

Matricial Realizations of the Solutions of the Carlson Problem *

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Abstract

The Klein theorem asserts the existence of a solution of the Carlson problem when a related Littelwood-Richardson sequence exists. Here we present an explicit construction of these matricial solutions.

1 Introduction

The classical Carlson's problem asks for the existence of a matrix Z such that the matrix

$$A = \begin{pmatrix} A_1 & Z \\ 0 & A_2 \end{pmatrix}$$

has prescribed Jordan form, being prescribed also the blocks A_1 and A_2 . It is easy to see that one can assume A_1 and A_2 nilpotent (Jordan) matrices, so that the prescribed data are the Segre characteristics of A_1 , A_2 and A .

A well-known theorem of T. Klein [5] relates the decomposition of p -modules with the existence of so-called LR-sequences. In the other hand, [4] proves the equivalence between the Carlson problem and the one of invariant factors of the product of polynomial matrices, which in turn is related by [8] with the decomposition of p -modules. Summarizing, one has the well-known theorem 2.1 which reduces the Carlson problem to the existence of LR-sequences.

Recently, the works in [6], [7] have led to complete solutions: see [2] and the review paper [3].

*Partially supported by DGICYT n.PB97-0599-C03-03

Howefer, as long as we know, there are no algorithms to construct explicit solutions Z . Here we present such a construction, based in a geometric proof of theorem 2.1, analogous to the one for pairs of matrices in [1].

As an interesting and immediate consequence of this construction, we obtain 4.3 that all the possible Segre characteristics appear in any neighbourhood of the simplest solutions, which we call “marked” 3.3. This is to say, they appear by perturbing the elementary marked solutions.

Section 2 is devoted to the geometric proof of the Klein theorem, which will be the pattern of our construction in 4. Previously, in section 3, we precise some definitions and notation. Then, in section 4, we detail the three steps to construct explicitly matricial solutions, we present an example, and we note the referred corollary.

2 A Constructive Proof of the Klein Theorem

We recall that a partition

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell(\alpha)}, 0, \dots)$$

is a finite non increasing sequence of non-negative integers

$$\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_{\ell(\alpha)} > 0$$

where $\ell(\alpha)$ is called the length. We note

$$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_{\ell(\alpha)}$$

(named its weight).

Its conjugate partition $\alpha^* = (\alpha_1^*, \alpha_2^*, \dots)$ is defined by means of

$$\alpha_j^* = \#\{1 \leq i \leq \ell(\alpha) : \alpha_i \geq j\}$$

where the symbol $\#$ means “cardinal”. Notice that $\alpha_1^* = \ell(\alpha)$, $\ell(\alpha^*) = \alpha_1$, $|\alpha^*| = |\alpha|$, $(\alpha^*)^* = \alpha$.

For example, if A is a nilpotent matrix, its Segre characteristic is the conjugate partition α^* of the one defined by

$$\alpha = (\dim \text{Ker } f, \dim \text{Ker } f^2 - \dim \text{Ker } f, \dim \text{Ker } f^3 - \dim \text{Ker } f^2, \dots)$$

With this notation, the Klein theorem can be enounced as follows

Theorem 2.1 [5] *Let be three partitions α , γ , β with $|\alpha| = n$, $|\gamma| = d$, $|\beta| = n - d$. The following conditions are equivalent:*

(I) For any nilpotent matrices $A_1 \in \mathbb{C}^{d \times d}$ and $A_2 \in \mathbb{C}^{(n-d) \times (n-d)}$ having Segre characteristic γ^* and β^* respectively, there is a matrix $Z \in \mathbb{C}^{d \times (n-d)}$ such that the matrix $A = \begin{pmatrix} A_1 & Z \\ 0 & A_2 \end{pmatrix}$ has Segre characteristic α^* .

(II) There is a finite sequence of partitions $\gamma^0, \gamma^1, \dots, \gamma^s$ ($s = \ell(\beta)$) such that $\gamma^0 = \gamma$, $\gamma^s = \alpha$, and for all $i, j \geq 1$:

$$(a) \quad |\gamma^j| - |\gamma^{j-1}| = \beta_j$$

$$(b) \quad \gamma_i^j \geq \gamma_i^{j-1} \geq \gamma_{i+1}^j$$

$$(c) \quad \sum_{\ell \leq i} (\gamma_\ell^{j+1} - \gamma_\ell^j) \leq \sum_{\ell \leq i-1} (\gamma_\ell^j - \gamma_\ell^{j-1})$$

taking $\gamma_0^{j-1} = 0, \forall j$.

