1, x_2, \dots, x_{k-1}, x_k, x_k, x_{k-1}, \dots, x_2, x_1).$$

Therefore,

$$\sigma(X) = 2 \sum_{i=1}^k x_i.$$

Since we have  $x_i \equiv 1 \pmod{2}$  for all  $i$ , with  $i \in [k]$ ,

$$\sum_{i=1}^k x_i \equiv k \pmod{2},$$

and

$$\sigma(X) \equiv 2k \equiv n \pmod{4}.$$

▽

**Lemma 3.2.2.** *Let  $X = (x_1, \dots, x_n)$  be a symmetric binary sequence of length  $n$ , with  $n$  odd. Let  $k = \lceil \frac{n}{2} \rceil = \frac{n+1}{2}$ , so  $x_k$  is the middle term of  $X$ . Then,  $\sigma(X) \equiv n + x_k - 1 \pmod{4}$ .*

*Proof.* Since  $k = \frac{n+1}{2}$ , we have  $n = 2k - 1$ . Then, since  $X$  is symmetric by hypothesis,

$$X = (x_1, x_2, \dots, x_{k-1}, x_k, x_{k-1}, \dots, x_2, x_1).$$

Therefore,

$$\sigma(X) = \sum_{i=1}^n x_i = x_k + 2 \sum_{i=1}^{k-1} x_i.$$

Since we have  $x_i \equiv 1 \pmod{2}$  for all  $i$ , with  $i \in [k-1]$ ,

$$\sum_{i=1}^{k-1} x_i \equiv k-1 \pmod{2},$$

and

$$\sigma(X) \equiv x_k + 2(k-1) \equiv x_k + n - 1 \pmod{4}.$$

▽

**Remark 3.2.3.** It is clear that if  $X$  is a symmetric binary sequence of length  $n$ , for all  $i \in [n-1]$ , the sequence  $\partial^i X$  is symmetric of length  $n-i$ .



**Lemma 3.2.4.** *Let  $X = (x_1, \dots, x_n)$  be a symmetric binary sequence of length  $n$ , and let  $i \in [n-1]$ . Then, the  $i$ th derived sequence of  $X$  fulfills  $\sigma(\partial^i X) \equiv n-i \pmod{4}$ .*

*Proof.* In order to prove this lemma, we will use the two previous ones, so we need to separate by cases.

- If  $n \equiv i \pmod{2}$ , their difference  $n-i$  is even. As observed in remark 3.2.3, we have a symmetric binary sequence with even length, thus we just need to apply lemma 3.2.1.
- Otherwise, the difference  $n-i$  is odd. Let  $k = \lceil \frac{n-i}{2} \rceil$ , then  $x_k$  is the middle term of  $\partial^i X$ , which is a symmetric binary sequence as we have seen in remark 3.2.3. Since the sequence  $\partial^{i-1} X$  is symmetric as well, and of even length, we know that  $x_k$  is the product of the two equal middle terms in  $\partial^{i-1} X$ , so  $x_k = 1$ . Therefore, if we use lemma 3.2.2,

$$\sigma(\partial^i X) \equiv n-i + x_k - 1 \equiv n-i \pmod{4}.$$

▽

Now we will focus on the sequences that are both balanced and symmetric. We start with a statement for even lengths.

**Proposition 3.2.5.** *Let  $X = (x_1, \dots, x_n)$  be a balanced symmetric binary sequence of length  $n$ , with  $n$  even. Then,  $n \equiv 0 \pmod{8}$ .*

*Proof.* Since  $X$  is symmetric, we know

$$\sigma(\Delta X) = \sum_{i=0}^{n-1} \sigma(\partial^i X) = 0.$$

Since  $X$  is balanced, using lemma 3.2.4, each summand with  $i \geq 1$  fulfills  $\sigma(\partial^i X) \equiv n-i \pmod{4}$ . Therefore,

$$\sigma(\Delta X) \equiv \sigma(X) + \frac{n(n-1)}{2} \pmod{4}.$$

Now, since  $n$  is even, we can use lemma 3.2.1, so we have

$$0 = \sigma(\Delta X) \equiv n + \frac{n(n-1)}{2} \equiv \frac{n(n+1)}{2} \pmod{4}.$$

Then, it follows that  $n \equiv 0 \pmod{8}$ .

▽

In the odd case, the valid length of  $X$  depends on the central term of the sequence, as shown in the next proposition.

**Proposition 3.2.6.** *Let  $X = (x_1, \dots, x_n)$  be a balanced symmetric binary sequence of length  $n$ , with  $n$  odd. Let  $k = \lceil \frac{n}{2} \rceil$ , so  $x_k$  is the middle term of  $X$ . Then,  $n \equiv 1 - 2x_k \pmod{8}$ . This is, if  $x_k = 1$ , then  $n \equiv 7 \pmod{8}$ ; else if  $x_k = -1$ , then  $n \equiv 3 \pmod{8}$ .*

*Proof.* Since  $X$  is balanced and symmetric, with the same arguments as in the previous proposition, we have

$$0 = \sigma(\Delta X) \equiv \sigma(X) + \frac{n(n-1)}{2} \pmod{4}.$$

By lemma 3.2.2 we have  $\sigma(X) \equiv n + x_k - 1 \pmod{4}$ . Then,

$$0 = \sigma(\Delta X) \equiv \frac{n(n+1)}{2} + x_k - 1 \pmod{4},$$

and we get that

$$2(1 - x_k) \equiv n(n + 1) \equiv n^2 + n \pmod{8}.$$

Since  $n$  is odd, we know  $n^2 \equiv 1 \pmod{8}$ , thus

$$2(1 - x_k) \equiv 1 + n \pmod{8} \implies n \equiv 1 - 2x_k \pmod{8}.$$

▽

### 3.2.2 The existence

We have seen that the unique lengths that can have a symmetric balanced binary sequence are those which fulfill  $n \equiv 0, 3$  or  $7 \pmod{8}$ . We will now see that for each of those lengths there is at least one symmetric balanced binary sequence, and we will prove that by giving a concrete example.

**Theorem 3.2.7.** *For every positive integer  $n \equiv 0, 3$  or  $7 \pmod{8}$ , there exists a binary sequence  $X$  of length  $n$  which is both symmetric and balanced.*

*Proof.*

#### Case $n \equiv 0 \pmod{8}$

Let  $X = + - - - - + + + + - - + + + + - - - - +$ . Then  $X$  is a symmetric binary sequence of length 24. Note that  $X$  is the concatenation of two antisymmetric binary sequences of length 12, named  $X_a$  and  $X_b$ . These sequences are balanced, as seen in figure 3.1.

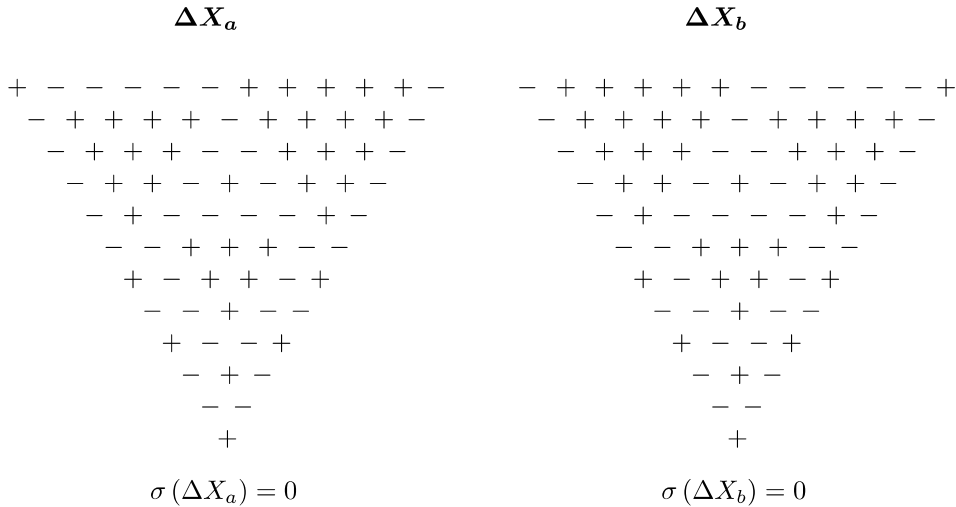


Figure 3.1: The Steinhaus triangles of the sequences  $X_a$  and  $X_b$

Actually, we just needed to check one of the triangles, since the sequences fulfill  $X_a = \overline{X_b}$  and they are antisymmetric, so  $X_a = -X_b$ , and the derived sequences are symmetric and equal, this is  $\partial X_a = \partial X_b$ , and therefore the rest of each triangle is the same as the rest of the other one.

Now, we write  $n = 24k + r$ , with  $k \geq 1$  and  $r = 0, 8$  or  $16$ , given that  $n \equiv 0 \pmod{8}$ . If  $r = 8$  or  $16$  we consider the case  $k = 0$  too, thinking of  $X^0$  like repeating  $X$  zero times, i.e.,

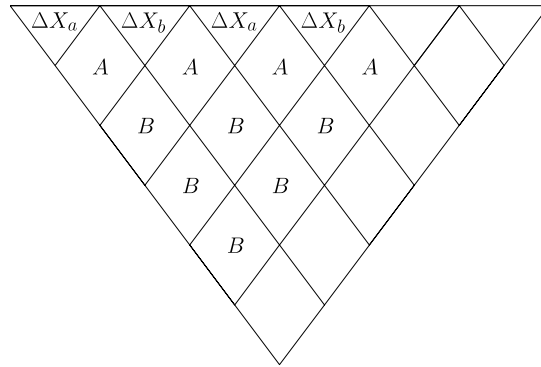


Figure 3.2: Structure of  $\Delta X^k$

the empty sequence. We will prove that the following constructions are both symmetric and balanced:

- $X^k$  for length  $24k$ ,  $k \geq 1$ ,
- $Y_k = +-++X^k++-+$  for length  $24k+8$ ,  $k \geq 0$ ,
- $Z_k = --+++-++X^k++-++--$  for length  $24k+16$ ,  $k \geq 0$ .

Let us prove these three cases individually.

Case  $r = 0$

We want to prove that  $X^k$  is both balanced and symmetric for every  $k \geq 1$ . First of all, we start checking that  $X$  is balanced, as we can see on figure 3.13.

Since  $X$  is also symmetric, we have seen that this construction is valid for  $k = 1$ . In order to prove that this is also valid for  $k \geq 2$ , we name  $A$  the diamond that we can find between  $\Delta X_a$  and  $\Delta X_b$  in  $\Delta X$ . As easily checked,  $\sigma(A) = 0$ .

Note that, as we said earlier, the two triangles  $\Delta X_a$  and  $\Delta X_b$  are equal except for their first rows. This is important because  $X^k$  is, by construction, the concatenation of  $k$  times  $X_a X_b$ , specifically

$$X^k = X_a X_b X_a X_b \dots X_a X_b.$$

We are therefore interested in the diamond between  $\Delta X_b$  and  $\Delta X_a$ , which we will see that is  $A$  again, just as between  $\Delta X_a$  and  $\Delta X_b$ .

We know that the triangles  $\Delta X_a$  and  $\Delta X_b$  are equal except for their first rows. This condition is not enough to guarantee that the diamond between  $\Delta X_b$  and  $\Delta X_a$  is  $A$ , but it is enough to affirm that, except for the upper element of the diamond, this is equal to  $A$ . However, it is not as important that the first rows are different as their first and last elements. The last element of  $X_b$  and the first element of  $X_a$  multiplied generate this upper element. Since them both are  $+$ , the upper element of the diamond is  $+$  too, just as the upper element of  $A$ . Therefore, the diamond between  $\Delta X_b$  and  $\Delta X_a$  is  $A$  again.

Then, between each pair of diamonds  $A$ , there is a diamond  $B$  (figure 3.14). Moreover, as we can see, the bottom triangle of  $B$  is exactly the same as the bottom triangle of  $A$ , thus between each pair of diamonds  $B$ , there is another diamond  $B$ .

We have proved that  $\Delta X^k$  follows the structure depicted in the figure 3.2. Then, we proceed now to prove that this structure provides that  $X^k$  is both symmetric and balanced for all  $k \geq 1$ .

We have seen previously the case  $k = 1$ , so let us suppose  $k \geq 2$  and that  $X^{k-1}$  is symmetric and balanced. The sequence  $X^k$  is clearly symmetric by construction. If we have  $\Delta X^{k-1}$ , we just need to glue a band to its right side in order to obtain  $\Delta X^k$ . This band is composed of a triangle  $\Delta X_a$ , a triangle  $\Delta X_b$ , two diamonds  $A$  and several diamonds  $B$ . Since  $X^{k-1}$  is balanced and the four different blocks  $\Delta X_a$ ,  $\Delta X_b$ ,  $A$  and  $B$  have entry sum zero, we have

$$\sigma(\Delta X^k) = \cancel{\sigma(X^{k-1})} + \sigma(\Delta X_a) + \sigma(\Delta X_b) + 2 \cdot \sigma(A) + (3 + 4(k-2)) \cdot \sigma(B) = 0,$$

and therefore  $X^k$  is balanced.

Case  $r = 8$

We want to prove that  $Y_k = + - + + X^k + + - +$  is both balanced and symmetric for every  $k \geq 0$ . First of all, we start checking that  $Y_0 = + - + + + - +$  is symmetric and balanced, as we can see in figure 3.3, and we have proved the case  $k = 0$ .

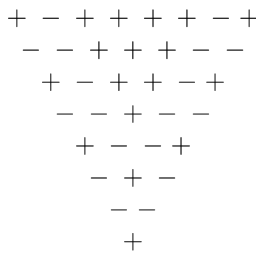


Figure 3.3:  $\Delta Y_0$ , that fulfills  $\sigma(\Delta Y_0) = 0$

Let us suppose  $k \geq 1$ . We can think of  $\Delta Y_k$  as  $\Delta X^k$  with two bands glued, one on the left side and the other one on the right side. Since we have seen in the previous case that  $X^k$  is balanced, we just need to see that these bands have sum 0. Given the clear symmetry of the case, actually it is sufficient to prove that one of them has sum 0, and by symmetry the other one will have sum 0 too.

We want to prove that  $\Delta Y_k$  has the periodic structure seen in figure 3.4. Since we have already proved the structure of  $\Delta X^k$ , we just need to see that the right band glued is composed by one trapezoid  $T_0$ ,  $(2k - 1)$  parallelograms  $P_0$  and a diamond  $C$ . These three blocks are shown in figure 3.15.

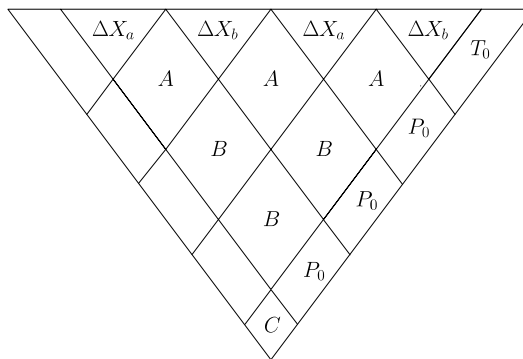


Figure 3.4: Structure of  $\Delta Y_k$

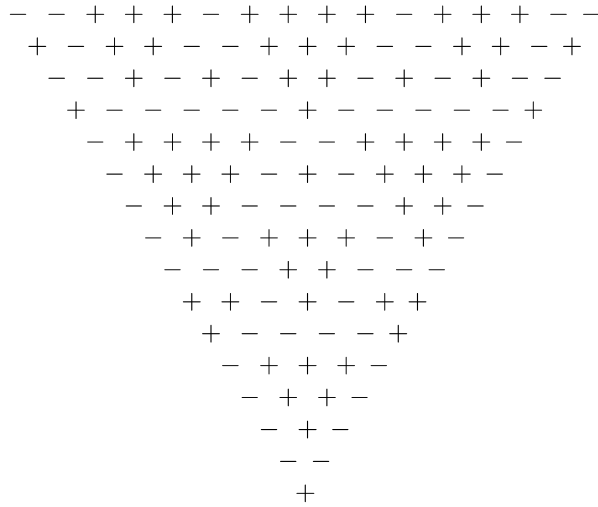


Figure 3.5:  $\Delta Z_0$ , that fulfills  $\sigma(\Delta Z_0) = 0$

First of all, let us recall proposition 2.2.1 and proposition 2.2.2. We know that a trapezoid is completely determined by its first row and by the antidiagonal just above its upper left side. We know that a parallelogram of length  $4 \times 12$  completely determined by the diagonal of length 4 just over its upper right side and the diagonal of length 12 just over its upper left side.

Then, we can guarantee that the trapezoid  $T_0$  is always the first element of the right band, since it is determined by its first row,  $++-$ , and the right side of  $\Delta X_b$ .

Note now that the bottom of  $T_0$  and the bottom of  $P_0$  are equal; and as we recall, the bottom triangle of  $A$  is the same as the bottom triangle of  $B$ . Thus, we know that between a diamond  $A$  and the trapezoid  $T_0$  or a diamond  $B$  and a parallelogram  $P_0$  there is always another parallelogram  $P_0$ . Since there are  $(2(k - 1))$   $B$  diamonds and a diamond  $A$  on the right side of  $\Delta X^k$ , there are  $(2k - 1)$  parallelograms  $P_0$ . The right band ends with the diamond  $C$ , of length  $4 \times 4$ , derived from  $P_0$  and its symmetric, and we have proved the structure of  $\Delta Y_k$ .

We check now that this structure provides a balanced Steinhaus triangle. In the following lines we use that, clearly, since the left band glued to  $\Delta X^k$  in order to obtain  $\Delta Y_k$  is the symmetric of the right band, they have the same entry sum.

$$\begin{aligned} \sigma(\Delta Y_k) &= \sigma(\Delta X^k) + 2 \cdot (\sigma(T_0) + (2k - 1) \cdot \sigma(P_0)) + \sigma(C) && \text{since } X^k \text{ is balanced,} \\ &= 2 \cdot (0 + (2k - 1) \cdot 0) + 0 && \text{since each block has entry sum 0,} \\ &= 0. && \text{And therefore it is balanced.} \end{aligned}$$

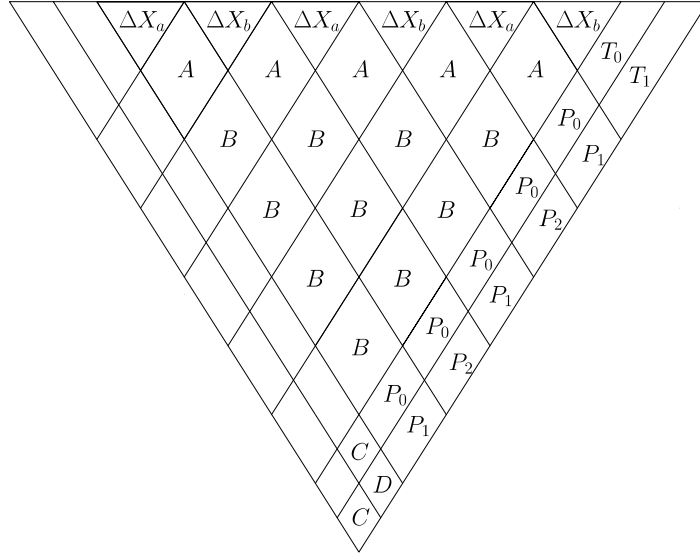
We have seen that for all integer  $k \geq 0$ ,  $Y_k$  is both symmetric and balanced.

