

On the spectrum of the normalized Laplacian of iterated triangulations of graphs

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Abstract

The eigenvalues of the normalized Laplacian of a graph provide information on its topological and structural characteristics and also on some relevant dynamical aspects, specifically in relation to random walks. In this paper we determine the spectra of the normalized Laplacian of iterated triangulations of a generic simple connected graph. As an application, we also find closed-forms for their multiplicative degree-Kirchhoff index, Kemeny's constant and number of spanning trees.

Keywords: Complex networks, Normalized Laplacian spectrum, Graph triangulations, Degree-Kirchhoff index, Kemeny constant, Spanning trees

1. Introduction

Important structural and dynamical properties of networked systems can be obtained from the eigenvalues and eigenvectors of matrices associated to their graph representations. The spectra of the adjacency, Laplacian and normalized Laplacian matrices of a graph provide information on the diameter, degree distribution, community structure, paths of a given length, local clustering, total number

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of links, number of spanning trees and many more invariants [6, 8, 27, 28, 36]. Also, dynamic aspects of a network, such as its synchronizability and random walk properties, can be obtained from the eigenvalues of the Laplacian and normalized Laplacian matrices from which it is possible to calculate some interesting graph invariants like the Estrada index and the Laplacian energy [2, 4, 12, 17, 22].

In the last years, there has been a particular interest in the study of the eigenvalues and eigenvectors of the normalized Laplacian matrix, since many measures for random walks on a network are linked to them. These include the hitting time, mixing time and Kemeny's constant which can be considered a measure of the efficiency of navigation on the network, see [19, 21, 24, 26, 41].

This paper is organized as follows. First, in Sec 2, we recall an operation called triangulation that can be applied to any simple connected graph. In Sec 3, we determine the spectra of the normalized Laplacians of iterated triangulations of any simple connected graph and we discuss their structure. Finally, in Sec 4, we use the results found to calculate three significant invariants for an iterated triangulation of a graph: the multiplicative degree-Kirchhoff index, its Kemeny's constant and the number of spanning trees.

2. Preliminaries

Let $G(V, E)$ be any simple connected graph with vertex set V and edge set E . Let $N = N_0 = |V|$ denote the number of vertices of G and $E_0 = |E|$ its number of edges.

We consider a graph operation known in different contexts as Henneberg 1 move [18], 0-extension [30], vertex addition [11, p. 63] and triangulation [33]. Here we will use the latter name as it is more common in recent literature.

Definition 2.1 . *The triangulation of G , denoted by $\tau(G)$, is the graph obtained by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end vertices of the edge corresponding to it .*

We denote $\tau^0(G) = G$. The n -triangulation of G is obtained through the iteration $\tau^n(G) = \tau(\tau^{n-1}(G))$ and N_n and E_n will be the total number of vertices and edges of $\tau^n(G)$.

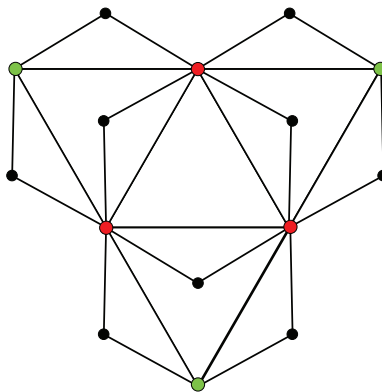


Figure 1: An example of $\tau^2(G)$. Red vertices denote the initial K_3 vertices while green and black vertices have been introduced to obtain $\tau(K_3)$ and $\tau^2(K_3)$, respectively.

From this definition clearly $E_n = 3E_{n-1}$ and $N_n = N_{n-1} + E_{n-1}$ and thus we have:

$$E_n = 3^n E_0, \quad N_n = N_0 + \frac{3^n - 1}{2} E_0. \quad (1)$$

Figure 1 shows an example of iterated triangulations where the initial graph is K_3 or the triangle graph. In this particular case, the resulting graph is known as scale-free pseudofractal graph [14] that exhibits a scale-free and small-world topology. The study of its structural and dynamic properties has produced an abundant literature, since it is a good deterministic model for many real-life networks, see [9, 10, 29, 40, 42] and references therein.

The triangulation operator τ turns each edge of a graph into a triangle, i.e. a cycle of length 3. As we will see in the next section, this property has a relevant impact on the structure of the spectrum of the normalized Laplacian matrix of $\tau^n(G)$.

