Graph Operations and Laplacian Eigenpolytopes

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Abstract. We introduce the *Laplacian eigenpolytopes* ("*L-polytopes*") associated to a simple undirected graph G, investigate how they change under basic operations such as taking the union, join, complement, line graph and cartesian product of graphs, and show how several "famous" polytopes arise as L-polytopes of "famous" graphs.

Eigenpolytopes have been previously introduced by Godsil, who studied them in detail in the context of distance-regular graphs. Our focus on the Laplacian matrix, as opposed to the adjacency matrix of G, permits simpler proofs and descriptions of the result of operations on not necessarily distance-regular graphs. Additionally, it motivates the study of new operations on polytopes, such as the Kronecker product.

Thus, we open the door to a detailed study of how combinatorial properties of G are reflected in its L-polytopes. Subsequent papers will use these tools to construct interesting polytopes from interesting graphs, and vice versa.

Key words: spectrum; Laplacian; polytope; eigenvalue

1 Introduction

Let $G = (V, E)$ be a simple graph with $n = |V|$ nodes. Its *adjacency matrix* A is the $n \times n$ matrix with $A_{ij} = 1$ when nodes i and j are adjacent and $A_{ij} = 0$ otherwise. Its *degree matrix* is the diagonal matrix D that collects the degrees of the nodes: $D_{ii} = \deg(i)$. The (ordinary) *spectrum* $\mathrm{Spec}(G)$ of G is the multiset of eigenvalues of A, and the *Laplacian spectrum* $\mathrm{LSpec}(G)$ of G is the multiset of eigenvalues of the *Laplacian Matrix* $L = D - A$ of G. We call these latter eigenvalues the *Laplacian eigenvalues* of G, or *L-values*[1] for short.

By definition, λ is an L-value of G if and only if there exists a non-zero vector $x \in \mathbb{R}^n$ such that $Lx = \lambda x$. In such a situation, we say that x is an *L-vector* of G corresponding to λ, or that x is a λ-*L-vector*. The vector subspace

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[1] No relation (yet) with their cousins from number theory.

$\mathrm{LE}_\lambda = \ker(L - \lambda\mathbf{I}) \subset \mathbb{R}^n$ of all L-vectors corresponding to λ is the *Laplacian eigenspace* or *L-space* of λ. Its dimension is the multiplicity of the L-value λ. The ordinary eigenspaces of G arise by replacing L by A and LE by E.

If G is regular of degree d, then $L = d\mathbf{I} - A$. Thus, if G has ordinary eigenvalues $d = \mu_1 \geq \cdots \geq \mu_n$, then $\mu_i = d - \lambda_i$ for $i = 1, \ldots, n$. Thus, for regular graphs, $\{E_\mu : \mu \in \mathrm{Spec}(G)\} = \{\mathrm{LE}_\lambda : \lambda \in \mathrm{LSpec}(G)\}$.

In 1978, Godsil [7] associated a so-called *eigenpolytope* to each ordinary eigenspace E_μ of a graph (see Section 1.3), and investigated the eigenpolytopes of distance-regular graphs [3,8]. In the present paper, we set out to study some of the relationships between the properties of arbitrary graphs G and their corresponding L-polytopes. In particular, we investigate how operations on G are reflected in the L-polytopes.

Eventually, we would like to obtain a dictionary that translates between graphs and polytopes, and then use this knowledge to describe polytopes with extremal characteristics parting from extremal graphs. As a first step in this direction, the present paper constructs examples such as simplices, cubes, crosspolytopes, the platonic solids and CUT polytopes as L-polytopes, and provides even more motivation (if necessary) to study exciting new constructions like the Kronecker product of polytopes (Definition 2 on page 495).

As for notation, we write $\mathbf{I}$ for the identity matrix, $\mathbf{J}$ and $\mathbf{O}$ for the matrices with all entries equal to 1 and 0 respectively, and $\mathbf{1}$ and $\mathbf{0}$ for the vectors with all entries equal to 1, respectively 0. If the size of these objects is not clear from the context, we indicate it using subindices.

The remainder of this section contains preliminary material on algebraic graph theory, polytopes and eigenpolytopes.

1.1 Algebraic Graph Theory

It is easy to see that the L-values of G satisfy $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Indeed, since L decomposes as $L = NN^\top$ and $L\mathbf{1} = 0$, where N is the directed incidence matrix of the directed graph obtained by arbitrarily orienting G's edges, we conclude that L is positive semidefinite and singular, which proves the assertion. Moreover, $\sum \lambda_i = \operatorname{tr} L = 2|E|$.