Case  $r = 16$

Let  $Z_k = - - + + + - + + X^k + + - + + + - - = - - + + Y_k + + - -$ .

Clearly, by construction,  $Z_k$  is symmetric. We want to prove that  $Z_k$  is both balanced and symmetric for every  $k \geq 0$ .

First of all, we start checking that the symmetric sequence  $Z_0 = - - + + + - + + + - + + + - -$  is balanced, as we can see in figure 3.5, so we have proved the case  $k = 0$ .

Figure 3.6: Structure of  $\Delta Z_k$ 

Now, let us suppose  $k \geq 1$ . In order to prove that the symmetric sequence  $Z_k$  is also balanced, we can think of  $\Delta Z_k$  as  $\Delta Y_k$  with a band glued on each side, just as we thought of  $\Delta Y_k$  as  $\Delta X^k$  with two bands glued on the previous case. In this case, again, we focus on showing that the right-side band has sum 0 as well, since we know that  $\Delta Y_k$  is balanced and that, by symmetry, the left-side band has the same sum as the right-side band.

We want to prove that  $\Delta Z_k$  has the periodic structure seen in figure 3.6. Since we have already proved the structure of  $\Delta Y_k$ , we just need to see that the right band glued is composed by one trapezoid  $T_1$ ,  $k$  parallelograms  $P_1$ ,  $(k-1)$  parallelograms  $P_2$ , a diamond  $C$  and a diamond  $D$ . These four blocks are shown in figure 3.16.

In order to justify this structure, we need to note that the bottom of  $T_1$  is equal to the bottom of  $P_2$  and that the lower left side of  $D$  is equal to the bottom of  $P_0$ .

Then, as in the previous case, we can recall proposition 2.2.1 and guarantee that the right band always starts with a  $T_1$  on its top. Moreover, if we recall again proposition 2.2.2, we know that, since between  $T_1$  and  $P_0$  there is a parallelogram  $P_1$ , between  $P_2$  and  $P_0$  there is also a parallelogram  $P_1$ . Furthermore, between a  $P_1$  and a  $P_0$  there is always a parallelogram  $P_2$ .

Thus, we have proved that the right band starts with a  $T_1$  and then continues with  $P_1$  and  $P_2$  interspersed, ending with  $P_1$ . The band ends with the little diamond  $D$  and, given its similarities with  $P_0$ , a little diamond  $C$ .

Then, we have proved that the triangle  $\Delta Z_k$  has the structure shown in figure 3.6. We just need to check now that it is balanced. In the following lines, we use the symmetry between the left-side band and the right-side band for saying that they have the same entry sum.

$$\begin{aligned}
 \sigma(\Delta Z_k) &= \cancel{\sigma(\Delta Y_k)} + 2 \cdot (\sigma(T_1) + k \cdot \sigma(P_1) + (k-1) \cdot \sigma(P_2)) + \sigma(D) + \sigma(C) \\
 &= 2 \cdot (0 + k \cdot 0 + (k-1) \cdot 0) + 0 + 0 && \text{since } Y_k \text{ is balanced,} \\
 &= 0. && \text{And therefore it is balanced.}
 \end{aligned}$$

Thus, we have shown that  $Z_k$  is both symmetric and balanced for all integer  $k \geq 0$ .

**Case  $n \equiv 7 \pmod{8}$**

First of all, we start with two special cases,  $n = 7$  and  $n = 11$ . Later, we will see the general case for  $n \geq 23$ .

Let  $X^* = ++-+-++$ , this is a symmetric balanced binary sequence of length 7, as seen in figure 3.7.

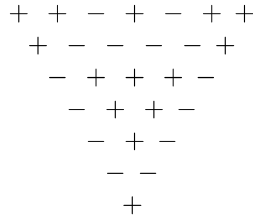


Figure 3.7:  $\Delta X^*$ , that fulfills  $\sigma(\Delta X^*) = 0$

Let  $X' = +-+++++-++++-+$ , a binary sequence of length 15 that is both symmetric and balanced, as seen in figure 3.8.

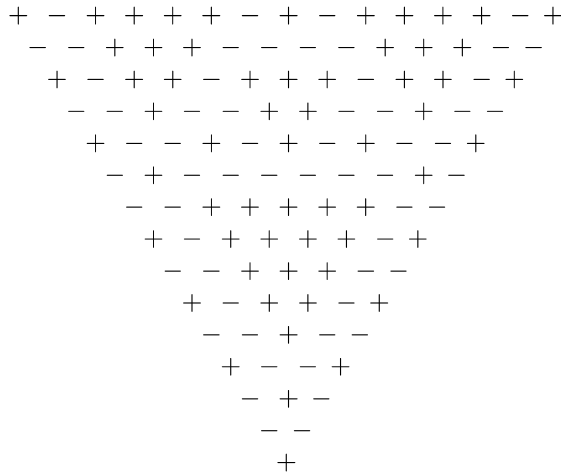


Figure 3.8:  $\Delta X'$ , that fulfills  $\sigma(\Delta X') = 0$

Now, for the general case with  $n \geq 23$ , let

$$X = +-----+++++-----+++++-----+,$$

as in the case  $n \equiv 0 \pmod{8}$ . We have seen that  $X$  is a symmetric balanced binary sequence of length 24.

We write  $n = 24k + r$ , with  $k \geq 1$  and  $r = -1, 7$  or  $15$ , given that  $n \equiv 7 \pmod{8}$ . We will

prove that the following constructions are both symmetric and balanced:

- $\partial(X^k)$  for length  $24k - 1$ ,  $k \geq 1$ ,
- $\hat{Y}_k = ++-+\partial(X^k)+-++$  for length  $24k + 7$ ,  $k \geq 1$ ,
- $\hat{Z}_k = +-++++-+\partial(X^k)+-++++-+$  for length  $24k + 16$ ,  $k \geq 1$ .

Let us prove these three cases individually.

*Case  $r = -1$*

We want to prove that  $\partial(X^k)$  is both balanced and symmetric for all  $k \geq 1$ . Given that  $X^k$  is symmetric, we know by remark 3.2.3 that  $\partial X^k$  is symmetric as well.

In order to prove that  $\partial(X^k)$  is balanced, note that  $X$  is such that  $\sigma(X) = 0$ , therefore  $X^k$  has entry sum 0 as well and

$$\sigma(\Delta X^k) = \sigma(X^k) + \sigma(\Delta \partial X^k) = 0.$$

Therefore,  $\sigma(\Delta \partial X^k) = 0$  and  $\partial(X^k)$  is balanced.

Note that the structure of  $\partial(X^k)$  is the same as in figure 3.2 with the unique change that the first row is removed. Clearly, this change only affects to  $\Delta X_a$  and  $\Delta X_b$  and it does not affect neither  $A$  nor  $B$ .

*Case  $r = 7$*

We want to prove that

$$\hat{Y}_k = +++-\partial(X^k)+-++$$

is both balanced and symmetric for all  $k \geq 1$ .

This case can be proved in the same way as the case  $r = 8$ , for  $k \geq 1$ , but now the blocks of the structure shown in figure 3.4 are different. As we have said on the previous case, the triangle  $\Delta \partial X^k$  differs to  $\Delta X^k$  by the absence of the first row, and the right band glued to  $\Delta \partial X^k$  in order to obtain  $\Delta \hat{Y}_k$  are those depicted on figure 3.17.

Analogously to the case  $r = 8$ , one can justify the periodic structure by observing that the bottom of  $\hat{T}_0$  and the bottom of  $\hat{P}_0$  are equal. Once seen the periodicity of the structure, one can easily prove that this provides an balanced Steinhau triangle since each block of the band has sum 0 and  $\Delta \partial X^k$  is balanced.

Therefore,  $\hat{Y}_k$  is both balanced and symmetric for all  $k \geq 1$ .

*Case  $r = 15$*

We want to prove that

$$\begin{aligned} \hat{Z}_k &= +-++++-\partial(X^k)+-++++-+ \\ &= +-++\hat{Y}_k+-++ \end{aligned}$$

is both balanced and symmetric for all  $k \geq 1$ .

Again, this case can be proved analogously to the case  $r = 16$ , for  $k \geq 1$ , with a structure similar to the one shown in figure 3.6. Since the bottom of  $\hat{T}_1$  is the same as the bottom of  $\hat{P}_1$ , the parallelogram  $\hat{P}_2$  is the same as the parallelogram  $\hat{P}_1$ . Then, there are only four different blocks in the right band,  $\hat{T}_1$ ,  $\hat{P}_1$ ,  $\hat{D}$  (depicted in figure 3.18) and  $\hat{C}$ . This, together with the fact that the structure of  $\Delta \hat{Y}_k$  has already been shown periodic, provides the periodicity of  $\Delta \hat{Z}_k$ .



Given that each block has sum 0 on this right-side band, by symmetry so do the blocks on the left-side band. Then, since we have seen that  $\tilde{Y}_k$  is balanced, we can conclude that  $\tilde{Z}_k$  is balanced too.

### Case $n \equiv 3 \pmod{8}$

This case is different than the two previous ones. In this case we define the sequences of each length  $n \equiv 3 \pmod{8}$  recursively.

Let  $\tilde{X}_3 = + - +$ , and let  $\tilde{X}_n$  for all  $n \geq 11$  be defined as follows:

- $\tilde{X}_n = + - + + \tilde{X}_{n-8} + + - +$  if  $n \equiv 3 \pmod{24}$ ,
- $\tilde{X}_n = - + + + \tilde{X}_{n-8} + + + -$  if  $n \equiv 11 \pmod{24}$ ,
- $\tilde{X}_n = + + - + \tilde{X}_{n-8} + - + +$  if  $n \equiv 19 \pmod{24}$ .

We want to prove by induction that every  $\tilde{X}_n$  is both symmetric and balanced. Since the symmetry is clear by construction, we just need to focus on proving that they are balanced.

First of all, let us check the base cases. In the figure 3.19 we can see  $\tilde{X}_{27}$ , but we can also see  $\tilde{X}_3$ ,  $\tilde{X}_{11}$  and  $\tilde{X}_{19}$ . Easily, we check

$$\sigma(\Delta\tilde{X}_3) = \sigma(\Delta\tilde{X}_{11}) = \sigma(\Delta\tilde{X}_{19}) = \sigma(\Delta\tilde{X}_{27}) = 0.$$

Note that we obtain  $\tilde{X}_{n+8}$  by gluing a v-shape band surrounding  $\tilde{X}_n$ . We can assume now  $n \geq 35$ . We want to prove that, if all the  $\tilde{X}_m$  with  $m < n$  and  $m \equiv 3 \pmod{8}$  are balanced, then  $\tilde{X}_n$  is balanced.

Let the triangles  $\alpha_1, \alpha_2, \alpha_3$  be defined as seen in figure 3.9.

$\alpha_1$	$\alpha_2$	$\alpha_3$
+ - +	+ + -	- + +
- -	+ -	- +
+	-	-

Figure 3.9: The three triangles  $\alpha_i$

Note that:

- The triangle  $\alpha_1$  is equal to  $\Delta\tilde{X}_3$ .
- Since it is symmetric,  $\alpha_1 = \bar{\alpha}_1$ .
- The other two triangles fulfill  $\alpha_2 = \bar{\alpha}_3$ .
- All of them are balanced, i.e.  $\sigma(\alpha_i) = 0$  for all  $i \in [3]$ .

Let  $a_1$ ,  $a_2$  and  $a_3$  be defined as seen in figure 3.10.

$$\begin{array}{ccc}
 \mathbf{a_1} & \mathbf{a_2} & \mathbf{a_3} \\
 + & + & + \\
 ++ & -- & ++ \\
 -++ & +++ & +-+ \\
 --+- & -++- & -+-+ \\
 +-+ & -+- & -++ \\
 -+ & -- & +- \\
 - & + & -
 \end{array}$$

Figure 3.10: The three diamonds  $a_i$

Note that:

- Since it is symmetric,  $a_2 = \bar{a}_2$ .
- The other two diamonds fulfill  $a_1 = \bar{a}_3$ .
- The three of them are such that  $\sigma(a_i) = 0$  for all  $i \in [3]$ .

And, at last, let  $b_1$ ,  $b_2$  and  $b_3$  be defined as seen in figure 3.11.

$$\begin{array}{ccc}
 \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \\
 + & + & + \\
 -- & -- & ++ \\
 -++ & +-+ & -+- \\
 +-++ & +-+ & +-+- \\
 --+ & +-+ & -+- \\
 +- & -+ & -- \\
 - & - & +
 \end{array}$$

Figure 3.11: The three diamonds  $b_i$

Note that:

- Since it is symmetric,  $b_3 = \bar{b}_3$ .
- The other two diamonds fulfill  $b_1 = \bar{b}_2$ .
- All of them are such that  $\sigma(b_i) = 0$  for all  $i \in [3]$ .

We want to prove that  $\tilde{X}_n$  has the structure depicted in figure 3.12. In this structure, the order of the blocks varies, so the figure shows general subindexes. The following table shows the value of the indexes  $i$ ,  $j$  and  $k$  in every case.

Case	$n \equiv 3 \pmod{24}$	$n \equiv 11 \pmod{24}$	$n \equiv 19 \pmod{24}$
$i$	3	2	1
$j$	2	1	3
$k$	1	3	2

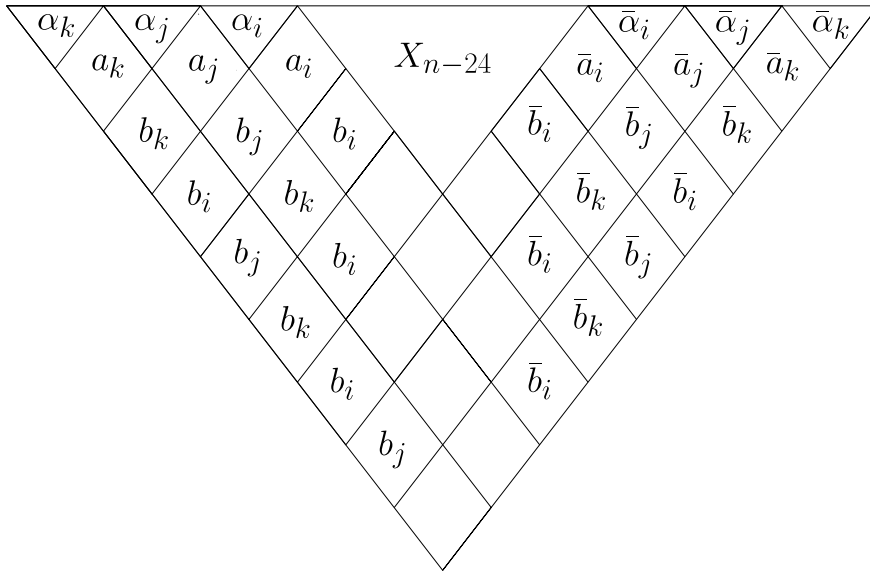


Figure 3.12: Structure of  $\Delta\tilde{X}_n$

We will prove that this structure is valid by induction. It is easy to check in figure 3.19 that it is correct for the cases  $n = 11, 19$  and  $27$ . Then, we suppose  $n \geq 35$  and we suppose the structure is fulfilled for all the triangles  $\Delta\tilde{X}_m$  with  $m < n$  and  $m \equiv 3 \pmod{8}$ .

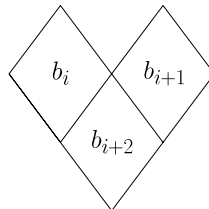
Now, let us prove this structure. First of all, given the symmetry of the triangle, it is clear that we just need to prove one half of it, and the other half is just the reversed of the first one. We prove the left half of  $\Delta\tilde{X}_n$ .

The triangles  $\alpha_i, \alpha_j$  and  $\alpha_k$  are clearly correct by inductive construction. Furthermore, if we once again recall proposition 2.2.2, we can justify that  $a_i, a_j$  and  $a_k$  are just as periodic as the triangles above them. It is easy to check that between  $\alpha_1$  and  $\alpha_2$  an  $a_1$  is generated, and analogously  $a_2$  and  $a_3$  are generated between  $\alpha_2$  and  $\alpha_3$  and between  $\alpha_3$  and  $\alpha_1$ , respectively.

The first line of  $b_i, b_j$  and  $b_k$  follows by the same argument.

Now, note that the last three rows of  $b_1$  are equal to the last three rows of  $a_3$ , and the same thing happens with  $b_2$  and  $a_1$  and with  $b_3$  and  $a_2$ . Note as well that the first and the last element of the fourth row of  $a_i$  are equal to  $-1$  for all  $i \in [3]$  and the first and the last element of the fourth row of  $b_i$  are equal to  $+1$  for all  $i \in [3]$ .

These two last facts guarantee that the upper element of a diamond generated between any pair  $a_i, a_j$  or  $b_i, b_j$  will be  $+1$ . Then, we can use proposition 2.2.2 for the rest of the diamond, and we get that just as between a  $a_3$  and a  $a_1$ , it is generated a  $b_3$  between  $b_1$  and  $b_2$ . Analogously, between  $b_2$  and  $b_3$ , there is a  $b_1$ ; and between  $b_3$  and  $b_1$ , a  $b_2$ . This can be pictured as: for all  $i \in \mathbb{Z}/3\mathbb{Z}$ ,



Therefore, we have proved the structure in figure 3.12. Since we have supposed that  $\Delta\tilde{X}_{n-24}$  is balanced, and we have seen that each block in this big v-shape band has sum 0, we can conclude that  $\Delta\tilde{X}_n$  has also sum 0 and thus,  $\tilde{X}_n$  is balanced for all  $n \equiv 3, 11$  or  $19 \pmod{24}$ .  $\nabla$

### 3.3 From the symmetric to the antisymmetric case

In this section we see how we can obtain an antisymmetric balanced binary sequence from a symmetric balanced binary sequence.

