to be *positive* if $C(i,j) \geq 0$ for every pair $(i,j)$ and $C(i,j)$ is strictly positive for at least one such pair. We prove that the maximum number of ones in an $n \times n$ zero-one matrix containing no positive orthogonal cycle is $O(n^{4/3})$. The order of magnitude of this bound cannot be improved.

Our results lead to a new proof of the celebrated theorem of Spencer, Szemerédi, and Trotter [5] stating that the number of times that the unit distance can occur among $n$ points in the plane is $O(n^{4/3})$. This is the first proof that does not use any tool other than a forbidden pattern argument. We present another geometric application, where the forbidden pattern $P$ is the adjacency matrix of an acyclic graph. A *hippodrome* is a $c \times d$ rectangle with two semicircles of diameter $d$ attached to its sides of length $d$. Improving a result of Efrat and Sharir [2], we show that the number of "free" placements of a convex $n$-gon in general position in a hippodrome $H$ such that simultaneously three vertices of the polygon lie on the boundary of $H$, is $O(n)$. This result is related to the Planar Segment-Center Problem.

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# Bounding the volume of facet-empty lattice tetrahedra


Julian Pfeifle

(joint work with Han Duong, Christian Haase, Bruce Reznick)


A lattice polytope is *empty* if it contains no lattice points except for its vertices. Already in 1957, Reeve [10] noticed that empty three-dimensional lattice simplices may have unbounded volume. In 1982, Zaks, Perles & Wills [12] constructed a family of $d$-dimensional lattice simplices, each member of which contains $k$ lattice points in total and has the rather large volume

$$\frac{k+1}{d!}\, 2^{2^{d-1}-1}\,.$$

In the following year, Hensley [4] proved that the volume of any $d$-dimensional lattice polytope containing $k \geq 1$ lattice points in its interior is bounded by a constant that depends only on $d$ and $k$. By sharpening Hensley's basic diophantine approximation lemma, Lagarias & Ziegler [6] in 1991 improved his bound and showed that the maximal volume $V(d,k)$ of a $d$-dimensional lattice polytope with $k$ interior lattice points is bounded by

$$V(d,k) \;\le\; k\big(7(k+1)\big)^{d2^{d+1}},$$

which for $d=3$ reads

$$V(3,k) \;\le\; k\big(7(k+1)\big)^{48}.$$

This bound was further sharpened by Pikhurko [8], who was able to prove an upper bound with only a linear dependence on $k$:

$$\begin{aligned} V(d,k) &\;\le\; (8d)^d \cdot 15^{d\cdot 2^{2d+1}} \cdot k\,,\\ V(3,k) &\;\le\; 24^3 \cdot 15^{384}\, k\,. \end{aligned}$$

A *facet-empty* or *clean* lattice polytope is a lattice polytope whose only lattice points on the boundary are its vertices. In our talk we focused on the special class of *facet-empty $k$-point lattice tetrahedra*, which contain exactly $k+4$ lattice points, $k$ of them in the relative interior. It is known [10], [11] that via unimodular transformations any facet-empty lattice tetrahedron may be brought into the normal form

$$T_{a,b,n} \;=\; \operatorname{conv}\big\{(0,0,0),(1,0,0),(0,1,0),(a,b,n)\big\},$$

where $(a,b,n)=(0,0,1)$, or $n\ge 2$, $0\le a,b\le n-1$, and $\gcd(a,n)=\gcd(b,n)=\gcd(1-a-b,n)=1$. Note that $\operatorname{vol} T_{a,b,n}=n$.

We sketched a proof of the following theorem, which significantly improves Pikhurko's bound for this special family of 3-dimensional lattice polytopes:

**Theorem 1.** *The maximal (normalized) volume of a clean lattice tetrahedron $\Delta$ with $k\ge 1$ interior lattice points is*

$$\operatorname{vol}\Delta \;\le\; 12k+8\,.$$

*This bound is attained by the family of clean $k$-point lattice tetrahedra*

$$\big\{T_{3,6k+1,12k+8} : k\ge 1\big\}.$$

The overall structure of the proof is as follows. We first show that the number $k$ of interior lattice points of $\Delta$ equals the number of times that the sum

$$f(z) \;=\; \left\lfloor \frac{(a+b-1)z}{n} \right\rfloor - \left\lfloor \frac{az}{n} \right\rfloor - \left\lfloor \frac{bz}{n} \right\rfloor$$

equals 1, as $z$ takes on integer values between 1 and $n$.

Next, we use that $f(z)\in\{0,\pm 1\}$ and $f(n-z)=-f(z)$ for $1\le z\le n-1$ to express $k$ as half the second moment of the sequence $\big(f(z): 1\le z\le n-1\big)$. This second moment is then expressed using *Dedekind sums* $s(a,n)$:

**Proposition 2.** *Set $c=1-a-b \bmod n$ and let $aa'=bb'=cc'=1 \bmod n$. Then*

$$\frac{1}{2}\sum_{z=1}^{n-1} f(z)^2 = \frac{n-3}{6}+\frac{1}{3n}-s(c,n)-s(a,n)-s(b,n)+s(a'b,n)+s(a'c,n)+s(b'c,n)\,.$$

**Remark 3.** *After obtaining this expression, we realized that we should have expected the appearance of Dedekind sums in this expression, because they appear in a formula of Pommersheim* [9] *for the Ehrhart polynomial of lattice tetrahedra. In fact, our derivation of Proposition 2 yields an elementary proof of this formula in the case of facet-empty tetrahedra; in particular, we do not need to evaluate the Todd class of the associated toric variety.*

There are now at least two ways to complete the proof of the theorem. First, we can express each Dedekind sum $s(a, n)$ as essentially the sum of digits of the *negative-regular continued fraction expansion* of $n/(n-a)$; see [1], [5], [7], [9].