The Laplacian spectrum $\mathrm{LSpec}(G)$ encodes a host of combinatorial information about G. For instance, using the Cauchy-Binet formula one can derive that the number of spanning trees of G equals $\det(L + \frac{1}{n^2}\mathbf{J}) = \frac{1}{n}\lambda_2 \cdots \lambda_n$. Also, if G is connected with diameter e, then G has at least $e+1$ distinct L-values, and equality holds for distance-regular graphs. Moreover, the multiplicity of the L-value 0 equals the number of connected components of G. There are also connections to the degree, the isoperimetric number, etc.

The L-values of graphs have been widely studied, for example in [4,9,13,14], while L-vectors and L-spaces of graphs are treated in [5,12]. As a particular example, L-vectors of graphs are used for graph clustering [11]. A very nice recent reference for algebraic graph theory is [2].

1.2 Polytopes

We very briefly recapitulate some basic concepts concerning convex polytopes. A very accessible treatment for this material is [15].

A *polytope* P is the convex hull of a finite set of points in $\mathbb{R}^d$. The dimension $\dim P$ of P is the dimension of its affine hull. Unless stated otherwise, we always assume that $\dim P = d$, and call P a *d-polytope*. A *face* of a polytope is the set of points in P that maximize some linear function on $\mathbb{R}^d$, and is itself a polytope. Faces of dimension 0 are called *vertices*, while those of dimension 1, respectively $d-1$, are called *edges*, respectively *facets*. Every face is the convex hull of the vertices that it contains, and it is also the intersection of all the facets that contain it. Faces of dimension $0 \leq k \leq d-1$ are called *proper*.

A *d-simplex* Δ_d is the convex hull of $d+1$ affinely independent points. Every polytope P with n vertices can be regarded as the projection of an $(n-1)$-simplex, by fixing any bijection between the respective vertex sets. In fact, any simplex of dimension at least $n-1$ can be projected into P, by mapping more than one vertex of the simplex into the same vertex of P. If k vertices of some simplex Δ_{m-1} with $m \geq n$ map into the vertex v of P, we say that $v = v^{(k)}$ has *multiplicity* k.

We now present some operations for constructing new polytopes from given ones. Let $P \in \mathbb{R}^d$ be a d-dimensional polytope with n vertices and m facets, and similarly for P', d', n' and m'.

Join: $P \star P'$ is the convex hull of $P \cup P'$, after embedding P and P' into mutually skew affine spaces. The faces of $P \star P'$ are the joins of faces of P and faces of P', including the empty face and the polytope itself.

Free sum: $P \oplus P' = \operatorname{conv}\big(P \times \{0\} \cup \{0\} \times P'\big) \subset \mathbb{R}^{d+d'}$. If both P and P' contain the origin in their interior, then each proper face of $P \oplus P'$ is the free sum of a proper face of P with a proper face of P'.

1.3 L-polytopes

Our results and proofs turn out to be much more convient to formulate using L-values and Laplacian eigenspaces, and this is why we exclusively consider those from now on. Therefore, we define L-polytopes directly in terms of L-spaces instead of the original definition in [7], and point out again that this changes nothing for regular graphs.

Let $\lambda \in \operatorname{LSpec}(G)$ be an L-value of an undirected graph G, let $\operatorname{LE}_\lambda(G)$ be the corresponding L-space, and put $\operatorname{LE}^0_\lambda(G) := \operatorname{LE}_\lambda(G) \cap \mathbf{1}^\perp$.

Definition 1. *Pick a basis* $u_1, \dots, u_m$ *of* $\operatorname{LE}^0_\lambda(G)$*, assemble these column vectors into the* $n \times m$ *matrix* $Z_\lambda(G)$*, and call its rows* $g_1, \dots, g_n$*, one for each node of* G*. The* L-polytope[2] *of* G *belonging to* λ *is the convex hull* $Q_\lambda(G) = \operatorname{conv}\{g_1, \dots, g_n\} \in \mathbb{R}^m$ *of the rows of* $Z_\lambda(G)$*.*

[2] cf. Remark 2 on page 490

Of course, the definition of $Q_\lambda(G)$ depends on the choice of basis. However, a change of basis in $\mathrm{LE}^0_\lambda(G)$ yields $\tilde{Z}_\lambda = Z_\lambda M$ and $\tilde{g}_i = M^\top g_i$ for some $M \in \mathrm{GL}_m(\mathbb{R})$, so that $Q_\lambda(G)$ only suffers a linear transformation.

Remark 1. Recall that $\mathbf{1}$ is always an L-vector corresponding to the L-value 0. Because eigenspaces to different eigenvalues are always orthogonal, we conclude that $\mathbf{1} \perp \mathrm{LE}_\lambda(G)$ if $\lambda \neq 0$, so that in this case $\mathrm{LE}_\lambda(G) = \mathrm{LE}^0_\lambda(G)$, and $m = \dim \mathrm{LE}^0_\lambda(G)$ coincides with the multiplicity of λ as an L-value.