**Remark 3.3.1.** Note that if  $X = (x_1, \dots, x_n)$  is an antisymmetric binary sequence, then  $\partial X$  is a symmetric binary sequence. This happens because in the construction of the derived sequence it does not matter the value of each element, but the fact that is equal or different to its neighbours.

**Proposition 3.3.2.** *Let  $X = (x_1, \dots, x_n)$  be a binary sequence of length  $n$ . Then,  $X$  is both antisymmetric and balanced if and only if  $n \equiv 4 \pmod{8}$  and  $\partial X$  is both symmetric and balanced.*

*Proof.*

- **$X$  antisymmetric and balanced  $\implies n \equiv 4 \pmod{8}$  and  $\partial X$  symmetric and balanced.**

First of all, we know that  $X$  is balanced, so  $\sigma(\Delta X) = 0$ . As seen in remark 3.1.6, we know that if  $X$  is antisymmetric, then  $\sigma(X) = 0$ . By definition,  $\Delta X$  is essentially  $\Delta\partial X$  with  $X$  as a new row above it, so

$$\sigma(\Delta\partial X) = \sigma(\Delta X) - \sigma(X) = 0.$$

Then,  $\partial X$  is balanced.

As discussed on remark 3.1.4, if  $X$  is antisymmetric, then  $n$  is even. We have also seen in remark 3.3.1 that  $\partial X$  is symmetric, furthermore, it is of odd length  $n - 1$  and its middle term is  $-1$ , since it is the product of two elements with different sign. Then, using that  $\partial X$  is symmetric and balanced with  $x_{(n/2)} = -1$  in proposition 3.2.6 we know that

$$n - 1 \equiv 3 \pmod{8} \implies n \equiv 4 \pmod{8}.$$

- **$\partial X$  symmetric and balanced and  $n \equiv 4 \pmod{8} \implies X$  antisymmetric and balanced.**

If  $X$  is of length  $n \equiv 4 \pmod{8}$ , then  $Y = \partial X$  has length  $n - 1 \equiv 3 \pmod{8}$ . Let  $k = \frac{n}{2}$ . Since  $Y$  is symmetric, we have

$$\partial X = Y = (y_1, \dots, y_{k-1}, y_k, y_{k-1}, \dots, y_1).$$

Then, the sequence  $X = (x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{n-1}, x_n)$  is one of the two *primitive sequences* of  $Y$  (the other one is  $-X$ ), and can be written as

$$X = \left( x_1, x_1 y_1, x_1 y_1 y_2, \dots, x_1 y_1 y_2 \cdots y_k, x_1 y_1 y_2 \cdots y_{k-2} y_{k-1}^2 y_k, \dots, x_1 y_1 y_k, x_1 y_k \right)$$

with  $x_1 = +1$  or  $x_1 = -1$ . Now, since  $\partial X$  is both symmetric and balanced, we use proposition 3.2.6 and we obtain that  $y_k = -1$ . Therefore,

$$X = (x_1, x_1 y_1, \dots, x_1 y_1 y_2 \cdots y_{k-2}, -x_1 y_1 y_2 \cdots y_{k-1}, -x_1 y_1 y_2 \cdots y_{k-2}, \dots, -x_1 y_1, -x_1)$$

and  $X$  is clearly antisymmetric. Then,  $\sigma(X) = 0$  and  $X$  is balanced since

$$\sigma(\Delta X) = \sigma(X) + \sigma(\Delta\partial X) = 0 + 0 = 0.$$

▽

### 3.4 The antisymmetric case

In this section we conclude the antisymmetric case, using the results obtained in the previous sections.

**Theorem 3.4.1.** *For every positive integer  $n \equiv 4 \pmod{8}$ , there exists a binary sequence  $X$  of length  $n$  which is both antisymmetric and balanced.*

*Proof.* Clear by theorem 3.2.7 and proposition 3.3.2. ▽

**Corollary 3.4.2.** *For every positive integer  $n \equiv 4 \pmod{8}$ , there exists a binary sequence  $X$  of length  $n$  which is both balanced and zero-sum, i.e.,  $\sigma(X) = 0$ .*

*Proof.* For theorem 3.4.1, exists an antisymmetric balanced binary sequence, and as seen in remark 3.1.6, the antisymmetric sequences are zero-sum. ▽

**Example 3.4.3.** We have seen that the sequence  $\tilde{X}_{11} = -++++-++++-$  is symmetric and balanced. The two sequences whose the derived sequence is  $X$  are antisymmetric and balanced. This sequences are  $\tilde{Y}^1 = +-----++++-$  and  $\tilde{Y}^2 = -+++++-----+$ . We can see  $\Delta\tilde{Y}^1$  and  $\Delta\tilde{Y}^2$  in figure 3.20.

**Theorem 3.4.4.** *For every positive integer  $n \equiv 0$  or  $3 \pmod{4}$ , there exists a balanced binary sequence  $X$  of length  $n$ .*

*Proof.* There are two separated cases:

- If  $n \equiv 4 \pmod{8}$ , trivial since theorem 3.4.1.
- Otherwise, trivial since theorem 3.2.7.

▽

And hence we have found an alternative proof of Steinhaus's problem.

### 3.5 Appendix: Figures

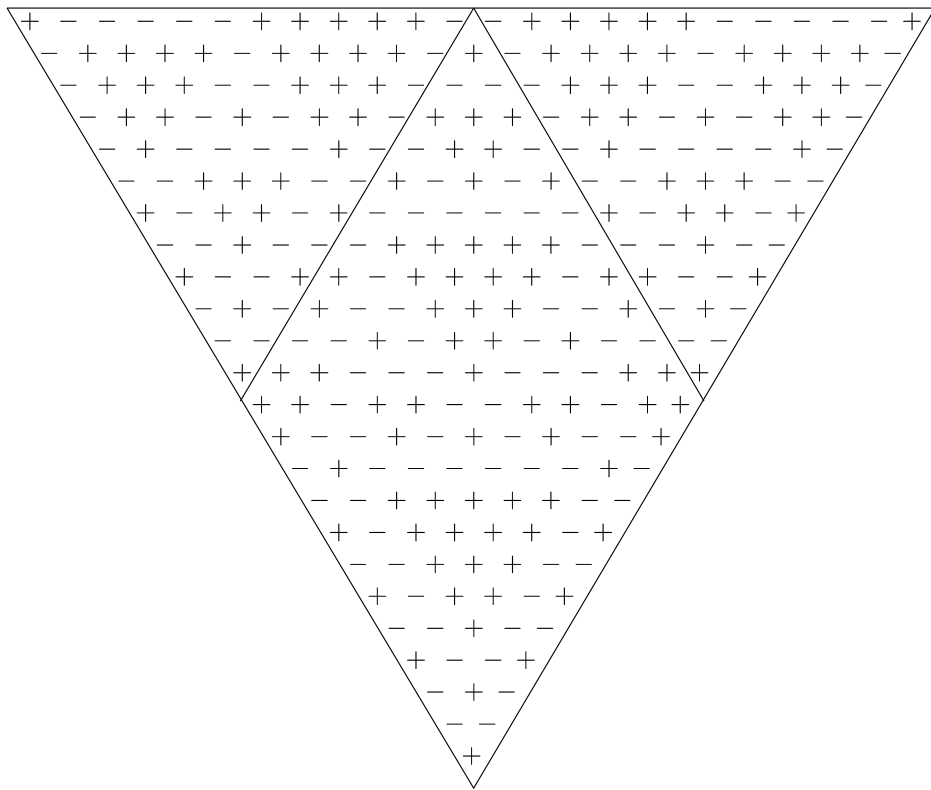


Figure 3.13:  $\Delta X$ , that fulfills  $\sigma(\Delta X) = 0$



<p><b><math>T_1</math></b></p> <pre>       + + - -       + + - +       - + - -       - - - +       + + + -       + + + -       + + + -       - + + -       - - + -       + + - -       - + - +       + - - -       - - + +       + + - +       + + - -       + + - +       - + - -       - - +       + -       -           </pre> <p><math>\sigma(T_1) = 0</math></p>	<p><b><math>P_1</math></b></p> <pre>       +       + -       - - -       - + + +       + - + +       - - - +       + + + -       - + + -       + - + -       + - - -       + - + +       - - - +       + + -       + -       -           </pre> <p><math>\sigma(P_1) = 0</math></p>
<p><b><math>P_2</math></b></p> <pre>       +       + +       - + +       - - + -       + + - -       - + - +       + - - -       - - + +       + + - +       + + - -       + + - +       - + - -       - - +       + -       -           </pre> <p><math>\sigma(P_2) = 0</math></p>	<p><b><math>D</math></b></p> <pre>       +       - +       + - +       + - - -       - + +       - +       -           </pre> <p><math>\sigma(D) = 0</math></p>

Figure 3.16: The four new different blocks of the right band of  $\Delta Z_k$



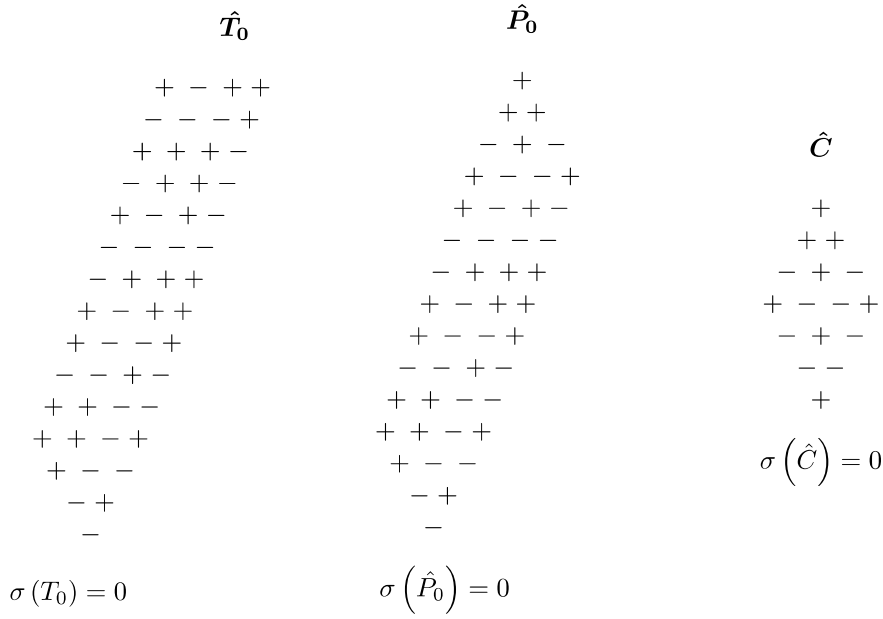


Figure 3.17: The constructing blocks of the right band of  $\Delta\hat{Y}_k$

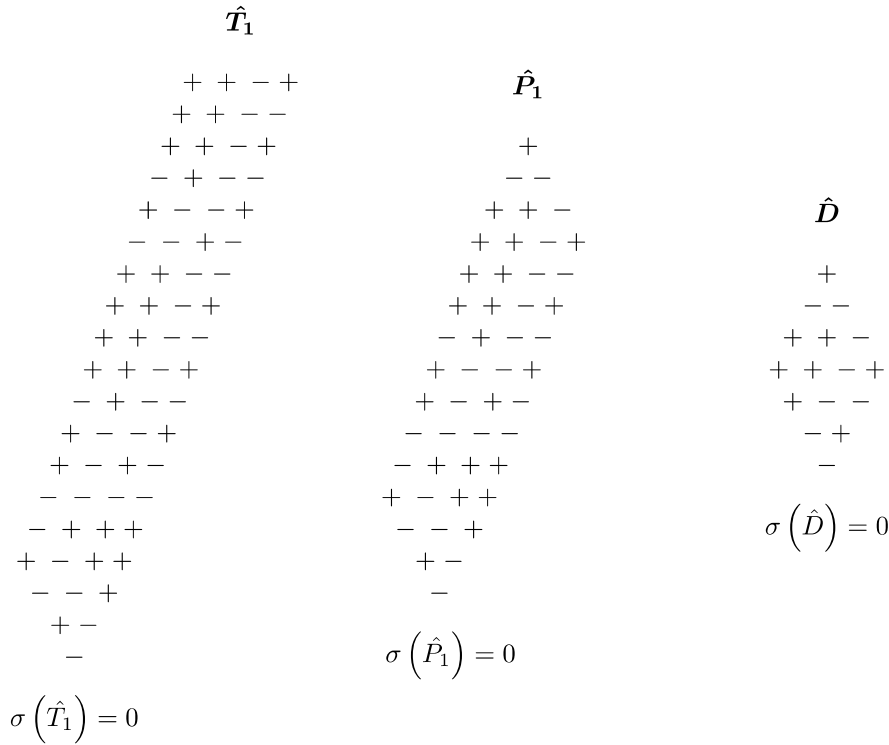


Figure 3.18: The four new different blocks of the right band of  $\Delta\hat{Z}_k$

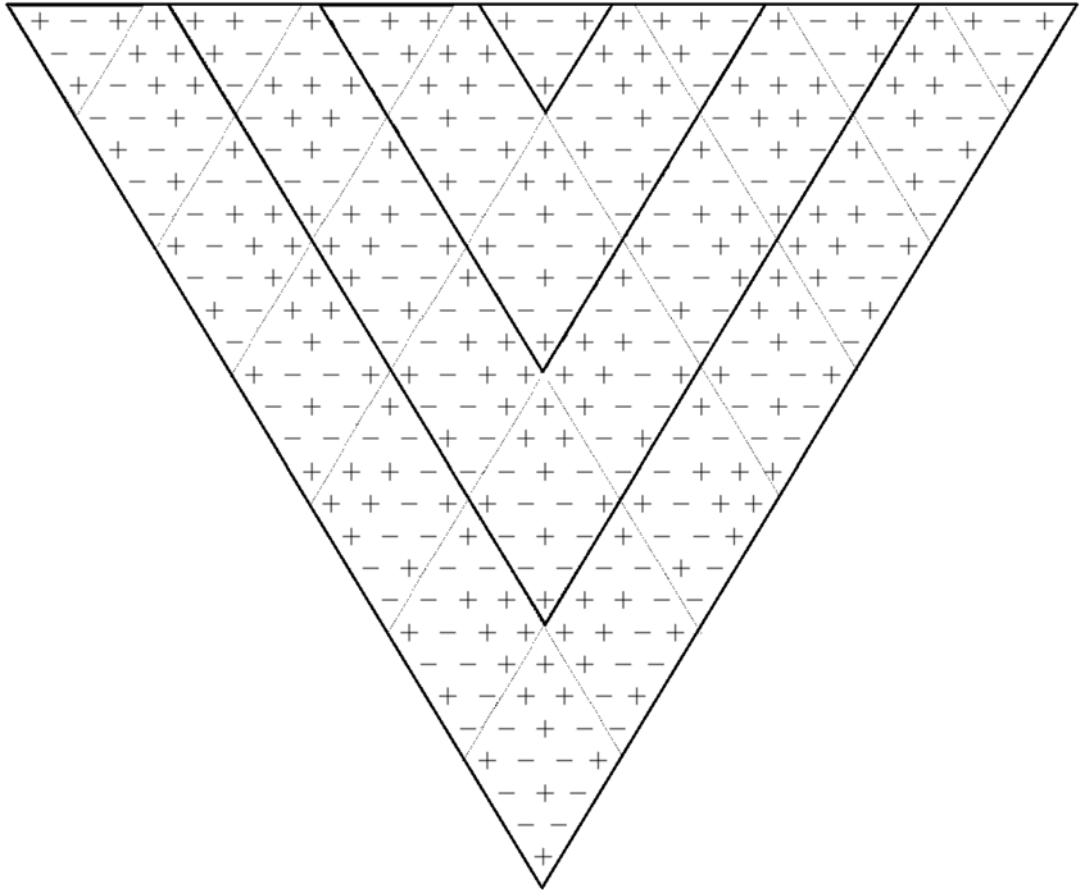


Figure 3.19: In this figure we can see  $\Delta\tilde{X}_{27}$ . The thick lines indicate the triangles  $\Delta\tilde{X}_3$ ,  $\Delta\tilde{X}_{11}$  and  $\Delta\tilde{X}_{19}$ ; while the thin lines indicate the blocks  $\alpha_i$ ,  $a_i$  and  $b_i$  for all  $i \in [3]$

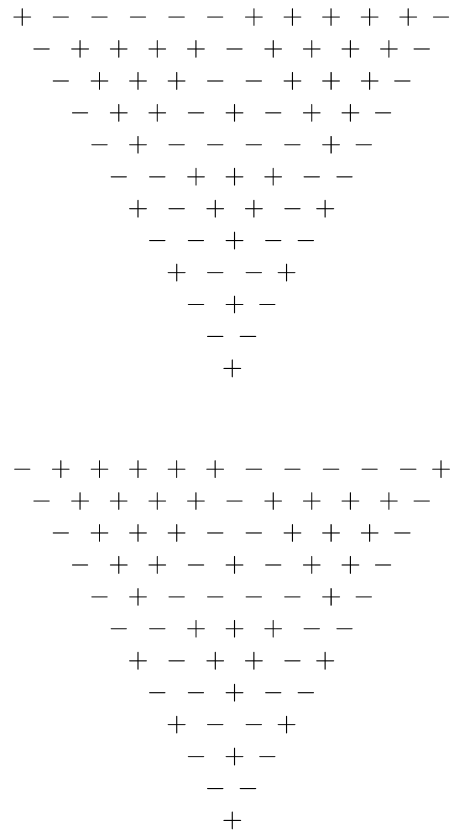


Figure 3.20: Two antisymmetric balanced binary sequences.



# Chapter 4

## Zero-sum balanced binary sequences

### 4.1 Introduction

In this chapter, we find balanced binary sequences of a given length  $n \in \mathbb{N}$  that have the same number of  $+1$  as  $-1$ . This derived question of Steinhaus's problem was discussed and solved in [7].

**Definition 4.1.1.** Let  $X$  be a finite binary sequence. We say that  $X = (x_1, x_2, \dots, x_n)$  is *zero-sum* if  $\sigma(X) = \sum_{i=1}^n x_i = 0$ .