Remark 2.2 (b) and (c) imply that $\gamma_i^j = \gamma_i^1$ if $j \geq 1$.

Proof of theorem 2.1. A geometric approach is presented in [1] by considering an endomorphism $f : X \rightarrow X$ and $W \subset X$ an invariant subspace (i.e., $f(W) \subset W$), so that the matrix of f has the form $\begin{pmatrix} A_1 & Z \\ 0 & A_2 \end{pmatrix}$ in any basis B of X adapted to W (i.e., $B \cap W$ is a basis of W), where A_1 and A_2 are the matrices of the natural endomorphisms $\hat{f} : W \rightarrow W$ and $f : X/W \rightarrow X/W$ respectively, in the bases induced by B in a natural way. Then, a proof of (I) \Rightarrow (II) is sketched.

Here, we are mainly interested in the converse: if (II) holds, we will build a nilpotent endomorphism $f : X \rightarrow X$ ($X = \mathbb{C}^n$) with Segre characteristic α^* and an invariant subspace $W \approx \mathbb{C}^d$ such that $\hat{f} : W \rightarrow W$ and $f : \frac{X}{W} \rightarrow \frac{X}{W}$ have γ^* and β^* as Segre characteristic respectively.

In order to do it we will build the following diagram

$$\begin{array}{ccccccc}
 W & = & W^0 & \subset & W^1 & \subset & \dots & \subset & X \\
 & & \parallel & & \parallel & & & & \parallel \\
 W_m & = & W_m^0 & \subset & W_m^1 & \subset & \dots & \subset & W_m^s \\
 & & \cup & & \cup & & & & \cup \\
 & & \vdots & & \vdots & & & & \vdots \\
 & & \cup & & \cup & & & & \cup \\
 W_1 & = & W_1^0 & \subset & W_1^1 & = & \dots & = & W_1^s \\
 & & \cup & & \cup & & & & \cup \\
 & & W_0^0 & = & W_0^1 & = & \dots & = & W_0^s \\
 & & \parallel & & \parallel & & & & \parallel \\
 & & 0 & & 0 & & & & 0
 \end{array}$$

with $W_i^j = W_{i+1}^j \cap W_i^{j+1}$ and $\dim W_i^j - \dim W_{i-1}^j = \gamma_i^j$.

We construct these subspaces, by recurrence, over $0 \leq j \leq s$, $1 \leq i \leq \ell(\gamma^j)$: we take

$$W_i^j = (W_i^{j-1} + W_{i-1}^j) \oplus [B_i^j]$$

where $[B_i^j]$ is the subspace spanned by a family of linear independent vectors $B_i^j = \{e_{i,k}^j\}_{1 \leq k \leq \gamma_i^j - \gamma_i^{j-1}}$ contained in some supplementary subspace of $W_i^{j-1} + W_{i-1}^j$. We will write $\delta_i^j = \gamma_i^j - \gamma_i^{j-1}$.

Notice that

$$W_i^j = \bigoplus_{\substack{0 \leq h \leq j \\ 1 \leq \ell \leq i}} [B_\ell^h].$$

We define a map $\widehat{f} : W \rightarrow W$ by means of

$$\begin{aligned} \widehat{f}(e_{1,k}^0) &= 0 \\ \widehat{f}(e_{i+1,k}^0) &= e_{i,k}^0, \quad i \geq 1 \end{aligned}$$

and it is obvious that γ^* is his Segre characteristic.

Then, we define two extensions $f_*, f^* : X \rightarrow X$ of \widehat{f} keeping the following conditions for all $0 \leq j \leq s-1$, $1 \leq i \leq \ell(\gamma^j) - 1$.