When $\lambda = 0$, then $\mathbf{1} \in \mathrm{LE}_0(G)$ and therefore the dimension m of $\mathrm{LE}^0_0(G)$ is one less than the multiplicity of 0. Notice that if we define $Z'_\lambda(G)$ instead of $Z_\lambda(G)$,z using $\mathrm{LE}_\lambda(G)$ instead of $\mathrm{LE}^0_\lambda(G)$, nothing changes for $\lambda \neq 0$. But for $\lambda = 0$ we obtain the matrix $Z'_0(G) = \big(\mathbf{1}, Z_0(G)\big)$ and the homogenized polytope $Q'_0(G) = \{1\} \times Q_0(G)$. We prefer to use $\mathrm{LE}^0_\lambda(G)$ because it creates less special cases, as the following elementary result shows. This homogenization will turn out to be convenient when we treat the Cartesian product.

Proposition 1. *For all $\lambda \in \mathrm{LSpec}(G)$, the polytope $Q_\lambda(G)$ has dimension $\dim Q_\lambda = \dim \mathrm{LE}^0_\lambda(G)$ and is centered at the origin.*

Proof. The first claim is clear. The second one follows from $\mathbf{1}^\top Z_\lambda(G) = 0$. □

L-polytopes are interesting because of the strong relationship to the combinatorics of the graph that defines them. Thus, tools from convex polytopes can be used to study graphs and vice versa.

2 A first example: The Cube Graph

The graph C_n of the n-dimensional cube $\square^n$ has 2^n nodes, corresponding to all binary words of length n; thus, $V = \{0,1\}^n$. Two nodes are joined by an edge if they share all bits but one. C_n is the Cartesian product of n edges[3]. The spectrum of C_n consists of the numbers $(n-2i)$ for $i = 0, 1, \dots, n$, each with multiplicity $\binom{n}{i}$.

Table 1 shows the L-polytopes of C_3. Besides simplices, suprisingly we get C_3 back, in the guise of the 1-skeleton of $Q_2(C_3)$. Actually, this is part of a pattern: Godsil [8] classifies all distance-regular graphs G that “reproduce themselves” in this way into 5 families and 4 individual graphs. One of these families includes the cubes; also, all platonic solids have this property.

Proposition 2. *The first L-polytopes of the cube graph C_n are:*

- $Q_0(C_n) = \{0\}^{(n)}$*, the n-fold copy of a point.*
- $Q_2(C_n) = [-1,1]^n$*, the n-cube itself again.*
- $Q_4(C_n) \simeq \mathrm{CUT}(n)^{(2)}$*, the cut polytope* [6] *of the complete graph K_n, with all vertices doubled.*

[3] See Section 6 for the definition of Cartesian product.

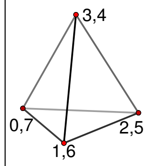
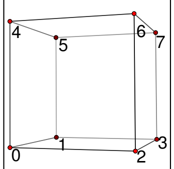
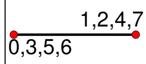
λ	$Z_\lambda(C_3)$	$Q_\lambda(C_3)$	$Q_\lambda(C_3)$	λ	$Z_\lambda(C_3)$	$Q_\lambda(C_3)$	$Q_\lambda(C_3)$
0	$\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\\0\end{bmatrix}$	$\{0\}^{(8)}$		4	$\begin{bmatrix}1&1&1\\1&-1&-1\\-1&1&-1\\-1&-1&1\\-1&-1&1\\-1&1&-1\\1&-1&-1\\1&1&1\end{bmatrix}$	$\Delta_3^{(2)}$	
2	$\begin{bmatrix}1&1&1\\1&1&-1\\1&-1&1\\1&-1&-1\\-1&1&1\\-1&1&-1\\-1&-1&1\\-1&-1&-1\end{bmatrix}$	$[-1,1]^3$		6	$\begin{bmatrix}1\\-1\\-1\\1\\-1\\1\\1\\-1\end{bmatrix}$	$\Delta_1^{(4)}$	

Table 1. Z_λ and Q_λ of the different L-values λ of the 3-cube

Proof. This is not completely evident, but not difficult. However, the calculations for the proof take up too much space for this paper. □

3 Basic Properties

Proposition 3. *Let G be a graph with n nodes and κ connected components. Then the L-polytope corresponding to the L-value* 0 *is a simplex with κ vertices, $Q_0(G) = \Delta_{\kappa-1}$. Each vertex of $\Delta_{\kappa-1}$ corresponds to a connected component, and its multiplicity equals the size of this component.*

Proof. Without loss of generality, assume that L is a diagonal block matrix with block sizes $n_1, \dots, n_\kappa$, corresponding to the cardinalities of the connected components of G. For $i = 2, \dots, \kappa$, let u_i be the column vector of size n that starts out as $-\mathbf{1}_{n_1}$, and whose only other non-zero entry is $\mathbf{1}_{n_i}$ in the i-th block. Then $u_2, \dots, u_\kappa$ form a basis of $\mathrm{LE}_0^0(G)$, and the description follows. □