**Proposition 4.1.2.** *If  $X = (x_1, \dots, x_n)$  is a binary sequence of length  $n$  such that  $X$  is both balanced and zero-sum, then  $n \equiv 0 \pmod{4}$ .*

*Proof.* Since it is necessary for  $X$  to be balanced that  $n \equiv 0$  or  $3 \pmod{4}$ , and for  $X$  to be zero-sum, i.e., have the same number of  $+1$  as  $-1$ , its length has to be even. Therefore,  $n \equiv 0 \pmod{4}$ .  $\nabla$

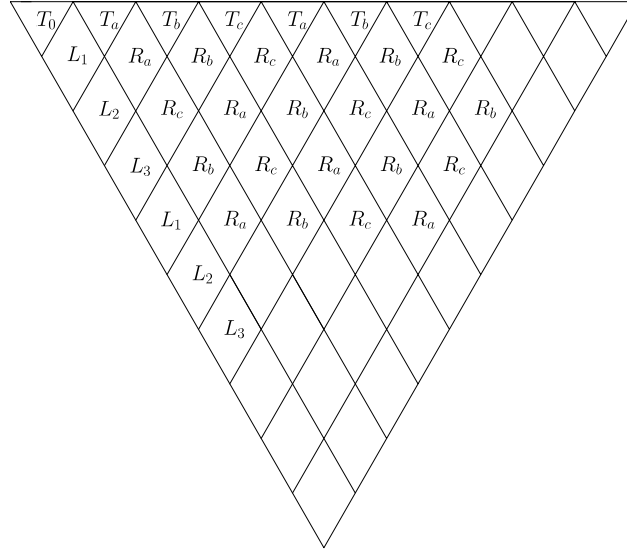
**Proposition 4.1.3.** *If  $X$  is both balanced and zero-sum, then  $\partial X$  is balanced.*

*Proof.* By construction, the triangle  $\Delta X$  is essentially  $\Delta \partial X$  with an added row on the top of it, the sequence  $X$ . Therefore,

$$\sigma(\Delta X) = \sigma(X) + \sigma(\Delta \partial X).$$

Since  $X$  is balanced, we have  $\sigma(\Delta X) = 0$ . Since  $X$  is zero-sum, we have  $\sigma(X) = 0$ . Thus,  $\sigma(\Delta \partial X) = 0$ , so  $\partial X$  is balanced.  $\nabla$

Therefore, if we are able to find a zero-sum balanced binary sequence for every length  $n \equiv 0 \pmod{4}$ , we will have found as well a balanced binary sequence of every length  $(n - 1) \equiv 3 \pmod{4}$  just by writing its derived sequence. In the following pages we will prove the existence of a binary sequence both balanced and zero-sum for every length  $n \equiv 0 \pmod{4}$  and thus we will provide an alternative proof of Steinhaus's problem.

Figure 4.1: Structure of  $\Delta S_0 [8m]$ 

## 4.2 The existence

**Theorem 4.2.1.** *Let  $I_0, I_4, P_0$  and  $P_4$ :*

$$I_0 = + - - + - + - +,$$

$$I_4 = - - + + + + - -,$$

$$P_0 = - - - + - + + + - - + - + - + + + + - - - - + +,$$

$$P_4 = - + - + + - - - - + + + - - + - + - + + - + + - .$$

Let  $S_0 = I_0 P_0^\infty$  and  $S_4 = I_4 P_4^\infty$ , eventually periodic binary sequences, with an initial segment of length 8 and a period of length 24. Then, for all  $m \geq 1$ , the initial sequences  $S_0 [8m]$  and  $S_4 [8m + 4]$  are both balanced and zero-sum. In the case of  $S_4$  this is also valid for  $m = 0$ .

*Proof.* We need to prove two different cases, the case with  $S_0 [8m]$  and the case with  $S_4 [8m + 4]$ , and, in addition, we need to prove two things in each case: that the sequences are zero-sum and that they are balanced. We will prove each case separately. In both cases we will find a periodic structure on each Steinhaus's triangle in order to prove that the sum of the elements will always be null and therefore the binary sequences will be always balanced. The periodic structure of the sequences will guarantee us that they are zero-sum.

### Case $S_0 [8m]$

First, we prove that for all  $m \geq 1$ , the sequence  $S_0 [8m]$  is zero-sum. We show this property by induction using the eventual periodicity of the  $S_0$ . To do so, we start by checking that the initial segment  $I_0$  is zero-sum.

$$\sigma(I_0) = \sigma(+ - - + - + - +) = 0.$$

Then, if we evaluate the period  $P_0$  in three concatenated pieces of length 8, named  $P_{0,a}$ ,  $P_{0,b}$  and  $P_{0,c}$ , we realize that all three of them are zero-sum.

$$\begin{aligned}\sigma(P_{0,a}) &= \sigma(- - - + - + ++ ) = 0, \\ \sigma(P_{0,b}) &= \sigma(- - + - + - ++ ) = 0, \\ \sigma(P_{0,c}) &= \sigma(+ + - - - - ++ ) = 0.\end{aligned}$$

Therefore, we can see that for all  $m \geq 1$ , the sequence  $S_0 [8m]$  is zero-sum, since given the sequence  $S_0 [8m]$  we are able to find the sequence  $S_0 [8(m+1)]$ .

Now we need to see that for all  $m \geq 1$ , the sequence  $S_0 [8m]$  is balanced. We denote by  $T_{8m}$  the Steinhaus triangles  $\Delta S_0 [8m]$ .

In order to prove the general case, we will show that  $T_{8m}$  has a periodic structure that guarantees that the sequence  $S_0 [8m]$  is balanced for all  $m \geq 1$ . This structure will correspond to the one shown in figure 4.1, which we will justify step by step.

We need to note that the triangle  $T_{8m+8}$  is obtained by gluing a band of width 8 to the right side of the triangle  $T_{8m}$ .

Therefore, if we can prove that the initial triangle  $\Delta I_0$  is balanced and that all these *band differences* have the same number of +1 as -1, we will have proved by induction that for all  $m \geq 1$  the sequences  $S_0 [8m]$  are balanced.

We begin showing the case with  $m = 1$ , the triangle  $\Delta I_0$  has to be balanced. Indeed, as we see in figure 4.2,  $\sigma(\Delta I_0) = \sigma(T_0) = 0$ .

$$\begin{array}{cccccccc} + & - & - & + & - & + & - & + \\ & - & + & - & - & - & - & - \\ & & - & - & + & + & + & + \\ & & & + & - & + & + & + \\ & & & & - & - & + & + \\ & & & & & + & - & + \\ & & & & & & - & - \\ & & & & & & & + \end{array}$$

Figure 4.2:  $T_0$

Then we present the blocks that form the general case  $T_{8m}$ .

First, there are the triangles  $T_a$ ,  $T_b$  and  $T_c$  that appear periodically on the top of the triangle given the periodic construction of  $S_0 [8m]$ . These three triangles are those depicted in figure 4.5. As we can see,

$$\begin{aligned}\sigma(T_a) &= \sigma(\Delta P_{0,a}) = -2, \\ \sigma(T_b) &= \sigma(\Delta P_{0,b}) = -2, \\ \sigma(T_c) &= \sigma(\Delta P_{0,c}) = +4.\end{aligned}$$

Second, in figure 4.6, there are the blocks  $L_1$ ,  $L_2$  and  $L_3$ , that appear periodically on the left side of the triangle. Their corresponding entry sums are

$$\sigma(L_1) = 2, \sigma(L_2) = 0, \sigma(L_3) = -2.$$

And last, in figure 4.7, there are the diamond-shaped blocks  $R_a$ ,  $R_b$  and  $R_c$ , that appear on the rest of the triangle. Their corresponding entry sums are

$$\sigma(R_a) = 2, \sigma(R_b) = 4, \sigma(R_c) = -6.$$

These blocks are obtained by the construction of the first triangles  $T_{8m}$ .

- The block  $L_1$  appears as of case  $m = 2$ , easily identified between  $T_0$  and  $T_a$ . This case is balanced since

$$\sigma(T_{16}) = \cancel{\sigma(T_0)} + \overset{0}{\sigma(T_a)} + \sigma(L_1) = (-2) + 2 = 0.$$

- The diamond-shaped blocks  $L_2$  and  $R_a$  appear as of the case  $m = 3$ . This case is balanced since

$$\sigma(T_{24}) = \cancel{\sigma(T_{16})} + \overset{0}{\sigma(T_b)} + \sigma(L_2) + \sigma(R_a) = (-2) + 0 + 2 = 0.$$

- And  $L_3$ ,  $R_b$  and  $R_a$  appear as of the case  $m = 4$ . This case is balanced since

$$\sigma(T_{32}) = \cancel{\sigma(T_{24})} + \overset{0}{\sigma(T_c)} + \sigma(L_3) + \sigma(R_b) + \sigma(R_c) = 4 + (-2) + 4 + (-6) = 0.$$

Once we have all these blocks, we realise that there are some similarities between them:

- The lower triangle of the diamond  $L_3$  is equal to  $T_0$ .
- The lower triangle of the diamond  $R_a$  is equal to  $T_c$ .
- The lower triangle of the diamond  $R_b$  is equal to  $T_a$ .
- The lower triangle of the diamond  $R_c$  is equal to  $T_b$ .

Now, suppose  $m \geq 5$ . For the induction step, we assume that all the triangles until  $T_{8(m-8)}$  are balanced. Now, we just need to see that all the band difference to obtain  $T_{8m}$ , has entry sum 0. But, if we observe the structure of the triangles, we can see that this band difference is the same that the band glued to  $T_{8(m-4)}$  to obtain  $T_{8(m-3)}$  with an extra  $R_a$ ,  $R_b$  and  $R_c$ . Since we have assumed that they both are balanced, their band difference must sum 0. In addition,

$$\sigma(R_a) + \sigma(R_b) + \sigma(R_c) = 2 + 4 + (-6) = 0,$$

thus the band difference has entry sum 0 and  $T_{8m}$  is balanced.

Therefore we have shown that, for all  $m \geq 1$ , the sequence  $S_0[8m]$  is both zero-sum and balanced.

**Case  $S_4[8m + 4]$  :**

In this case, we prove that for all  $m \geq 0$ , the sequence  $S_4[8m + 4]$  is zero-sum. Let us consider that  $I_4$  is the concatenation of two segments of length 4, named  $I_{4,1}$  and  $I_{4,2}$ . First of all, we check that  $I_{4,1}$ , the initial segment of  $I_4$  is zero-sum. This is the case  $m = 0$ .

$$\sigma(S_4[4]) = \sigma(I_{4,1}) = \sigma(- - + +) = 0.$$

Then, if we consider the concatenation  $I_{4,2}P_4(P_4[4])$ , we will be able to cut it in four pieces of length 8, named  $P_{4,1}$ ,  $P_{4,a}$ ,  $P_{4,b}$  and  $P_{4,c}$ , that will correspond to the four possible segments that can be concatenated to  $S_4[8m + 4]$  in order to get  $S_4[8(m + 1) + 4]$ . Then, as we had in the previous case,

$$\sigma(P_{4,1}) = \sigma(+ + - - - + - +) = 0, \quad \sigma(P_{4,a}) = \sigma(+ - - - - + + +) = 0,$$

$$\sigma(P_{4,b}) = \sigma(- - + - + - + +) = 0, \quad \sigma(P_{4,c}) = \sigma(- + + - - - + +) = 0.$$

Therefore, we can see that for all  $m \geq 0$ , the sequence  $S_4[8m + 4]$  is zero-sum.



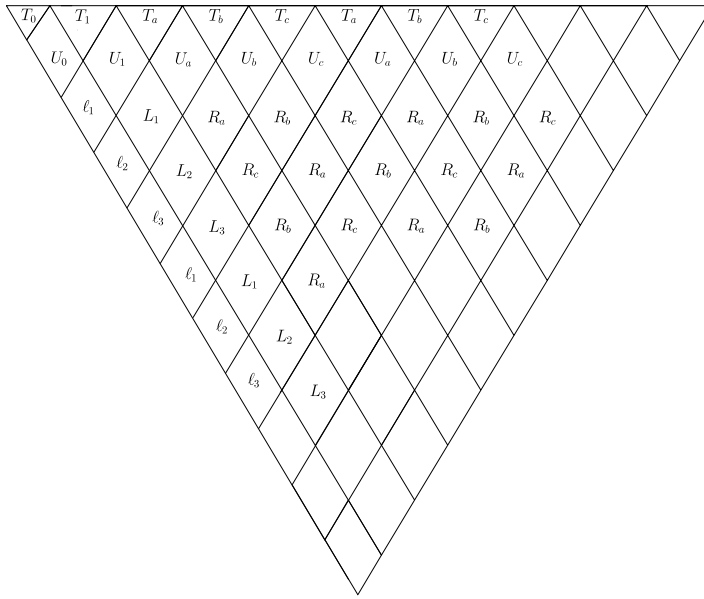


Figure 4.3: Structure of  $\Delta S_4 [8m + 4]$

In order to prove that, for all  $m \geq 0$ , the sequence  $S_4 [8m + 4]$  is balanced, we can follow the same techniques seen in the previous case. This time the structure of the triangle  $T_{8m+4} = \Delta S_4 [8m + 4]$ , as seen in figure 4.3, is more complicated and requires 19 different blocks instead of the 10 of the previous case.

The first block constitutes the base case of the induction, the case  $m = 0$ , i.e., the triangle  $T_0 = \Delta S_4 [4]$ . This is easily checked to be balanced as seen below.

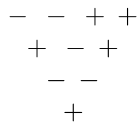


Figure 4.4:  $T_0$

Then, there are the four different triangles of sidelength 8: the one that completes the contribution of initial segment, the triangle  $T_1$ , and the three that appear periodically direct consequence of the periodicity of the sequence, triangles  $T_a$ ,  $T_b$  and  $T_c$ . They are depicted in figures 4.8. We can see that

$$\begin{aligned} \sigma(T_1) &= \sigma(\Delta P_{4,1}) = -4, & \sigma(T_b) &= \sigma(\Delta P_{4,b}) = -2, \\ \sigma(T_a) &= \sigma(\Delta P_{4,a}) = +4, & \sigma(T_c) &= \sigma(\Delta P_{4,c}) = -2. \end{aligned}$$

Then, there are four parallelograms of size  $4 \times 8$ , named  $U_0$ ,  $\ell_1$ ,  $\ell_2$  and  $\ell_3$ . They are placed on the left side of  $T_{8m+4}$ . They contain all the influence of the presence of the initial triangle  $T_0$ . They fulfill

$$\sigma(U_0) = 4, \quad \sigma(\ell_1) = 0, \quad \sigma(\ell_2) = -2, \quad \sigma(\ell_3) = -2,$$

as can be observed in figure 4.9.

And finally, there are as well 10 different diamonds of sidelength 8:

- Four that appear on the upper side, just below the triangles, named  $U_1, U_a, U_b$  and  $U_c$  (depicted in figure 4.10). The last three appear periodically given the periodicity presence of  $T_a, T_b$  and  $T_c$ . They fulfill

$$\sigma(U_1) = -4, \quad \sigma(U_a) = -6, \quad \sigma(U_b) = -10, \quad \sigma(U_c) = -4.$$

- Three on the left side, next to the parallelograms, named  $L_1, L_2$  and  $L_3$  (can be observed in figure 4.11). They appear periodically, always on the left side. They fulfill

$$\sigma(L_1) = 10, \quad \sigma(L_2) = 18, \quad \sigma(L_3) = -4.$$

- On the rest of the triangle, there are three different diamonds, named  $R_a, R_b$  and  $R_c$  (depicted in figure 4.12). They appear periodically as well. They fulfill

$$\sigma(R_a) = -4, \quad \sigma(R_b) = 2, \quad \sigma(R_c) = 2.$$

These blocks are easily obtained in the cases with small  $m$ :

- In the case  $m = 1$ , we obtain  $U_0$ . The triangle  $T_{12}$  is balanced since

$$\sigma(T_{12}) = \cancel{\sigma(T_0)} + \overset{0}{\sigma(T_1)} + \sigma(U_0) = 4 + (-4) = 0.$$

- In the case  $m = 2$ , we obtain  $U_1$  and  $\ell_1$ . The triangle  $T_{20}$  is balanced since

$$\sigma(T_{20}) = \cancel{\sigma(T_{12})} + \overset{0}{\sigma(T_a)} + \sigma(U_1) + \sigma(\ell_1) = 4 + (-4) + 0 = 0.$$

- If  $m = 3$ , we can get  $U_a, L_1$  and  $\ell_2$ . The triangle  $T_{28}$  is balanced since

$$\sigma(T_{28}) = \cancel{\sigma(T_{20})} + \overset{0}{\sigma(T_b)} + \sigma(U_a) + \sigma(L_1) + \sigma(\ell_2) = (-2) + (-6) + 10 + (-2) = 0.$$

- Now we consider  $m = 4$ , and we get  $U_b, R_a, L_2$  and  $\ell_3$ . The triangle  $T_{36}$  is balanced since

$$\begin{aligned} \sigma(T_{36}) &= \cancel{\sigma(T_{28})} + \overset{0}{\sigma(T_c)} + \sigma(U_b) + \sigma(R_a) + \sigma(L_2) + \sigma(\ell_3) \\ &= (-2) + (-10) + (-4) + 18 + (-2) = 0. \end{aligned}$$

- Lastly, the case  $m = 5$  provides us  $U_c, R_b, R_a$  and  $L_3$ . The triangle  $T_{44}$  is balanced since

$$\begin{aligned} \sigma(T_{44}) &= \cancel{\sigma(T_{36})} + \overset{0}{\sigma(T_a)} + \sigma(U_c) + \sigma(R_b) + \sigma(R_c) + \sigma(L_3) + \sigma(\ell_1) \\ &= 4 + (-4) + 2 + 2 + (-4) + 0 = 0. \end{aligned}$$

The periodic structure observed in figure can be justified observing the similarities between blocks and applying proposition 2.2.2.

Now, suppose  $m \geq 6$ . For the induction step, we assume that all the triangles until  $T_{8(m-1)+4}$  are balanced. Now, we just need to see that all the band difference, has entry sum 0. But, if we observe the structure of the triangles, we can see that this band difference is the same that

the band glued to  $T_{8(m-4)+4}$  to obtain  $T_{8(m-3)+4}$  with an extra  $R_a$ ,  $R_b$  and  $R_c$ . Since we have assumed that they both are balanced, their band difference must sum 0. In addition,

$$\sigma(R_a) + \sigma(R_b) + \sigma(R_c) = 2 + 2 + (-4) = 0,$$

thus the band difference has entry sum 0 and  $T_{8m+4}$  is balanced.

