On polytopality of Cartesian products of graphs

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Abstract. We study the polytopality of Cartesian products of non-polytopal graphs. On the one hand, we prove that a product of graphs is the graph of a simple polytope if and only if its factors are. On the other hand, we provide a general construction of polytopal products of a polytopal graph by a non-polytopal graph.

Key words: Graphs of polytopes, Cartesian product

1 Introduction

When we consider a polytope (the convex hull of a finite point set in an Euclidean space), we are interested in its faces (its intersections with its supporting hyperplanes), in the inclusion relations between its faces, and in its graph in particular. Polytopality problems form in a sense the reciprocal question: we want to determine whether a given graph can be realized by some polytope, and in what dimension. This question is totally understood until dimension 3 but is difficult in higher dimension.

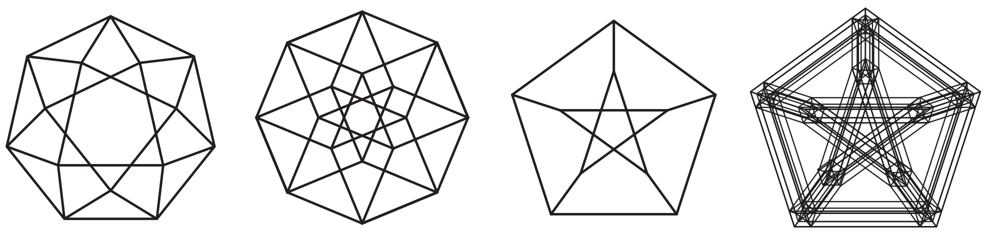

Fig. 1. One of these graphs is the graph of a simple polytope. Any guess?

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In this paper, we study this general question for a special class of graphs, namely those obtained as Cartesian products of other graphs. The Cartesian product of graphs is defined to be coherent with the Cartesian product of polytopes: the graph of a product of polytopes is the product of their graphs. In particular, the product of two polytopal graphs is automatically polytopal. Two questions then naturally arise:

1. *Dimensional ambiguity of products*: What is the minimal dimension of a realizing polytope of a product of graphs?
2. *Polytopality of non-polytopal graphs*: Are the two factors of a polytopal product of graphs necessarily polytopal?

The first question received much attention in recent literature with the construction of cubical neighborly polytopes [8], prodsimplicial neighborly polytopes [12], the techniques of deformed products of polytopes [14], and the topological obstruction method of [13]. In this paper, we provide partial answers to the second question. On the one hand, we prove that a product of graphs is the graph of a simple polytope if and only if its factors are. On the other hand, we provide a general construction of polytopal products of a polytopal graph by a non-polytopal graph.

2 Polytopality of graphs

In this section, we recall the classical knowledge concerning polytopality of general graphs. We refer to the excellent treatments in [7] and [18] as well as the survey paper [10] for more details.

Definition 1. *A graph G is* polytopal *if it is the* 1*-skeleton of some polytope P. If P has dimension d, we say that G is d-polytopal.*

The fundamental result on polytopality of graphs is Steinitz' Theorem which characterizes 3-polytopality:

Theorem 1 (Steinitz [16]). *A graph G is the graph of a* 3*-polytope P if and only if G is planar and* 3*-connected. Moreover, the combinatorial type of P is uniquely determined by G.*

In contrast to the easy 2- and 3-dimensional worlds, d-polytopality becomes much more involved as soon as $d \geq 4$. As an illustration, the existence of neighborly polytopes [6] proves that all possible edges can be present in the graph of a 4-polytope. Starting from a neighborly polytope, and stacking vertices on undesired edges, Perles even observed that every graph is an induced subgraph of the graph of a 4-polytope. It is a long-standing question of polytope theory how to determine whether a graph is d-polytopal or not. In the next section, we recall some general necessary conditions and apply them to discuss polytopality of small examples.