Proposition 4. *Let G be a graph with n nodes, such that the complementary graph $\overline{G}$ has $\overline{\kappa}$ connected components. Then the L-polytope corresponding to the L-value n is a simplex with $\overline{\kappa}$ vertices, $Q_n(G) = \Delta_{\overline{\kappa}-1}$. Each vertex of $\Delta_{\overline{\kappa}-1}$ corresponds to a connected component of $\overline{G}$, and its multiplicity equals the size of this component.*

Proof. This follows from Proposition 3 and the upcoming Proposition 8. □

Proposition 5. [7] *Every automorphism σ of G induces an isometry on each eigenpolytope $Q_\lambda(G)$.*

Proposition 6. [3] *Let G be a vertex-transitive graph. For $\lambda \in \mathrm{LSpec}(G)$, the vectors g_i corresponding to a row of $Z_\lambda(G)$ all have the same length. In particular, they lie on a sphere and they cannot be interior points of $Q_\lambda(G)$.*

Notice that point multiplicities greater than 1 are not excluded by this.

Remark 2. This is as good a place as any to point out that the appearance of possibly repeated points both on and inside the convex hull suggests that the story of eigenpolytopes and L-polytopes is really about point configurations and their oriented matroid. We would also like to mention that there is at least one other promising candidate for the definition of eigenpolytopes, namely the (rows or columns of the) Gale dual configuration $L - \lambda\mathbf{I}$: all these points lie on a sphere even if the graph is only regular, but not vertex transitive. Moreover, it has the intriguing feature $(L - \lambda\mathbf{I})^\top = L - \lambda\mathbf{I}$.

Proposition 7. [3] *For $v \in \mathrm{LE}_\lambda(G)$ let $I_v = \{i \mid v_i \geq v_j \text{ for all } j\}$ index the maximal entries of v. The set of vertices of $Q_\lambda(G)$ indexed by I_v forms a face, and all faces of $Q_\lambda(G)$ can be obtained this way.*

Proof. Applying a linear functional a to each vertex of $Q_\lambda(G)$ and assembling the result in a vector v corresponds to forming the product $v = Z_\lambda a$. This yields a linear combination of the columns of Z_λ, and thus an L-vector. □

4 Graph Complementation, Union and Join

4.1 Complementation

Let $G = (V, E)$ be a graph on n nodes. Two nodes in the *complement graph* $\overline{G}$ are adjacent if and only if they are not so in G; thus, $\overline{G} = \big(V, \binom{V}{2} \setminus E\big)$. If L is the Laplacian of G, then $\overline{L} = n\mathbf{I} - \mathbf{J} - L$ is the Laplacian of $\overline{G}$. Because eigenvectors of L are also eigenvectors of $\mathbf{J}$, the L-values of $\overline{L}$ are $0 \leq n - \lambda_n \leq \cdots \leq n - \lambda_2$; in particular, $\lambda_n \leq n$.

Proposition 8. *For all $\overline{\lambda} \in \mathrm{LSpec}(\overline{G})$, we have $Q_{\overline{\lambda}}(\overline{G}) = Q_{n-\overline{\lambda}}(G)$.*

Proof. Let $v \in \mathbf{1}^\perp$ satisfy $\overline{L}v = \lambda v$. Then $Lv = (n - \overline{\lambda})v$ because $v \in \ker(\mathbf{J})$. Thus, $v \in E_{\overline{\lambda}}(\overline{G}) \cap \mathbf{1}^\perp$ if and only if $v \in E_{n-\overline{\lambda}}(G) \cap \mathbf{1}^\perp$. The result follows. □

4.2 Union

The union of two graphs $G_i = (V_i, E_i)$ with Laplacian L_i, for $i = 1, 2$, is the graph[4] $G = G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ with Laplacian $L = \begin{pmatrix} L_1 & \mathbf{O} \\ \mathbf{O} & L_2 \end{pmatrix}$. With this information, we can describe the corresponding L-polytopes:

[4] Graph theorists also write $G = G_1 + G_2$ for the union. Also, notice the implicit relabelings.