Therefore we have shown that, for all  $m \geq 0$ , the sequence  $S_0[8m]$  is both zero-sum and balanced.

▽

**Theorem 4.2.2.** *For every positive integer  $n \equiv 0 \pmod{4}$ , there exists a binary sequence  $X$  of length  $n$  which is both zero-sum and balanced.*

*Proof.* Trivial since theorem 4.2.1.

▽

**Corollary 4.2.3.** *For every positive integer  $n \equiv 0$  or  $3 \pmod{4}$ , there exists a balanced binary sequence  $X$  of length  $n$ .*

*Proof.* There are two separated cases:

- If  $n \equiv 0 \pmod{4}$ , trivial since theorem 4.2.2.
- Otherwise,  $n \equiv 3 \pmod{4}$ . Then, let  $m = n + 1$ , with  $m \equiv 0 \pmod{4}$ . Since theorem 4.2.2 there exists  $Y$  a binary sequence of length  $m$  which is both zero-sum and balanced. Then, as we have seen in proposition 4.1.3, the derived sequence  $X = \partial Y$  is balanced and of length  $n$ .

▽

And hence we have found an alternative proof of Steinhaus's problem.

### 4.3 Appendix: Figures

In the following pages we can see the blocks described on the proof of theorem 4.2.1.

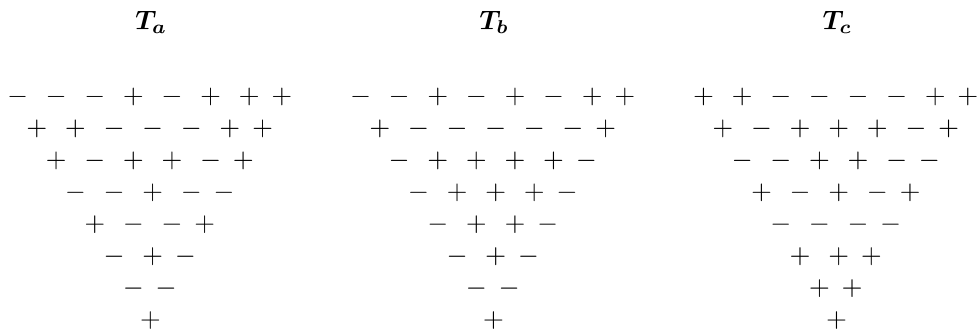


Figure 4.5: The three triangles that appear on the top of  $T_{8m}$

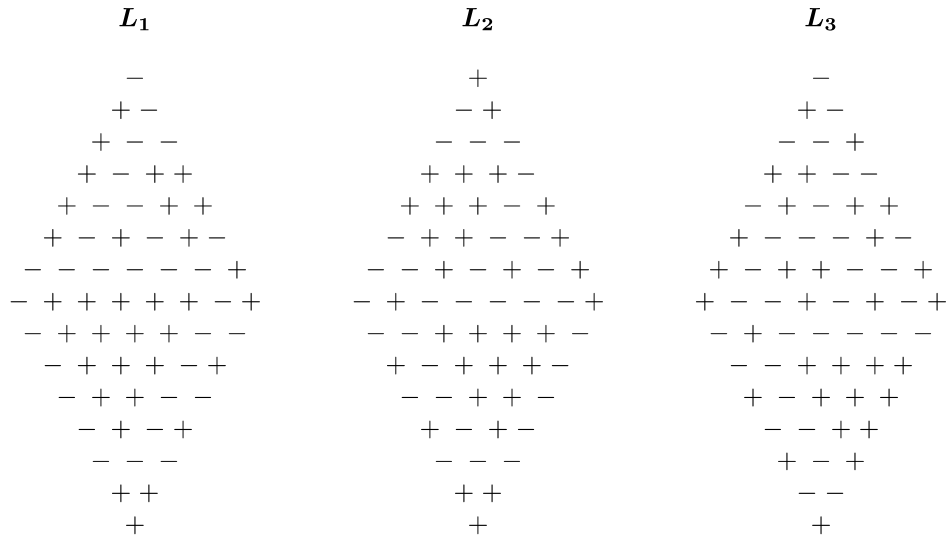


Figure 4.6: The diamonds appearing on the left side of  $T_{8m}$

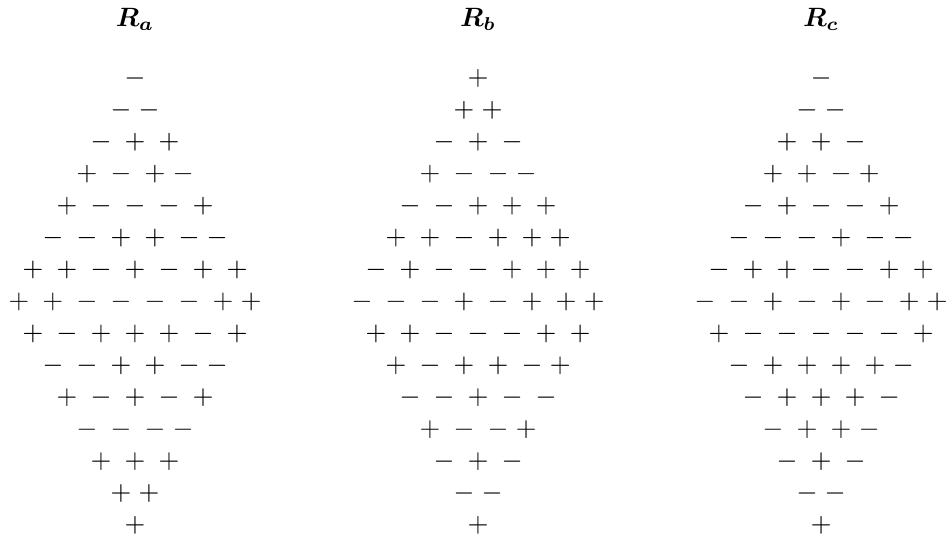


Figure 4.7: The diamonds that appear on the general part of  $T_{8m}$

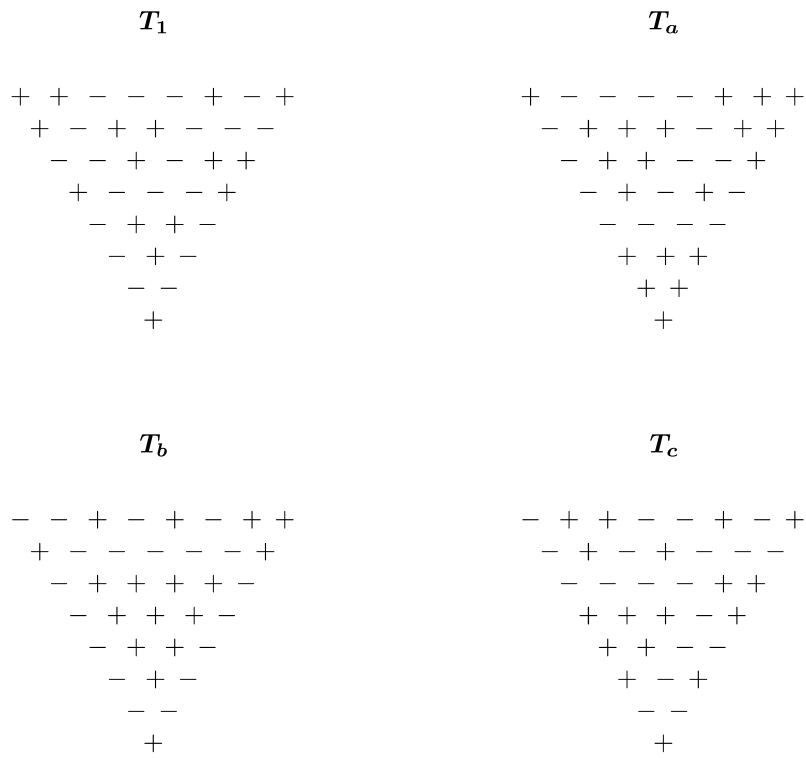


Figure 4.8: The four triangles that appear on the top of  $T_{8m+4}$

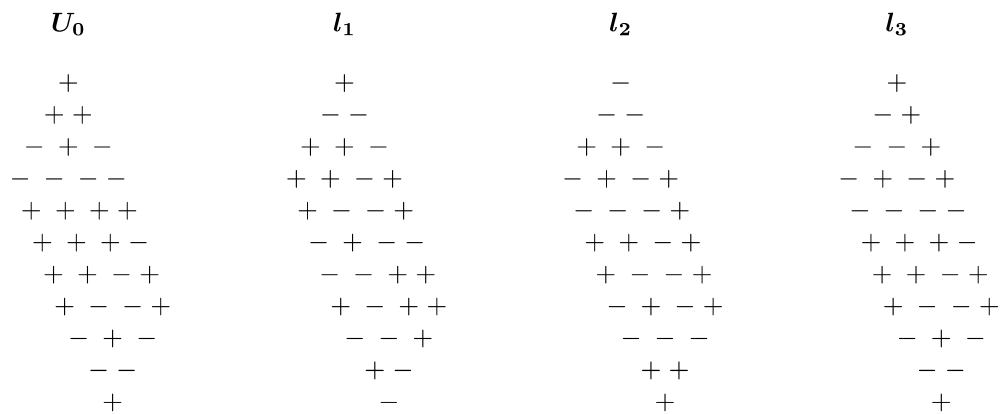
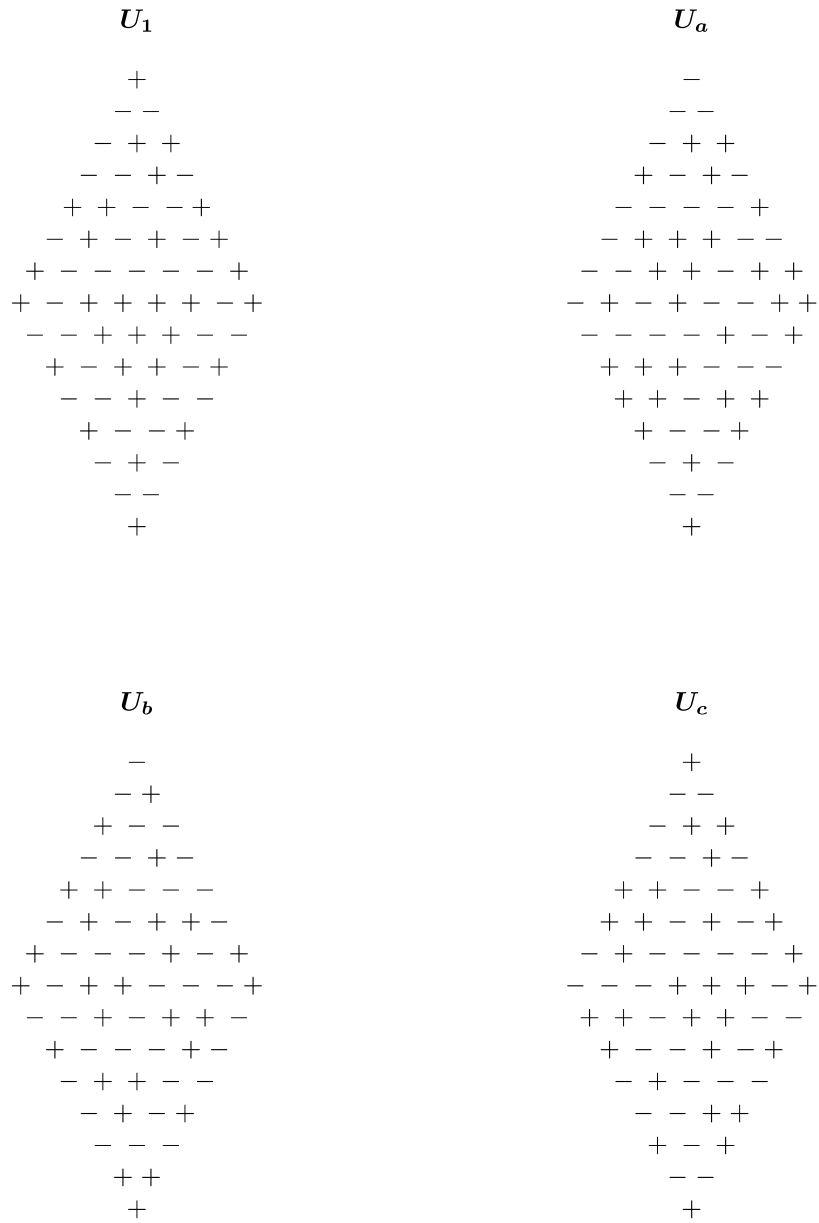


Figure 4.9: The four parallelograms of  $\Delta S_4[8m + 4]$

Figure 4.10: The four upper diamonds of  $S_4[8m + 4]$

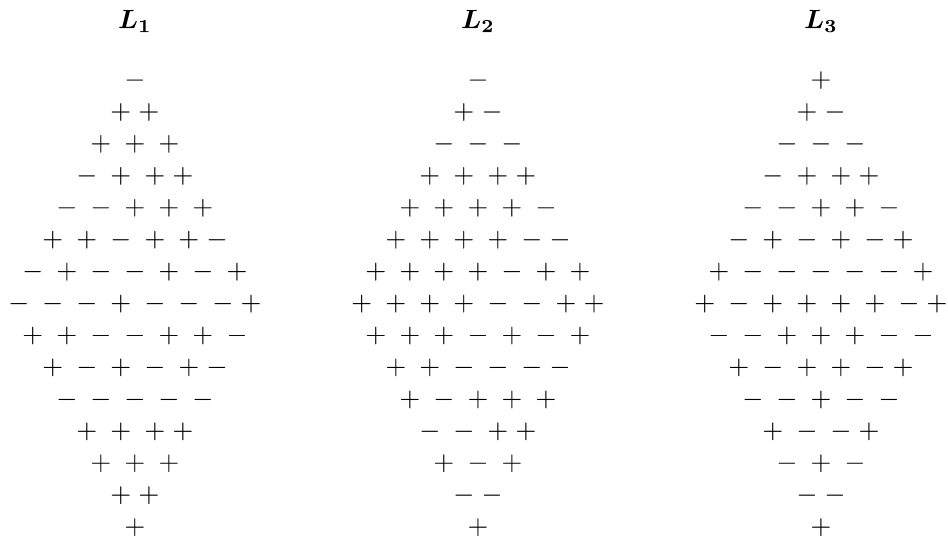


Figure 4.11: The three left-side diamonds of  $S_4[8m + 4]$

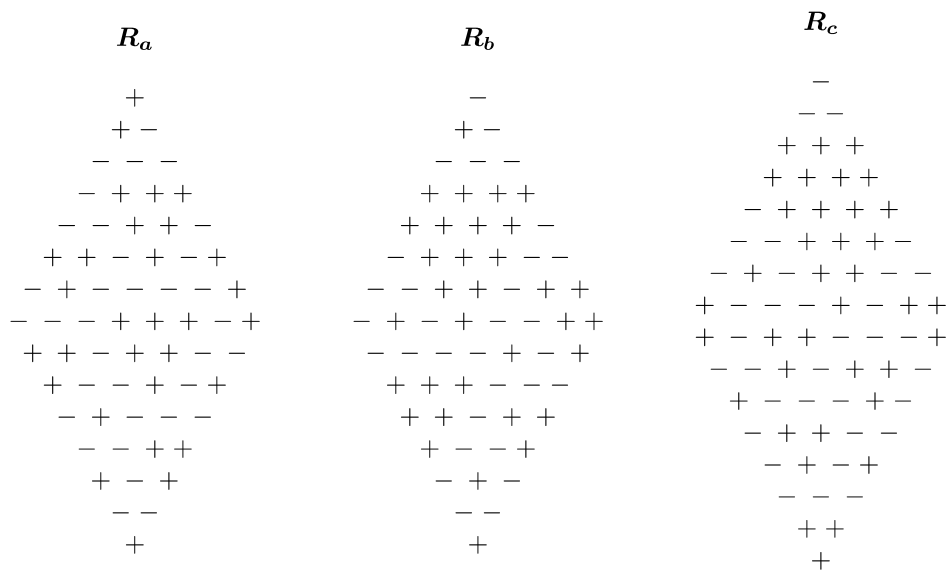


Figure 4.12: The three general diamonds of  $S_4[8m + 4]$





## Chapter 5

# First approach to Molluzzo's problem

### 5.1 Introduction

In 1976, John C. Molluzzo posed one of the most interesting, and therefore difficult, derived problems of Steinhaus's problem.

In his paper [4], he took Harborth's alternative notation to Steinhaus's problem, the elements of the triangle are zeros and ones and the derived sequences are made by sums modulo 2; and wondered what would happen if one made sums modulo  $m$ , with  $m \geq 2$ . He asked whether there would be a sequence of any length such that its generalized Steinhaus triangle would have the same number elements equal to 0, as of 1, as of 2,..., as of  $(m - 1)$  or not.

In other words, if  $n$  and  $m$  are positive integers, with  $m \geq 2$ , is it possible to find a sequence  $X = (x_1, x_2, \dots, x_n)$  with  $x_i \in \mathbb{Z}/m\mathbb{Z}$  for all  $i$ , such that each element in the finite cyclic group  $\mathbb{Z}/m\mathbb{Z}$  appears the same number of times in the generalized Steinhaus triangle  $\Delta X$ ?

**Example 5.1.1.** A balanced generalized Steinhaus triangle in  $\mathbb{Z}/4\mathbb{Z}$  of initial length 7 is depicted in fig. 5.1.

$$\begin{array}{cccccccc} 3 & 3 & 1 & 0 & 3 & 3 & 1 & \\ & 2 & 0 & 1 & 3 & 2 & 0 & \\ & & 2 & 1 & 0 & 1 & 2 & \\ & & & 3 & 1 & 1 & 3 & \\ & & & & 0 & 2 & 0 & \\ & & & & & 2 & 2 & \\ & & & & & & 0 & \end{array}$$

Figure 5.1: A generalized triangle in  $\mathbb{Z}/4\mathbb{Z}$ .

In this chapter, we will introduce some partial solutions of Molluzzo's problem, following the steps of [8] and our own computer simulations.

## 5.2 Generalities on balanced sequences

First of all, let us define the terms of our generalization. These definitions are the intuitive generalization of the definitions that we have been using throughout the thesis.