- i) $f_*(e_{i+1,k}^{j+1}) \in \bigcup_{0 \leq h \leq j} B_i^h$, $\text{Ker } f_* = W_1^1$.
- ii) $f^*(e_{i+1,k}^{j+1}) \in \bigcup_{1 \leq \ell \leq i} B_\ell^j$, $\text{Ker } f^* = W_1^1$.
- iii) $f_*(e_{i+1,k}^1) = f^*(e_{i+1,k}^1) = e_{i,k+\gamma_{i+1}}^0$, for all $k \geq 1$.
- iv) $f_*(e_{i+1,k}^{j+1}) = f^*(e_{i+1,k}^{j+1}) = e_{i,k}^j$, if $k \leq \min(\delta_{i+1}^{j+1}, \delta_i^j)$.

Condition i) is possible if

$$\text{card} \left(\bigcup_{0 \leq h \leq j} B_i^h \right) \geq \text{card} \left(\bigcup_{0 \leq h \leq j+1} B_{i+1}^h \right),$$

but $\text{card} \left(\bigcup_{0 \leq h \leq j} B_i^h \right) = \dim W_i^j - \dim W_{i-1}^j = \gamma_i^j$, so that the inequality holds because of the second inequality of (b). In an analogous way one sees that condition ii) is possible because of (c).

Finally, let $f \doteq \frac{1}{2}(f_* + f^*)$. It is obviously an extension of \widehat{f} and the sufficiency would be proved if

$$\begin{cases} \text{Ker } f^i = W_i^s \\ f^{-j}(W) = W^j \end{cases}$$

Previously, we note that

$$e_{i,k}^j \in f_*(B) \cap f^*(B)$$

if and only if

$$e_{i,k}^j = f_*(e_{i+1,k}^{j+1}) = f^*(e_{i+1,k}^{j+1}) \quad \text{with} \quad k \leq \min(\delta_{i+1}^{j+1}, \delta_i^j)$$

or there are unique $h, \ell > 1$ such that $e_{i,k}^j \in f_*(B_{i+1}^{j+h}) \cap f^*(B_{i+\ell}^{j+1})$.

It is straightforward that $W_i^s \subset \text{Ker } f^i$. We will prove by induction the other inclusion:

If $i = 0$, $\text{Ker } f^0 = W_0^s = \{0\}$.

If $x \notin W_i^s$ $x = x_0 + \sum_{\substack{i < \ell \leq i_0 \\ 0 \leq j \leq s}} \lambda_{\ell,k}^j e_{\ell,k}^j$, with $x_0 \in W_i^s$ and $\lambda_{i_0,k_0}^{j_0} \neq 0$ for any j_0 and k_0 .

Then $\lambda_{i_0,k_0}^{j_0} f_*(e_{i_0,k_0}^{j_0}) \notin W_{i-1}^s$ can not be cancelled by any image of the other components of x and therefore $f(x) \notin W_{i-1}^s$.

Hence, by the induction hypothesis, $f(x) \notin \text{Ker } f^{i-1}$ and we can conclude that $x \notin \text{Ker } f^i$. We have proved that $\text{Ker } f^i = W_i^s$.

We would prove the other equality in an analogous way.

3 Matricial Realizations

The constructive proof in section 2 allows to obtain explicit solutions Z in (I) of the theorem 2.1, when condition (II) is verified. Let precise some definitions and notation before doing that in section 4,

Definition 3.1 *If three partitions α, β, γ verify the conditions in theorem 2.1, we say that α is Carlson compatible with β, γ .*

Then, we call a matrix realization of them any matrix of the form

$$M(\alpha, \beta, \gamma) = \left(\begin{array}{c|c} J(\gamma^*) & Z \\ \hline 0 & J(\beta^*) \end{array} \right)$$

where

$$\begin{aligned} J(\gamma^*) &= \text{diag}(J(\gamma_1^*), J(\gamma_2^*), \dots) \\ J(\beta^*) &= \text{diag}(J(\beta_1^*), J(\beta_2^*), \dots) \end{aligned}$$

are nilpotent Jordan matrices having Segre characteristic γ^ and β^* respectively, and such that $M(\alpha, \beta, \gamma)$ has Segre characteristic α^* .*

We call it condensed if the only non-zero entries of Z are placed in the rows corresponding to null rows of $J(\gamma^)$. In addition, we say that it is reduced if, in the submatrix of Z corresponding to null columns of $J(\beta^*)$ and to null rows of $J(\gamma^*)$, the only non zero entries are 1-valued and placed in different columns and different rows.*

It is well known (and easy to prove) that a reduced condensed matrix realization always exists:

Lemma 3.2 *If the partition α is Carlson compatible with β, γ , there exist reduced condensed matricial realizations $M(\alpha, \beta, \gamma)$.*

In our case, we will construct explicitly in section 4 one of these realizations $M(\alpha, \beta, \gamma)$, which in addition can be seen as a local perturbation of a so-called (see example 3.3) “marked” one. We recall that an invariant subspace is called marked if there is some Jordan basis of it which can be extended to a Jordan basis of the whole space. In our case, it means that the Jordan blocks for α are obtained by joining those of β and γ , as we precise in the following example.