Proposition 9. *Let G_1 and G_2 be graphs, and let $\lambda \in \mathrm{LSpec}(G_1 \cup G_2)$. Then*

$$Q_\lambda(G_1 \cup G_2) \;=\; \begin{cases} Q_\lambda(G_1) \oplus Q_\lambda(G_2) & \text{if } \lambda \neq 0, \\ Q_0(G_1) \star Q_0(G_2) \;=\; \Delta_{\kappa_1+\kappa_2-1} & \text{if } \lambda = 0. \end{cases}$$

Proof. First assume $\lambda \neq 0$, so that all eigenvectors in $\mathrm{LE}_\lambda(G_i)$ are orthogonal to $\mathbf{1}$. Since forming eigenspaces commutes with forming direct sums, the matrix Z_λ is a block matrix $Z_\lambda(G) = \begin{pmatrix} Z_\lambda(G_1) & \mathbf{O} \\ \mathbf{O} & Z_\lambda(G_2) \end{pmatrix}$, and the description of $Q_\lambda(G)$ follows. For $\lambda = 0$, we can construct a basis of $\mathrm{LE}_0(G)$ using vectors of the form $\binom{v}{\mathbf{0}}$ and $\binom{\mathbf{0}}{w}$ with $v \in \mathrm{LE}_0(G_1)$ and $w \in \mathrm{LE}_0(G_2)$. All of them are orthogonal to $\mathbf{1}$, except for $\binom{\mathbf{1}}{\mathbf{0}}$ and $\binom{\mathbf{0}}{\mathbf{1}}$. We replace these with $\binom{\mathbf{1}}{\mathbf{1}}$ and $\binom{\mathbf{1}}{-\mathbf{1}}$, and recognize the construction of a join from Section 1.2. □

4.3 Join

The *join* $G = G_1 \star G_2$ of two graphs is constructed from $G_1 \cup G_2$ by adding all edges that join one vertex of G_1 with one vertex of G_2. Its Laplacian spectrum is well known: If $\mathrm{LSpec}(G_1) = \{0, \lambda_1, \dots \lambda_{n_1-1}\}$ and $\mathrm{LSpec}(G_2) = \{0, \lambda'_1, \dots, \lambda'_{n_2-1}\}$ then

$$\begin{aligned} \mathrm{LSpec}(G_1 \star G_2) = \{0,\, &\lambda_1 + n_2, \dots, \lambda_{n_1-1} + n_2, \\ &\lambda'_1 + n_1, \dots, \lambda'_{n_2-1} + n_1,\, n_1 + n_2\}. \end{aligned}$$

The join of graphs can be decomposed in terms of complements and unions:

$$G_1 \star G_2 \;=\; \overline{\overline{G_1} \cup \overline{G_2}}. \tag{1}$$

This lets us use Propositions 8 and 9 to determine the L-polytopes of joins.

Proposition 10. *Let $G = G_1 \star G_2$ be the join of size $n = n_1 + n_2$ of graphs G_1, G_2 of size n_1 and n_2, respectively, and let $\lambda \in \mathrm{LSpec}(G)$. Then*

$$Q_\lambda(G_1 \star G_2) \;=\; \begin{cases} Q_{\lambda-n_2}(G_1) \oplus Q_{\lambda-n_1}(G_2) & \text{if } \lambda \neq n, \\ Q_{n_1}(G_1) \star Q_{n_2}(G_2) & \text{if } \lambda = n. \end{cases}$$

Proof. This follows directly from (1) and Propositions 8 and 9. If $\lambda \neq n$,

$$\begin{aligned} Q_\lambda(G_1 \star G_2) = Q_\lambda(\overline{\overline{G}_1 \cup \overline{G}_2}) \;&=\; Q_{n-\lambda}(\overline{G}_1 \cup \overline{G}_2) \\ &= Q_{n-\lambda}(\overline{G}_1) \oplus Q_{n-\lambda}(\overline{G}_2) = Q_{\lambda-n_2}(G_1) \oplus Q_{\lambda-n_1}(G_2). \end{aligned}$$

Here we use that $Q_\lambda(G) = \{0\}^{(n)}$ if $\lambda \notin \mathrm{LSpec}(G)$. For example, if $\lambda = 0$ then $Q_{-n_2}(G_1) = \{0\}^{(n_1)}$ and $Q_{-n_1}(G_2) = \{0\}^{(n_2)}$. Therefore $Q_0(G) = \{0\}^{(n)}$, which agrees with the fact that the join is always a connected graph.

For $\lambda = n$, we only need to slightly change how we treat the union:

$$Q_0(\overline{G}_1 \cup \overline{G}_2) \;=\; Q_0(\overline{G}_1) \star Q_0(\overline{G}_2) \;=\; Q_{n_1}(G_1) \star Q_{n_2}(G_2).$$

This concludes the proof. □

Remark 3. Both union and join are associative and commutative operations, therefore Propositions 9 and 10 directly extend to the union and join of an arbitrary number of graphs:

$$Q_\lambda\Big(\bigcup_i G_i\Big) = \begin{cases} \bigoplus_i Q_\lambda(G_i) & \text{if } \lambda \neq 0, \\ \bigstar_i Q_\lambda(G_i) & \text{if } \lambda = 0, \end{cases}$$

$$Q_\lambda\Big(\bigstar_i G_i\Big) = \begin{cases} \bigoplus_i Q_{\lambda-n+n_i}(G_i) & \text{if } \lambda \neq n. \\ \bigstar_i Q_{n_i}(G_i) & \text{if } \lambda = n. \end{cases}$$

Example 1 (Complete multipartite graphs). The complete multipartite graph $M_{s,k}$ is described either as the complement of the union of s complete graphs with k nodes $\overline{\cup_s K_k}$; or as the join of s empty graphs with k nodes $\star^s \overline{K_k}$. Its L-spectrum is $\mathrm{LSpec}(M_{s,k}) = \{0^{(1)}, (s-1)k^{(sk-k)}, ks^{(s-1)}\}$, where as usual exponents in parenthesis denote multiplicities of the L-values.