We label the vertices of $\tau^n(G)$ from 1 to N_n . Let d_i be the degree of vertex i of $\tau^n(G)$, then $D_n = \text{diag}(d_1, d_2, \dots, d_{N_n})$ denotes its diagonal degree matrix and A_n the adjacency matrix, defined as a matrix with the (i, j) -entry equal to 1 if vertices i and j are adjacent and 0 otherwise.

The Laplacian matrix of $\tau^n(G)$ is $L_n = D_n - A_n$. The probability transition matrix for random walks on $\tau^n(G)$ or Markov matrix is $M_n = D_n^{-1} A_n$. M_n can be normalized to obtain a symmetric matrix

P_n .

$$P_n = D_n^{-\frac{1}{2}} A_n D_n^{-\frac{1}{2}} = D_n^{\frac{1}{2}} M_n D_n^{-\frac{1}{2}}. \quad (2)$$

The (i, j) th entry of P_n is $P_n(i, j) = \frac{A_n(i, j)}{\sqrt{d_i d_j}}$. Matrices P_n and M_n share the same eigenvalue spectrum.

Definition 2.2 *The normalized Laplacian matrix of $\tau^n(G)$ is*

$$\mathcal{L}_n = I - D_n^{\frac{1}{2}} M_n D_n^{-\frac{1}{2}} = I - P_n, \quad (3)$$

where I is the identity matrix with the same order as P_n .

We denote the spectrum of \mathcal{L}_n by $\sigma_n = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \dots, \lambda_{N_n}^{(n)}\}$ where $0 = \lambda_1^{(n)} < \lambda_2^{(n)} \leq \dots \leq \lambda_{N_n-1}^{(n)} \leq \lambda_{N_n}^{(n)} \leq 2$. The zero eigenvalue is unique, due to the existence of a stationary distribution for random walks. We denote the multiplicity of λ_i as $m_{\mathcal{L}_n}(\lambda_i)$. Eq. (3) shows a one-to-one correspondence between the spectra of \mathcal{L}_n and P_n .

The spectrum of the normalized Laplacian matrix of a graph can provide us with important structural and dynamical information about the graph, see for example [5, 8].

Definition 2.3 *If we replace each edge of a simple connected graph G by a unit resistor, we obtain an electrical network G^* associated with G . The resistance distance r_{ij} between vertices i and j of G is equal to the effective resistance between the two corresponding vertices of G^* [23].*

Definition 2.4 *The multiplicative degree-Kirchhoff index of G is defined, as in [7]:*

$$Kf^*(G) = \sum_{i < j} d_i d_j r_{ij}, \quad i, j = 1, 2, \dots, N. \quad (4)$$

This index is different from the classical Kirchhoff index [3], $Kf(G) = \sum_{i < j} r_{ij}$, as it considers also the degree distribution of the graph. See also [15, 25, 32] for recent results on these and related indices. For a k -regular graph $Kf^* = k^2 Kf$.

It is known [7] that $Kf^*(G)$ can be expressed in terms of the spectrum $\sigma_0 = \{\lambda_1, \dots, \lambda_{N_0}\}$ of the normalized Laplacian matrix \mathcal{L}_0 of G . Thus

$$Kf^*(G) = 2E_0 \sum_{k=2}^{N_0} \frac{1}{\lambda_k}, \quad (5)$$

where $0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_{N_0} \leq 2$.

For $n \geq 0$, we can obtain the degree-Kirchhoff index of the n -triangulation of G as:

$$Kf^*(\tau^n(G)) = 2E_n \sum_{k=2}^{N_n} \frac{1}{\lambda_k^{(n)}}, \quad (6)$$

where $0 = \lambda_1^{(n)} < \lambda_2^{(n)} \leq \dots \leq \lambda_{N_n-1}^{(n)} \leq \lambda_{N_n}^{(n)} \leq 2$ are the eigenvalues of \mathcal{L}_n .

The knowledge of σ_n also allows the calculation of an important graph invariant associated with random walks.

Definition 2.5 *Given a graph G , the Kemeny's constant $K(G)$, or average hitting time, is the expected number of steps required for the transition from a starting vertex i to a destination vertex, which is chosen randomly according to a stationary distribution of unbiased random walks on G (see [20] for more details).*

$K(G)$ is a constant which is independent of the selection of the starting vertex i [24]. Moreover, the Kemeny's constant can be computed as the sum of all reciprocal eigenvalues of the normalized Laplacian of G , except $1/\lambda_1$, see [5].