2.1 Necessary conditions of polytopality

Proposition 1. *A d-polytopal graph G satisfies the following properties:*

1. Balinski's Theorem*: G is d-connected [1].*
2. Principal Subdivision Property *(d-PSP): Every vertex of G is the principal vertex of a principal subdivision of* K_{d+1}*. Here, a* subdivision *of* K_{d+1} *is obtained by replacing edges by paths, and a* principal subdivision *of* K_{d+1} *is a subdivision in which all edges incident to a distinguished* principal vertex *are not subdivided [2].*
3. Separation Property*: The maximal number of components into which G may be separated by removing* $n > d$ *vertices equals* $f_{d-1}\big(C_d(n)\big)$*, the maximum number of facets of a d-polytope with n vertices [11].*

Remark 1. The principal minor property together with Steinitz' Theorem ensure that no graph of a 3-polytope is d-polytopal for $d \neq 3$. In other words, any 3-polytope is the unique polytopal realization of its graph. This property is also obviously true in dimension 0, 1 or 2. In contrast, it is strongly wrong in dimension 4 and higher.

Before providing examples of application of Proposition 1, let us insist on the fact that these necessary conditions are not sufficient:

Example 1 (Non-polytopality of the complete bipartite graph [2]). For any two integers $m, n \geq 3$, the complete bipartite graph $K_{m,n}$ is not polytopal, although $K_{n,n}$ satisfies all properties of Proposition 1 to be 4-polytopal as soon as $n \geq 7$.

Example 2 (Some circulant graphs). For an integer n and a subset S of $\big\{1, \dots, \lfloor \frac{n}{2} \rfloor\big\}$, we denote by $\Gamma_n(S)$ the circulant graph whose vertex set is $\mathbb{Z}_n$ and whose edges are pairs of vertices whose difference lies in $S \cup (-S)$. We can apply Proposition 1 to study the polytopality of circulant graphs: for example, for any integer $m \geq 2$,

1. the circulant graph $\Gamma_{2m+1}(1, 2)$ is not polytopal: it is not planar and does not satisfy the principal subdivision property for dimension 4. In contrast, $\Gamma_{2m}(1, 2)$ is the graph of an antiprism over an m-gon.

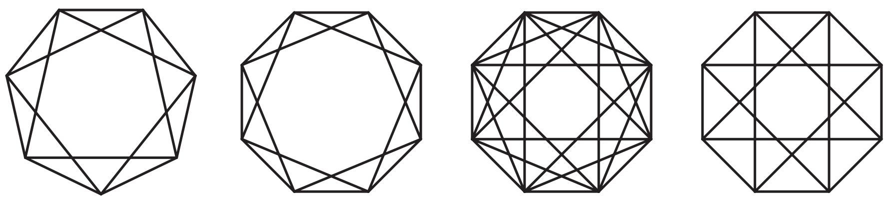

Fig. 2. The circulant graphs $\Gamma_7(1, 2)$, $\Gamma_8(1, 2)$, $\Gamma_8(1, 2, 3)$ and $\Gamma_8(1, 3)$.

2. the circulant graph $\Gamma_{2m}(1, 2, \dots, m-1)$ (that is, the complete graph on $2m$ vertices minus a perfect matching) is not $(2m-2)$-polytopal since it does not satisfy the principal subdivision property in this dimension. However, it is always the graph of the m-dimensional cross-polytope, and when m is even, it is also the graph of the join of two $(m/2)$-dimensional cross-polytopes.

Example 3 (A graph whose polytopality range is $\{d\}$ [11]). An interesting application of the separation property of Proposition 1 is the possibility to construct, for any integer d, a polytope whose polytopality range is exactly the singleton $\{d\}$. The construction, proposed by Klee [11], consists in stacking a vertex on all facets of the cyclic polytope $C_d(n)$ (for example on all facets of a simplex). The graph of the resulting polytope can be separated into $f_{d-1}(C_d(n))$ isolated points by removing the n initial vertices, and thus is not d'-polytopal for $d' < d$, by the separation property. It can not be d'-polytopal for $d' > d$ either, since the stacked vertices have degree d (because the cyclic polytope is simplicial). Thus, the dimension of the resulting graph is not ambiguous.

