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A pseudo-normal form for planar vector fields

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The pseudo-normal form is presented as an alternative to the Birkhoff normal form for a planar vector field with the origin as an equilibrium point. Its convergence is proved for non-zero critical exponents $\lambda, -\lambda$, and some consequences for the center-focus problem are presented.

Key Words: Normal Forms, integrability, Hamiltonian and reversible systems, limit cycles.

1. INTRODUCTION

Given a "planar" general ordinary differential equation (o.d.e.) of the type

$$\begin{cases} \dot{z} = F(z,\theta) & z \in \mathbb{R}^2 \text{ or } \mathbb{C}^2 \\ \dot{\theta} = \omega & \theta \in \mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d, \quad d \geq 0, \end{cases} \tag{1}$$

where F is analytic in z and vanishes for $z = 0$: $F(0,\theta) = 0$ for all $\theta \in \mathbb{T}^d$, an important problem is to find a transformation $(z,\theta) = (\Phi(\zeta,\theta),\theta)$, with $\zeta = (\xi,\eta)$, such that in the new variables the system takes the simpler *normal form*

$$\begin{cases} \dot{\xi} = \xi A_1(\xi\eta) \\ \dot{\eta} = \eta A_2(\xi\eta) \\ \dot{\theta} = \omega. \end{cases} \tag{2}$$

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The existence and convergence of such change depends not only on the dimension d, but also on the dynamical character exhibited by the invariant torus $z=0$. In order to achieve an affirmative answer to this problem, we will restrict ourselves to normal forms of the type

$$\begin{aligned} \begin{pmatrix} \dot\xi \\ \dot\eta \end{pmatrix} &= N(\xi,\eta) = \begin{pmatrix} \xi A(\xi\eta) \\ -\eta A(\xi\eta) \end{pmatrix}, \\ \dot\theta &= \omega \end{aligned} \tag{3}$$

that is, with $A_2 = -A_1$ in (2). A system written in this way is said to be in *Birkhoff normal form.* It is straightforward to verify that it is *Hamiltonian*, with Hamilton function $H(\xi\eta)$, where $H(u) = \int A(u)\,du$. Thus, given a (Hamiltonian) system like (1), our original situation has become the problem of seeking for a transformation $(\zeta,\theta)\mapsto(z,\theta)=(\Phi(\zeta,\theta),\theta)$ leading it into its corresponding Birkhoff normal form (3). The existence and convergence of the transformation to Birkhoff normal form has been proved, for analytic Hamiltonian systems, in several cases:

The autonomous case $(d=0)$. Here, the equilibrium point $z=0$ can be hyperbolic or elliptic, and the first results were due to Poincaré and Birkhoff.

The periodic case $(d=1)$. In this situation, $z=0$ is a periodic orbit $\gamma = \{(0,\theta),\theta\in\mathbb{T}\}$. The convergence of the transformation to Birkhoff normal form is achieved provided γ is hyperbolic, that is, with real characteristic exponents $\pm\lambda$, $\lambda>0$. The dependence of F with respect to the angle θ does not need to be analytic, and it suffices to consider F to be $\mathcal{C}^1$ with respect to θ. This result was obtained by J. Moser [5] in 1956.

The quasi-periodic case $(d=2)$. The invariant object is now a 2-dimensional torus $\mathcal{T}=\{(0,\theta),\theta\in\mathbb{T}^2\}$, assumed to be hyperbolic, that is, with 3-dimensional stable and unstable associated invariant manifolds. F is assumed analytic in $\theta\in\mathbb{T}^2$ and the frequency ω of the invariant torus is assumed to be Diophantine, that is, there exist $C>0$ and $\tau\geq 1$ such that

$$|k\cdot\omega| \geq C|k|^{-\tau}, \quad \forall k\in\mathbb{Z}^2\setminus\{0\}.$$

The existence and convergence of such transformation was provided by A. Delshams *et al.* [2] in 1997, and can be easily generalized for $d\geq 2$.

Similar results hold for analytic reversible systems. Indeed, the Birkhoff normal form (3) is just the form obtained when the normal form (1) is required to be invariant under the involution $(\xi,\eta,t)\longrightarrow(\eta,\xi,-t)$. Thus, the Birkhoff normal form (3) is the normal form that arises in Hamiltonian or reversible systems in a neighborhood of $z=0$.

In the present paper, we will restrict ourselves to the case $d = 0$, that is, we will consider a general analytic two-dimensional system $\dot{z} = F(z)$ where

$$F(z) = \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}.$$

We recall that the process leading to a Birkhoff normal form for this system is equivalent to the existence of a transformation $z = \Phi(\zeta)$, close to the identity and analytic in $z = (x, y)$, such that the new vector field $N = \Phi^* F = [D\Phi]^{-1} F(\Phi)$ is of the form given in (3). In other words, F, Φ and N must satisfy the equation

$$D\Phi \cdot N = F(\Phi). \tag{4}$$

Since the functions f and g involved in this process are general (they are just assumed to be analytic in x, y), the study of its transformation to Birkhoff normal form is equivalent to the problem of determining if a given system is Hamiltonian or not.

Our approach, which follows an idea of D. DeLatte [1], and J. Moser, consists of looking for vector fields Φ, N and B satisfying the equation

$$D\Phi \cdot N + B = F(\Phi), \tag{5}$$

with

$$B(\zeta) = \begin{pmatrix} \xi b_1(\xi\eta) \\ \eta b_2(\xi\eta) \end{pmatrix}. \tag{6}$$

Condition (5) is weaker than the one in (4) and, in particular, implies that Φ does not have to be a change of variables, unless B is of the form $D\Phi \cdot W$. Indeed, the new system is not necessarily written in Birkhoff normal form. In a naïve way, the *remainder term* B contains the *obstructions* of the original system to be Hamiltonian and, therefore, in the planar context, integrable. From now on, we will say that the transformation Φ leads system $\dot{z} = F(z)$ into its Birkhoff pseudo-normal form (or shorter, its pseudo-normal form) if there exists a vector field N of the form (3) and a vector field B of the form (6) such that equation (5) is satisfied.

It is worth mentioning that, during the procedure of calculating the transformation Φ and the vector fields N and B corresponding to a given system, the connection between the two scalar functions b_1 and b_2 of the *remainder term* $\mathcal{B}$ is not completely determined. That is, its particular aspect can be established *a priori* with a certain degree of freedom. Thus, we may, depending on the context, consider different forms for the vector field B like, for instance [1]

$$B(\xi,\eta) = \begin{pmatrix} 0 \\ \eta b(\xi\eta) \end{pmatrix},$$

which provides an easy triangular scheme to obtain $A(\xi\eta)$ and $b(\xi\eta)$, or

$$B(\xi,\eta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix},$$

which will be the one used in this paper, since it will be more useful to preserve the geometrical properties of the original system..

We now present our main result on the convergence of the pseudo-normal form. In its statement, as well as along the paper, the following notation will be used: $\hat{h}(x,y)$ will denote the terms of order equal or greater than 2 in the variables x, y of a function $h(x,y)$ of these two variables.

THEOREM 1.1 (Pseudo-Normal Form Theorem).
Let us consider a general system of the form,

$$\dot{z} = F(z), \qquad z = (x,y) \in \mathbb{R}^2 \quad \text{or} \quad \mathbb{C}^2, \tag{7}$$

where

$$F(z) = \Lambda z + \hat{F}(z) = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} + \begin{pmatrix} \hat{f}(x,y) \\ \hat{g}(x,y) \end{pmatrix}$$

is analytic in z, with $\lambda \neq 0$ and such that $z = 0$ is an equilibrium solution. Then, there exist vector fields

$$N(\zeta) = \begin{pmatrix} \xi A(\xi\eta) \\ -\eta A(\xi\eta) \end{pmatrix}, \qquad B(\zeta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix}, \tag{8}$$

and an analytic in ζ transformation, convergent in a neighborhood of $\zeta = 0$,

$$z = \Phi(\zeta), \qquad \zeta = (\xi,\eta),$$

leading system (7) into its pseudo-normal form*, that is, satisfying*

$$D\Phi(\zeta) \cdot N(\zeta) + B(\zeta) = F(\Phi(\zeta)). \tag{9}$$

Remark 1.1. A similar result is given in [1], where the convergence of the transformation leading to the pseudo-normal form (9) is proved for the case of a non-necessarily area-preserving mapping, with λ real, $|\lambda| > 1$ and B of type $(0, \eta b(\xi\eta))$. Our approach is slightly different since we focus on planar vector fields and we deal in a unified way, by means of hypothesis $\lambda \neq 0$, with both the hyperbolic and the elliptic (linear center) equilibrium point case.

There are several cases, depending on the vector field F, where the remainder term B is known to be zero, and therefore Theorem 1.1 ensures the convergence of the normal form. One of them is the case of a Hamiltonian vector field F. Another one is the case of a reversible vector field F.

We recall that, given an involution G, ($\mathsf{G}^2 = \mathcal{I}d$ and $\mathsf{G} \neq \mathcal{I}d$), a system $\dot{X} = \mathcal{F}(X)$ is called G-*reversible* if it is invariant under the action $(X,t) \mapsto (\mathsf{G}(X), -t)$. Equivalently, the transformation G conjugates $\mathcal{F}$ with $-\mathcal{F}$, that is $\mathsf{G}^*\mathcal{F} = -\mathcal{F}$, where $\mathsf{G}^*\mathcal{F} = (D\mathsf{G})^{-1}\mathcal{F}(\mathsf{G})$. The involution G is called a *reversing involution* of the system $\dot{X} = \mathcal{F}(X)$. In the particular case that G is linear, we will denote it by R and therefore the last equality becomes just $\mathsf{R} \circ \mathcal{F} \circ \mathsf{R} = -\mathcal{F}$. For instance, $\mathsf{R}(\xi, \eta) = (\eta, \xi)$ is a reversing involution for the Birkhoff normal form (3). For more details about reversible systems, see [7].

Remark 1.2. The reversing involutions G of a reversible system do not need to be linear. However, it is known that any reversible system is conjugate, in a neighborhood of symmetric object, to a linear reversible system (Bochner Theorem, see [4]). Moreover, having in mind that the invariance under an involution is preserved by coordinate transformations, it is straightforward to construct families of G-reversible systems with a non-linear G. Namely, if the system $\dot{X} = \mathcal{F}(X)$ satisfies $\mathsf{R} \circ \mathcal{F} \circ \mathsf{R} = -\mathcal{F}$, where R is linear, applying a transformation of the form $W = S(X)$, the new system $\dot{W} = \mathcal{H}(W)$ is reversible with respect the reversing (not-necessarily linear) involution $\mathsf{G} = S \circ \mathsf{R} \circ S^{-1}$.

We now summarize these results about convergence of the Birkhoff normal form in the following corollary of the Pseudo-Normal Form Theorem.

COROLLARY 1.1. *Given a Hamiltonian or reversible analytic system*

$$\dot{z} = F(z) = \Lambda z + \hat{F}(z), \qquad z \in \mathbb{C}^2,$$

with Λ *a diagonal matrix* $(\lambda, -\lambda)$, $\lambda \neq 0$, *there exists an analytical change of variables* $z = \Phi(\zeta)$, *convergent in a neighborhood of the origin, which leads it into its Birkhoff Normal Form (i.e., with* $B \equiv 0$*).*

The result above simply requires an additional study of the form of the transformations Φ of Theorem 1.1 leading to pseudo-normal form, that will be performed in section 5. Interestingly enough, that result for the Hamiltonian case can be derived from a Criterium of Integrability (Theorem 5.1), which establishes that the vanishing of the remainder term B in the pseudo-normal form is equivalent to the existence of an analytical first integral of system (7) around the origin. Therefore, for the vector field F of system (7), its Hamiltonian character is equivalent to its reversibility and also to its integrability.