To study the corresponding L-polytopes, we decompose $M_{s,k}$ as the join of s empty graphs on k nodes. The empty graph has $\mathrm{LSpec}(\overline{K_k}) = \{0^{(k)}\}$, so that its only L-polytope is $Q_0(\overline{K_k}) = \Delta_{k-1}$. Now Proposition 10 yields

$$Q_0(M_{s,k}) = \{0\}^{(sk)}; \quad Q_{(s-1)k}(M_{s,k}) = \bigoplus^{s} \Delta_{k-1}; \quad Q_{sk}(M_{s,k}) = \Delta_{s-1}^{(k)}.$$

For example, $Q_{2(s-1)}(M_{s,2})$ is the s-dimensional crosspolytope, and $Q_k(M_{2,k})$ is the cyclic polytope $\mathcal{C}(2k, 2k-2)$ with $2k$ vertices in dimension $2k-2$.

5 Line Graph

The line graph $G_l := \mathrm{line}(G)$ of a graph G with n nodes and m edges is the graph that has one node for each edge of G, and where two nodes are adjacent when the corresponding edges in G have a vertex in common. Let A and L be the adjacency and Laplacian matrices of G, and A_l and L_l those of G_l. Moreover, let M be the $n \times m$ undirected incidence matrix[5] of G, and let D be the degree matrix of G. Then,

$$MM^\top = D + A; \qquad M^\top M = 2\mathbf{I} + A_l.$$

If G is d-regular, then G_l is $(2d-2)$-regular and we have

$$L = 2d\mathbf{I} - MM^\top; \qquad L_l = 2d\mathbf{I} - M^\top M.$$

From now to the end of the section we restrict our attention to d-regular graphs. In this case, the L-values (and their multiplicities) of G and its line graph G_l coincide, except possibly for $\lambda = 2d$. Let $\lambda \in \mathrm{LSpec}(G_l)$, and let

[5] Its entries are the absolute value of those of the directed incidence matrix.

$\mathrm{LE}_\lambda(G)$ and $\mathrm{LE}_\lambda(G_l)$ be the corresponding L-spaces of G and G_l. We distinguish two cases:
$\boldsymbol{\lambda \neq 2d}$. In this case, $\mathrm{LE}_\lambda(G)$ and $\mathrm{LE}_\lambda(G_l)$ are related [5] by

$$\mathrm{LE}_\lambda(G_l) = M^\top \, \mathrm{LE}_\lambda(G); \qquad \mathrm{LE}_\lambda(G) = M \; \mathrm{LE}_\lambda(G_l).$$

Because $M^\top \mathbf{1} = 2\mathbf{1}$ and $M\mathbf{1} = d\mathbf{1}$, the same relations hold for LE^0_λ, and hence

$$Z_\lambda(G_l) = M^\top Z_\lambda(G); \qquad Z_\lambda(G) = M \, Z_\lambda(G_l). \tag{2}$$

How does this operation affect the eigenpolytopes? $Q_\lambda(G_l)$ is the convex hull of the rows of $M^\top Z_\lambda(G)$. Each row of $M^\top$ corresponds to an edge e of G and is $e_i + e_j$, where i and j are the two endpoints of e. Therefore the row of $M^\top Z_\lambda(G)$ indexed by e is the sum of the two rows of $Z_\lambda(G)$ indexed by the endpoints of e. After rescaling, this corresponds to taking their midpoint.

In short, to construct $Q_\lambda(G_l)$, one must take $Q_\lambda(G)$, draw all segments that connect points representing adjacent nodes in G, and take the convex hull of the middle points of all these segments. In the graphs studied in [8], which coincide with the 1-skeleton of one of its L-polytopes, this gives the geometrical operation of *deep vertex truncation*[6].

$\boldsymbol{\lambda = 2d}$. It is an L-value of G if and only if G has a bipartite connected component. In this case, its multiplicity is the number b of bipartite connected components. If $d \neq 1$ then $2d \in \mathrm{LSpec}(G_l)$, with multiplicity $m - n + b$, and it can be easily shown that $\mathrm{LE}_{2d}(G_l) = \ker(M)$.