2.2 Simple polytopes

A d-polytope is *simple* if its vertex figures are simplices. In other words, its facet-defining hyperplanes are in general position, so that a vertex is contained in exactly d facets, and also in exactly d edges (and thus the graph of a simple d-polytope is d-regular). Surprisingly, a d-regular graph can be realized by at most one simple polytope:

Theorem 2 ([3,9]). *Two simple polytopes are combinatorially equivalent if and only if they have the same graph.*

This property, conjectured by Perles, was first proved by Blind and Mani [3]. Kalai [9] gave a simple way of reconstructing the face lattice from the graph, and Friedman [5] showed that this can even be done in polynomial time.

The first step to realize a graph is often to understand the possible face lattice of a polytopal realization. Theorem 2 ensures that if the realization is simple, there is only one choice. This motivates to temporarily restrict the study of realization of regular graphs only to simple polytopes:

Definition 2. *A graph is* simply d-polytopal *if it is the* 1*-skeleton of a simple* d*-dimensional polytope.*

We can exploit properties of simple polytopes to obtain results on the simple polytopality of graphs. For us, the key property turns out to be that any k-tuple of edges incident to a vertex of a simple polytope is contained in a k-face. For example, this implies the following result:

Proposition 2. *All induced cycles of length* 3*,* 4 *and* 5 *in the graph of a simple* d*-polytope* P *are graphs of* 2*-faces of* P*.*

Proof. For 3-cycles, the result is immediate: any two adjacent edges of a 3-cycle induce a 2-face, which must be a triangle because the graph is induced.

Next, let $\{a, b, c, d\}$ be consecutive vertices of an induced 4-cycle in the graph of a simple polytope P. Any pair of edges emanating from a vertex lies in a 2-face of P. Let C_a be the 2-face of P that contains the edges $\text{conv}\{a, b\}$ and $\text{conv}\{a, d\}$. Similarly, let C_c be the 2-face of P that contains $\text{conv}\{b, c\}$ and $\text{conv}\{c, d\}$. If C_a and C_c were distinct, they would intersect improperly, at least in the two vertices b and d. Thus, $C_a = C_c = \text{conv}\{a, b, c, d\}$ is a 2-face.

The case of 5-cycles is a little more involved. We first address the case of 3-polytopes. If a 5-cycle C in the graph G of a simple 3-polytope does not define a 2-face, it separates G into two nonempty subgraphs A and B (Whitney's Theorem [17]). Since G is 3-connected, both A and B are connected to C by at least three edges. But the endpoints of these six edges must be distributed among the five vertices of C, so one vertex of C receives two additional edges, and this contradicts simplicity.

For the general case, we show that any 5-cycle C in a simple polytope is contained in some 3-face, and apply the previous argument (a face of a simple polytope is simple). First observe that any three consecutive edges in the graph of a simple polytope lie in a common 3-face. This is true because any two adjacent edges define a 2-face, and a 2-face together with another adjacent edge defines a 3-face. Thus, four of the vertices of C are already contained in a 3-face F. If the fifth vertex w of C lies outside F, then the 2-face defined by the two edges of C incident to w intersects improperly with F.

Remark 2. Observe that there is an induced 6-cycle in the graph of the cube (resp. an induced p-cycle in the graph of a double pyramid over a p-cycle, for $p \geq 3$) which is not the graph of a 2-face. It is also interesting to notice that contrarily to dimension 3 (Whitney's Theorem [17]), the 2-faces of a 4-polytope are not characterized by a separation property: a pyramid over a cube has a non-separating induced 6-cycle which does not define a 2-face.