**Proposition 4.** *Let* $n/(n-a) = b_1 - 1/(b_2 - 1/(\cdots - 1/b_r))$ *be the* negative-regular continued fraction expansion *of* $n/(n-a)$*, where* $0 \le a < n$ *are coprime and we require* $b_i \ge 2$ *for* $i = 1, 2, \ldots, r$*. Moreover, define* $a' \in \mathbb{N}$ *by* $aa' = 1 \bmod n$ *and* $0 \le a' < n$*. Then*

$$s(a, n) \;=\; \frac{1}{12}\left(\sum_{i=1}^{r}(3 - b_i) + \frac{a + a'}{n} - 2\right).$$

We then use a detailed analysis of the behavior of digit sums of negative-regular continued fraction expansions to bound $k$ from below. This approach requires a fairly substantial amount of case distinctions.

The other way of proving the theorem is not by passing to continued fractions, but instead by bounding the individual Dedekind sums directly and controlling the interaction between the six summands in Proposition 2. This alternative proof of the theorem is still work in progress; we have reason to hope that it requires a substantially smaller number of case distinctions.

It has been observed several times that $s(a, n)$ changes drastically if $a$ is close to numbers of the form $n \cdot c/d$. The behaviour of Dedekind sums in the neighborhood of such values was studied by Girstmair [2], [3], who introduced the notion of "$F$-neighbors". First, define a *Farey point* to be a real number of the form $n \cdot c/d$, where $d$ is "small"; more precisely, $1 \le d \le \sqrt{n}$, $0 \le c \le d$ and $\gcd(c, d) = 1$. (Note that this $c$ is different from the one used before.) The denominator $d$ is called the *order* of the Farey point. The *$F$-neighbors of order $d$* are all real numbers $x$ such that $0 \le x \le n$ and $|x - n \cdot c/d| \le \sqrt{n}/d^2$, for some $0 \le c \le d$ with $\gcd(c, n) = 1$. An integer $x \in [0, n]$ that is not an $F$-neighbor of order $1 \le d \le \sqrt{n}$ is called an *ordinary integer.*

**Proposition 5.** [3, Theorem 1 and Section 3]

(a) *If* $n \ge 15$ *and* $x \in [0, n]$ *is an ordinary integer, then* $|s(x, n)| \le \frac{1}{4}\sqrt{n} + \frac{5}{12}$.

(b) *Let* $x$ *be a* $F$*-neighbor of order* $d$*, let* $n \cdot c/d$ *be the corresponding* $F$*-point, and put* $q = xd - cn$*, so that* $|q| \le \sqrt{n}/d$*. Then*

$$s(x, n) \;=\; \frac{n}{12dq} + \frac{1}{12}\,E(d + |q| + 4)\,,$$

*where* $E$ *denotes an error term such that* $|E(z)| \le z$*.*

The next proposition is crucial for analyzing the sum from Proposition 2.

**Proposition 6.** *Let $x = (cn + q)/d$ with $\gcd(c, d) = \gcd(x, n) = 1$ be a Farey neighbor of order $d$ to $cn/d$. Then for any $0 \leq c' < q$ relatively prime to $q$, there exists a parametrization $n = \alpha s + \beta$ with $\alpha, \beta \in \mathbb{Z}$ and $s \in \mathbb{N}$ such that the inverse $x'$ of $x$ modulo $n$ has the form $x' = (c'n + d)/q$.*

*In particular, inverting $x$ leaves the product $dq$ invariant.*

It turns out that we may assume all of the six first arguments $\{a, b, c, a'b, b'c, c'a\}$ of the Dedekind sums in Proposition 2 to be $F$-neighbors. Moreover, by the estimates from Proposition 5 we only need to consider those 20 sets of arguments such that the associated values $d_i q_i$ satisfy $\sum_{i=1}^{6} 1/(d_i q_i) > 1$. We leave the completion of this argument for further study.

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## On Geometric Graphs with no Pair of Parallel Edges

Rom Pinchasi

A *geometric graph* is a graph drawn in the plane with its vertices as points and its edges as straight line segments connecting corresponding points. A topological graph is defined similarly except that its edges are simple Jordan arcs connecting corresponding points. Two edges in a geometric graph are said to be *parallel*, if they are two opposite edges of a convex quadrilateral.

In [2, 3] Katchalski, Last, and Valtr prove a conjecture of Kupits and obtain the following result:

**Theorem 1.** *A geometric graph on $n$ vertices with no pair of parallel edges has at most $2n - 2$ edges.*