We can construct a basis of $\ker M$ using the even closed walks of G. Let $\mathcal{C} = (v_0, v_1, \ldots, v_{2k-1}, v_0)$ be a closed walk on G with an even number of steps, with no repeated edges[7]. For each such cycle of G, we create the vector $v_\mathcal{C}$ assigning 1 to the entries corresponding to edges (v_{2l}, v_{2l+1}), -1 to the edges (v_{2l-1}, v_{2l}), and leaving 0 everywhere else. We can see it as walking along $\mathcal{C}$ and alternately painting its edges with 1 and -1. As each node is incident to the same number of edges with 1 and -1, $v_\mathcal{C}$ belongs to $\ker(M)$.

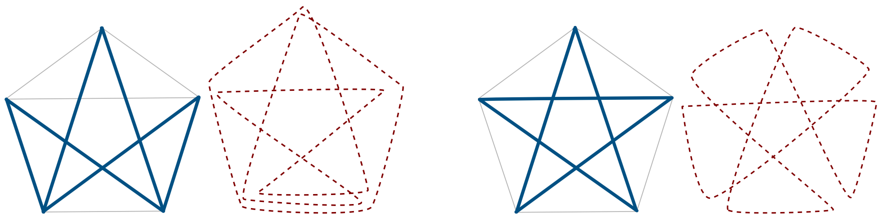
The space of even cycles is an $(m-n+b)$-dimensional subspace of the *cycle space* of G. Hence, taking a basis of the even cycles we have found a complete basis of $\ker(M)$. In particular, we can view each eigenvector in $\mathrm{LE}_{2d}(G_l)$, and by Proposition 7 even each face of $Q_{2d}(G_l)$, as an even closed walk on G.

Example 2 (The Petersen Graph). The Petersen Graph $\mathbf{P} = \overline{\mathrm{line}(K_5)}$, with 10 vertices and L-spectrum $\mathrm{LSpec}(\mathbf{P}) = \{0^{(1)}, 5^{(4)}, 2^{(5)}\}$, is one of the most well known graphs. We study the L-polytope for each non-zero L-value λ in turn. We exploit the relation $\overline{\mathbf{P}} = \mathrm{line}(K_5)$, denoting $\overline{\lambda} = n - \lambda$ the L-values of $\overline{\mathbf{P}}$, and making use of the fact that K_5 is regular with $d = 4$:

[6] For example, this produces the cuboctahedron and icosidodecahedron from the 3-cube and the dodecahedron, respectively.

[7] We construct a basis without using repeated edges, but they will arise from sums of walks.

- $\lambda = 5$: Because $\overline{\lambda} \neq 2d$, we obtain $Z_5(\mathbf{P}) = Z_5(\overline{\mathbf{P}}) = Z_5(\text{line}(K_5)) = M^\top Z_5(K_5)$, the last equation by (2). Since $Q_5(K_5) = \Delta_4$ with skeleton K_5, the corresponding L-polytope is the deep vertex truncation[8] $Q_5(\mathbf{P}) = \text{dvt}(\Delta_4) = \Delta_4(2)$, a hypersimplex. In fact, the relation $Q_n(\text{line}(K_n)) = \Delta_n(2)$ holds in general: the matrix M contains all the combinations $u_{ij} = e_i + e_j$, which lie in the hyperplane $\langle x, \mathbf{1}\rangle = 2$ in $\mathbb{R}^5$; the rows of $Z_n(\text{line}(K_n))$ consist in applying the linear map $Z_n(K_n)^\top$ on each u_{ij}; but the map $Z_n(K_n)^\top$ is just the orthogonal projection along $\mathbf{1}$, and we are just translating the hyperplane that contains the points.
- $\lambda = 2$: Here, $\overline{\lambda} = 2d$, so we can identify the facets of $Q_2(\mathbf{P}) = Q_8(\text{line}(K_5))$ with even closed walks on K_5. This polytope is a 5-dimensional, 2-neighborly 0/1-polytope, and according to Aichholzer's classification [1] has the maximal number of vertices among all such polytopes. Its complete f-vector is $(10, 45, 90, 75, 22)$. There are two symmetry classes of facets, one consisting of 10 facets with 6 vertices, and the other of 12 simplices with 5 vertices. They are determined by the following even closed walks:



In this picture, dark wide lines highlight the edges belonging to the facet, while the dashed line represents the corresponding closed walk.

6 Cartesian Product

The cartesian product $G = G_1 \times G_2$ of $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ has vertex set $V(G) = V_1 \times V_2$ and edge set $\big(V(G_1) \times E(G_2)\big) \cup \big(E(G_1) \times V(G_2)\big)$. The Laplacian of G is $L = L_1 \otimes \mathbf{I} + \mathbf{I} \otimes L_2$, where L_i is the Laplacian of G_i, and $\otimes$ represents the Kronecker product of matrices:

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix},$$

where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$, so that $A \otimes B \in \mathbb{R}^{mp\times nq}$.