Corollary 1. *A simply polytopal graph cannot:*

1. *be separated by an induced cycle of length* 3*,* 4 *or* 5*.*
2. *contain two induced cycles of length* 4 *or* 5 *which share* 3 *vertices.*

Example 4. For example, the circulant graph $\Gamma_m(1, 3)$ (see the rightmost graph in Figure 2) is not polytopal since it contains induced 4-cycles which share two edges. Similarily, the graph of a simple polytope cannot have an induced Petersen subgraph. Thus both graphs on the right of Figure 1 are not simply polytopal. A more careful application of Proposition 2 proves that the leftmost graph of Figure 1 is also not polytopal. The only simply polytopal graph on Figure 1 is the second one which is the graph of the 4-dimensional cube.

2.3 Truncation and star-clique operation

We consider the polytope $\tau_v(P)$ obtained by cutting off a single vertex v in a polytope P: the set of inequalities defining $\tau_v(P)$ is that of P together with a new inequality satisfied by all the vertices of P except v. The faces of $\tau_v(P)$ are: (i) all the faces of P which do not contain v; (ii) the truncations $\tau_v(F)$ of all faces F of P containing v; and (iii) the vertex figure of v in P together with all its faces. In particular, if v is a simple vertex in P, then the truncation of v in P replaces v by a simplex. On the graph of P, it translates into the following transformation:

Definition 3. *Let G be a graph and v be a vertex of degree d of G. The* star-clique operation *(at v) replaces vertex v by a d-clique K, and assigns one edge incident to v to each vertex of K. The resulting graph $\sigma_v(G)$ has $d-1$ more vertices and $\binom{d}{2}$ more edges.*

Proposition 3. *Let v be a vertex of degree d in a graph G. Then $\sigma_v(G)$ is d-polytopal if and only if G is d-polytopal.*

Proof. If a d-polytope P realizes G, then the truncation $\tau_v(P)$ realizes $\sigma_v(G)$. For the other direction, consider a d-polytope Q which realizes $\sigma_v(G)$. We use simplicity to assert that the d-clique replacing v forms a facet F of Q. Up to a projective transformation, we can assume that the d facets of Q adjacent to F intersect behind F. Then, removing the inequality defining F from the facet description of Q creates a polytope which realizes G.

We can exploit Proposition 3 to construct several families of non-polytopal graphs for non-trivial reasons.

Corollary 2. *Any graph obtained from the graph of a* 4*-regular* 3*-polytope by a finite nonempty sequence of star-clique operations is non-polytopal.*

Proof. No such graph can be 3-polytopal since it is not planar. If the resulting graph were 4-polytopal, Proposition 3 would assert that the original graph was also 4-polytopal, which would contradict Remark 1.

Example 5 (An infinite family of non-polytopal graphs for non-trivial reasons). For $n \geq 3$, consider the family of graphs suggested by Figure 3. They are constructed as follows: place a regular $2n$-gon C_{2n} into the plane, centered at the origin. Draw a copy C'_{2n} of C_{2n} scaled by $\frac{1}{2}$ and rotated by $\frac{\pi}{2n}$, and lift the vertices of C'_{2n} alternately to heights 1 and -1 into the third dimension. The graph $\diamond_n$ is the graph of the convex hull of the result.

Let $\diamond_n^\star$ be the result of successively applying the star-clique operation to all vertices on the external cycle C_{2n}. Corollary 2 ensures that $\diamond_n^\star$ is not polytopal, although it satisfies all necessary conditions of Proposition 1 to be 4-polytopal.