Example 3.3 Given two partitions β and γ , trivial examples (which we call *marked*) of partitions Carlson compatible with them are obtained taking each $\alpha_1^*, \alpha_2^*, \dots$ either some element of β^* , or some element of γ^* , or the sum of some of each one. More precisely, if σ and τ are permutations of $\{1, 2, \dots, \ell(\alpha^*)\}$, then

$$\alpha_i^* = \beta_{\sigma(i)}^* + \gamma_{\tau(i)}^*$$

where $\beta_{\sigma(i)}^* = 0$ if $\sigma(i) > \ell(\beta^*)$ and $\gamma_{\tau(i)}^* = 0$ if $\tau(i) > \ell(\gamma^*)$.

Then, a reduced condensed matrix realization is obtained by taking Z with the only non-zero entries corresponding to the indices $1 \leq i \leq \ell(\alpha^*)$ such that $\beta_{\sigma(i)}^*, \gamma_{\tau(i)}^* \neq 0$ as follows: for each i of this kind, a 1-valued entry is placed in the column corresponding to the null column of $J(\beta_{\sigma(i)}^*)$ and in the row corresponding to the null row of $J(\gamma_{\tau(i)}^*)$.

For example, for $\beta^* = (3, 1)$, $\gamma^* = (4, 2)$ a marked Carlson compatible partition is $\alpha^* = (5, 4, 1)$. A matrix realization of them is

0	0	0	0				
1	0	0	0				
0	1	0	0				
0	0	1	0				
				0	0		1
				1	0		
0				0	0	0	
				1	0	0	
				0	1	0	
							0

4 Construction of Matricial Realizations

Since in our proof of the sufficiency in section 2 we have constructed an explicit endomorphism f verifying (I) in theorem 2.1, its matrix in a convenient basis will be a matricial

realization of the involved Carlson compatible partition. We organize the algorithm into 3 steps.

Step 1. In order to clarify the construction, we will slightly modify the considered extension f of \widehat{f} (it is immediate that the proof is always valid). For all $1 \leq j \leq s-1$, $1 \leq i \leq \ell(\gamma^j) - 1$, we maintain the conditions:

- i') $f(e_{1,k}^1) = 0$, for all $k \geq 1$.
- ii') $f(e_{i+1,k}^1) = f_*(e_{i+1,k}^1) = f^*(e_{i+1,k}^1) = e_{i,k+\gamma_{i+1}}^0$, for all $k \geq 1$.
- iii') $f(e_{i+1,k}^{j+1}) = f_*(e_{i+1,k}^{j+1}) = f^*(e_{i+1,k}^{j+1}) = e_i^j$, if $k \leq \min(\delta_{i+1}^{j+1}, \delta_i^j)$.

But, for $\delta_i^j < k \leq \delta_{i+1}^{j+1}$, instead of $\frac{1}{2}(f_* + f^*)$ we define

$$\text{iv')} \quad f(e_{i+1,k}^{j+1}) = \lambda f_*(e_{i+1,k}^{j+1}) + f^*(e_{i+1,k}^{j+1}), \text{ where } \lambda \neq 0.$$

Step 2. For the moment, we consider the basis of X , adapted to $W \subset X$, formed by the vectors

$$\{e_{i,k}^j, 0 \leq j \leq s, 1 \leq j \leq \ell(\gamma^j), 1 \leq k \leq \delta_i^j\}$$

ordered as follows: firstly, the basis of W formed by those vectors with $j = 0$, ordered as a Jordan basis of \widehat{f} ; next, the remainder vectors ($1 \leq j \leq s$) ordered in such a way that its classes in X/W form a Jordan basis of $\widetilde{f}^* : X/W \rightarrow X/W$.