Let  $n, m$  be two positive integers with  $m \geq 2$ , and let  $X = (x_1, x_2, \dots, x_n)$  be a sequence of length  $n$ , with  $x_i \in \mathbb{Z}/m\mathbb{Z}$  for all  $i$ .

**Definition 5.2.1.** The *generalized derived sequence* of  $X$  is a sequence  $\partial X = (y_1, \dots, y_{n-1})$  such that  $y_i = x_i + x_{i+1}$  in  $\mathbb{Z}/m\mathbb{Z}$ . In the case  $n = 1$ , we will consider  $\partial X = \emptyset$ .

Therefore, for every  $0 \leq i \leq n - 1$  we can define  $\partial^i X$  as the  $i$ th derived sequence of  $X$ , that recursively seen is

$$\partial^i X = \begin{cases} X & \text{if } i = 0; \\ \partial(\partial^{i-1} X) & \text{if } i \geq 1. \end{cases}$$

Then, the  $i$ th derived sequence of  $X$  is

$$\partial^i X = \left( \sum_{k=0}^i \binom{i}{k} x_{1+k}, \sum_{k=0}^i \binom{i}{k} x_{2+k}, \dots, \sum_{k=0}^i \binom{i}{k} x_{n-i+k} \right).$$

**Definition 5.2.2.** The *generalized Steinhaus triangle*, or just *generalized triangle*, of  $X$  is the sequence  $\Delta X = (X, \partial X, \dots, \partial^{n-1} X)$ .

**Definition 5.2.3.** We say that a generalized triangle  $\Delta X$  of a sequence  $X$  is *balanced* if all the elements in  $\mathbb{Z}/m\mathbb{Z}$  appear the same number of times.

**Definition 5.2.4.** We say that a sequence  $X$  is *balanced* if its generalized triangle  $\Delta X$  is balanced.

**Example 5.2.5.** The generalized Steinhaus triangle shown in example 5.1.1 is a balanced generalized triangle, and therefore its initial row is a balanced sequence.

In the following, we denote by  $N$  the total number of elements of the generalized triangle, i.e  $N = \frac{n(n+1)}{2}$ .

Note that, if  $X = (x_1, \dots, x_n)$  is a sequence of length  $n$  with  $x_i \in \mathbb{Z}/m\mathbb{Z}$  for all  $i \in [n]$ , then  $X \in (\mathbb{Z}/m\mathbb{Z})^n$ . However, from now on we will make an abuse of notation and write that  $X$  is in  $\mathbb{Z}/m\mathbb{Z}$ .

**Remark 5.2.6.** As in the Steinhaus's original problem, there is a clear necessary condition: the total number of elements has to be a multiple of the number of different elements. This is, if the sequence  $X = (x_1, \dots, x_n)$  is balanced, then  $m|N$ .

So far, the notation for Molluzzo's problem is very similar to the notation for Steinhaus's. However, talking about  $\sigma(\Delta X)$  does not make sense anymore, and it is necessary to define a multiplicity function in order to have a compact way to denote that a sequence is balanced.

Given a finite multiset  $M$  of  $\mathbb{Z}/m\mathbb{Z}$ , we denote its multiplicity function of  $M$  by

$$\mathbf{m}_M : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{N},$$

where  $\mathbf{m}_M(x)$  is the number of occurrence of  $x \in \mathbb{Z}/m\mathbb{Z}$  in the multiset  $M$ . If  $x$  is not in  $M$ , then  $\mathbf{m}_M(x) = 0$ .

The total number of elements of  $M$  fulfills

$$|M| = \sum_{x \in \mathbb{Z}/m\mathbb{Z}} \mathbf{m}_M(x).$$

We extend this definition to the generalized triangles  $\Delta X$ .

**Remark 5.2.7.** Therefore, clearly, the sequence  $X$  is balanced if, and only if, for all  $x \in \mathbb{Z}/m\mathbb{Z}$ ,

$$\mathbf{m}_{\Delta X}(x) = \frac{1}{m}N.$$

For a given factor  $q$  of  $m$ , we denote the canonical surjective morphism

$$\pi_q : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/q\mathbb{Z}.$$

Then, if  $X = (x_1, \dots, x_n)$ , we define

$$\pi_q(X) = (\pi_q(x_1), \dots, \pi_q(x_n)),$$

its projected sequence in  $\mathbb{Z}/q\mathbb{Z}$ .

Let us study the relation between the balanced sequences in  $\mathbb{Z}/m\mathbb{Z}$  and the projection morphism  $\pi_q$ .

**Theorem 5.2.8.** *Let  $m, n, q$  be positive integers such that  $q|m$ , and let  $X = (x_1, \dots, x_n)$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then,  $X$  is balanced if, and only if,  $\pi_q(X)$  is balanced and  $\mathbf{m}_{\Delta X}$  is constant on each coset of the subgroup  $q\mathbb{Z}/m\mathbb{Z}$ .*

*Proof.* For a  $x \in \mathbb{Z}/m\mathbb{Z}$ , the multiplicity of its projected element  $\pi_q(x)$  in the triangle  $\Delta\pi_q(X)$  is

$$\mathbf{m}_{\Delta\pi_q(X)}(\pi_q(x)) = \sum_{i=0}^{\frac{m}{q}-1} \mathbf{m}_{\Delta X}(x + iq),$$

the sum of the multiplicities of the elements of the coset  $x + q\mathbb{Z}/m\mathbb{Z}$  in the triangle  $\Delta X$ . Then, the result follows by remark 5.2.7.  $\nabla$

## 5.3 Arithmetic progressions

In this section, we describe the structure of a generalized triangle derived from a sequence that is an arithmetic progression of  $\mathbb{Z}/m\mathbb{Z}$ .

Let  $n, m$  be positive integers, and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Then, we denote by

$$AP(a, d, n) = (a, a + d, a + 2d, \dots, a + (n - 1)d)$$

the arithmetic progression of initial value  $a$ , common difference  $d$  and of length  $n$ .

Now, let us see the relation between an arithmetic progression and its derived sequences.

**Proposition 5.3.1.** *Let  $n, m$  be positive integers, and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Let  $X = AP(a, d, n)$  an arithmetic progression as defined above. Then, the  $i$ th derived sequence of  $X$ , with  $i \in [n - 1]$ , is*

$$\partial^i X = \partial^i AP(a, d, n) = AP(2^{i-1}(2a + id), 2^i d, n - i),$$

*another arithmetic progression.*

*Proof.* Let  $\partial^i X = (y_1, \dots, y_{n-i})$ . Then, for all  $j \in [n-i]$ ,

$$\begin{aligned} y_j &= \sum_{k=0}^i \binom{i}{k} x_{j+k} = \sum_{k=0}^i \binom{i}{k} (a + (j+k-1)d) \\ &= \sum_{k=0}^i \binom{i}{k} (a + (j-1)d) + \sum_{k=0}^i \binom{i}{k} kd. \end{aligned}$$

Now, using that  $\sum_{k=0}^i \binom{i}{k} = 2^i$  and that  $\sum_{k=0}^i \binom{i}{k} k = 2^{i-1}i$ , we have that

$$\begin{aligned} y_j &= 2^i(a + (j-1)d) + 2^{i-1}id, \\ &= 2^{i-1}(2a + id) + (j-1)2^i d. \end{aligned}$$

▽

Then, in terms of the elements of the triangle  $\Delta X(i, j)$ :

**Proposition 5.3.2.** *Let  $n, m$  be positive integers, and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Let  $X = AP(a, d, n)$  an arithmetic progression in  $\mathbb{Z}/m\mathbb{Z}$ . Then, for all  $i \in [n]$ ,  $j \in [n-i+1]$ :*

$$\Delta X(i, j) = \begin{cases} a + (j-1)d & \text{if } i = 1, \\ 2^{i-1}a + 2^{i-2}(2j+i-3)d & \text{otherwise.} \end{cases}$$

*Proof.* Remember that  $\Delta X(i, j)$  is the  $j$ th element of the  $i$ th row of  $\Delta X$ , i.e., the  $(i-1)$ th derived of  $X$ . Then, it is clear since proposition 5.3.1. ▽

Now let us discuss about primitive sequences.

**Definition 5.3.3.** A *primitive sequence* of  $X$  is any sequence  $Y = (y_1, \dots, y_{n+1})$  such that  $\partial Y = X$ .

It is clear that  $X$  has  $m$  different primitive sequences in  $\mathbb{Z}/m\mathbb{Z}$ . For instance, in the original Steinhaus's problem, with the plus and minus notation, any binary sequence has two primitive sequences, being one the opposite of the other.

But, if  $m$  is odd, there is only one primitive of an arithmetic progression that is itself an arithmetic progression.

**Proposition 5.3.4.** *Let  $n, m$  be positive integers, with  $m$  odd, and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Let  $AP(a, d, n)$  an arithmetic progression in  $\mathbb{Z}/m\mathbb{Z}$ . Then, the sequence  $AP(2^{-1}a - 2^{-2}d, 2^{-1}d, n+1)$  is the only arithmetic progression such that its derived sequence is  $AP(a, d, n)$ .*

*Proof.* By proposition 5.3.1,

$$\partial AP(2^{-1}a - 2^{-2}d, 2^{-1}d, n+1) = AP((2(2^{-1}a - 2^{-2}d) + 2^{-1}d), 2 \cdot 2^{-1}d, n) = AP(a, d, n).$$

Now, let us see that it is unique. Suppose now that  $AP(a_1, d_1, n+1)$  and  $AP(a_2, d_2, n+1)$  are two different arithmetic progressions that have the same derived sequence

$$\partial AP(a_1, d_1, n+1) = \partial AP(a_2, d_2, n+1).$$

Then, by proposition 5.3.1, we have

$$AP(2a_1 + d_1, 2d_1, n) = AP(2a_2 + d_2, 2d_2, n).$$

Therefore, we have that

$$\begin{cases} 2a_1 + d_1 = 2a_2 + d_2; \\ 2d_1 = 2d_2. \end{cases}$$

Since  $m$  is odd, 2 is invertible in  $\mathbb{Z}/m\mathbb{Z}$ . It follows that  $d_1 = d_2$ , and therefore  $a_1 = a_2$ . Thus, the two arithmetic progressions are equal, and we have seen its the uniqueness.  $\nabla$

However, the hypothesis  $n$  is necessary, as we can see in the following example.

**Example 5.3.5.** Let  $n = 7$  and  $m = 4$ . Let  $X$  and  $Y$  be sequences of length 7 in  $\mathbb{Z}/4\mathbb{Z}$ , defined as

$$\begin{aligned} X &= AP(3, 2, 7) = (3, 1, 3, 1, 3, 1, 3), \\ Y &= AP(2, 0, 7) = (2, 2, 2, 2, 2, 2, 2). \end{aligned}$$

Then,  $X$  and  $Y$  are both arithmetic progressions, and

$$\partial X = (0, 0, 0, 0, 0, 0) = \partial Y,$$

which is the arithmetic progression  $AP(0, 0, 6)$ .

### 5.3.1 Balanced arithmetic progressions in $\mathbb{Z}/m\mathbb{Z}$ for $m$ odd

In this section, we will assume  $m$  odd. Now, we will see that the common difference of a balanced arithmetic progression in  $\mathbb{Z}/m\mathbb{Z}$  must be invertible.

**Theorem 5.3.6.** *Let  $n, m$  be positive integers, with  $m$  odd, and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Then, if  $d$  is non-invertible, the arithmetic progression  $AP(a, d, n)$  is not balanced.*

*Proof.* Suppose that exists a balanced sequence  $X = AP(a, d, n)$  with  $d$  non-invertible in  $\mathbb{Z}/m\mathbb{Z}$  and we will find a contradiction.

Let  $q = \gcd(m, d') > 1$ , where  $d'$  is any integer such that  $d' \equiv d \pmod{m}$ . Now, consider the canonical surjective morphism

$$\pi_q : \mathbb{Z}/m\mathbb{Z} \longrightarrow \mathbb{Z}/q\mathbb{Z},$$

and the arithmetic progression

$$\pi_q(X) = AP(\pi_q(a), \pi_q(d), n) = AP(\pi_q(a), 0, n),$$

a constant sequence in  $\mathbb{Z}/q\mathbb{Z}$ .

Since we have supposed that  $X$  is balanced, by theorem 5.2.8, the sequence  $\pi_q(X)$  is balanced. Therefore, there is at least one element of the triangle  $\Delta\pi_q(X)$  equal to 0. Let  $\Delta\pi_q(i, j) = 0$ . Then, by proposition 5.3.2,

$$\begin{aligned} 0 &= \Delta\pi_q(X)(i, j) = 2^{i-1}\pi_q(a) + 2^{i-2}(2j + i - 3)\pi_q(d) \\ &= 2^{i-1}\pi_q(a). \end{aligned}$$

Since  $m$  is odd, we know that 2 is invertible, so we have  $\pi_q(a) = 0$ . Therefore,  $\pi_q(X)$  is the zero-sequence of length  $n$  in  $\mathbb{Z}/q\mathbb{Z}$ . Contradiction with the fact that  $\pi_q(X)$  is balanced.  $\nabla$

Therefore, if in order to find balanced arithmetic progressions, we focus on the invertible common differences.

Now, let us introduce some notation. For every odd number  $m$ , we denote by:

- $\alpha(m)$  the *multiplicative order of  $2^m$  modulo  $m$* , this is,

$$\alpha(m) = \min \{e \in \mathbb{N}^* \mid 2^{em} \equiv 1 \pmod{m}\}.$$

- $\varphi(m)$  the *Euler's totient of  $m$* , i.e. the number of positive integers  $\ell \in [m]$  such that  $\gcd(\ell, m) = 1$ .

Since Euler's theorem states that if  $m$  and  $\ell$  are relatively prime then  $\ell^{\varphi(m)} \equiv 1 \pmod{m}$ , note that, for  $m$  odd,  $\alpha(m)$  divides  $\varphi(m)$ .

- $\text{rad}(m)$  the *radical of  $m$* , i.e. the product of the distinct prime factors of  $m$ . If  $\mathcal{P}$  is the set of all prime numbers, then

$$\text{rad}(m) = \prod_{\substack{p \in \mathcal{P} \\ p|m}} p.$$

- $\beta(m)$  the *projective multiplicative order of  $2^m$  modulo  $m$* , this is,

$$\beta(m) = \min \{e \in \mathbb{N}^* \mid 2^{em} \equiv \pm 1 \pmod{m}\}.$$

There are a few interesting properties about  $\alpha(m)$  and  $\beta(m)$ .

**Remark 5.3.7.** Note that, either  $\alpha(m) = \beta(m)$  or  $\alpha(m) = 2\beta(m)$ . Furthermore,  $\alpha(m) = 2\beta(m)$  if and only if there exists a positive integer  $e$  such that  $2^{em} \equiv -1 \pmod{m}$ .

**Proposition 5.3.8.** *Let  $m$  be an odd number. Then,  $\alpha(m)$  divides  $\alpha(\text{rad}(m))$ .*

*Proof.* By induction:

- If  $m = \text{rad}(m)$ , it is trivial.
- Otherwise, there is at least a prime factor  $p$  such that  $p^2|m$ . We will show that  $\alpha(m)$  divides  $\alpha\left(\frac{m}{p}\right)$  and the result will follow by induction.

By definition of  $\alpha(m)$ , there exists a positive integer  $\ell$  such that

$$2^{\alpha\left(\frac{m}{p}\right)\frac{m}{p}} = 1 + \ell\frac{m}{p}.$$

Then,

$$\begin{aligned} 2^{\alpha\left(\frac{m}{p}\right)m} &= \left(2^{\alpha\left(\frac{m}{p}\right)\frac{m}{p}}\right)^p = \left(1 + \ell\frac{m}{p}\right)^p && \text{by the binomial theorem,} \\ &= 1 + \sum_{k=1}^{p-1} \binom{p}{k} \ell^k \left(\frac{m}{p}\right)^k + \ell^p \left(\frac{m}{p}\right)^p \\ &\equiv 1 \pmod{m}. \end{aligned}$$

Therefore,  $\alpha(m)$  divides  $\alpha\left(\frac{m}{p}\right)$  and, by induction,  $\alpha(m)$  divides  $\alpha(\text{rad}(m))$ .

▽

**Proposition 5.3.9.** *Let  $p$  be an odd number and  $k$  a positive integer. Then,  $\alpha(p^k) = \alpha(p)$  and  $\beta(p^k) = \beta(p)$ .*

*Proof.* We start with proving that  $\alpha(p^k) = \alpha(p)$  by showing that  $\alpha(p^k) | \alpha(p)$  and  $\alpha(p) | \alpha(p^k)$ .

The fact that  $\alpha(p^k)$  divides  $\alpha(p)$  is clear by proposition 5.3.8.

Now, we have that  $2^{\alpha(p^k)p^k} \equiv 1 \pmod{p^k}$ , so  $2^{\alpha(p^k)p^k} \equiv 1 \pmod{p}$ . If we recall Fermat's little theorem, for any  $p$  prime and  $\ell$  an integer, we have  $\ell^p \equiv \ell \pmod{p}$ . Then,

$$2^{\alpha(p^k)p} \equiv \left(2^{\alpha(p^k)p}\right)^p \equiv 2^{\alpha(p^k)p^3} \equiv \dots \equiv 2^{\alpha(p^k)p^k} \equiv 1 \pmod{p}.$$

Therefore,  $\alpha(p)$  divides  $\alpha(p^k)$  and thus,  $\alpha(p^k) = \alpha(p)$ .

Now, let us show that  $\beta(p^k) = \beta(p)$ . We will see that

$$\alpha(p^k) = 2\beta(p^k) \Leftrightarrow \alpha(p) = 2\beta(p).$$

$\Rightarrow$  If  $\alpha(p^k) = 2\beta(p^k)$ , then  $2^{\beta(p^k)p^k} \equiv -1 \pmod{p^k}$ , so  $2^{\beta(p^k)p^k} \equiv -1 \pmod{p}$ . By Fermat's little theorem, we have  $2^{\beta(p^k)p} \equiv -1 \pmod{p}$ . Then,  $\alpha(p) = 2\beta(p)$  and  $\beta(p) | \beta(p^k)$ .