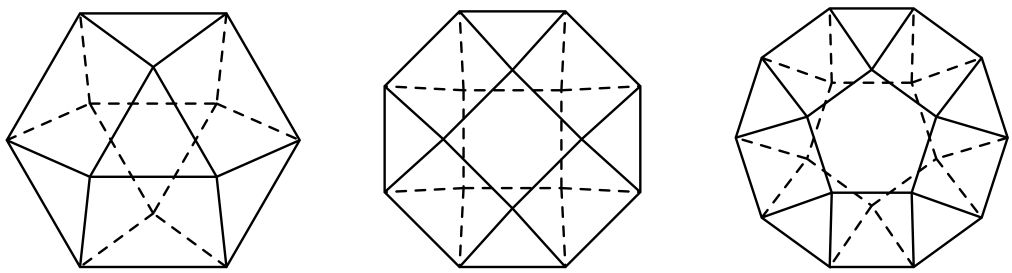

Fig. 3. The graphs $\diamond_n$ for $n \in \{3, 4, 5\}$.

3 Polytopality of products of graphs

We consider the *Cartesian product* $G \times H$ of two graphs G and H, that is, the graph whose vertex set is $V(G \times H) := V(G) \times V(H)$, and whose edge set is $E(G \times H) := \big(V(G) \times E(H)\big) \cup \big(E(G) \times V(H)\big)$. In other words, for $a, c \in V(G)$ and $b, d \in V(H)$, the vertices (a, b) and (c, d) of $G \times H$ are adjacent if either $a = c$ and $\{b, d\} \in E(H)$, or $b = d$ and $\{a, c\} \in E(G)$. Notice that this product is usually denoted by $G \square H$ in graph theory. We choose to use the notation $G \times H$ to be consistent with the Cartesian product of polytopes: if G and H are the graphs of the polytopes P and Q respectively, then the product $G \times H$ is the graph of the product $P \times Q$. In this section, we focus on the polytopality of products of non-polytopal graphs.

The factors of a polytopal product are not necessarily polytopal: consider for example the product of a triangle by a path, or the product of a segment by two glued triangles (see Figure 4 and more generally Proposition 5). We neutralize these elementary examples by furthermore requiring the product $G \times H$, or equivalently the factors G and H, to be regular. In this case, it is natural to investigate when such regular products can be simply polytopal. The answer is given by Theorem 3.

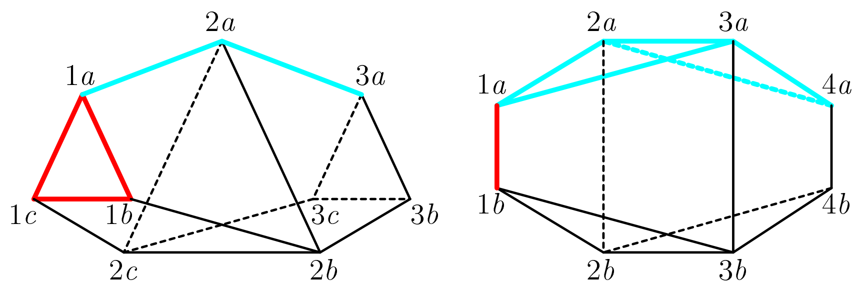

Fig. 4. Polytopal products of non-polytopal graphs: a triangle abc by a path 123 (left) and a segment ab by two glued triangles 123 and 234 (right).

Before starting, let us observe that the necessary conditions of Proposition 1 are preserved under Cartesian products in the following sense:

Proposition 4. *If two graphs G and H are respectively d- and e-connected, and respectively satisfy d- and e-PSP, then their product $G \times H$ is $(d+e)$-connected and satisfies $(d+e)$-PSP.*

Proof. The connectivity of a Cartesian product of graphs was studied in [4]. In fact, it is even proved in [15] that

$$\kappa(G \times H) = \min(\kappa(G)|H|, \kappa(H)|G|, \delta(G) + \delta(H)) \geq \kappa(G) + \kappa(H),$$

where $\kappa(G)$ and $\delta(G)$ denote the connectivity and the minimum degree of G.