For a non-integrable vector field F, the vector field N in the pseudo-normal form (9) plays the rôle of its integrable part, whereas the vector field B plays the rôle of the obstruction to integrability. Since they have the form

$$N(\zeta) = \begin{pmatrix} \xi A(\xi\eta) \\ -\eta A(\xi\eta) \end{pmatrix}, \qquad B(\zeta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix},$$

each one of them is determined by an analytic function A, b of the variable $u = \xi\eta$. It is worth noticing that each zero $u_* = \xi_*\eta_*$ of the function b gives rise to a solution of system (7) given explicitly by

$$z(t) = \Phi(\xi_* \exp(tA(u_*)), \eta_* \exp(-tA(u_*))). \tag{10}$$

If in system (7), F is a real vector field with both critical exponents $\lambda, -\lambda$ real (and non-zero: the saddle case), the transformation Φ and the vector fields N, B can be chosen also real, as well as the functions A, b, which are real analytic functions of the real variable $u = \xi\eta$. In accordance with the Criterium of Integrability, the function $b(u)$ measures the non-integrability of the vector field F. On the other hand, the non-constant character of the function $A(u)$ measures the anisochronicity of the vector field F. Indeed, given an integrable vector field F, it will be conjugated to its normal form N, whose solutions are of the form (10), which will be linear if and only if A is a constant function. On the contrary, we may speak about an (integrable) isochronous saddle when $b = 0$ and A is constant. Of course, this notion is simply a particular case of the one given by Christopher *et al.* [3], which is also valid for more general critical exponents.

If the vector field F in system (7) is the complexification of a real vector field with both critical exponents $\lambda, -\lambda$ imaginary (and non-zero: the linear center case, with its application to the *center-focus problem*), it turns out that to come back to the real variables, one has to consider ξ, η as conjugate variables (that is, $\bar{\xi} = \eta$), and b is a real analytic function of the variable $r^2 = \xi\eta$, whereas A is a pure imaginary function of the same variable. Alternatively, introducing $A = ia$, a is a real analytic function of the variable $r^2 = \xi\eta$,

By equation (10), each zero $r_*^2 = \xi_*\eta_*$ of the function b gives rise to a periodic solution of system (7) with period $T = 2\pi/a(r_*^2)$. Unless $b \equiv 0$ (the integrable case, that is the origin is a center), each zero of a will be isolated, and will give rise to a limit cycle of system (7). One has, in this way, a new tool to locate limit cycles close to linear centers of analytic systems in the plane. Of course, in the case of a center ($b \equiv 0$), it is clear by equation (10) the rôle of anisochronicity played by the function a.

Indeed, from the series of a:

$$a(r^2) = a_0 + a_1 r^2 + a_2 r^4 + a_3 r^6 + \cdots$$

convergent for, say, $|r^2| < u_0$ and where $\lambda = ia_0$, one obtains straightforwardly the so-called *period constants* T_m of the series expansion

$$T = 2\pi / a(r^2) = 1 + T_1 r^2 + T_2 r^4 + T_3 r^6 + \cdots$$

We summarize these results in the following corollary.

COROLLARY 1.2. *For any planar vector field (7) under the assumptions of Theorem 1.1, with a function b in (8) not identically zero, we have*

(i) For each zero $u_* = \xi_* \eta_*$ of b, there exists a solution of the form (10) of system (7).

(ii) In the linear center case ($\lambda = ia_0$, $a_0 > 0$), for each zero r_*^2 of the analytic function b, there exists a limit cycle of period $T = 2\pi/a(r_*^2)$, with $A = ia$ in (8).

Coming back to the non-center case, the series expansion of b

$$b(r^2) = b_1 r^2 + b_2 r^4 + b_3 r^6 + \cdots$$

gives rise to a kind of *focal values.* In contrast with the center case, where the constants a_m are uniquely determined, in the non-center case the constants b_m (and therefore a_m) are not uniquely determined, but it turns out that each of them is uniquely determined modulo the ideal generated by the previous ones, as the so-called *Lyapunov constants* do. This common property gives strong evidences that the constants b_m are, in fact, the Lyapunov constants modulo some constant term.

The rest of the paper is organized as follows. In the next section, the proof of the Pseudo-Normal Form Theorem is begun, at least at a formal level. Later on, in section 3 we detail the inductive process, and finally, in section 4, the proof of Theorem 1.1 is finished. Section 5 deals with the Criterium of Integrability and the recursive computation of the constants b_m. As a corollary, the proof of the Corollary 1.1 is readily performed.

2. PSEUDO-NORMAL FORM THEOREM: FORMAL SOLUTION

In order to solve, formally, the so-called *homological equation*

$$D\Phi \cdot N + B = F(\Phi), \tag{11}$$

we assume that the remainder function B begins with terms of order at least 2 in ξ, η. Introducing $N = \Lambda\mathcal{I}d + \hat{N}$, $\Phi = \mathcal{I}d + \hat{\Phi}$, $B = \hat{B}$ and $F = \Lambda\mathcal{I}d + \hat{F}$ (see the notation introduced before Theorem 1.1), we obtain

$$D(\mathcal{I}d + \hat{\Phi}) \cdot (\Lambda\mathcal{I}d + \hat{N}) + \hat{B} = \Lambda\Phi + \hat{F}(\Phi),$$

which is equivalent to

$$D\hat{\Phi} \cdot N - \Lambda\hat{\Phi} = \hat{F}(\Phi) - \hat{N} - \hat{B}.$$

Thus, by defining the functional operator,

$$\mathcal{L}_N\Psi := D\Psi \cdot N - \Lambda\Psi,$$

equation (11) is equivalent to

$$\mathcal{L}_N\hat{\Phi} = \hat{F}(\Phi) - \hat{N} - \hat{B}. \tag{12}$$

2.1. The homological equation

Before dealing with the resolution of (12), we study first the formal solution of the linear equation

$$\mathcal{L}_N\hat{\Phi} = H, \tag{13}$$

that is, the formal invertibility of the functional operator $\mathcal{L}_N$. First of all, notice that $\mathcal{L}_N\Psi$ is of order equal or greater than 2 in $\zeta = (\xi, \eta)$ if Ψ is. This means that we can consider $H = \hat{H} = (\hat{h}_1(\xi,\eta), \hat{h}_2(\xi,\eta))$. Introducing $\hat{\Phi} = (\hat{\phi}(\xi,\eta), \hat{\psi}(\xi,\eta))$, we write the series expansions for the components of $\hat{\Phi}$ and $\hat{H}$,

$$\hat{\phi}(\xi,\eta) = \sum_{j+k\geq 2} \phi_{jk}\xi^j\eta^k \ , \qquad \hat{\psi}(\xi,\eta) = \sum_{j+k\geq 2} \psi_{jk}\xi^j\eta^k$$

and $\hat{h}_m(\xi,\eta) = \sum_{j+k\geq 2} h_{jk}^{(m)}\xi^j\eta^k$, $m = 1, 2$. Hence,

$$\begin{aligned}
\mathcal{L}_N\hat{\Phi} &= D\hat{\Phi} \cdot N - \Lambda\hat{\Phi} \\
&= \begin{pmatrix} \hat{\phi}_\xi & \hat{\phi}_\eta \\ \hat{\psi}_\xi & \hat{\psi}_\eta \end{pmatrix} \begin{pmatrix} \xi A(\xi\eta) \\ -\eta A(\xi\eta) \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} \hat{\phi} \\ \hat{\psi} \end{pmatrix} \\
&= \begin{pmatrix} (\xi\hat{\phi}_\xi - \eta\hat{\phi}_\eta)A(\xi\eta) - \lambda\hat{\phi} \\ (\xi\hat{\psi}_\xi - \eta\hat{\psi}_\eta)A(\xi\eta) + \lambda\hat{\psi} \end{pmatrix} = \begin{pmatrix} \hat{h}_1(\xi,\eta) \\ \hat{h}_2(\xi,\eta) \end{pmatrix}.
\end{aligned}$$

Equating the first components of the last expression we get

$$\hat{h}_1(\xi,\eta) = \sum_{j+k\geq 2} h_{jk}^{(1)} \xi^j \eta^k = \sum_{j+k\geq 2} \{(j-k)A(\xi\eta) - \lambda\} \phi_{jk} \xi^j \eta^k,$$

which, using that

$$A(\xi\eta) = \lambda + \hat{A}(\xi\eta) = \lambda + \sum_{m\geq 1} \alpha_m (\xi\eta)^m,$$

is equal to

$$\begin{aligned} &\sum_{j+k\geq 2} (j-k-1)\lambda\phi_{jk}\xi^j\eta^k + \sum_{m\geq 1}\sum_{j+k\geq 2} \alpha_m (j-k)\phi_{jk}\xi^{j+m}\eta^{k+m} \\ = &\sum_{j+k\geq 2} (j-k-1)\lambda\phi_{jk}\xi^j\eta^k + \sum_{m\geq 1}\sum_{j+k\geq 2(m+1)} \alpha_m (j-k)\phi_{j-m,k-m}\xi^j\eta^k. \end{aligned}$$

We see that the corresponding coefficient ϕ_{jk} of a given order in ξ, η is computed, iteratively, as a function of the coefficient $h_{jk}^{(1)}$ of the same order and coefficients $\phi_{j_* k_*}$ of a lower order, (which are known from previous steps of the process) provided that $j \neq k+1$. The terms of the expansion of $\hat{\Phi}$ that we are not able to determine from our system are of the type

$$\xi \sum_{k\geq 1} \phi_{k+1,k} (\xi\eta)^k,$$

and are known as *resonant terms.* So, at the end, following this iterative scheme, we are able to determine the coefficients ϕ_{jk}, of the function $\phi(\xi,\eta)$ if $j \neq k+1$, with arbitrary values for $\phi_{k+1,k}$, $k \geq 0$. Analogously, for the second component, such coefficients ψ_{jk} can be obtained, provided that $k \neq j+1$ and for any fixed value of $\psi_{j,j+1}$, $j \geq 0$. It is worth stressing that the only condition we need to carry out this (formal) procedure is that λ does not vanish.

2.2. Definition of the projections

In solving the linear equation (13), we have seen that it is only possible to compute the coefficients of the series expansion of $\Phi = (\phi, \psi)$ corresponding to non-resonant terms. This fact leads to the following

DEFINITION 2.1. Given a function f of the form

$$f(x,y) = \sum_{j+k\geq 1} f_{jk} x^j y^k,$$

we define the following two *projectors*, P_1 and P_2, as

$$\begin{aligned} P_1 f(x,y) &= x\sum_{j\geq 0} f_{j+1,j}(xy)^j \\ P_2 f(x,y) &= y\sum_{j\geq 0} f_{j,j+1}(xy)^j. \end{aligned} \tag{14}$$

In the same way, if $F(x,y) = (f(x,y), g(x,y))$ is a vector field, we define

$$\mathcal{P}F(x,y) = \begin{pmatrix} P_1 f(x,y) \\ P_2 g(x,y) \end{pmatrix}. \tag{15}$$

Moreover, we will denote $\mathcal{R} = \mathcal{I}d - \mathcal{P}$, where $\mathcal{I}d$ is the identity.