$\Leftarrow$  If  $\alpha(p) = 2\beta(p)$ , then  $2^{\beta(p)p} \equiv -1 \pmod{p}$ . Then, there exists a positive integer  $\ell$  such that

$$2^{\beta(p)p} = \ell p - 1.$$

Then,

$$\begin{aligned} 2^{\beta(p)p^2} &= (\ell p - 1)^p && \text{and by the binomial theorem,} \\ &= (\ell p)^p + \sum_{i=1}^{p-1} \binom{p}{i} (-1)^{p-i} (\ell p)^i + (-1)^p \\ &\equiv (-1)^p = (-1) \pmod{p^2} && \text{since } p \text{ is odd.} \end{aligned}$$

Then, we can iterate and obtain

$$2^{\beta(p)p^k} \equiv -1 \pmod{p^k}.$$

Therefore, we have  $\alpha(p^k) = 2\beta(p^k)$  and  $\beta(p^k) | \beta(p)$ .

Thus, we have seen that  $\alpha(p^k) = 2\beta(p^k) \Leftrightarrow \alpha(p) = 2\beta(p)$ . Now,  $\beta(p^k) = \beta(p)$  follows directly from the result  $\alpha(p^k) = \alpha(p)$ . \(\nabla\)

**Proposition 5.3.10.** *Let  $m_1$  and  $m_2$  be two coprime odd numbers. Then,  $\alpha(m_1 m_2)$  divides  $\text{lcm}(\alpha(m_1), \alpha(m_2))$ .*

*Proof.* We have

$$\begin{cases} 2^{\alpha(m_1)m_1} \equiv 1 \pmod{m_1}, \\ 2^{\alpha(m_2)m_2} \equiv 1 \pmod{m_2}. \end{cases}$$

Then,

$$\begin{cases} 2^{\text{lcm}(\alpha(m_1), \alpha(m_2))m_1 m_2} \equiv 1 \pmod{m_1}, \\ 2^{\text{lcm}(\alpha(m_1), \alpha(m_2))m_1 m_2} \equiv 1 \pmod{m_2}. \end{cases}$$

The result follows by the Chinese remainder theorem. \(\nabla\)

Now, we get to the following theorem, that guarantees the existence of infinite balanced arithmetic progressions in  $\mathbb{Z}/m\mathbb{Z}$ , with  $m$  odd, of any length  $n$  that fulfills  $n \equiv 0$  or  $-1 \pmod{\alpha(m)}$ .

**Lemma 5.3.11.** *Let  $n, m$  be positive integers, with  $m$  odd and  $n \geq m$ , and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ , with  $d$  invertible. Let  $X$  be the sequence  $AP(a, d, n) = (x_1, \dots, x_n)$ . Then, every  $m$  consecutive terms of  $AP(a, d, n)$  are distinct. This is, for all  $i \in [n - m + 1]$ ,*

$$\{x_i, x_{i+1}, \dots, x_{i+m-1}\} = \mathbb{Z}/m\mathbb{Z}.$$

*Proof.* Let  $i_1, i_2$  be positive integers. Then, if  $x_{i_1} = x_{i_2}$ , we have

$$a + (i_1 - 1)d = a + (i_2 - 1)d.$$

Then,  $(i_1 - 1)d = (i_2 - 1)d$  and, since  $d$  is invertible in  $\mathbb{Z}/m\mathbb{Z}$ ,

$$i_1 \equiv i_2 \pmod{m}.$$

▽

**Theorem 5.3.12.** *Let  $n, m$  be positive integers, with  $m$  odd, and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ , with  $d$  invertible. Then, the arithmetic progression  $AP(a, d, n)$  is balanced if  $n \equiv 0$  or  $-1 \pmod{\alpha(m)m}$ .*

*Sketch of proof:* We do not show the complete proof of this theorem, that can be found in [8], but we explain its outline.

First of all, the case  $n \equiv -1 \pmod{\alpha(m)m}$  is derived from the case  $n \equiv 0 \pmod{\alpha(m)m}$ , using lemma 5.3.11, the multiplicity function of a sequence with length  $n \equiv -1 \pmod{\alpha(m)m}$  is constant if, and only if, the multiplicity of a sequence with length  $n \equiv 0 \pmod{\alpha(m)m}$  is.

Then, the case  $n \equiv 0 \pmod{\alpha(m)m}$  is shown by induction on  $m$ . If  $m = 1$ , all the sequences in  $\mathbb{Z}/m\mathbb{Z} = \{0\}$  are balanced. If  $m > 1$ , let  $p$  the greatest prime factor of  $m$ . Suppose that for  $q = \frac{m}{p}$ , the statement its true.

At this point, one can show that, with  $d$  invertible,  $m$  odd and  $k$  a positive integer, the arithmetic progression  $AP(a, d, k\alpha(m)m)$  is balanced if, and only if,  $AP(a, d, \alpha(m)m)$  is balanced. Therefore, it will be sufficient to see the statement for one length  $n$  multiple of  $\alpha(m)m$ . The chosen length is

$$\lambda\alpha(p) = \varphi\left(\frac{\text{rad}(m)}{p}\right)\alpha(p),$$

that is multiple of  $\alpha(m)m$  by proposition 5.3.10.

The proof that  $X = AP(a, d, \lambda\alpha(p)m)$  is balanced uses theorem 5.2.8. First, it is shown that  $\mathbf{m}_{\Delta X}$  is constant on each coset of the subgroup  $q\mathbb{Z}/m\mathbb{Z}$ , by considering the structure of  $\Delta X$  depicted in figure 5.6, constituted by the blocks  $A_r, B_{(s,t)}$  and  $C_u$ . These blocks are defined as

$$A_r = \left\{ \Delta X(i, j) \mid (r-1)\lambda\frac{m}{p} + 1 \leq i \leq r\lambda\frac{m}{p}, j \in \left[ r\lambda\frac{m}{p} - i + 1 \right] \right\},$$

$$B_{(s,t)} = \left\{ \Delta X(i, j) \mid \begin{array}{l} ((s-1)p+t-1)\lambda m/p + 1 \leq i \leq ((s-1)p+t)\lambda m/p \\ ((s-1)p+t)\lambda m/p - i + 2 \leq j \leq s\lambda m - i + 1 \end{array} \right\},$$

$$C_u = \{ \Delta X(i, j) \mid (u-1)\lambda m + 1 \leq i \leq u\lambda m, u\lambda n - i + 2 \leq j \leq \lambda\alpha(p)m - i + 1 \}.$$

The blocks are depicted in figure 5.7.

With this, it is shown that

- (1) The multiplicity function  $\mathbf{m}_{C_u}$  is constant for each  $C_u$ ,
- (2) the multiplicity function of the union of the  $B_{(s,t)}$  is constant,



(3) and the multiplicity function of the union of the  $A_r$  is constant on each coset of the subgroup  $q\mathbb{Z}/m\mathbb{Z}$ .

Therefore, the multiplicity function  $\mathbf{m}_{\Delta X}$  is constant on each coset of the subgroup  $\frac{m}{p}\mathbb{Z}/m\mathbb{Z} = q\mathbb{Z}/m\mathbb{Z}$ .

Now, it is shown that the arithmetic progression with invertible common difference  $\pi_q(X)$  is balanced.

Then, by theorem 5.2.8, the sequence  $X$  is balanced.

**Corollary 5.3.13.** *Let  $m$  be an odd number. Then, there exist at least  $\varphi(m)m$  balanced sequences of every length  $n \equiv 0$  or  $-1 \pmod{\varphi(\text{rad}(m))m}$ .*

*Proof.* Every arithmetic progression  $AP(a, d, n)$  is determined by its initial element  $a$  and its common difference  $d$ . There are  $m$  possible initial elements in  $\mathbb{Z}/m\mathbb{Z}$ , while the number of possible invertible common differences is  $\varphi(m)$ . Then, there exist exactly  $\varphi(m)m$  distinct arithmetic progressions  $AP(a, d, n)$  in  $\mathbb{Z}/m\mathbb{Z}$ .

Since  $m$  is odd, by proposition 5.3.8 we know that  $\alpha(m)$  divides  $\alpha(\text{rad}(m))$ , who itself divides  $\varphi(\text{rad}(n))$ . Therefore, by theorem 5.3.12, there exist at least  $\varphi(m)m$  balanced sequences of every length  $n \equiv 0$  or  $-1 \pmod{\alpha(m)m}$ .  $\nabla$

However, we will see that Molluzzo's problem is not solved by these results, as can be seen in the following proposition.

**Proposition 5.3.14.** *Let  $m > 1$  be an odd number. Then  $\alpha(n) \geq 2$ .*

*Proof.* Let  $m = p_1^{r_1} \cdots p_k^{r_k}$  be the prime factorization of  $m$ , with  $p_1 < p_2 < \cdots < p_k$ .

Suppose  $\alpha(m) = 1$ . Then, by definition,

$$2^{\alpha(m)m} = 2^m \equiv 1 \pmod{m},$$

and

$$2^m \equiv 1 \pmod{p_1}.$$

Then, the order of 2 in  $\mathbb{Z}/p_1\mathbb{Z}$  divides  $m$ . But, since  $m$  and  $(p_1 - 1)$  are coprimes, this is in contradiction with the fact that, by Euler's theorem, it divides  $(p_1 - 1)$ .

Therefore,  $\alpha(m) \geq 2$  for all  $m > 1$  odd.  $\nabla$

### 5.3.2 The antisymmetric case

In this section, we use the results of the previous section in order to solve completely Molluzzo's problem for any  $m = 3^k$ .

First of all, let us define the concept of antisymmetry in generalized sequences.

**Definition 5.3.15.** Let  $X = (x_1, \dots, x_n)$  be a sequence of length  $n$  in  $\mathbb{Z}/m\mathbb{Z}$ . The sequence  $X$  is *antisymmetric* if  $x_{n-i+1} = -x_i$  for all  $i \in [n]$ .

Now, we will see the relation between the antisymmetry and the derivation process.

**Proposition 5.3.16.** *Let  $m, n$  be positive integers. Let  $X = (x_1, \dots, x_n)$  a finite sequence in  $\mathbb{Z}/m\mathbb{Z}$ . Let  $k = \lceil \frac{n}{2} \rceil$ . Then, the sequence  $X$  is antisymmetric if, and only if, its derived sequence  $\partial X = (y_1, \dots, y_{n-1})$  is antisymmetric and  $x_k + x_{n-k+1} = 0$ .*

*Proof.*

- **$X$  antisymmetric  $\implies \partial X$  antisymmetric and  $x_k + x_{n-k+1} = 0$ .**

Clearly, since  $X$  is antisymmetric,  $x_k + x_{n-k+1} = 0$ .

For all  $i \in [n-1]$ , we have

$$y_{n-i} + y_i = (x_{n-i} + x_{n-i+1}) + (x_i + x_{i+1}) = (x_i + x_{n-i+1}) + (x_{i+1} + x_{n-i}) = 0.$$

Therefore,  $y_{n-i} = -y_i$  for all  $i \in [n-1]$ , and  $\partial X$  is antisymmetric.

- **$\partial X$  antisymmetric and  $x_k + x_{n-k+1} = 0 \implies X$  antisymmetric.**

If  $n = 2$ , the fact that  $\partial X$  is antisymmetric implies  $\partial X = (0)$ , so clearly  $X$  is antisymmetric.

Now, assume  $n \geq 3$ . Since  $\partial X$  is antisymmetric, we know that  $y_{n-i} + y_i = 0$  for all  $i \in [n-1]$ . We want to see that  $x_{n-i+1} + x_i = 0$ .

Now, by construction of the derivative, we have  $y_i = x_i + x_{i+1}$  for all  $i \in [n-1]$ . Thus,  $x_i = y_i - x_{i+1}$  for all  $i \in [n-1]$ . Moreover, for all  $2 \leq i \leq n$ , we have  $x_i = y_{i-1} - x_{i-1}$ . Then, for all  $i \in [n-1]$ ,

$$x_{n-i+1} + x_i = (y_{n-i} - x_{n-i}) + (y_i - x_{i+1}) = (y_{n-i} + y_i) - (x_{n-i} + x_{i+1}) = -(x_{n-i} + x_{i+1}).$$

If we chose  $i = (k-1)$ , we have the equality

$$x_{n-k} + x_{k-1} = -(x_{n-k+1} + x_k) = 0.$$

Recursively,  $x_{n-i+1} + x_i = 0$  for all  $i \in [n-1]$ , and  $X$  is antisymmetric.

▽

**Proposition 5.3.17.** *Let  $m, n$  be positive integers, with  $m$  odd and  $n \geq 2$ . Let  $a, d$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then, the arithmetic progression  $X = AP(a, d, n)$  is antisymmetric if, and only if, its derived sequence  $\partial X = AP(2a + d, 2d, n-1)$  is antisymmetric.*

*Proof.* Let  $X = AP(a, d, n) = (x_1, \dots, x_n)$  and  $\partial X = AP(2a + d, 2d, n-1) = (y_1, \dots, y_{n-1})$ . Then, for all  $i \in [n-1]$ ,  $j \in [n]$ ,

$$\begin{aligned} y_{n-i} + y_i &= (2a + d) + (n-i-1)2d + (2a + d) + (i-1)2d \\ &= 2(2a + (n-1)d) \\ &= 2(a + (n-j)d + a + (j-1)d) \\ &= 2(x_{n-j+1} + x_j). \end{aligned}$$

Then, since  $m$  is odd, 2 is invertible and  $x_{n-j+1} + x_j = 0 \Leftrightarrow y_{n-i} + y_i = 0$  for all  $i \in [n-1]$ ,  $j \in [n]$ . Thus,  $X$  is antisymmetric if, and only if,  $\partial X$  is antisymmetric. ▽

**Proposition 5.3.18.** *Let  $m, n$  be positive integers, with  $m$  odd. Let  $d$  be in  $\mathbb{Z}/m\mathbb{Z}$ . Then, there exists a unique antisymmetric arithmetic progression  $X$  of length  $n$  with common difference  $d$ . Moreover,*

- if  $n \equiv 0 \pmod{m}$ , then  $X = AP(2^{-1}d, d, n)$ ;*
- if  $n \equiv -1 \pmod{m}$ , then  $X = AP(d, d, n)$ .*

*Proof.* Let  $X = AP(a, d, n) = (x_1, \dots, x_n)$ . On one side, if  $X$  is antisymmetric,  $x_{n-i+1} + x_i = 0$  for all  $i \in [n]$ . On the other side, for all  $i \in [n]$ ,

$$x_{n-i+1} + x_i = (a + (n-i)d) + (a + (i-1)d) = 2a + (n-1)d.$$

Then, the arithmetic progression  $X$  is antisymmetric if, and only if,

$$2a + (n-1)d = 0.$$

Therefore, the sequence  $AP(2^{-1}(1-n)d, d, n)$  is the only antisymmetric arithmetic progression of length  $n$  and common difference  $d$ . ▽

**Lemma 5.3.19.** *Let  $n, m$  be positive integers. Let  $X$  be an antisymmetric sequence of length  $n$  in  $\mathbb{Z}/m\mathbb{Z}$ . Then, for all  $x \in \mathbb{Z}/m\mathbb{Z}$ ,*

$$\mathbf{m}_{\Delta X}(x) = \mathbf{m}_{\Delta X}(-x).$$

*Proof.* By proposition 5.3.17, since  $X$  is antisymmetric, the sequence  $\partial^i X$  is antisymmetric for all  $i \in [n-1]$ . Then,

$$\mathbf{m}_{\Delta X}(x) = \sum_{i=0}^{n-1} \mathbf{m}_{\partial^i X}(x) = \sum_{i=0}^{n-1} \mathbf{m}_{\partial^i X}(-x) = \mathbf{m}_{\Delta X}(-x),$$

for all  $x \in \mathbb{Z}/m\mathbb{Z}$ . ▽

**Theorem 5.3.20.** *Let  $m$  be an odd number, and let  $d \in \mathbb{Z}/m\mathbb{Z}$  invertible. Then,*

- (i) *for every  $n \equiv 0 \pmod{\beta(m)m}$ , the arithmetic progression  $AP(2^{-1}d, d, n)$  is balanced;*
- (ii) *for every  $n \equiv -1 \pmod{\beta(m)m}$ , the arithmetic progression  $AP(d, d, n)$  is balanced.*

*Proof.*

**Case  $n \equiv -1 \pmod{\beta(m)m}$**

We begin deriving the case  $n \equiv -1 \pmod{\beta(m)m}$  from the case  $n \equiv 0 \pmod{\beta(m)m}$ .

First of all, we write  $n = k\beta(m)m - 1$  for some positive integer  $k$ . Let  $X = AP(d, d, n)$ . By proposition 5.3.4,

$$Y = AP(2^{-2}d, 2^{-1}d, n+1) = AP(2^{-2}d, 2^{-1}d, k\beta(m)m)$$

is a primitive sequence of  $X$ .

The sequence  $Y$  is an arithmetic progression of length  $k\beta(m)m$  with invertible common difference. Then, by lemma 5.3.11 each element of  $\mathbb{Z}/m\mathbb{Z}$  occurs  $k\beta(m)$  times in  $Y$ . Then, for all  $x \in \mathbb{Z}/m\mathbb{Z}$ ,

$$\begin{aligned} \mathbf{m}_{\Delta X}(x) &= \mathbf{m}_{\Delta \partial Y}(x) && \text{since } X \text{ is the derived sequence of } Y, \\ &= \mathbf{m}_{\Delta Y}(x) - \mathbf{m}_Y(x) && \text{since } \Delta X \text{ is } \Delta Y \text{ without its first row,} \\ &= \mathbf{m}_{\Delta Y}(x) - k\beta(m) && \text{as we have seen, each } x \text{ occurs } k\beta(m) \text{ times in } Y. \end{aligned}$$

Then, the sequence  $X = AP(d, d, n)$  is balanced if, and only if, its primitive arithmetic progression sequence  $Y = AP(2^{-2}d, 2^{-1}d, n+1)$  is balanced. Therefore, if we can prove the case  $n \equiv 0 \pmod{\beta(m)m}$ , we will have proved the case  $n \equiv -1 \pmod{\beta(m)m}$ .

**Case  $n \equiv 0 \pmod{\beta(m)m}$** 

Case  $\alpha(m) = \beta(m)$

By theorem 5.3.12, we are done.

Case  $\alpha(m) = 2\beta(m)$

Then, we know that  $2^{\beta(m)m} \equiv -1 \pmod{m}$ .

First, we write  $n = k\beta(m)m$  for some positive integer  $k$ . We want to see that the sequence  $Y = AP(2^{-1}d, d, k\beta(m)m)$  is balanced.

Now, let  $X = AP(2^{-1}d, d, 2k\beta(m)m)$ . The structure of the generalized triangle  $\Delta X$  corresponds to the one shown in figure 5.2.