For the principal subdivision property, consider a vertex (v, w) of $G \times H$. Choose a principal subdivision of K_{d+1} in G with principal vertex v and neighbors N_v, and a principal subdivision of K_{e+1} in H with principal vertex w and neighbors N_w. This gives rise to a principal subdivision of K_{d+e+1} in $G \times H$ with principal vertex (v, w) and neighbors $(N_v \times \{w\}) \cup (\{v\} \times N_w)$. Indeed, for $x, x' \in N_v$, the vertices (x, w) and (x', w) are connected by a path in $G \times w$ by construction; similarly, for $y, y' \in N_w$, the vertices (v, y) and (v, y') are connected by a path in $v \times H$. Finally, for each $x \in N_v$ and $y \in N_w$, connect (x, w) to (v, y) via the path of length 2 that passes through (x, y). All these paths are disjoint by construction.

3.1 Simply polytopal products

A product of simply polytopal graphs is automatically simply polytopal. We prove that the reciprocal statement is also true:

Theorem 3. *A product of graphs is the graph of a simple polytope if and only if its factors are.*

Applying Theorem 2, we obtain a strong characterization of the simply polytopal products:

Corollary 3. *Products of simple polytopes are the only simple polytopes whose graph is a product.*

Let G and H be two connected regular graphs of degree d and e, and assume that the graph $G \times H$ is the graph of a simple $(d+e)$-polytope P. By Proposition 2, for all edges a of G and b of H, the 4-cycle $a \times b$ is the graph of a 2-face of P.

Lemma 1. *Let F be any facet of P, let v be a vertex of G, and let $\{x, y\}$ be an edge of H such that $(v, x) \in F$ and $(v, y) \notin F$. Then $G \times \{x\} \subset F$ and $G \times \{y\} \cap F = \emptyset$.*

Proof. Since the polytope is simple, all neighbors of (v,x) except (v,y) are connected to (v,x) by an edge of F. Let v' be a neighbor of v in G, and let C be the 2-face $\text{conv}\{v,v'\} \times \text{conv}\{x,y\}$ of P. If (v',y) were a vertex of F, the intersection $C \cap F$ would consist of exactly three vertices (because $(v,y) \notin F$), a contradiction. In summary, $(v',x) \in F$ and $(v',y) \notin F$, for all neighbors v' of v. Repeating this argument and using the fact that G is connected yields $G \times \{x\} \subset F$ and $G \times \{y\} \cap F = \emptyset$.

Lemma 2. *The graph of any facet of P is either of the form $G' \times H$ for a $(d-1)$-regular induced subgraph G' of G, or of the form $G \times H'$ for an $(e-1)$-regular induced subgraph H' of H.*

Proof. Assume that the graph of a facet F is not of the form $G' \times H$. Then there exists a vertex v of G and an edge $\{x,y\}$ of H such that $(v,x) \in F$ and $(v,y) \notin F$. By Lemma 1, the subgraph H' of H induced by the vertices $y \in H$ such that $G \times \{y\} \subset F$ is nonempty. We now prove that the graph $\text{gr}(F)$ of F is exactly $G \times H'$.

The inclusion $G \times H' \subset \text{gr}(F)$ is clear: by definition, $G \times \{y\}$ is a subgraph of $\text{gr}(F)$ for any vertex $y \in H'$. For any edge $\{x,y\}$ of H' and any vertex $v \in G$, the two vertices (v,x) and (v,y) are contained in F, so the edge between them is an edge of F; if not, we would have an improper intersection between F and this edge.

For the other inclusion, let $H'' := \{y \in H \mid G \times \{y\} \cap F = \emptyset\}$ and let $H''' := H \setminus (H' \cup H'')$. If $H''' \neq \emptyset$, the fact that H is connected ensures that there is an edge between some vertex of H''' and either a vertex of H' or H''. This contradicts Lemma 1.

We have proved that $G \times H' = \text{gr}(F)$. Since F is a simple $(d+e-1)$-polytope and since G is d-regular, the subgraph H' is $(e-1)$-regular.