These operators satisfy very nice properties. For instance, as stated in the following lemma, the three operators $\mathcal{P}$, $\mathcal{R}$ and $\mathcal{L}_N$ commute.

LEMMA 2.1. *The projections $\mathcal{P}$ and $\mathcal{R}$ commute with $\mathcal{L}_N$, that is,*

$$\mathcal{P}(\mathcal{L}_N\Phi) = \mathcal{L}_N(\mathcal{P}\Phi), \qquad \mathcal{R}(\mathcal{L}_N\Phi) = \mathcal{L}_N(\mathcal{R}\Phi).$$

We omit its proof, since it consists on straightforward computations, and come back to the solution of the homological equation (12). Since $\mathcal{P}\hat{N} = \hat{N}$ and $\mathcal{P}\hat{B} = \hat{\mathcal{B}}$, applying $\mathcal{P}$ onto both sides of equation (12) we get

$$\mathcal{P}(\mathcal{L}_N(\hat{\Phi})) = \mathcal{P}\hat{F}(\Phi) - \hat{N} - \hat{B}. \tag{16}$$

In the same way, applying $\mathcal{R}$ and taking into account the previous lemma it follows that

$$\mathcal{L}_N(\mathcal{R}\hat{\Phi}) = \mathcal{R}\hat{F}(\Phi).$$

Looking for an iterative scheme solving, formally, the homological equation (12) we present a first attempt which derives directly from the properties above. We take as initial values

$$\Phi^{(1)} = \mathcal{I}d, \qquad N^{(1)} = \Lambda\mathcal{I}d, \qquad B^{(1)} = 0. \tag{17}$$

Recursively, the corresponding $(K+1)$-iterate of the transformation $\hat{\Phi}$, i.e. $\hat{\Phi}^{(K+1)}$, will be chosen as the (infinite) formal series satisfying

$$\mathcal{L}_{N^{(K)}}(\mathcal{R}\hat{\Phi}^{(K+1)}) = \mathcal{R}\hat{F}(\Phi^{(K)}). \tag{18}$$

Moreover, if we write

$$\mathcal{P}(\mathcal{L}_{N^{(K)}}\hat{\Phi}^{(K)}) = \mathcal{P}\hat{F}(\Phi^{(K)}) - \hat{N}^{(K+1)} - \hat{B}^{(K+1)},$$

using that

$$\mathcal{P}(\mathcal{L}_{N^{(K)}}\hat{\Phi}^{(K+1)}) = \mathcal{L}_{N^{(K)}}(\mathcal{P}\hat{\Phi}^{(K+1)}) =: \hat{\Psi}^{(K+1)} = \begin{pmatrix} \xi\hat{\psi}_1^{(K+1)}(\xi\eta) \\ \eta\hat{\psi}_2^{(K+1)}(\xi\eta) \end{pmatrix}$$

is known, we finally get

$$\hat{N}^{(K+1)} + \hat{B}^{(K+1)} = \mathcal{P}\hat{F}(\Phi^{(K+1)}) - \hat{\Psi}^{(K+1)}. \tag{19}$$

3. THE ITERATIVE SCHEME: AN IMPROVEMENT

Equation (18) will determine $\hat{\Phi}^{(K+1)}$ except for its resonant terms, i.e., for $\mathcal{P}\hat{\Phi}^{(K+1)}$ that, for the moment, we leave undetermined. For computational reasons, we will not follow exactly the iterative scheme presented in the previous section. Before going on with this question, we introduce some notation that will be used along the section. Indeed, we will denote $G(\xi,\eta) = \mathcal{O}_{[K]}$ if G is a (formal) homogeneous polynomial of order exactly K in the spatial variables ξ, η. Besides, we will write $G(\xi,\eta) = \mathcal{O}_K$ if G contains only terms of order greater or equal than K in ξ, η and $G(\xi,\eta) = \mathcal{O}_{\leq K}$ if all the terms in G are of order less or equal that K. Now, we present a result that will be crucial to refine the initial iterative process.

Lemma 3.2. *The vector fields Φ, N and B provided by the inductive scheme (17) –(19) satisfy, for $K > 1$,*

$$\begin{aligned} \Phi^{(K+1)} - \Phi^{(K)} &= \mathcal{O}_{K+1} \\ \hat{B}^{(K+1)} - \hat{B}^{(K)} &= \mathcal{O}_{K+1} \\ \hat{N}^{(K+1)} - \hat{N}^{(K)} &= \mathcal{O}_{K+1}. \end{aligned} \tag{20}$$

Proof. We proceed inductively. For $K = 1$ it is straightforward to verify them, since $\Phi^{(2)} = \mathcal{I}d + \hat{\Phi}^{(2)} = \Phi^{(1)} + \hat{\Phi}^{(2)}$, where $\hat{\Phi}^{(2)} = \mathcal{O}_2$ and $\hat{B}^{(2)}$, $\hat{N}^{(2)}$ are, in fact, $\mathcal{O}_3$. Thus, assume that the following equations

$$\begin{aligned} \Phi^{(K)} - \Phi^{(K-1)} &= \mathcal{O}_K \\ \hat{B}^{(K)} - \hat{B}^{(K-1)} &= \mathcal{O}_K \\ \hat{N}^{(K)} - \hat{N}^{(K-1)} &= \mathcal{O}_K \end{aligned}$$

hold. To prove $\Phi^{(K+1)} - \Phi^{(K)} = \mathcal{O}_{K+1}$, we compare $\mathcal{L}_{N^{(K)}}(\mathcal{R}\hat{\Phi}^{(K+1)}) = \mathcal{R}\hat{F}(\Phi^{(K)})$ with $\mathcal{L}_{N^{(K-1)}}(\mathcal{R}\hat{\Phi}^{(K)}) = \mathcal{R}\hat{F}(\Phi^{(K-1)})$. Subtracting them we have the following equation,

$$\mathcal{L}_{N^{(K)}}(\mathcal{R}\hat{\Phi}^{(K+1)}) - \mathcal{L}_{N^{(K-1)}}(\mathcal{R}\hat{\Phi}^{(K)}) = \mathcal{R}\hat{F}(\Phi^{(K)}) - \mathcal{R}\hat{F}(\Phi^{(K-1)}),$$

that we study in two steps. Namely,

(i) First, expanding the right-hand side in Taylor series up to order one, using that $\mathcal{R}(D\hat{F}) = \mathcal{O}_1$ and the induction hypothesis, it turns out that

$$\begin{aligned}
&\mathcal{R}\hat{F}(\Phi^{(K)}) - \mathcal{R}\hat{F}(\Phi^{(K-1)}) \\
&= \mathcal{R}D\hat{F}(\mu\Phi^{(K-1)} + (1-\mu)\Phi^{(K)}) \left[\Phi^{(K)} - \Phi^{(K-1)}\right] \\
&= \mathcal{R}D\hat{F}(\mathcal{I}d + \mu\hat{\Phi}^{(K-1)} + (1-\mu)\hat{\Phi}^{(K)}) \left[\Phi^{(K)} - \Phi^{(K-1)}\right] = \mathcal{O}_{K+1},
\end{aligned}$$

where $0 < \mu < 1$.

(ii) Second, taking into account that

$$\mathcal{L}_{N_1+N_2}\Psi = \mathcal{L}_{N_1}\Psi + D\Psi \cdot N_2,$$

it follows that

$$\begin{aligned}
&\mathcal{L}_{N^{(K)}}(\mathcal{R}\hat{\Phi}^{(K+1)}) - \mathcal{L}_{N^{(K-1)}}(\mathcal{R}\hat{\Phi}^{(K)}) \\
&= \mathcal{L}_{N^{(K-1)}}(\mathcal{R}\hat{\Phi}^{(K+1)}) + D(\mathcal{R}\hat{\Phi}^{(K+1)}) \left[N^{(K)} - N^{(K-1)}\right] \\
&\qquad -\mathcal{L}_{N^{(K-1)}}(\mathcal{R}\hat{\Phi}^{(K)}) \\
&= \mathcal{L}_{N^{(K-1)}}(\mathcal{R}\hat{\Phi}^{(K+1)} - \mathcal{R}\hat{\Phi}^{(K)}) + D(\mathcal{R}\hat{\Phi}^{(K+1)}) \left[N^{(K)} - N^{(K-1)}\right].
\end{aligned}$$

It is clear that

$$D(\mathcal{R}\hat{\Phi}^{(K+1)}) \left[N^{(K)} - N^{(K-1)}\right] = \mathcal{O}_{K+1}$$

so, at the end, using [(i)], we obtain an equality of the form

$$\mathcal{L}_{N^{(K-1)}}(\mathcal{R}\hat{\Phi}^{(K+1)} - \mathcal{R}\hat{\Phi}^{(K)}) = H^{(K+1)},$$

where $H^{(K+1)} = \mathcal{O}_{K+1}$. Since $\mathcal{L}_N$ preserves the order in ξ, η, that is, $\Psi = \mathcal{O}_{K+1}$ if and only if $\mathcal{L}_N\Psi = \mathcal{O}_{K+1}$, it follows that $\mathcal{R}\hat{\Phi}^{(K+1)} - \mathcal{R}\hat{\Phi}^{(K)} = \mathcal{O}_{K+1}$ and finally

$$\Phi^{(K+1)} - \Phi^{(K)} = \mathcal{O}_{K+1}.$$

The proof of the estimates

$$\begin{aligned}
\hat{B}^{(K)} - \hat{B}^{(K-1)} &= \mathcal{O}_K \\
\hat{N}^{(K)} - \hat{N}^{(K-1)} &= \mathcal{O}_K
\end{aligned}$$

is completely analogous. ∎

One of the most important consequences of this lemma is that it shows how to decrease enormously the computational effort, not only in terms of the CPU time but also in terms of the amount of memory employed. For this reason, from now on, the corresponding K-iterates, $\Phi^{(K)}$, $N^{(K)}$, $B^{(K)}$ will be assumed to contain only terms up to order K in ξ, η.