It is easy to see that the matrix of f^* , in this basis, is a marked matrix Carlson compatible with β, γ . Then, the matrix of f result by adding λ entries corresponding to condition iv') of the definition of f (see step 1). Therefore, the matrix of f results from a marked matrix

$$\left(\begin{array}{c|c} J(\gamma^*) & Z^* \\ \hline 0 & J(\beta^*) \end{array} \right)$$

by adding some λ entries placed in columns corresponding to non-null ones of $J(\beta^*)$, and in rows not corresponding to non-null ones of $J(\gamma^*)$.

Step 3. Finally, by means of elementary transformations we reduce to $J(\beta^*)$ the right down block of the matrix of f in step (2). In this process, non zero entries, depending of λ , are added in the right-up block, placed in columns corresponding to non-null ones of $J(\beta^*)$ and in rows corresponding to null ones of $J(\gamma^*)$. Summarizing:

Proposition 4.1 *Given α, β, γ Carlson compatible partitions and a LR-sequence as in (II) of theorem 2.1, a reduced condensed matrix realization*

$$M(\alpha, \beta, \gamma) = \left(\begin{array}{c|c} J(\gamma^*) & Z \\ \hline 0 & J(\beta^*) \end{array} \right)$$

is obtained by means of the aboves steps 1, 2, 3.

Example 4.2 For

$$\begin{aligned}\gamma &= (2, 2, 2, 1, 1) \\ \beta &= (2, 2, 1, 1) \\ \alpha &= (3, 3, 2, 2, 2, 1, 1)\end{aligned}$$

let consider the LR-sequence

$$\begin{aligned}\gamma^1 &= (3, 2, 2, 2, 1) \\ \gamma^2 &= (3, 3, 2, 2, 1, 1) \\ \gamma^3 &= (3, 3, 2, 2, 2, 1, 1)\end{aligned}$$

so that $\delta_i^j = 0$ except for

$$\delta_2^1 = \delta_4^1 = \delta_2^2 = \delta_6^2 = \delta_5^3 = \delta_7^4 = 1.$$

Let be Jordan chains for \widehat{f}

$$\begin{aligned}e_{5,1}^0 &\rightarrow e_{4,1}^0 \rightarrow e_{3,1}^0 \rightarrow e_{2,1}^0 \rightarrow e_{1,1}^0 \rightarrow 0 \\ e_{3,2}^0 &\rightarrow e_{2,2}^0 \rightarrow e_{1,2}^0 \rightarrow 0\end{aligned}$$

In the step 1 we define the extension f by means of:

$$\begin{aligned}\text{i')} \quad f(e_1^1) &= 0. \\ \text{ii')} \quad f(e_4^1) &= e_{3,2}^0. \\ \text{iii')} \quad f(e_2^2) &= e_1^1. \\ \text{iv')} \quad f(e_6^2) &= \lambda e_{5,1}^0 + e_4^1. \\ f(e_5^3) &= \lambda e_4^1 + e_2^2. \\ f(e_7^4) &= \lambda e_6^2 + e_5^3.\end{aligned}$$

where we have written simply e_i^j instead of $e_{i,1}^j$.

In the step 2, we consider the basis

$$\begin{aligned}e_{5,1}^0, e_{4,1}^0, e_{3,1}^0, e_{2,1}^0, e_{1,1}^0 \\ e_{3,2}^0, e_{2,2}^0, e_{1,2}^0 \\ e_7^4, e_5^3, e_2^2, e_1^1 \\ e_6^2, e_4^1\end{aligned}$$

The matrix of f in this basis is

1			λ
1	1		
		1	
			1
0		1	
		1	
0		λ	
		λ	1

In the step 3, from the change of basis

$$\bar{e}_5^3 = e_5^3 + \lambda e_6^2, \quad \bar{e}_2^2 = e_2^2 + 2\lambda e_4^1$$

we obtain the matrix

1		λ^2	λ
1	1		
		1	
			1
0		1	
		1	
0		2λ	
		2λ	1

As an application, notice that if $\lambda \rightarrow 0$ then one obtains a marked matrix realization of the partitions β, γ . Therefore:

Corollary 4.3 *Given partitions β and γ , matrix realizations of any partition Carlson compatible with them exist in any neighbourhood of the marked ones.*

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