Therefore, given the spectrum σ_n of $\tau^n(G)$ we can write:

$$K(\tau^n(G)) = \sum_{k=2}^{N_n} \frac{1}{\lambda_k^{(n)}}. \quad (7)$$

Moreover, we have:

$$Kf^*(\tau^n(G)) = 2E_n \cdot K(\tau^n(G)). \quad (8)$$

This equation reflects the fact that, for a connected graph, the resistance distance can be related to random walks [31].

The last graph invariant considered is the number of spanning trees of a graph G . A spanning tree is a subgraph of G that includes all the vertices of G and is a tree. By using a result from Chung [8] we can find the total number of spanning trees of $\tau^n(G)$ in terms of σ_n , its normalized Laplacian spectrum, and the degrees of all the

vertices:

$$N_{\text{st}}^{(n)} = \frac{\prod_{i=1}^{N_n} d_i \prod_{i=2}^{N_n} \lambda_i^{(n)}}{\sum_{i=1}^{N_n} d_i}. \quad (9)$$

In the next section we give a closed analytical expression for this invariant for any value of $n \geq 0$.

3. The normalized Laplacian spectrum of the triangulation graph $\tau^n(G)$

In this section we find an exact analytical expression for the spectrum σ_n of \mathcal{L}_n , the normalized Laplacian of the n -triangulation graph $\tau^n(G)$. We show that this spectrum can be obtained iteratively from the spectrum of any simple connected graph G .

Lemma 3.1 *Let λ be any eigenvalue of \mathcal{L}_n , $n > 0$, such that $\lambda \neq 1$ and $\lambda \neq \frac{3}{2}$. Then 2λ is an eigenvalue of \mathcal{L}_{n-1} and its multiplicity, denoted $m_{\mathcal{L}_{n-1}}(2\lambda)$, is the same than the multiplicity $m_{\mathcal{L}_n}(\lambda)$ of the eigenvalue λ of \mathcal{L}_n . Moreover, $m_{\mathcal{L}_n}(\frac{3}{2}) = N_{n-1}$.*

Proof. Let V_{new}^n be the set of all the newly added vertices in $\tau^n(G)$ and V_{old}^n be its complement. This is also equivalent to saying that V_{old}^n contains all the vertices inherited from $\tau^{n-1}(G)$. For convenience, in the following when we refer to any vertex of V_{old}^n , we also refer to the corresponding vertex of $\tau^{n-1}(G)$.

Let $\psi = (\psi_1, \psi_2, \dots, \psi_{N_n})^\top$ be an eigenvector with respect to the eigenvalue λ of \mathcal{L}_n . Then, as $\mathcal{L}_n = I - P_n$, ψ is also an eigenvector of P_n corresponding to the eigenvalue $(1 - \lambda)$. Hence

$$P_n \psi = (1 - \lambda) \psi. \quad (10)$$

For any vertex $i \in V_{\text{old}}^n$, let $N \subset V_{\text{new}}^n$ denote the set of the new neighbors of vertex i in $\tau^n(G)$ and $N' \subset V_{\text{old}}^n$ the set of its old neighbors. There exists, from the definition of triangulation, a bijection between N and N' .

Eq. (10) can be written as

$$\sum_{j=1}^{N_n} P_n(i, j) \psi_j = (1 - \lambda) \psi_i. \quad (11)$$

If $d_n(i)$ denotes the degree of vertex i of $\tau^n(G)$, this equation leads to

$$\begin{aligned} (1-\lambda)\psi_i &= \sum_{j \in N} \frac{1}{\sqrt{d_n(i)d_n(j)}} \psi_j + \sum_{j' \in N'} \frac{1}{\sqrt{d_n(i)d_n(j')}} \psi'_{j'} \\ &= \sum_{j \in N} \frac{1}{2\sqrt{d_{n-1}(i)}} \psi_j + \sum_{j' \in N'} \frac{1}{2\sqrt{d_{n-1}(i)d_{n-1}(j')}} \psi'_{j'}. \end{aligned} \quad (12)$$