Proof (of Theorem 3). One direction is clear. For the other direction, proceed by induction on $d+e$, the cases $d=0$ and $e=0$ being trivial. Now assume that $d,e \geq 1$, that $G \times H = \text{gr}(P)$, and that G is not the graph of a d-polytope. By Lemma 2, all facets of P are of the form $G' \times H$ or $G \times H'$, where G' (resp. H') is an induced $(d-1)$-regular (resp. $(e-1)$-regular) subgraph of G (resp. H). By induction, the second case does not arise. We fix a vertex w of H. Then induction tell us that $F_w := G' \times \{w\}$ is a face of P, and $G' \times H$ is the only facet of P that contains F_w by Lemma 2. This cannot occur unless F_w is a facet, but this only happens in the base case $H = \{w\}$.

3.2 Polytopal products of non-polytopal graphs

In this section, we give a general construction to obtain polytopal products starting from a polytopal graph G and a non-polytopal one H. We need the graph H to be the graph of a *regular subdivision* of a polytope Q, that is, the graph of the upper[4] envelope (the set of all upper facets with respect to the last coordinate) of the convex hull of the point set $\{(q, \omega(q)) \mid q \in V(Q)\} \subset \mathbb{R}^{e+1}$ obtained by lifting the vertices of $Q \subset \mathbb{R}^e$ according to a *lifting function* $\omega : V(Q) \to \mathbb{R}$.

Proposition 5. *If G is the graph of a d-polytope P, and H is the graph of a regular subdivision of an e-polytope Q, then $G \times H$ is $(d+e)$-polytopal. In the case $d > 1$, the regular subdivision of Q can even have internal vertices.*

Proof. Let $\omega : V(Q) \to \mathbb{R}_{>0}$ be a lifting function that induces a regular subdivision of Q with graph H. Assume without loss of generality that the origin of $\mathbb{R}^d$ lies in the interior of P. For each $p \in V(P)$ and $q \in V(Q)$, we define the point $\rho(p,q) := (\omega(q)p, q) \in \mathbb{R}^{d+e}$. Consider

$$R := \operatorname{conv}\{\rho(p,q) \mid p \in V(P), q \in V(Q)\}.$$

Let g be a facet of Q defined by the linear inequality $\langle \psi \,|\, y \rangle \leq 1$. Then $\langle (0,\psi) \,|\, (x,y) \rangle \leq 1$ defines a facet of R, with vertex set $\{\rho(p,q) \mid p \in P, q \in g\}$, and isomorphic to $P \times g$.

Let f be a facet of P defined by the linear inequality $\langle \phi \,|\, x \rangle \leq 1$. Let c be a cell of the subdivision of Q, and let $\psi_0 h + \langle \psi \,|\, y \rangle \leq 1$ be the linear inequality that defines the upper facet corresponding to c in the lifting. Then we claim that the linear inequality

$$\chi(x,y) = \psi_0 \langle \phi \,|\, x \rangle + \langle \psi \,|\, y \rangle \leq 1$$

selects a facet of R with vertex set $\{\rho(p,q) \mid p \in f, q \in c\}$ that is isomorphic to $f \times c$. Indeed,

$$\chi\big(\rho(p,q)\big) = \chi(\omega(q)p, q) = \psi_0 \omega(q) \langle \phi \,|\, p \rangle + \langle \psi \,|\, q \rangle \leq 1$$

where equality holds if and only if $\langle \phi \,|\, p \rangle = 1$ and $\psi_0 \omega(q) + \langle \psi \,|\, q \rangle = 1$, so that $p \in f$ and $q \in c$.

The above set $\mathcal{F}$ of facets of R in fact contains all facets: indeed, any $(d+e-2)$-face of a facet in $\mathcal{F}$ is contained in precisely two facets in $\mathcal{F}$. Since the union of the edge sets of the facets in $\mathcal{F}$ is precisely $G \times H$, it follows that the graph of R equals $G \times H$.

A similar argument proves the same statement in the case when $d > 1$ and H is a regular subdivision of Q with internal vertices (meaning that not only the vertices of Q are lifted, but also a finite number of interior points).

