been verified for $n = 2$ and also the weaker inequality $G_\Lambda(K) \le \lfloor 2/\lambda_1(K,\Lambda)\rfloor^n$ was shown. In [5] it was proven that

$$G_\Lambda(K) < 2^{n-1} \prod_{i=1}^{n} \left\lfloor \frac{2}{\lambda_i(K,\Lambda)} \right\rfloor .$$

For more information on bounds on the lattice point enumerator as well as for references of the presented inequalities we refer to the survey [4] and the book [3].

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Basis expansions and roots of Ehrhart polynomials

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(joint work with M. Beck, J. De Loera, M. Develin, and R. P. Stanley)

The Ehrhart polynomial i_P of a d-dimensional lattice polytope $P \subset \mathbb{R}^d$ is usually written in the *power basis* of the vector space of polynomials of degree d:

$$i_P(n) \ = \ \sum_{i=0}^{d} c_i \, n^i \, .$$

In this talk, we argued that comparing this representation with the basis expansion

$$i_P(n) \ = \ \sum_{i=0}^{d} a_i \binom{n+d-i}{d}$$

yields useful information about i_P. Note that in the literature sometimes the notation h_i^* is used instead of a_i.

(1) The inequalities $a_i \ge 0$ (that follow from the fact that i_P is the Hilbert function of a semi-standard graded Cohen-Macaulay algebra) are used to derive all other known inequalities [1] [2] [3] [6] for the coefficients c_i, with the exception of the inequality $c_{d-1} \ge \frac{1}{2} \cdot$ (normalized surface area) that comes from geometry.

(2) Some of the coefficients in this representation have nice interpretations:

$$\begin{aligned} a_1 &= i_P(1) - (d+1)\,, \\ a_2 &= i_P(2) - (d+1) i_P(1)\,, \\ a_{d-1} &= (-1)^d \big(i_P(-2) - (d+1)\, i_P(-1)\big)\,, \\ a_d &= (-1)^d\, i_P(-1) \;=\; \#\{\text{inner points}\}\,. \end{aligned}$$

(3) Expressing the Ehrhart polynomial in this basis makes it easy to prove relations such as

$$\binom{d}{\ell}\, \Delta^k i_P(0) \;\leq\; \binom{d}{k}\, \Delta^\ell i_P(0),$$

where $\Delta^k i_P$ is the k-th difference of i_P.

We also present new linear inequalities satisfied by the coefficients of Ehrhart polynomials and relate them to known inequalities.

Next, we investigated the roots of Ehrhart polynomials:

Theorem.

(a) *The complex roots of Ehrhart polynomials of lattice d-polytopes are bounded in norm for fixed d.*

(b) *All real roots of Ehrhart polynomials of d-dimensional lattice polytopes lie in the half-open interval $[-d, \lfloor d/2 \rfloor)$. For $d = 4$, the real roots lie in the interval $[-4, 1)$.*

(c) *For any positive real number t, there exists an Ehrhart polynomial of sufficiently large degree with a real root strictly larger than t. In fact, for every d there is a d-dimensional $0/1$-polytope whose Ehrhart polynomial has a real zero α_d such that $\lim_{d\to\infty} \alpha_d/d = 1/(2\pi e) = 0.0585\cdots$. In particular, the upper bound in* (b) is tight up to a constant.

An experimental investigation of the Ehrhart polynomials of cyclic polytopes leads to the following conjecture:

Conjecture. *Let $P = C_d(n)$ be any cyclic polytope realized with integer vertices on the standard moment curve $t \mapsto (t, t^2, \ldots, t^d)$ in $\mathbb{R}^d$. Then the Ehrhart polynomial of P reads*

$$i_P(n) \;=\; \sum_{k=0}^{d} \mathrm{vol}_k\big(\pi_k(P)\big)\, n^k,$$

where $vol_d(\cdot)$ is the standard d-dimensional volume, $vol_k(\cdot)$ for $k = 0, 1, \ldots, d-1$ is the normalized lattice volume, and $\pi_k : \mathbb{R}^d \to \mathbb{R}^k$ is the projection to the first k coordinates.

Problem. *Find an explicit expression for the Todd class of the toric variety associated to the outer normal fan of $P = C_d(n)$.*

This problem has been solved for $0 \leq d \leq 3$ by using the expressions for the codimension ≤ 3 parts of the Todd class from [4] and the techniques of [5]. In particular, the conjecture has been proven for $0 \leq d \leq 3$.

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On normal polytopes without regular unimodular triangulations

Francisco Santos

A lattice polytope $P \subset \mathbb{R}^d$ is *normal* if $nP \cap \mathbb{Z}^d = n(P \cap \mathbb{Z}^d)$ for every $n \in \mathbb{N}$. Normal polytopes arise naturally in algebraic geometry and in combinatorial optimization [8]. Starting with [6], it has been repeatedly observed that normality of a polytope is closely related to its being covered by unimodular simplices. More precisely, from [6, 3, 5] one can extract the following sequence of properties, each of which implies the next one. In all of them, $S = P \cap \mathbb{Z}$. A simplex is unimodular if its vertices are a basis for the affine lattice $\mathbb{Z}^d$. A triangulation is unimodular if all its simplices are.

(1) All simplices with vertices in S are unimodular. (P is *totally unimodular*).
(2) P is *compressed*. (All its *pulling* triangulations are unimodular).
(3) P has a unimodular *regular triangulation*.
(4) P has a unimodular triangulation.
(5) P has a unimodular *binary cover*. This is a property introduced by Firla and Ziegler [3], whose significance comes from the fact that it is much easier to check algorithmically than any other of the properties (3) to (8).
(6) P has a *unimodular cover*. (Every $x \in P$ lies in some unimodular simplex).
(7) For every n, every integer point in nP is an integer positive combination of an affinely independent subset of points of S. (This is called the *Free Hilbert Cover* property in [1])
(8) For every n, every integer point in nP is an integer positive combination of at most $d+1$ points of S. (The *Integral Carathéodory Property* of [3]).
(9) For every n, every integer point in nP is an integer positive combination of an affinely independent subset of points of S. (P is normal).

It is very easy to find examples that prove $3 \not\Rightarrow 2$ and $2 \not\Rightarrow 1$, but not so easy for any of the other implications. Ohsugi and Hibi [5] found the first normal polytope without regular unimodular triangulations, which turned out to give $4 \not\Rightarrow 3$. Then Bruns and Gubeladze [1] proved $8 \Leftrightarrow 7$ and found an example for $9 \not\Rightarrow 8$ [2]. The