For any vertex $j \in N$, we have a similar relationship

$$(1-\lambda)\psi_j = \sum_{k=1}^{N_n} P_n(j,k)\psi_k = \frac{1}{2\sqrt{d_{n-1}(i)}} \psi_i + \frac{1}{2\sqrt{d_{n-1}(j')}} \psi_{j'}, \quad (13)$$

where vertex $j' \in N'$ is the other neighbor of vertex j in $\tau^n(G)$ (excluding the vertex i). Combining Eq. (12) and Eq. (13) we have:

$$\begin{aligned} (1-\lambda)^2\psi_i &= \frac{1}{2\sqrt{d_{n-1}(i)}} \cdot \sum_{j' \in N'} \left(\frac{1}{2\sqrt{d_{n-1}(i)}} \psi_i + \frac{1}{2\sqrt{d_{n-1}(j')}} \psi_{j'} \right) \\ &\quad + \frac{1-\lambda}{2} \cdot \sum_{j' \in N} \frac{1}{\sqrt{d_{n-1}(i)d_{n-1}(j')}} \psi'_{j'} \\ &= \frac{\psi_i}{4} + \frac{3-2\lambda}{4} \cdot \sum_{j' \in N'} \frac{1}{\sqrt{d_{n-1}(i)d_{n-1}(j')}} \psi'_{j'}. \end{aligned} \quad (14)$$

Therefore,

$$(1-2\lambda)\psi_i = \sum_{j' \in N'} \frac{1}{\sqrt{d_{n-1}(i)d_{n-1}(j')}} \psi_{j'} \quad (15)$$

holds for $\lambda \neq \frac{3}{2}$.

Eq. (15) shows that $(1-2\lambda)$ is an eigenvalue of P_{n-1} . Thus 2λ is an eigenvalue of \mathcal{L}_{n-1} and $\psi_o = (\psi_i)_{i \in V_{\text{old}}^n}$ one associated eigenvector. ψ can be totally determined by ψ_o using Eq. (13) and $m_{\mathcal{L}_{n-1}}(2\lambda) \geq m_{\mathcal{L}_n}(\lambda)$.

Suppose now that $m_{\mathcal{L}_{n-1}}(2\lambda) > m_{\mathcal{L}_n}(\lambda)$. This means that there should exist an extra eigenvector ψ_e associated to 2λ without a corresponding eigenvector in \mathcal{L}_n . But Eq. (13) provides ψ_e with an associated eigenvector of P_n since $\lambda \neq 1$, and this contradicts our assumption. Therefore, $m_{\mathcal{L}_{n-1}}(2\lambda) = m_{\mathcal{L}_n}(\lambda)$.

When $\lambda = \frac{3}{2}$, ψ_j can be determined by Eq. (13) and we find that Eq. (12) holds for any ψ_i and $\psi'_{j'}$, and thus $m_{\mathcal{L}_n}(\frac{3}{2}) = N_{n-1}$. \square

Lemma 3.2 *Let λ be any eigenvalue of \mathcal{L}_{n-1} such that $\lambda \neq 2$. Then $\frac{\lambda}{2}$ is an eigenvalue of \mathcal{L}_n . Besides, $m_{\mathcal{L}_n}\left(\frac{\lambda}{2}\right) = m_{\mathcal{L}_{n-1}}(\lambda)$.*

Proof. This is a direct consequence of Lemma 3.1 \square

Remark 3.3 *Lemmas 3.1 and 3.2 show that a large fraction of the eigenvalues of \mathcal{L}_n are obtained from the spectrum of \mathcal{L}_{n-1} as each eigenvalue, except 2, of \mathcal{L}_{n-1} will generate a unique eigenvalue of \mathcal{L}_n . As we know the multiplicity of $\frac{3}{2}$, we conclude that we have $(2N_{n-1} - m_{\mathcal{L}_{n-1}}(2))$ eigenvalues of σ_n . The rest of the spectrum consists of 1s.*

From the Perron-Frobenius theorem [16] we know that the largest absolute value of the eigenvalues of P_n is 1. Since there exists a unique stationary distribution for random walks on $\tau^n(G)$, the multiplicity of eigenvalue 1 of P_n is 1. Thus the multiplicity of the smallest eigenvalue of \mathcal{L}_n is also 1, i.e. $m_{\mathcal{L}_n}(0) = 1$.