To apply this result on our procedure, we start with a general **even step**, say $2M$ (where M is assumed to be greater than 1). Indeed the $2M$-approximation to the function Φ comes from

$$\mathcal{L}_{N^{(2M-1)}}(\mathcal{R}\hat{\Phi}^{(2M)}) = \mathcal{R}\hat{F}(\Phi^{(2M-1)}),$$

where $\Phi^{(2M)} = \Phi^{(2M-1)} + \Delta\Phi^{(2M-1)}$, $\Phi^{(2M-1)} = \mathcal{O}_{\leq 2M-1}$ and $\Delta\Phi^{(2M-1)}$ being a homogeneous polynomial containing only terms of order $2M$ in ξ, η, that is, $\Delta\Phi^{(2M-1)} = \mathcal{O}_{[2M]}$. By comparison with the corresponding equation from the previous step, namely,

$$\mathcal{L}_{N^{(2M-2)}}(\mathcal{R}\hat{\Phi}^{(2M-1)}) = \mathcal{R}\hat{F}(\Phi^{(2M-2)}),$$

using that $N^{(2M-2)} = N^{(2M-3)}$ and writing

$$N^{(2M-1)} = N^{(2M-3)} + \Delta N^{(2M-3)},$$

where $N^{(2M-3)} = \mathcal{O}_{\leq 2M-3}$ and $\Delta N^{(2M-3)} = \mathcal{O}_{[2M-1]}$, we arrive at

$$\begin{aligned} &\mathcal{L}_{N^{(2M-1)}}(\mathcal{R}\hat{\Phi}^{(2M)}) - \mathcal{L}_{N^{(2M-2)}}(\mathcal{R}\hat{\Phi}^{(2M-1)}) = \\ &= \mathcal{R}\hat{F}(\Phi^{(2M-1)}) - \mathcal{R}\hat{F}(\Phi^{(2M-2)}). \end{aligned} \tag{21}$$

Using that $\mathcal{L}_{N^{(2M-1)}}\Psi = \mathcal{L}_{N^{(2M-3)}}\Psi + D\Psi \cdot \Delta N^{(2M-3)}$ and the linearity of $\mathcal{L}_N$ and $\mathcal{R}$, the left-hand side of (20) can be checked to be equal to

$$D(\mathcal{R}\Delta\Phi^{(1)}) \cdot \Delta N^{(2M-3)} + \left[\mathcal{R}\Delta\Phi^{(2M-1)}, \Lambda\mathcal{I}d\right] + \mathcal{O}_{2M+1},$$

where $[G,H] = DG \cdot H - DH \cdot G$ is the *Lie bracket*. Concerning the right-hand side of the same equation, expanding in Taylor series around $\Phi^{(2M-2)}$, and denoting $F = F_2 + F_3 + \cdots$, F_j being homogeneous polynomials of order exactly j, it follows that

$$\begin{aligned} &\mathcal{R}\hat{F}(\Phi^{(2M-1)}) - \mathcal{R}\hat{F}(\Phi^{(2M-2)}) = \\ &= \mathcal{R}\left\{\hat{F}(\Phi^{(2M-2)})\right\}_{[2M]} + \mathcal{R}\left(DF_2 \cdot \Delta\Phi^{(2M-2)}\right) + \mathcal{O}_{2M+1}. \end{aligned}$$

So finally, we get for $M \geq 2$ the following iterative equation, which provides the incremental term $\Delta\Phi^{(2M-1)}$ becomes

$$\left[\mathcal{R}\Delta\Phi^{(2M-1)}, \Lambda\mathcal{I}d\right] = \mathcal{R}\left\{\hat{F}(\Phi^{(2M-2)})\right\}_{[2M]}$$

$$+\mathcal{R}\left(DF_2 \cdot \Delta\Phi^{(2M-2)}\right) - D(\mathcal{R}\Delta\Phi^{(1)}) \cdot \Delta N^{(2M-3)}.$$

For the case $M = 1$, corresponding to the first step of the process, it is straightforward to derive the equation

$$\left[\mathcal{R}\Delta\Phi^{(1)}, \Lambda\mathcal{I}d\right] = \mathcal{R}F_2.$$

With respect to the **odd steps** of the process, we should distinguish between two kind of equations: a first type, providing the new term $\Delta\Phi^{(2M)}$ and a second one which gives the increments related to N and B, namely $\Delta N^{(2M-1)}$ and $\Delta B^{(2M-1)}$, respectively. About the former, it follows the same argument that was used for the even steps case, coming now from the comparison between the current step equation

$$\mathcal{L}_{N^{(2M)}}(\mathcal{R}\hat{\Phi}^{(2M+1)}) = \mathcal{R}\left\{\hat{F}(\Phi^{(2M)})\right\}_{\leq 2M+1}$$

with the previous one

$$\mathcal{L}_{N^{(2M-1)}}(\mathcal{R}\hat{\Phi}^{(2M)}) = \mathcal{R}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{\leq 2M}.$$

Notice that, since $\Delta\Phi^{(2M)} = \mathcal{O}_{[2M+1]}$, it is only necessary to consider terms of order less or equal than $2M+1$. Then we obtain the following iterative equation

$$\begin{aligned}&\left[\mathcal{R}\Delta\Phi^{(2M)}, \Lambda\mathcal{I}d\right]\\ &\quad = \mathcal{R}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{[2M+1]} + \mathcal{R}(DF_2 \cdot \Delta\Phi^{(2M-1)}).\end{aligned}$$

Concerning the second type, which provides the new approximation to the vector fields N and B, it comes from

$$\hat{N}^{(2M+1)} + \hat{B}^{(2M+1)} = \mathcal{P}\left\{\hat{F}(\Phi^{(2M+1)})\right\}_{\leq 2M+1} - \hat{\Psi}^{(2M+1)},$$

where $\hat{\Psi}^{(2M+1)} = \mathcal{L}_{N^{(2M-1)}}(\mathcal{P}\hat{\Phi}^{(2M+1)})$ and we denote

$$\begin{aligned}\hat{N}^{(2M+1)} &= \hat{N}^{(2M-1)} + \Delta N^{(2M-1)}\\ \hat{B}^{(2M+1)} &= \hat{B}^{(2M-1)} + \Delta B^{(2M-1)}\\ \hat{\Psi}^{(2M+1)} &= \hat{\Psi}^{(2M-1)} + \Delta\Psi^{(2M-1)},\end{aligned}$$

with $\Delta N^{(2M-1)}$, $\Delta B^{(2M-1)}$, $\Delta\Psi^{(2M-1)}$ homogeneous polynomials of order exactly $2M+1$ in ξ, η (that is, they are $\mathcal{O}_{[2M+1]}$). We compare it with

$$\hat{N}^{(2M-1)} + \hat{B}^{(2M-1)} = \mathcal{P}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{\leq 2M-1} - \hat{\Psi}^{(2M-1)},$$

where $\hat{\Psi}^{(2M-1)} = \mathcal{L}_{N^{(2M-3)}}(\mathcal{P}\hat{\Phi}^{(2M-1)})$, and then apply on it the following lemma, whose proof is a consequence of the linearity of the operator $\mathcal{L}_N$ and the expansion on Taylor series, and is left to the reader.

Lemma 3.3. *Having in mind the definitions above, we have*

$$\hat{\Psi}^{(2M+1)} = \hat{\Psi}^{(2M-1)} + D(\mathcal{P}\Delta\Phi^{(2)}) \cdot \Delta N^{(2M-3)} + \mathcal{O}_{2M+2}$$

and

$$\begin{aligned} &\mathcal{P}\left\{\hat{F}(\Phi^{(2M+1)})\right\}_{\leq 2M+1} - \mathcal{P}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{\leq 2M-1} \\ &\quad = \mathcal{P}\left(DF_2 \cdot \Delta\Phi^{(2M-1)}\right) + \mathcal{P}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{[2M+1]} + \mathcal{O}_{2M+3}. \end{aligned}$$

Indeed, it turns out that

$$\begin{aligned} &\Delta N^{(2M-1)} + \Delta B^{(2M-1)} \\ &\quad = \mathcal{P}\left(DF_2 \cdot \Delta\Phi^{(2M-1)}\right) + \mathcal{P}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{[2M+1]} \\ &\quad - D\left(\mathcal{P}\Delta\Phi^{(2)}\right) \cdot \Delta N^{(2M-3)} + \mathcal{O}_{2M+2}. \end{aligned}$$

Because of the freedom we have in the choice of $\mathcal{P}\Phi$ and in order to simplify the final scheme, we can take $\mathcal{P}\Delta\Phi^{(2)} = 0$. Therefore, the final equation involving $\Delta N^{(2M-1)}$ and $\Delta B^{(2M-1)}$, for $M \geq 2$, becomes

$$\begin{aligned} &\Delta N^{(2M-1)} + \Delta B^{(2M-1)} \\ &\quad = \mathcal{P}\left(DF_2 \cdot \Delta\Phi^{(2M-1)}\right) + \mathcal{P}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{[2M+1]} + \mathcal{O}_{2M+2}. \end{aligned}$$

Concerning the first iterates $\Delta N^{(1)}$ and $\Delta B^{(1)}$, it is straightforward to verify that they come from

$$\Delta N^{(1)} + \Delta B^{(1)} = \mathcal{P}F_3 + \mathcal{P}(DF_2 \cdot \Delta\Phi^{(1)}).$$

This equation completes the *final iterative scheme*, which can be summarized in the following way. Start the process with initial values

$$\Phi^{(1)} = \mathcal{I}d, \qquad N^{(1)} = \Lambda\mathcal{I}d, \qquad B^{(1)} = 0,$$

and fix a value for $\mathcal{P}\Phi$ such that $\mathcal{P}\Delta\Phi^{(2)} = 0$. Then,

$$\begin{aligned}
\left[\mathcal{R}\Delta\Phi^{(1)}, \Lambda\mathcal{I}d\right] &= F_2, \\
\left[\mathcal{R}\Delta\Phi^{(2)}, \Lambda\mathcal{I}d\right] &= \mathcal{R}\left\{\hat{F}(\Phi^{(1)})\right\}_{[3]} + \mathcal{R}(DF_2 \cdot \Delta\Phi^{(1)}), \\
\Delta N^{(1)} + \Delta B^{(1)} &= \mathcal{P}F_3 + \mathcal{P}(DF_2 \cdot \Delta\Phi^{(1)}),
\end{aligned}$$

and, for $K \geq 3$,

$$\begin{aligned}
\left[\mathcal{R}\Delta\Phi^{(K)}, \Lambda\mathcal{I}d\right] &= \mathcal{R}\left\{\hat{F}(\Phi^{(K-1)})\right\}_{[K+1]} \\
&\quad + \mathcal{R}(DF_2 \cdot \Delta\Phi^{(K-1)}) - \delta_K D(\mathcal{R}\Delta\Phi^{(1)}) \cdot \Delta N^{(K-2)},
\end{aligned}$$

where $\delta_K = 0$ if K is even and $\delta_K = 1$ for K odd. Moreover, about N and B, if we write $K = 2M - 1$ with $M > 1$, we have

$$\Delta N^{(2M-1)} + \Delta B^{(2M-1)} = \mathcal{P}\left(DF_2 \cdot \Delta\Phi^{(2M-1)}\right) + \mathcal{P}\left\{\hat{F}(\Phi^{(2M-1)})\right\}_{[2M+1]}.$$

We want to stress the fact that, besides the simplicity of the scheme above, its solution at any step is given by a linear equation. Concretely, we must solve

$$\left[\mathcal{R}\Delta\Phi^{(K)}, \Lambda\mathcal{I}d\right] = \mathcal{R}\Delta H^{(K)}, \tag{22}$$

where the term on the right-hand side is known from the previous steps of the process. Thus, if we write

$$\Delta\Phi^{(K)} = \begin{pmatrix} \phi(\xi,\eta) \\ \psi(\xi,\eta) \end{pmatrix}, \qquad \Delta H^{(K)} = \begin{pmatrix} h_1(\xi,\eta) \\ h_2(\xi,\eta) \end{pmatrix},$$

where

$$\phi(\xi,\eta) = \sum_{\substack{j+k=K+1 \\ j\neq k+1}} \phi_{jk}\xi^j\eta^k, \qquad \psi(\xi,\eta) = \sum_{\substack{j+k=K+1 \\ k\neq j+1}} \psi_{jk}\xi^j\eta^k,$$

and

$$h_\ell(\xi,\eta) = \sum_{\substack{j+k=K+1 \\ j\neq k+1}} h_{jk}^{(\ell)}\xi^j\eta^k, \qquad \ell = 1, 2,$$

the explicit solution for (22) is given by

$$\phi_{jk} \;=\; \frac{h_{jk}^{(1)}}{\lambda(j-k-1)} \quad \text{where } j+k = K+1 \text{ and } j \neq k+1,$$

$$\psi_{jk} \;=\; \frac{h^{(2)}_{jk}}{\lambda(j-k+1)} \qquad \text{where } j+k=K+1 \text{ and } k\neq j+1.$$

4. PROOF OF THE CONVERGENCE

4.1. Definition of the norms

Let us consider the following domain around the origin

$$\mathcal{D}_r = \left\{ z=(z_1,z_2,\dots,z_n)\in\mathbb{C}^n : \quad \|z\|_\infty = \max_{j=1,\dots,n} |z_j| \le r \right\}$$

and let $f(z)$ be an analytic function

$$f(z) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha.$$

Writing $z=|z|e^{i\varphi}$ we can also express it in multi-index notation as

$$f(z) = f(|z|,\varphi(z)) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha |z|^\alpha e^{i\alpha\cdot\varphi}, \tag{23}$$

being $\alpha\cdot\varphi = \alpha_1\varphi_1+\alpha_2\varphi_2+\cdots+\alpha_n\varphi_n$ and $\varphi=\varphi(z)=\arg z$. At first, for such kind of functions we consider the *supremum norm*

$$\|f\|_{\infty,r} = \sup_{z\in\mathcal{D}_r} |f(z)|.$$

However, this is not the norm we are going to deal with. This new norm, closely related to the ℓ^2-norm, will be defined thanks to the following result.