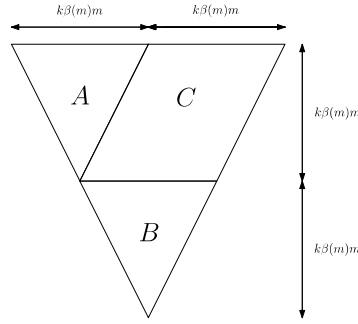


Figure 5.2: Structure of  $\Delta X$

In this structure, we can see two triangles, named  $A$  and  $B$ , and a rhomboid  $C$ . Let us define each block.

The triangle  $A$  is the generalized triangle generated by the first half of  $X$ , i.e.

$$\begin{aligned} A &= \{\Delta X(i, j) \mid i \in [k\beta(m)m], j \in [k\beta(m)m - i + 1]\} \\ &= \Delta AP(2^{-1}d, d, k\beta(m)m). \end{aligned}$$

On the other hand, the triangle  $B$  is the generalized triangle generated by the  $(k\beta(m)m + 1)$ th row of  $\Delta X$ , i.e.

$$\begin{aligned} B &= \{\Delta X(i, j) \mid k\beta(m)m + 1 \leq i \leq 2k\beta(m)m, j \in [2k\beta(m)m - i + 1]\} \\ &= \Delta \partial^{k\beta(m)m} X \\ &= \Delta \partial^{k\beta(m)m} AP(2^{-1}d, d, 2k\beta(m)m) \\ &= \Delta AP(2^{k\beta(m)m-1}(1 + k\beta(m)m)d, 2^{k\beta(m)m}d, k\beta(m)m) \end{aligned} \quad \text{by proposition 5.3.1.}$$

Then, since  $2^{\beta(m)m} \equiv -1 \pmod{m}$ , we have that

$$B = \Delta AP((-1)^k 2^{-1}d, (-1)^k d, k\beta(m)m).$$

If  $k$  is even,  $B = \Delta AP(2^{-1}d, d, k\beta(m)m) = A$ .

If  $k$  is odd,  $B = \Delta AP(-2^{-1}d, -d, k\beta(m)m)$ . By proposition 5.3.18, this arithmetic progression is antisymmetric. Then, for all  $x \in \mathbb{Z}/m\mathbb{Z}$ ,

$$\begin{aligned} \mathbf{m}_B(x) &= \mathbf{m}_B(-x) && \text{by lemma 5.3.19,} \\ &= \mathbf{m}_{\Delta AP(-2^{-1}d, -d, k\beta(m)m)}(-x) \\ &= \mathbf{m}_{\Delta AP(2^{-1}d, d, k\beta(m)m)}(x) \\ &= \mathbf{m}_A(x). \end{aligned}$$

Then, for any  $k$  we obtain  $\mathbf{m}_B(x) = \mathbf{m}_A(x)$  for all  $x \in \mathbb{Z}/m\mathbb{Z}$ .

Finally, the rhomboid  $C$  is defined by

$$C = \{\Delta X(i, j) \mid i \in [k\beta(m)m], k\beta(m)m - i + 2 \leq j \leq 2k\beta(m)m - i + 1\}.$$

Each row of  $C$  is, by proposition 5.3.1, an arithmetic progression of length  $k\beta(m)m$  and invertible common difference in  $\mathbb{Z}/m\mathbb{Z}$ . Then, by lemma 5.3.11, we know that each element of  $\mathbb{Z}/m\mathbb{Z}$  appears exactly  $k\beta(m)$  times in each row. Thus, we know that

$$\mathbf{m}_C(x) = k^2\beta(m)^2m,$$

for all  $x \in \mathbb{Z}/m\mathbb{Z}$ .

Therefore, we have that, for all  $x \in \mathbb{Z}/m\mathbb{Z}$ ,

$$\mathbf{m}_{\Delta X}(x) = \mathbf{m}_A(x) + \mathbf{m}_B(x) + \mathbf{m}_C(x) = 2\mathbf{m}_A(x) + k^2\beta(m)^2m.$$

Thus, the sequence  $Y = AP(2^{-1}d, d, k\beta(m)m)$  is balanced if, and only if, the sequence  $X = AP(2^{-1}d, d, 2k\beta(m)m) = AP(2^{-1}d, d, k\alpha(m)m)$  is balanced. Then, by theorem 5.3.12, we are done.  $\nabla$

**Corollary 5.3.21.** *There exists a balanced sequence of length  $m$  in  $\mathbb{Z}/3^k\mathbb{Z}$  if, and only if, the number of elements  $N = \frac{n(n+1)}{2}$  is divisible by  $3^k$ , with  $k$  a positive integer.*

*Proof.* We know that  $3^k$  divides  $N$  if, and only if,  $n \equiv 0$  or  $-1 \pmod{3^k}$ .

By proposition 5.3.9, we have

$$\beta(3^k) = \beta(3) = 1.$$

Let  $d \in \mathbb{Z}/3^k\mathbb{Z}$  invertible. Then, by theorem 5.3.20,  $AP(2^{-1}d, d, n)$  is balanced for every  $n \equiv 0 \pmod{3^k}$  and  $AP(d, d, n)$  is balanced for every  $n \equiv -1 \pmod{3^k}$ . Thus, there are balanced sequences for all admissible lengths in  $\mathbb{Z}/3^k\mathbb{Z}$ .  $\nabla$

### 5.3.3 Balanced arithmetic progressions in $\mathbb{Z}/m\mathbb{Z}$ for $m$ even

In this section, we will assume that the positive integer  $m$  is even and we will see that, in general, there is no balanced arithmetic progression in  $\mathbb{Z}/m\mathbb{Z}$ .

**Theorem 5.3.22.** *Let  $m$  be a positive even integer, and  $a, d \in \mathbb{Z}/m\mathbb{Z}$ . Then, the arithmetic progression  $X = AP(a, d, n)$  is balanced if, and only if,*

- $m = 2$  and  $X \in \{(0, 1, 0), (1, 1, 1), (0, 1, 0, 1), (1, 0, 1, 0)\}$ ,
- or  $m = 6$  and  $X \in \{(1, 3, 5), (2, 3, 4), (4, 3, 2), (5, 3, 1)\}$ .

*Proof.* It is easy to check that all the sequences described are balanced arithmetic progressions.

Now, let us suppose that  $X = AP(a, d, n)$  is balanced, and we will see that  $X$  is one of the described sequences.

First of all, consider the canonical surjective morphism

$$\pi_2 : \mathbb{Z}/m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z},$$

and the projected sequence

$$\pi_2(X) = AP(\pi_2(a), \pi_2(d), n).$$

By theorem 5.2.8,  $\pi_2(X)$  is balanced. Then, by proposition 5.3.2, the elements of the generalized triangle are

$$\Delta\pi_2(X)(i, j) = 2^{i-2}(2\pi_2(a) + (2j + i - 3)\pi_2(d)).$$

Clearly, for  $i \geq 3$ , we have  $\Delta\pi_2(X)(i, j) = 0 \in \mathbb{Z}/2\mathbb{Z}$ .

Therefore, from the third row of  $\Delta\pi_2(X)$ , every element will be 0. Since  $\pi_2(X)$  is balanced,  $\Delta\pi_2(X)$  has to contain at least twice as many elements as  $\Delta\partial^2\pi_2(X)$ , which is the triangle without its first two rows. Then, if

$$N = \frac{n(n+1)}{2}$$

is the number of elements of  $\Delta\pi_2(X)$  and

$$N_3 = \frac{(n-2)(n-1)}{2}$$

is the number of elements of  $\Delta\pi_2(X)$  without the two first rows, we know that

$$N \geq 2N_3.$$

In terms of  $n$ ,

$$\frac{n(n+1)}{2} \geq 2 \cdot \frac{(n-2)(n-1)}{2}.$$

Thus,  $n$  has to fulfill

$$n^2 - 7n + 4 \leq 0.$$

Then, we have that  $n \in [6]$ .

Furthermore, as we have seen, we need that  $2|N$  in order to get  $\pi_2(X)$  to be balanced, and therefore we have that  $n = 3$  or  $n = 4$ .

Now, let us solve the different cases.

- If the length of  $X$  is  $n = 3$ , the total number of elements in the triangle is  $N = 6$ . The only even positive integers that divide  $N$  are  $m = 2$  and  $m = 6$ .
  - Let  $m = 2$ . There are exactly four arithmetic progressions of length 3 in  $\mathbb{Z}/2\mathbb{Z}$ , the sequences  $(0, 0, 0)$ ,  $(0, 1, 0)$ ,  $(1, 0, 1)$ ,  $(1, 1, 1)$ , but only two of them,  $X_1 = (0, 1, 0)$  and  $X_2 = (1, 1, 1)$  are balanced.

$$\begin{array}{ccc} 0 & 1 & 0 \\ 1 & 1 & \\ 0 & & \end{array} \qquad \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 0 & \\ 0 & & \end{array}$$

Figure 5.3:  $\Delta X_1$  and  $\Delta X_2$ , respectively

- Let  $m = 6$ . Since  $N = 6$  and  $m = 6$ , each element of  $\mathbb{Z}/6\mathbb{Z}$  has to occur exactly once. Since each element is the sum of the two elements above it, the only element that can be equal to zero is  $\Delta X(3, 1)$ , or we would have two elements with the same value. Since  $\Delta X(3, 1) = 0$ , by proposition 5.3.2 we have

$$0 = \Delta X(3, 1) = 4(a + d) = 4 \cdot \Delta X(1, 2).$$

Then, we know that  $\Delta X(1, 2) = 3$ , and the sequence  $X$  is  $AP(a, 3 - a, n)$ , with  $a \in \{1, 2, 4, 5\}$ . This provides us four arithmetic progressions of length 3 in  $\mathbb{Z}/6\mathbb{Z}$ , the sequences  $X_3 = (1, 3, 5)$ ,  $X_4 = (2, 3, 4)$ ,  $X_5 = (4, 3, 2)$  and  $X_6 = (5, 3, 1)$ , and all of them are balanced.

$$\begin{array}{ccc} 1 & 3 & 5 \\ 4 & 2 & \\ 0 & & \end{array} \quad \begin{array}{ccc} 2 & 3 & 4 \\ 5 & 1 & \\ 0 & & \end{array} \quad \begin{array}{ccc} 4 & 3 & 2 \\ 1 & 5 & \\ 0 & & \end{array} \quad \begin{array}{ccc} 5 & 3 & 1 \\ 2 & 4 & \\ 0 & & \end{array}$$

Figure 5.4:  $\Delta X_3$ ,  $\Delta X_4$ ,  $\Delta X_5$  and  $\Delta X_6$ , respectively

- If the length of  $X$  is  $n = 4$ , the total number of elements in the triangle is  $N = 10$ . The only even positive integers that divide  $N$  are  $m = 2$  and  $m = 10$ .
  - Let  $m = 2$ . There are four arithmetic progressions of length 4 in  $\mathbb{Z}/2\mathbb{Z}$  and two of them are balanced, the sequences  $X_7 = (0, 1, 0, 1)$  and  $X_8 = (1, 0, 1, 0)$ .

$$\begin{array}{cccc} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & \\ 0 & 0 & & \\ 0 & & & \end{array} \quad \begin{array}{cccc} 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & \\ 0 & 0 & & \\ 0 & & & \end{array}$$

Figure 5.5:  $\Delta X_7$  and  $\Delta X_8$ , respectively

- Let  $m = 10$ . Since  $N = 10$  and  $m = 10$ , each element of  $\mathbb{Z}/10\mathbb{Z}$  has to occur exactly once. Since each element is the sum of the two elements above it, the only element that can be equal to zero is  $\Delta X(4, 1)$ , or we would have two elements with the same value.

Since  $\Delta X(4, 1) = 0$ , by proposition 5.3.2 we have

$$0 = \Delta X(3, 1) = 4(2a + 3d) = 4 \cdot \Delta X(2, 2).$$

Then, we know that  $\Delta X(2, 2) = 5$ . Then, given that  $2a + 3d = 5$ , we have that

$$d = 3^{-1}(5 - 2a) = 7(5 - 2a) = 5 - 4a.$$

Then, the sequence  $X$  is  $AP(a, 5 - 4a, n)$ , with  $a \in \{1, 2, 3, 4, 6, 7, 8, 9\}$ . But, none of these sequences is balanced, so there is no balanced arithmetic progression in  $\mathbb{Z}/10\mathbb{Z}$ .

▽

## 5.4 Computer simulation

Through brute force computer simulations, we have found that the answer to Molluzzo's question is no. No, it is not always possible to find a balanced sequence modulo  $m$  of a given length  $n$ , even if  $m|N$ .

Our simulation tests every possible sequence  $X = (x_1, \dots, x_n)$  with  $x_i \in \mathbb{Z}/m\mathbb{Z}$  and has found that there is no possible balanced sequences in some cases, even when the necessary condition  $m|N$  is fulfilled. We have found some examples. There is no balanced sequence for:

length $n$	$\mathbb{Z}/m\mathbb{Z}$ , with $m$
5	15
6	21
7	14
7	28
8	18

Therefore, we can affirm that the necessary condition of Molluzzo's problem is not a sufficient condition, contrary to its particularization in Steinhaus's problem ( $m = 2$ ).



### 5.5 Appendix: Figures

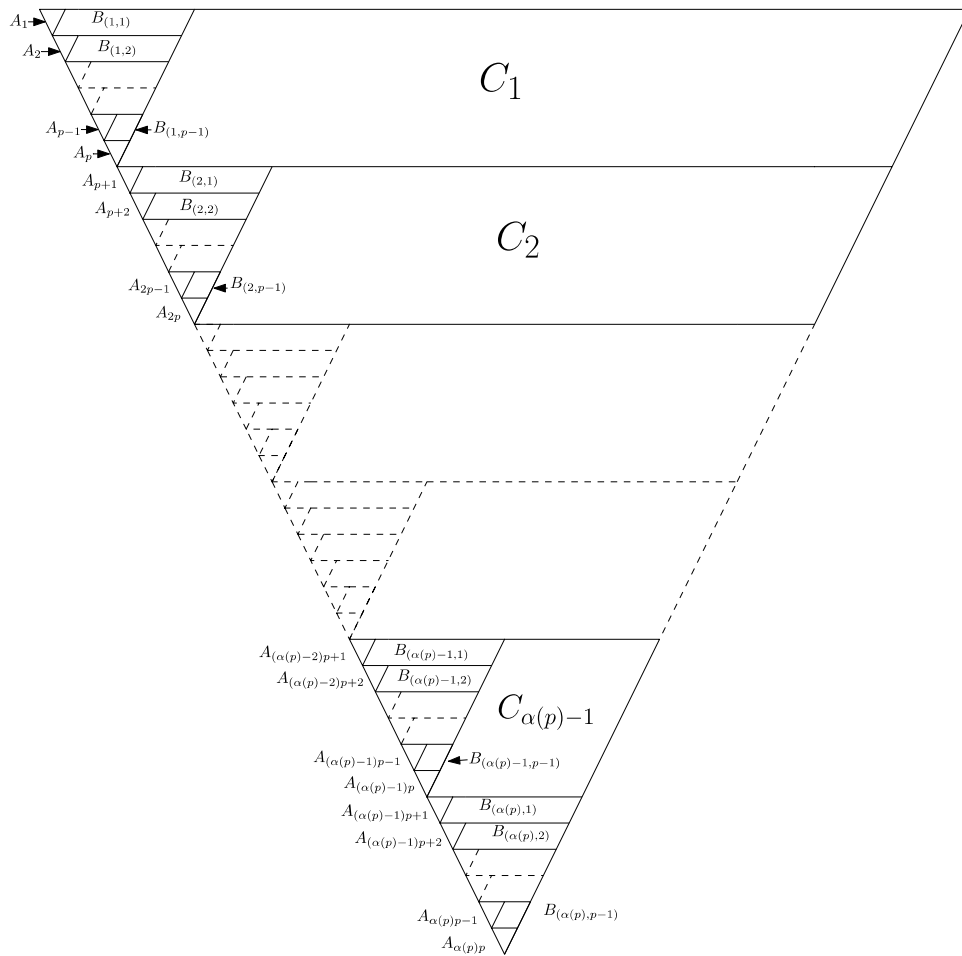
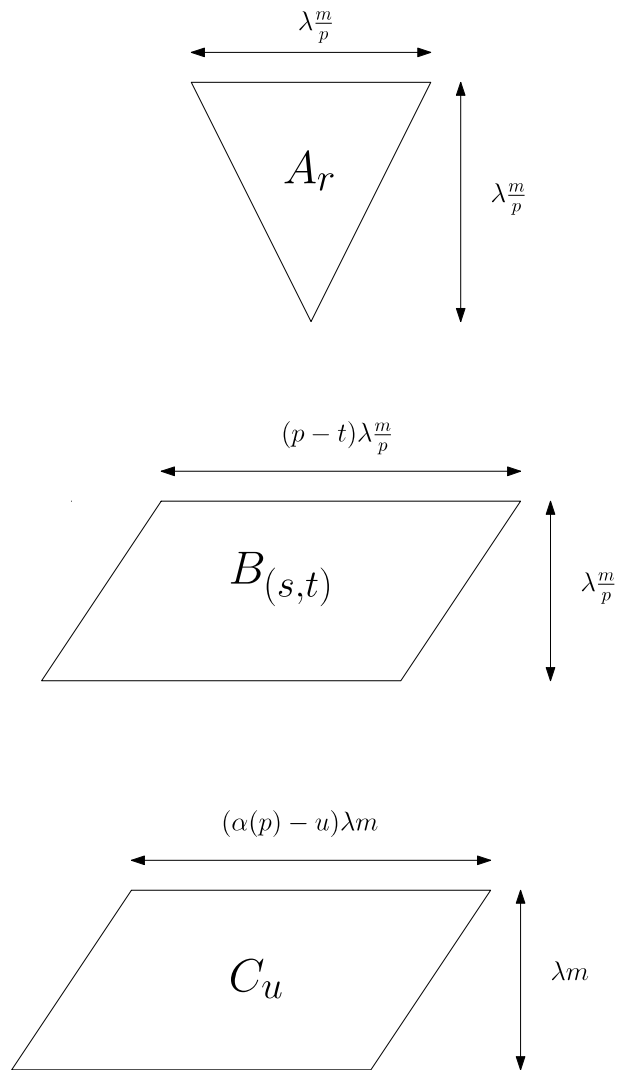


Figure 5.6: Structure of  $\Delta X$

Figure 5.7: The blocks  $A_r$ ,  $B_{(s,t)}$  and  $C_u$

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