We wish to find the multiplicity of 2, the largest eigenvalue of \mathcal{L}_n , in the context of Markov chains. A Markov chain is aperiodic if and only if the smallest eigenvalue of the Markov matrix is different from -1 . Thus, the largest eigenvalue of \mathcal{L}_n is not equal to 2 if and only if random walks on $\tau^n(G)$ are aperiodic. Since each edge of $\tau^n(G)$, $n > 0$, belongs to an odd-length cycle (a triangle), the graph is aperiodic [35, 13]. Thus $m_{\mathcal{L}_n}(2) = 0$ holds for $n > 0$. Note, however, that the value of $m_{\mathcal{L}_0}(2)$ depends on the structure of the initial graph G , since it may be periodic [38].

Definition 3.4 *Let $U = \{u_1, u_2, \dots, u_k\}$ be any finite multiset of real number. The multiset $\mathcal{R}^{-1}(U)$ is defined as*

$$\mathcal{R}^{-1}(U) = \left\{ \frac{u_1}{2}, \frac{u_2}{2}, \dots, \frac{u_k}{2} \right\}. \quad (16)$$

Theorem 3.5 *The spectrum σ_n of \mathcal{L}_n is*

$$\sigma_n = \begin{cases} \mathcal{R}^{-1}(\sigma_0 \setminus \{2\}) \cup \underbrace{\left\{ \frac{3}{2}, \dots, \frac{3}{2} \right\}}_{N_0} \cup \underbrace{\{1, 1, \dots, 1\}}_{r_1 + m_{\mathcal{L}_0}(2)} & n = 1 \\ \mathcal{R}^{-1}(\sigma_{n-1}) \cup \underbrace{\left\{ \frac{3}{2}, \dots, \frac{3}{2} \right\}}_{N_{n-1}} \cup \underbrace{\{1, 1, \dots, 1\}}_{r_n} & n > 1, \end{cases} \quad (17)$$

where

$$r_n = \frac{3^{n-1} + 1}{2} E_0 - N_0. \quad (18)$$

Proof. From results obtained so far, the multiplicity of the eigenvalue 1 of \mathcal{L}_n can be determined indirectly:

$$\begin{aligned} m_{\mathcal{L}_n}(1) &= N_n - (2N_{n-1} - m_{\mathcal{L}_{n-1}}(2)) \\ &= \frac{3^{n-1} + 1}{2} E_0 - N_0 + m_{\mathcal{L}_{n-1}}(2). \end{aligned} \quad (19)$$

Since $m_{\mathcal{L}_{n-1}}(2) = 0$ for $n > 1$, the theorem is true. \square

The two particular eigenvalues 1 and $\frac{3}{2}$ of \mathcal{L}_n are called exceptional eigenvalues for the family of matrices $\{\mathcal{L}_n\}$, for which the spectra shows self-similarity characteristics [1, 34].

Theorem 3.5 confirms a previous result on the spectrum of the scale-free pseudofractal graph, obtained with other related methods by one of the authors [39].

4. Applications of the normalized Laplacian spectra of an n -triangulation of a graph

From the spectrum of \mathcal{L}_n , the normalized Laplacian of the n -triangulation graph $\tau^n(G)$, we compute in this section some relevant invariants related to its structure. We give closed formulas for the multiplicative degree-Kirchhoff index, Kemeny's constant and the number of spanning trees of $\tau^n(G)$. The results depend only on n and some invariants of graph G .

4.1. Multiplicative degree-Kirchhoff index

Theorem 4.1 *For a simple connected graph G , the multiplicative degree-Kirchhoff indices of $\tau^n(G)$ and $\tau^{n-1}(G)$, $n > 0$, are related as follows:*

$$Kf^*(\tau^n(G)) = 6Kf^*(\tau^{n-1}(G)) - 2 \cdot 3^{n-1} N_0 E_0 + (5 \cdot 3^{2n-2} + 3^{n-1}) E_0^2. \quad (20)$$

Thus, the general expression for $Kf^*(\tau^n(G))$ is

$$Kf^*(\tau^n(G)) = 6^n Kf^*(G) + (2 \cdot 3^{n-1} - 4 \cdot 6^{n-1}) N_0 E_0 + (5 \cdot 3^{2n-1} - 8 \cdot 6^{n-1} - 3^{n-1}) E_0^2. \quad (21)$$