LEMMA 4.4. *Given a positive real number r, there exits a unique $r_*\ge 0$ such that $r=r_*e^{r_*}$.*

Hence, we define

$$\|f\|_r = \|f(|z|,\varphi(z))\|_r = \sup_{|z_*|,|\eta|\le r_*} \left(\frac{1}{(2\pi)^n}\int_{T^n} |f(|z_*|,\varphi+i\eta)|^2 d\varphi \right)^{1/2},$$

with $\varphi=\varphi(z_*)=\arg z_*$ and where r_* satisfies $r=r_*e^{r_*}$. Notice that it is well-defined, since $|z_*e^{i(\varphi+i\eta)}| \le |z_*|e^{|\eta|} \le r_*e^{r_*} = r$. Moreover, we can express the Fourier coefficients of $f(|z_*|,\varphi+i\eta)$ in terms of the corresponding ones of $f(|z_*|,\varphi)$. Namely, applying a shift $\varphi\longmapsto\varphi+i\eta$

with $|\eta| \le r_*$, it follows that

$$\begin{aligned} f_\alpha |z_*|^\alpha &= \frac{1}{(2\pi)^n} \int_{T^n} f(|z_*|, \varphi) e^{-i\alpha\cdot\varphi} d\varphi \\ &= \frac{1}{(2\pi)^n} \int_{T^n} f(|z_*|, \varphi + i\eta) e^{-i\alpha\cdot(\varphi+i\eta)} d\varphi \\ &= e^{\alpha\cdot\eta} \frac{1}{(2\pi)^n} \int_{T^n} f(|z_*|, \varphi + i\eta) e^{-i\alpha\cdot\varphi} d\varphi \end{aligned}$$

and then

$$\frac{1}{(2\pi)^n} \int_{T^n} f(|z_*|, \varphi + i\eta) e^{-i\alpha\cdot\varphi} d\varphi = f_\alpha |z_*|^\alpha e^{-\alpha\cdot\eta}.$$

Using the isometry between $\mathcal{L}^2$ and ℓ^2 norms, we have

$$\int_{T^n} |f(|z_*|, \varphi + i\eta)|^2 \, d\varphi = \sum_{\alpha\in\mathbb{N}^n} |f_\alpha|^2 |z_*|^{2\alpha} e^{-2\alpha\cdot\eta},$$

so

$$\begin{aligned} &\sup_{|z_*|,|\eta|\le r_*} \left(\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |f(|z_*|, \varphi + i\eta)|^2 d\varphi \right)^{1/2} \\ &\qquad = \left(\frac{1}{(2\pi)^n} \sum_{\alpha\in\mathbb{N}^n} |f_\alpha|^2 r_*^{2\alpha} e^{2|\alpha| r_*} \right)^{1/2} \end{aligned}$$

where $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$. Indeed we can write the norm defined above, in a equivalent way, as an slightly weighted ℓ^2-norm,

$$\|f\|_r = \|f(|z|, \varphi(z))\|_r = \left(\frac{1}{(2\pi)^n} \sum_{\alpha\in\mathbb{N}^n} |f_\alpha|^2 r_*^{2\alpha} e^{2|\alpha| r_*} \right)^{1/2}$$

with $r = r_* e^{r_*}$ and $\varphi(z) = \arg\, z$. Such norm satisfies some useful properties collected in the following lemma, whose proof is standard.

LEMMA 4.5.

(i) For any positive number r we have

$$\|f\|_r \le \|f\|_{\infty,r}.$$

(ii) Given a positive number R, for any ρ, r satisfying $0 < \rho < r \le R$, the following estimate

$$\|f\|_{\infty,\rho} \le \frac{c_0}{(r-\rho)^{n/2}} \|f\|_r,$$

holds, where $c_0 = c_0(n)$ *is a constant which depends on* n *and* R.

(iii)Let us consider an analytic function $\Psi : \mathcal{D}_\rho \mapsto \mathcal{D}_s$, *satisfying that* $\|\Psi\|_\rho \le s$. *Then, if* $s < r$ *we have*

$$\|f \circ \Psi\|_\rho \le \frac{c_0}{(r-s)^{n/2}} \|f\|_r.$$

(iv)Let $f_M(z)$ *be an homogeneous polynomial of order* M,

$$f_M(z) = \sum_{|\alpha|=M} f_\alpha z^\alpha.$$

Then, if $0 \le r \le R$, *we have the following bound*

$$\|f_M\|_r \le \left(\frac{r_*}{R_*}\right)^M \|f_M\|_R,$$

where s_* *means the unique positive real number satisfying* $s = s_* e^{s_*}$.

This norm can be easily extended to a norm for vector fields. More precisely, we have

DEFINITION 4.1. Let us consider a vector field $F(z) = (f(z), g(z))$, analytic in $\mathcal{D}_r$. Then, we define

$$\|F\|_r = \left(\|f\|_r^2 + \|g\|_r^2\right)^{1/2}.$$

It is straightforward to verify that this norm satisfies analogous results to the ones achieved in the previous lemma.

4.2. Convergence of the iterative scheme

The scheme we will follow to prove the convergence of Φ, N and B is supported on simple ideas. Namely, since $\Phi = \mathcal{I}d + \sum_{K\ge1} \Delta\Phi^{(K)}$, $N = \Lambda\mathcal{I}d + \sum_{K\ge2} \Delta N^{(2K-1)}$ and $\hat{B} = \sum_{K\ge2} \Delta B^{(2K-1)}$, we will get estimates for $\|\Delta\Phi^{(K)}\|_\rho$, $\|\Delta N^{(2K-1)}\|_\rho$ and $\|\Delta B^{(2K-1)}\|_\rho$ in a suitable domain $\mathcal{D}_\rho$ a bit smaller than the original one, $\mathcal{D}_{r_0}$. More precisely, we will obtain bounds of type

$$a_{2M-1} \le k_1 \varepsilon^{2M} + k_2 \varepsilon a_{2M-3}, \quad a_{2M-2} \le k_3 \varepsilon^{2M-1} + k_4 \varepsilon a_{2M-3},$$

where $a_K := \|\Delta\Phi^{(k)}\|_\rho$ and $0 < \varepsilon < 1$, $k_1, \dots, k_4$ are suitable constants. It will be derived from these expressions the convergence of $\|\Phi\|_\rho$. The corresponding estimates for N and B come from the fact that $\|\Phi\|_\rho$ majorates them.

Let us be more precise. First, assume that the following quantities are finite on $\mathcal{D}_{r_0}$

$$L_0 := \|\hat{F}\|_{r_0}, \qquad L_1 := \|DF_2\|_{r_0},$$

Moreover, consider $r_j = r_{j-1} - \delta$, $j = 1, 2, 3$, intermediate radii and define

$$0 < \gamma_* := \frac{r_{3*}}{r_{0*}} < 1,$$

with r_{3*} and r_{0*} the unique positive numbers satisfying $r_3 = r_{3*}e^{r_{3*}}$ and $r_0 = r_{0*}e^{r_{0*}}$, respectively. Since the final domain of convergence will be $\mathcal{D}_{r_3}$, we define

$$a_K := \|\Delta\Phi^{(K)}\|_{r_3}.$$

Since $\Delta\Phi^{(2K-1)} = \mathcal{O}_{[2K]}$, we have that $\mathcal{P}\Delta\Phi^{(2K-1)} = 0$ and, consequently, $\mathcal{R}\Delta\Phi^{(2K-1)} = \Delta\Phi^{(2K-1)}$. Furthermore, remind that during the iterative scheme the value of the projection $\mathcal{P}\hat{\Phi}$ can be chosen arbitrary, so we will need to impose some conditions on this term. Concretely, we want it to be absolutely convergent in such norm and bounded by the norm of the vector field $\hat{F}$, more precisely,

$$\sum_{K\geq 1} \|\mathcal{P}\Delta\Phi^{(K)}\|_{r_0} \leq c_3\|\hat{F}\|_{r_0}, \tag{24}$$

where c_3 is a positive constant (that could be large).

Let us focus our attention on the former estimates. From equation $\left[\mathcal{R}\Delta\Phi^{(1)}, \Lambda\mathcal{I}d\right] = F_2$ it turns out that $|\lambda|a_1 \leq L_0$. From

$$\left[\mathcal{R}\Delta\Phi^{(2)}, \Lambda\mathcal{I}d\right] = \mathcal{R}F_3 + \mathcal{R}\left(DF_2 \cdot \Delta\Phi^{(1)}\right),$$

and hypothesis (24), one gets $|\lambda|a_2 \leq (c_3|\lambda| + 1)L_0 + L_1a_1$. Finally, $\|\Delta N^{(1)} + \Delta B^{(1)}\|_{r_3} \leq L_0 + L_1a_1$. It is now important to remark the concrete shape of the vector fields N and B. Since $N(\xi,\eta) = (\xi A(\xi\eta), -\eta A(\xi\eta))$ and $B(\xi,\eta) = (\xi b(\xi\eta), \eta b(\xi\eta))$, it is not difficult to check that

$$\|\Delta N^{(2K-1)}\|_{r_3}\ ,\ \|\Delta B^{(2K-1)}\|_{r_3} \leq c_2\|\Delta N^{(2K-1)} + \Delta B^{(2K-1)}\|_{r_3},$$

where c_2 is a constant depending only on r_0. Therefore, the previous bounds become $\|\Delta N^{(1)}\|_{r_3} \leq L_0 + L_1a_1$ and $\|\Delta B^{(1)}\|_{r_3} \leq L_0 + L_1a_1$.