For any L-vectors $v \in \text{LE}_\lambda(G_1)$ and $w \in \text{LE}_\mu(G_2)$, $v \otimes w \in \text{LE}_{\lambda+\mu}(G)$. Indeed, it is well known that $\text{LSpec}(G) = \text{LSpec}(G_1) + \text{LSpec}(G_2)$, and that for $\theta \in \text{LSpec}(G)$, $\text{LE}_\theta(G) = \bigoplus_{\lambda_i+\mu_j=\theta} \text{LE}_{\lambda_i}(G_1) \otimes \text{LE}_{\mu_j}(G_2)$. So we can

[8] Apparently, this polytope was originally discovered by Gosset [10], who called it *Tetroctahedric*, because its facets are tetrahedra and octahedra.

construct a basis of $\mathrm{LE}_\theta(G)$ by joining together bases of each $\mathrm{LE}_{\lambda_i}(G_1) \otimes \mathrm{LE}_{\mu_j}(G_2)$. In terms of matrices, this means that for each pair $\lambda_i + \mu_j = \theta$ we build the matrices $Z_{\lambda_i}(G_1) \otimes Z_{\mu_j}(G_2)$. Whenever neither λ_i nor μ_j is 0, then Z_θ is just the concatenation of these matrices. If one L-value is 0, we have not added yet the columns $\mathbf{1} \otimes w_j$ (or $v_i \otimes \mathbf{1}$) which are also valid basic L-vectors. To overcome this problem, we just need to use $Z'_0(G_i)$ instead of $Z_0(G_i)$.

When there is only one combination such that $\lambda + \mu = \theta$, then $Z'_\theta(G) = Z'_\lambda(G_1) \otimes Z'_\mu(G_2)$. The fact that transposition is distributive over $\otimes$ says that points of $Q_\theta(G)$ are products of points of $Q_\lambda(G_1)$ and $Q_\mu(G_2)$. This motivates the following definition.

Definition 2 (Kronecker Product of sets). *Let $X \subset \mathbb{R}^d$ and $Y \subset \mathbb{R}^{d'}$ be subsets of $\mathbb{R}^d$ and $\mathbb{R}^{d'}$. We define the Kronecker product of X and Y as*

$$X \otimes Y = \mathrm{conv}(x \otimes y \mid x \in X, y \in Y) \subset \mathbb{R}^{dd'},$$

where $\otimes$ represents the Kronecker product of the vectors.

Notice that $X \otimes Y = \mathrm{conv}\, X \otimes \mathrm{conv}\, Y$ by bilinearity; in particular, the Kronecker product of polytopes is the convex hull of the Kronecker product of their vertex sets.

When there is more than one possible combination $\lambda_i + \mu_j = \theta$, we need to concatenate the corresponding matrices $Z'_{\lambda_i}(G_1) \otimes Z'_{\mu_j}(G_2)$.

Definition 3 (Composition of points and point configurations). *The* composition *of m points $x_i \in \mathbb{R}^{d_i}$, $i = 1, \ldots, m$, is* $\mathrm{concat}(x_1, \ldots, x_m) := (x_1, \ldots, x_m) \in \mathbb{R}^{\sum d_i}$. *The composition of labelled point configurations $X_i = (x_{ij} : j \in [n]) \subset \mathbb{R}^{d_i}$, $i = 1, \ldots, m$, of the same cardinality is the labelled point configuration* $\mathrm{concat}(X_1, \ldots, X_m) = \big(\mathrm{concat}(x_{1j}, \ldots, x_{mj}) : j \in [n]\big) \subset \mathbb{R}^{\sum d_i}$.

Taking these definitions into account, we can describe the L-polytopes of products of graphs.

Proposition 11. *Let G_1 and G_2 be graphs with L-spectra $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$. Identifying the matrices $Z'_{\lambda_i}(G_1)$ and $Z'_{\mu_j}(G_2)$ with their sets of rows, and using the labelling induced by the indices of the nodes of the G_i, the L-polytope of $G_1 \times G_2$ corresponding to $\lambda + \mu \neq 0$ is*

$$Q_{\lambda+\mu}(G_1 \times G_2) = \mathrm{conv}\left\{ \operatorname*{concat}_{\lambda_i+\mu_j=\lambda+\mu} Z'_{\lambda_i}(G_1) \otimes Z'_{\mu_j}(G_2) \right\},$$

When $\lambda = \mu = 0$, the L-polytope is $Q_0(G_1 \times G_2) = \Delta_{\kappa_1\kappa_2 - 1}$.

Proof. This follows from $\mathrm{LE}_\theta(G) = \bigoplus_{\lambda_i+\mu_j=\lambda+\mu} \mathrm{LE}_{\lambda_i}(G_1) \otimes \mathrm{LE}_{\mu_j}(G_2)$. □

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