We already mentioned two examples obtained by such a construction in the beginning of this section (see Figure 4): the product of a polytopal graph by a path and the product of a segment by a subdivision of an n-gon with no internal vertex. Proposition 5 even produces examples of regular polytopal products which are not simply polytopal:

[4] The unusual convention to define a subdivision as the projection of the upper facets of the lifting simplifies the presentation of the construction.

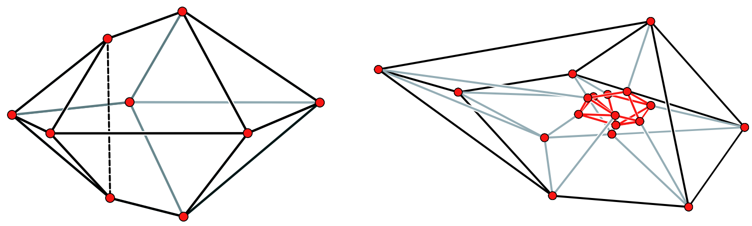

Fig. 5. A non-polytopal 4-regular graph H which is the graph of a regular subdivision of a 3-polytope (left) and the Schlegel diagram of a 4-polytope whose graph is the product of H by a segment (right).

Example 6. Let H be the graph obtained by a star-clique operation from the graph of an octahedron. It is non-polytopal (Corollary 2), but it is the graph of a regular subdivision of a 3-polytope (see Figure 5). Consequently, the product of H by any regular polytopal graph is polytopal. Thus, there exist regular polytopal products which are not simply polytopal.

Similarly, although the polytopes $\diamond_n^\star$ of the family of Example 5 are not polytopal, their product by any polytopal graph is polytopal.

Example 7 (Product of dominos). Define the *p-domino graph* D_p to be the product of a path P_p of length p by a segment. Let $p, q \geq 2$. Observe that D_p and D_q are not polytopal and that $D_p \times P_q$ is a regular subdivision of a 3-polytope. Consequently, the product of dominos $D_p \times D_q$ is a 4-polytopal product of two non-polytopal graphs (see Figure 6).

Finally, let us observe that the product $D_p \times D_q = P_p \times P_q \times (K_2)^2$ can be decomposed in different ways into a product of two graphs. However, in any such decomposition, at least one of the factors is non-polytopal.

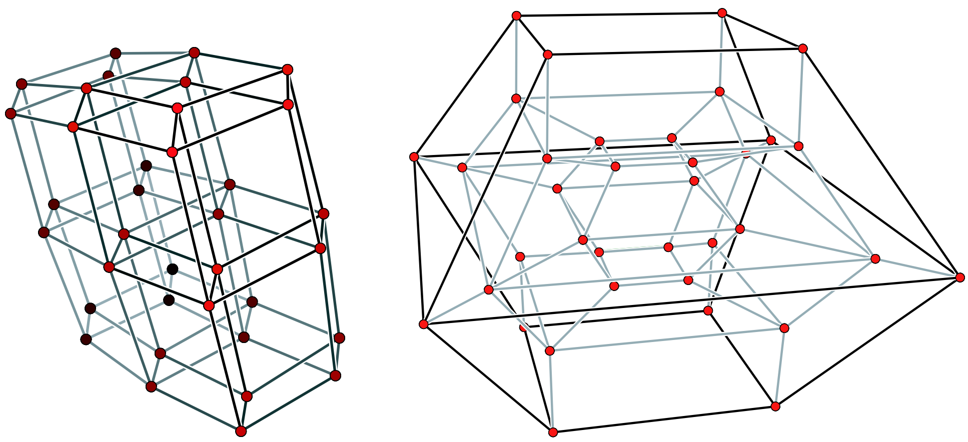

Fig. 6. The graph of the product of two 2-dominos (left) and the Schlegel diagram of a realizing 4-polytope (right).

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