Our aim is to get recurrent estimates on a_{2K-1}. To this purpose let us consider the equation

$$\begin{aligned}\left[\mathcal{R}\Delta\Phi^{(2M-1)}, \Lambda\mathcal{I}d\right] &= \mathcal{R}\left\{\hat{F}(\Phi^{(2M-2)})\right\}_{[2M]} \\ &+\mathcal{R}(DF_2 \cdot \Delta\Phi^{(2M-2)}) - D(\mathcal{R}\Delta\Phi^{(1)}) \cdot \Delta N^{(2M-3)}.\end{aligned}$$

A first naïve estimate reads

$$|\lambda|\|\mathcal{R}\Delta\Phi^{(2M-1)}\|_{r_3} \leq \| \left\{ \hat{F}(\Phi^{(2M-2)}) \right\}_{[2M]} \|_{r_3} \\ + \|DF_2\|_{r_3}\|\Delta\Phi^{(2M-2)}\|_{r_3} + \|D(\mathcal{R}\Delta\Phi^{(1)})\|_{r_3}\|\Delta N^{(2M-3)}\|_{r_3}, \quad (25)$$

but can be refined by dealing with its terms separately. Namely,

(i) Since $D(\mathcal{R}\Delta\Phi^{(1)}) = \mathcal{O}_{[1]}$, applying the last lemma, it follows that

$$\|D(\mathcal{R}\Delta\Phi^{(1)})\|_{r_3} \leq \gamma_* \|D(\mathcal{R}\Delta\Phi^{(1)})\|_{r_0} \leq \frac{2\gamma_* L_0}{|\lambda|}.$$

(ii) We have

$$\|F \circ \Phi^{(2M-2)}\|_{r_3} \leq \frac{c_1}{\delta}\|F\|_{r_1}$$

provided $\|\Phi^{(2M-2)}\|_{r_3} < r_2$. So, using again the same lemma, it turns out

$$\| \left\{ \hat{F}(\Phi^{(2M-2)}) \right\}_{[2M]} \|_{r_3} \leq \gamma_*^{2M} \| \left\{ \hat{F}(\Phi^{(2M-2)}) \right\}_{[2M]} \|_{r_2} \leq \frac{c_1 L_0}{\delta}\gamma_*^{2M}.$$

(iii) Having in mind that $DF_2 = \mathcal{O}_{[1]}$, it follows that

$$\|DF_2\|_{r_3} \leq \gamma_* \|DF_2\|_{r_0} \leq \gamma_* L_1.$$

(iv) Since

$$\|\Delta N^{(2J-1)} + \Delta B^{(2J-1)}\|_{r_3} \\ \leq \|DF_2\|_{r_3}\|\Delta\Phi^{(2J-1)}\|_{r_3} + \|\mathcal{P}\left\{ \hat{F}(\Phi^{(2J-1)}) \right\}_{[2J-1]} \|_{r_3}$$

we have

$$\|\Delta N^{(2M-3)}\|_{r_3} \leq \frac{c_1 c_2 L_0}{\delta}\gamma_*^{2M-1} + c_2 L_1 \gamma_* a_{2M-3}.$$

(v) From

$$\left[\mathcal{R}\Delta\Phi^{(2M-2)}, \Lambda\mathcal{I}d\right] = \mathcal{R}\left\{ \hat{F}(\Phi^{(2M-3)}) \right\}_{[2M-1]} + \mathcal{R}\left(DF_2 \cdot \Delta\Phi^{(2M-3)}\right)$$

and assumption (24), it is deduced that

$$|\lambda| a_{2M-2} \leq \left(\frac{c_1}{\delta} + c_3|\lambda|\right) L_0 \gamma_*^{2M-1} + L_1 \gamma_* a_{2M-3}.$$

Applying together bounds $(i)-(v)$ onto inequality (25), we arrive at the following estimate

$$
\begin{aligned}
&|\lambda| a_{2M-1} \\
&\quad \leq L_0 \left(\frac{c_1|\lambda| + 2c_1c_2L_0 + c_1L_1 + c_3\delta|\lambda|L_1}{\delta|\lambda|} \right) \gamma_*^{2M} \\
&\quad + \frac{L_1}{|\lambda|} \left(L_1 + 2L_0 \right) \gamma_*^2 a_{2M-3},
\end{aligned}
$$

which involves only odd terms of the sequence $\{a_K\}_K$. Refining the constants we reach the final expression

$$a_{2M-1} \leq \frac{K_1}{\delta|\lambda|^2} \gamma_*^{2M} + \frac{K_2}{|\lambda|^2} \gamma_*^2 a_{2M-3},$$

where

$$
\begin{aligned}
K_1 &= |\lambda|(c_1 + c_3L_1) + 2c_1c_2L_0 + c_1L_1 \\
K_2 &= L_1(L_1 + 2L_0),
\end{aligned}
$$

depend on $\|F\|_{r_0}$, $\|DF_2\|_{r_0}$ and $|\lambda|$. In the same way, from (v),

$$a_{2M-2} \leq \frac{K_3}{\delta|\lambda|} \gamma_*^{2M-1} + \frac{K_4}{|\lambda|} \gamma_* a_{2M-3},$$

with $K_3 = c_1 + c_3|\lambda|$ and $K_4 = L_1$ also depend only on $\|F\|_{r_0}$, $\|DF_2\|_{r_0}$ and $|\lambda|$. Choosing γ_* such that

$$\gamma_* \leq \frac{|\lambda|^2}{L_1(L_1 + 2L_0)},$$

the inequality satisfied by a_{2M-1}, for $M \geq 2$, becomes

$$a_{2M-1} \leq \frac{K_1}{\delta|\lambda|^2} \gamma_*^{2M} + \gamma_* a_{2M-3}. \tag{26}$$

The convergence of the series $\sum_{M\geq 1} a_{2M-1}$ (and, therefore, of $\sum_{M\geq 1} a_{2M}$), comes directly from the application of the following result.

LEMMA 4.6. *Let us consider a sequence $\{a_k\}_k$, with $a_k \geq 0 \quad \forall k$, such that the following recurrent inequalities are satisfied, for $m \geq 2$ and $0 < \varepsilon < 1$,*

$$
\begin{aligned}
a_{2m-1} &\leq k\varepsilon^{2m} + \varepsilon a_{2m-3}, \\
a_{2m-2} &\leq k_1\varepsilon^{2m-1} + k_2\varepsilon a_{2m-3}.
\end{aligned} \tag{27}
$$

Then, $\sum_{m\geq 1} a_m$ *is convergent.*

Proof. We start by checking that $\sum_{m\geq 1} a_{2m-1}$ is convergent. To this end, we apply recursively the first equation in (27), getting

$$a_{2m-1} \leq k\left(\varepsilon^{2m} + \varepsilon^{2m-1} + \cdots + \varepsilon^{m+1}\right) + \varepsilon^{m-1}a_1 \leq km\varepsilon^{m+1} + \varepsilon^{m-1}a_1.$$

So,

$$\sum_{m\geq 1} a_{2m-1} \leq a_1 + k\sum_{m\geq 2}(m-1)\varepsilon^{m+1} + a_1\sum_{m\geq 2}\varepsilon^{m-1},$$

which is not difficult to see that is equal to

$$\frac{a_1}{1-\varepsilon} + k\frac{\varepsilon^3 + 2\varepsilon(1-\varepsilon)}{(1-\varepsilon)^2}.$$

Finally, the convergence of the even part comes from

$$\sum_{m\geq 2} a_{2m-2} \leq k_1\sum_{m\geq 2}\varepsilon^{2m-1} + k_2\varepsilon\sum_{m\geq 2} a_{2m-3}.$$

∎

Notice that this lemma gives the convergence of $\|\Phi\|_{r_3}$, provided we take $\varepsilon = \gamma_*$, with γ_* satisfying condition (26). Concretely,

$$\|\Phi\|_{r_3} \leq \|\mathcal{I}d\|_{r_3} + \sum_{m\geq 1}\|\Delta\Phi^{(m)}\|_{r_3} = \frac{r_3}{\pi\sqrt{2}} + \sum_{m\geq 1} a_m.$$

Moreover, because of the restriction imposed by (ii) on $\|\Phi^{(2M-2)}\|_{r_3}$ we choose γ_* in such a way that $\sum a_m$ is less than r_2 $(= r_0 - 2\delta)$.

The convergence, in $\|\cdot\|_{r_3}$-norm, of N and B is easily derived from the estimates,

$$\|\Delta N^{(2M-1)}\|_{r_3}\ ,\ \|\Delta B^{(2M-1)}\|_{r_3} \leq \frac{c_1L_0}{\delta}\gamma_*^{2M+1} + L_1\gamma_* a_{2M-1}.$$

In this way we get analiticity of the transformation Φ and the vector fields N and B in a domain $\mathcal{D}_{r_3}$, where $r_3 = r_{3*}e^{r_{3*}}$ and $r_{3*} = \gamma_* r_{0*}$. This concludes the proof of the Pseudo-Normal Form Theorem.

5. INTEGRABILITY AND PSEUDO-NORMAL FORMS

Integrability is closely related to an special pseudo-normal form of the system. Precisely, the existence of a first integral will depend on the fact that $b(\xi\eta)$ vanishes.

THEOREM 5.1 (Criterium of integrability).
Let us consider a system

$$\dot{z} = F(z) = \Lambda z + \hat{F}(z), \tag{28}$$

with $\Lambda = diag\ (\lambda, -\lambda)$ *and* $\lambda \neq 0$, *verifying that there exist vector fields* N, B *and a transformation* $z = \Phi(\zeta)$,

$$N(\xi, \eta) = \begin{pmatrix} \xi A(\xi\eta) \\ -\eta A(\xi\eta) \end{pmatrix}, \qquad B(\xi, \eta) = \hat{B}(\xi, \eta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix},$$

leading it to its pseudo-normal form, i.e. satisfying $D\Phi \cdot N + B = F(\Phi)$.

Then, $\dot{z} = F(z)$ *has a first integral* $h(z)$ *if and only if* $B \equiv 0$. *Moreover, if* $h(z) = h(x, y)$ *is a first integral of this system, then it has the form* $h = \tilde{h} \circ \Phi^{-1}$, *where* $\tilde{h}(\zeta) = \tilde{h}(\xi\eta)$, *depending only on the product* $\xi\eta$.

Remark 5.1. Since $B(\xi, \eta) = (\xi b(\xi\eta), \eta b(\xi\eta))$, this result also holds for the scalar function $b(\xi\eta)$.

Proof. It is clear that if $B \equiv 0$, this is, our system can be written in Birkhoff normal form, any function of the form $\tilde{h}(\xi\eta)$ is a first integral. Moreover, this is the unique kind of first integrals it has. Then, it is straightforward to obtain one for the initial system.

To prove the theorem in the other sense, we apply some ideas given by C.L. Siegel and J.K. Moser (see [8, §30]). Indeed, assuming that system (28) has a first integral h and that $B \neq 0$ we arrive at a contradiction. Performing the transformation $z = \Phi(\zeta)$, system (28) becomes

$$\dot{\zeta} = N(\zeta) + [D\Phi(\zeta)]^{-1} B(\zeta). \tag{29}$$

It is easy to verify that $h(z)$ is a first integral of $\dot{z} = F(z)$ if and only if $(h \circ \Phi)(\zeta)$ is a first integral of (29). This first integral $\tilde{h} = h \circ \Phi$ can be written as

$$\tilde{h}(\xi, \eta) = \tilde{h}_M(\xi, \eta) + \tilde{h}_{M+1}(\xi, \eta) + \cdots$$

with $\tilde{h}_M(\xi, \eta) \neq 0$, $M \geq 1$, $\tilde{h}_J(\xi, \eta)$ being homogeneous polynomials of order J in ξ, η. Then, since $\tilde{h}$ is a first integral, we have that the equation

$$D\tilde{h} \cdot \left\{ N + [D\Phi]^{-1} B \right\} = 0 \tag{30}$$

holds for any order in the variables ξ, η. On the other hand, we know that Φ begins with the identity and that B is of order greater or equal than 3 in ξ, η. Therefore, the homogeneous polynomial of minimal order we get from the left-hand side of (30) comes from $D\tilde{h} \cdot N$

$$\begin{pmatrix} \frac{\partial}{\partial \xi} \tilde{h}_M(\xi, \eta) & \frac{\partial}{\partial \eta} \tilde{h}_M(\xi, \eta) \end{pmatrix} \cdot \begin{pmatrix} \lambda\xi + \cdots \\ -\lambda\eta - \cdots \end{pmatrix}, \tag{31}$$

where $(\lambda\xi, -\lambda\eta)$ is the linear part of $N(\xi,\eta) = (\xi A(\xi\eta), -\eta A(\xi\eta))$. If we write

$$\tilde{h}_M(\xi,\eta) = \sum_{j+k=M} h_{jk}^{(M)} \xi^j \eta^k,$$

equations (30) and (31) lead to

$$\lambda \sum_{j+k=M} (j-k) h_{jk}^{(M)} \xi^j \eta^k = 0$$

so, if $j \neq k$, we have that $h_{jk}^{(M)} = 0$. In other words, if $\tilde{h}(\xi,\eta)$ is a first integral of system (29), then it starts with a term $h_m(\xi\eta)^m$, where $m = M/2$ and $h_m \neq 0$.

Once we know how the first integral $\tilde{h}$ begins, we seek for the term of type $(\xi\eta)^s$ on the left-hand side of equation (30), having minimal order in ξ, η. First, notice that $D\tilde{h}(\xi,\eta) \cdot N(\xi,\eta)$ does not contribute to this kind of terms, because if

$$\tilde{h}(\xi,\eta) = \cdots + c_\ell(\xi\eta)^\ell + \cdots + d_{jk}\xi^j\eta^k,$$

with $j \neq k$, it follows that

$$\begin{aligned}
&D\tilde{h}(\xi,\eta) \cdot N(\xi,\eta) \\
&\quad = \big(\cdots + \ell c_\ell(\xi\eta)^{\ell-1}\eta + \cdots + j d_{jk}\xi^{j-1}\eta^k \\
&\qquad + \cdots + \ell c_\ell(\xi\eta)^{\ell-1}\xi + \cdots + k d_{jk}\xi^j\eta^{k-1} + \cdots\big) \cdot \begin{pmatrix} \xi A(\xi\eta) \\ -\eta A(\xi\eta) \end{pmatrix} \\
&\quad = \cdots + c_\ell(\ell-\ell)(\xi\eta)^\ell + \cdots + d_{jk}(j-k)\xi^j\eta^k + \cdots \\
&\quad = d_{jk}(j-k)\xi^j\eta^k + \cdots
\end{aligned}$$

Concerning the second part of (30), since we are assuming $B \neq 0$, there must exist a non-zero constant β_ℓ, in such a way that $b(\xi\eta) = \beta_\ell(\xi\eta)^\ell +$ h.o.t. Moreover, since $[D\Phi]^{-1} B = B + \cdots$, the $(\xi\eta)^s$-term of minimal order provided by $D\tilde{h} \cdot [D\Phi]^{-1} B$ comes from

$$\begin{aligned}
&\begin{pmatrix} m h_m(\xi\eta)^{m-1}\eta + \cdots & m h_m(\xi\eta)^{m-1}\xi + \cdots \end{pmatrix} \cdot \begin{pmatrix} \xi\beta_\ell(\xi\eta)^\ell + \cdots \\ \eta\beta_\ell(\xi\eta)^\ell + \cdots \end{pmatrix} \\
&\quad = 2 m h_m \beta_\ell (\xi\eta)^{m+\ell} + \cdots,
\end{aligned}$$

where $+\cdots$ means terms of higher order. From equation (30), it follows that $m h_m \beta_\ell = 0$, which is a contradiction since m, h_m and β_ℓ do not vanish. ∎

To extract from $b(\xi\eta)$ the obstructions to integrability, we will define a kind of invariants, which will be derived from its coefficients. This set of invariants must be independent of the pseudo-normal form exhibited by the system. Since it is not uniquely determined (unless we fix the projection of the transformation, $\mathcal{P}\Phi$), we need to know, first, the family of transformations preserving the aspect of the pseudo-normal form of the system and, second, how they affect the coefficients of $b(\xi\eta)$. In this sense, we have the following result about existence.

PROPOSITION 5.1. *Given a vector field F, assume that we have vector fields N, B and a transformation $X = \Phi(\zeta)$ leading it into pseudo-normal form, that is,*

$$D\Phi \cdot N + B = F(\Phi).$$

Then, if we perform a close to the identity transformation, $\zeta = \Psi(\chi)$,

$$\left\{ \begin{array}{rcl} \xi & = & \mathsf{x}\psi(\mathsf{xy}) \\ \eta & = & \mathsf{y}\psi(\mathsf{xy}) \end{array} \right. \tag{32}$$

it follows that the aspect of the pseudo-normal form corresponding to F is preserved, i.e.

$$D(\Phi \circ \Psi)(\chi) \cdot N'(\chi) + B'(\chi) = F((\Phi \circ \Psi)(\chi)).$$

Moreover, the new vector fields $\mathcal{N}'$ and $\mathcal{B}'$ are given by $\mathcal{N} \circ \Psi$ and $\mathcal{B} \circ \Psi$, respectively, and it is straightforward to verify that $\mathcal{P}(\Phi \circ \Psi) = (\mathcal{P}\Phi) \circ \Psi$.

Proof. Using that the change $z = \Phi(\zeta)$ satisfies

$$D\Phi(\zeta) \cdot \mathcal{N}(\zeta) + \mathcal{B}(\zeta) = F(\Phi(\zeta)),$$

it follows that

$$\dot{\zeta} = (\Phi^* F)(\zeta) = [D\Phi(\zeta)]^{-1} F(\Phi(\zeta)) = N(\zeta) + [D\Phi(\zeta)]^{-1} B(\zeta). \tag{33}$$

Hence, by performing a change $\zeta = \Psi(\chi)$ of the form (32), the latter equation becomes

$$\begin{aligned} \dot{\chi} &= [D\Psi(\chi)]^{-1} N(\Psi(\chi)) + [D\Psi(\chi)]^{-1} [(D\Phi)(\Psi(\chi))]^{-1} B(\Psi(\chi)) \\ &= (\Psi^* N)(\chi) + [D(\Phi \circ \Psi)]^{-1} (\chi) B(\Psi(\chi)). \end{aligned}$$

If we want it to be of the form (33) it must verify

$$(\Psi^* N)(\chi) = N'(\chi) = \begin{pmatrix} \mathsf{x}\mathcal{A}(\mathsf{xy}) \\ -\mathsf{y}\mathcal{A}(\mathsf{xy}) \end{pmatrix}$$

and

$$B(\Psi(\chi)) = B'(\chi) = \begin{pmatrix} \mathsf{x}\beta(\mathsf{xy}) \\ \mathsf{y}\beta(\mathsf{xy}) \end{pmatrix}.$$

Thus, by using the explicit definition of Ψ, it follows that

$$D\Psi(\mathsf{x},\mathsf{y}) = \begin{pmatrix} \psi + \mathsf{xy}\psi' & \mathsf{x}^2\psi' \\ \mathsf{y}^2\psi' & \psi + \mathsf{xy}\psi' \end{pmatrix}$$

and then det $D\Psi(\mathsf{x},\mathsf{y}) = \psi^2 + (\mathsf{xy})(\psi^2)' := J(\mathsf{xy})$. So,

$$[D\Psi(\mathsf{x},\mathsf{y})]^{-1} = \frac{1}{J(\mathsf{xy})} \begin{pmatrix} \psi + \mathsf{xy}\psi' & -\mathsf{x}^2\psi' \\ -\mathsf{y}^2\psi' & \psi + \mathsf{xy}\psi' \end{pmatrix}.$$

Besides,

$$N(\Psi(\chi)) = \begin{pmatrix} \mathsf{x}\psi(\mathsf{xy})A(\mathsf{xy}\psi^2(\mathsf{xy})) \\ -\mathsf{y}\psi(\mathsf{xy})A(\mathsf{xy}\psi^2(\mathsf{xy})) \end{pmatrix} = \begin{pmatrix} \mathsf{x}\psi\mathsf{A}(\mathsf{xy}) \\ -\mathsf{y}\psi\mathsf{A}(\mathsf{xy}) \end{pmatrix}.$$

Indeed,

$$\begin{aligned} (\Psi^*N)(\chi) &= [D\Psi(\chi)]^{-1}\, N(\Psi(\chi)) \\ &= \frac{1}{J(\mathsf{xy})} \begin{pmatrix} \psi + \mathsf{xy}\psi' & -\mathsf{x}^2\psi' \\ -\mathsf{y}^2\psi' & \psi + \mathsf{xy}\psi' \end{pmatrix} \begin{pmatrix} \mathsf{x}\psi\mathsf{A}(\mathsf{xy}) \\ -\mathsf{y}\psi\mathsf{A}(\mathsf{xy}) \end{pmatrix} \\ &= \frac{1}{J(\mathsf{xy})} \begin{pmatrix} \mathsf{x}\psi^2\mathsf{A} + \mathsf{x}^2\mathsf{y}\psi\psi'\mathsf{A} + \mathsf{x}^2\mathsf{y}\psi'\psi\mathsf{A} \\ -\mathsf{xy}^2\psi\psi'\mathsf{A} - \mathsf{y}\psi^2\mathsf{A} - \mathsf{xy}^2\psi'\psi\mathsf{A} \end{pmatrix} \\ &= \frac{1}{J(\mathsf{xy})} \begin{pmatrix} \mathsf{x}J(\mathsf{xy})\mathsf{A}(\mathsf{xy}) \\ -\mathsf{y}J(\mathsf{xy})\mathsf{A}(\mathsf{xy}) \end{pmatrix} = \begin{pmatrix} \mathsf{x}\mathcal{A}(\mathsf{xy}) \\ -\mathsf{y}\mathcal{A}(\mathsf{xy}) \end{pmatrix}, \end{aligned}$$

where $\mathcal{A}(\mathsf{xy}) = A(\mathsf{xy}\psi^2(\mathsf{xy}))$. With respect to the *remainder term* B', we get

$$B'(\chi) = \begin{pmatrix} \mathsf{x}\psi(\mathsf{xy})b(\mathsf{xy}\psi^2(\mathsf{xy})) \\ \mathsf{y}\psi(\mathsf{xy})b(\mathsf{xy}\psi^2(\mathsf{xy})) \end{pmatrix} = \begin{pmatrix} \mathsf{x}\beta_1(\mathsf{xy}) \\ \mathsf{y}\beta_2(\mathsf{xy}) \end{pmatrix}.$$

Finally, to check that $\mathcal{P}(\Phi \circ \Psi) = (\mathcal{P}\Phi) \circ \Psi$, we consider $\Phi = (\phi_1, \phi_2)$ where

$$\phi_\ell = \sum_{j+k\geq 1} \phi_{jk}^{(\ell)} \xi^j \eta^k.$$

Then, with respect to the projection P_1 it turns out that

$$\begin{aligned} P_1(\phi_1 \circ \Psi)(\mathsf{x},\mathsf{y}) &= P_1 \left\{ \sum_{j+k\geq 1} \phi_{jk}^{(1)} \mathsf{x}^j \mathsf{y}^k (\psi_1(\mathsf{xy}))^j (\psi_2(\mathsf{xy}))^k \right\} \\ &= \sum_{k\geq 0} \phi_{k+1,k}^{(1)} (\mathsf{x}\psi_1(\mathsf{xy}))^{k+1} (\mathsf{y}\psi_2(\mathsf{xy}))^k = ((P_1\phi_1) \circ \Psi)(\mathsf{x},\mathsf{y}). \end{aligned}$$

In the same way it is proved that $P_2(\phi_2 \circ \Psi)(\mathsf{x},\mathsf{y}) = (P_2\phi_2)\circ\Psi$. ∎

We are going to prove that this family is completely determined, that is, any transformation $\zeta = \Psi(\chi)$ preserving the pseudo-normal form has to be necessarily of the form (32). As it is easy to verify, this family presents a group-like structure so, in this way, it constitutes a generalization of the *group of self-transformations* of the normal norm given by Moser in [5].

PROPOSITION 5.2. *If* $\zeta = \Psi(\chi)$ *is a transformation which preserves the pseudo-normal form then it is of the type* $(\mathsf{x}\psi(\mathsf{xy}), \mathsf{y}\psi(\mathsf{xy}))$.

Proof. Let us consider $\Phi(\zeta)$, a vector field leading our system into pseudo-normal form, and $\Psi(\chi)$ the one defining the transformation, written in the form

$$\Phi(\xi,\eta) = \begin{pmatrix} \phi_1(\xi,\eta) \\ \phi_2(\xi,\eta) \end{pmatrix}, \qquad \Psi(\mathsf{x},\mathsf{y}) = \begin{pmatrix} \psi_1(\mathsf{x},\mathsf{y}) \\ \psi_2(\mathsf{x},\mathsf{y}) \end{pmatrix},$$

where

$$\phi_1(\xi,\eta) = \xi + \sum_{j+k\geq 2} \phi^{(1)}_{jk}\xi^j\eta^k,$$

$$\phi_2(\xi,\eta) = \eta + \sum_{j+k\geq 2} \phi^{(2)}_{jk}\xi^j\eta^k.$$

Assume that $\mathcal{P}\Psi \neq \Psi$, i.e. $\mathcal{R}\Psi \neq 0$, so that

$$\psi_1(\mathsf{x},\mathsf{y}) = P_1\psi_1(\mathsf{x},\mathsf{y}) + R_1\psi_1(\mathsf{x},\mathsf{y}) = \sum_{k\geq 0}\psi^{(1)}_{k+1,k}\mathsf{x}^{k+1}\mathsf{y}^k + c_{\ell m}\mathsf{x}^\ell\mathsf{y}^m + \cdots,$$

with $\ell \neq m+1$, $c_{\ell m} \neq 0$ and $\ell + m = M$ the minimal order of this form, or analogously,

$$\psi_2(\mathsf{x},\mathsf{y}) = P_2\psi_2(\mathsf{x},\mathsf{y}) + R_2\psi_2(\mathsf{x},\mathsf{y}) = \sum_{j\geq 0}\psi^{(2)}_{j,j+1}\mathsf{x}^j\mathsf{y}^{j+1} + c_{\ell m}\mathsf{x}^{\ell'}\mathsf{y}^{m'} + \cdots,$$

where $\ell' \neq m'-1$ and $c_{\ell' m'} \neq 0$. Suppose we are in the first situation, that is, $\ell \neq m+1$, $c_{\ell m} \neq 0$. Then, if $\zeta = \Psi(\chi)$ preserves the pseudo-normal form, we have that

$$D(\Phi\circ\Psi)\cdot\mathcal{N} + \mathcal{B} = F(\Phi\circ\Psi). \tag{34}$$

Writing $F = (f,g)$, $\mathcal{N} = (\mathsf{x}\mathcal{A}(\mathsf{xy}), -\mathsf{y}\mathcal{A}(\mathsf{xy}))$ and $\mathcal{B} = (\mathsf{x}\tilde{b}(\mathsf{xy}), \mathsf{y}\tilde{b}(\mathsf{xy}))$, the first component of (34) becomes

$$< \nabla(\phi_1\circ\Psi), \mathcal{N} > (\mathsf{x},\mathsf{y}) + \mathsf{x}\tilde{b}(\mathsf{xy}) = f(\Phi\circ\Psi)(\mathsf{x},\mathsf{y}).$$

Using that $f(x,y) = \lambda x + \mathcal{O}_2(x,y)$, and the expressions for Φ, Ψ introduced above, it follows that

$$f(\Phi \circ \Psi)(\mathsf{x},\mathsf{y}) = \lambda \sum_{\substack{k\geq 0\\ 2k+1\leq M}} d_{k+1,k}\mathsf{x}^{k+1}\mathsf{y}^k + \lambda c_{\ell m}\mathsf{x}^\ell \mathsf{y}^m + \cdots$$

where $+\cdots$ means terms of order greater or equal than $\ell + m = M$ in x, y. In the same way, bearing in mind that $\mathcal{A}(\mathsf{xy}) = \lambda + \hat{\mathcal{A}}(\mathsf{xy})$, it turns out that

$$\begin{aligned} &< \nabla(\phi_1 \circ \Psi), \mathcal{N} > (\mathsf{x},\mathsf{y}) + \mathsf{x}\tilde{b}(\mathsf{xy}) \\ &= \lambda \sum_{\substack{k\geq 0\\ 2k+1\leq M}} d_{k+1,k}\mathsf{x}^{k+1}\mathsf{y}^k + \lambda(\ell - m)c_{\ell m}\mathsf{x}^\ell\mathsf{y}^m + \cdots + \mathsf{x}\tilde{b}(\mathsf{xy}). \end{aligned}$$

Equating both expressions, we have

$$\lambda c_{\ell m}\mathsf{x}^\ell \mathsf{y}^m = \lambda(\ell - m)c_{\ell m}\mathsf{x}^\ell\mathsf{y}^m,$$

that is only true if $\ell = m+1$ or $c_{\ell m} = 0$, contradicting the initial assumptions. ▮

These results lead us to define a new set of invariants. Concretely,

THEOREM 5.2. *Let us consider a system $\dot{z} = F(z)$ into pseudo-normal form,*

$$D\Phi \cdot N + B = F(\Phi),$$

and write

$$B(\xi,\eta) = \begin{pmatrix} \xi b(\xi\eta) \\ \eta b(\xi\eta) \end{pmatrix},$$

where

$$b(\xi\eta) = \sum_{k\geq 1} b_k (\xi\eta)^k. \tag{35}$$

Then, we define the set $\{\beta_m\}_m$, by means of the relation

$$\beta_m = b_m \quad (mod\ b_1 = b_2 = \ldots = b_{m-1} = 0)\,,$$

Assume now that we have a transformation Ψ leading our system into pseudo-normal form and let $\tilde{B}(\mathsf{x},\mathsf{y})$ be the corresponding new remainder term. If we write

$$\tilde{b}(\mathsf{xy}) = \sum_{k\geq 1} \tilde{b}_k (\mathsf{xy})^k \tag{36}$$

and take again

$$\tilde{\beta}_m = \tilde{b}_m \quad \left(mod\ \tilde{b}_1 = \tilde{b}_2 = \ldots = \tilde{b}_{m-1} = 0 \right),$$

it turns out that

$$\tilde{\beta}_m \equiv \beta_m \quad (mod\ \mathcal{J}_{m-1}),$$

where $\mathcal{J}_m$ *is the ideal generated by* $\beta_1, \beta_2, \ldots, \beta_m$.

Proof. Notice that this definition for β_m is not trivial since (and it is easily verified by looking at the iterative scheme presented in preceding sections) any term b_m is determined not only by the coefficients forming the vector field F but also by $b_1, b_2, \ldots, b_{m-1}$. Suppose now that the corresponding function $b^*(\mathsf{xy})$ obtained by means of a transformation $(\xi, \eta) = (\mathsf{x}\psi_1(\mathsf{xy}), \mathsf{y}\psi_2(\mathsf{xy}))$ is written as follows

$$b^*(\mathsf{xy}) = \sum_{m\geq 1} b^*_m (\mathsf{xy})^m.$$

Thus, using that $(\mathsf{x}\psi(\mathsf{xy}),\ \mathsf{y}\psi(\mathsf{xy}))$ begins with the identity,

$$\psi(\mathsf{xy}) = 1 + \psi_1(\mathsf{xy}) + \psi_2(\mathsf{xy})^2 + \cdots,,$$

it is easily derived that $\beta^*_1 = b^*_1 = b_1 = \beta_1$. Then, let us assume that the following induction hypothesis

$$\beta^*_s = \beta_s$$

holds, for $s = 1, 2, \ldots, m-1$ or, in other words,

$$b^*_s \ \ (\text{mod } b^*_1 = b^*_2 = \ldots = b^*_{s-1} = 0) = b_s \ \ (\text{mod } b_1 = b_2 = \ldots = b_{s-1} = 0),$$

where $s = 1, 2, \ldots, m-1$ and $b^*_0 = b_0 = 0$. Since

$$b^*(\mathsf{xy}) = b(\mathsf{x}\psi(\mathsf{xy}), \mathsf{y}\psi(\mathsf{xy})) = \sum_{s\geq 1} b_s\, (\mathsf{xy})^s \left\{\psi(\mathsf{xy})\psi(\mathsf{xy})\right\}^s =$$

$$= \sum_{s\geq 1} b_s\, (\mathsf{xy})^s \left\{ \sum_{n\geq 0} \sum_{j=0}^{n} \psi_j \psi_{n-j} (\mathsf{xy})^n \right\}^s,$$

we have that the contribution of the latter expression to the term of type $(\mathsf{xy})^m$ is given by

$$b^*_m \ = \ b_1 \varphi_{m-1} + b_2 \left(\sum_{i_1+i_2=m-2} \varphi_{i_1}\varphi_{i_2} \right) + b_3 \left(\sum_{i_1+i_2+i_3=m-3} \varphi_{i_1}\varphi_{i_2}\varphi_{i_3} \right)$$

$$+\cdots+b_r\left(\sum_{i_1+i_2+\cdots+i_r=m-r}\varphi_{i_1}\varphi_{i_2}\cdots\varphi_{i_r}\right)+b_{m-1}(m-1)\varphi_1+b_m,$$

where

$$\varphi_u=\sum_{\ell+k=u}\psi_\ell\psi_k.$$

Therefore, since mod $(b_1^*=b_2^*=\ldots=b_{m-1}^*=0)$ is equivalent to mod $(b_1=b_2=\ldots=b_{m-1}=0)$, we get the assertion

$$\beta_m^*=\beta_m,$$

which completes the proof. ▌

Proof of Corollary 1.1

Proof. For the Hamiltonian case, this result is derived from the Criterium of Integrability (Theorem 5.1), introduced above.

To prove it for the reversible case, we use the following result. A transformation $z=\Psi(\zeta)$ satisfying $\mathsf{R}\circ\Psi\circ\mathsf{R}=\Psi$ preserves the R-reversibility. For this reason, we call them R-symmetric. In our case, it is straightforward to verify that the transformation $z=\Phi(\zeta)$ provided by the iterative scheme (see section 3) is R-symmetric, with $\mathsf{R}(\xi,\eta)=(\eta,\xi)$. As a consequence, it follows that the vector field B is R-reversible and, therefore, $b(\xi\eta)\equiv 0$. ▌

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