Proof. We use Eq. (6) and Theorem 3.5:

$$\begin{aligned}
Kf^*(\tau^n(G)) &= 2E_n \left(\sum_{k=2}^{N_{n-1}} \frac{1}{\lambda_k^{(n-1)}/2} + \frac{1}{3/2} \times N_{n-1} + \frac{1}{1} \times r_n \right) \\
&= 2E_n \left(\frac{Kf^*(\tau^{n-1}(G))}{E_{n-1}} + \frac{2N_{n-1}}{3} + r_n \right) \\
&= 6Kf^*(\tau^{n-1}(G)) - 2 \cdot 3^{n-1} N_0 E_0 + (5 \cdot 3^{2n-2} + 3^{n-1}) E_0^2,
\end{aligned} \tag{22}$$

where $\lambda_k^{(n)}$ are eigenvalues of \mathcal{L}_n . Finally:

$$Kf^*(\tau^n(G)) = 6^n Kf^*(G) + (2 \cdot 3^{n-1} - 4 \cdot 6^{n-1}) N_0 E_0 + (5 \cdot 3^{2n-1} - 8 \cdot 6^{n-1} - 3^{n-1}) E_0^2. \tag{23}$$

□

4.2. Kemeny's constant

Theorem 4.2 *The Kemeny's constant for random walks on $\tau^n(G)$ can be obtained from $K(\tau^{n-1}(G))$ through*

$$K(\tau^n(G)) = 2K(\tau^{n-1}(G)) - \frac{N_0}{3} + \frac{5^{n-1} + 1}{6} E_0. \tag{24}$$

The general expression is

$$K(\tau^n(G)) = 2^n K(G) + \frac{1 - 2^n}{3} N_0 + \frac{5 \cdot 3^n - 2^{n+2} - 1}{6} E_0. \tag{25}$$

Proof. This result is a direct consequence of Theorem 4.1 and Eq. (8).

□

4.3. Spanning trees

Theorem 4.3 *The number of spanning trees of $\tau^n(G)$, $n > 0$, is:*

$$N_{\text{st}}^{(n)} = 3^{\kappa-n} 2^{\kappa-n(2N_0-E_0-1)} N_{\text{st}}^{(0)} \tag{26}$$

where $\kappa = \sum_{i=0}^{n-1} N_i = \frac{3^n-1}{4} E_0 + n \left(N_0 - \frac{E_0}{2} \right)$.

Proof. From Eq. (9) and the definition of triangulation of a graph:

$$\frac{N_{\text{st}}^{(n)}}{N_{\text{st}}^{(n-1)}} = \frac{2^{N_n}}{3} \cdot \frac{\prod_{i=2}^{N_n} \lambda_i^{(n)}}{\prod_{i=2}^{N_{n-1}} \lambda_i^{(n-1)}}, \quad (27)$$

where $\lambda_i^{(n)}$ are the eigenvalues of L_n . We obtain, for $n > 0$:

$$\begin{aligned} \prod_{i=2}^{N_n} \lambda_i^{(n)} &= \left(\frac{3}{2}\right)^{N_{n-1}} \cdot \prod_{i=2}^{N_{n-1}} \frac{\lambda_i^{(n-1)}}{2} \\ &= \frac{3^{N_{n-1}}}{2^{2N_{n-1}-1}} \prod_{i=2}^{N_{n-1}} \lambda_i^{(n-1)}. \end{aligned} \quad (28)$$

Therefore, the following equality

$$N_{\text{st}}^{(n)} = \frac{3^{N_{n-1}-1}}{2^{2N_{n-1}-N_n-1}} \cdot N_{\text{st}}^{(n-1)} = 3^{N_{n-1}-1} 2^{N_{n-1}-2N_0+E_0+1} N_{\text{st}}^{(n-1)}, \quad (29)$$

holds for any $n > 0$, and finally we have:

$$N_{\text{st}}^{(n)} = 3^{\kappa-n} 2^{\kappa-n(2N_0-E_0-1)} N_{\text{st}}^{(0)}, \quad (30)$$

where $\kappa = \sum_{i=0}^{n-1} N_i$. \square

We recall that former expressions for the multiplicative degree-Kirchhoff index, Kemeny's constant and the total number of spanning trees of an n -triangulation graph are valid given any initial simple connected graph G .

We note here that the multiplicative degree-Kirchhoff index of $\tau^n(G)$ has been obtained recently by Yang and Klein [37] by using a counting methodology not related with spectral techniques and also that some results for the particular case of k -regular graphs have been obtained in [19] from the characteristic polynomials of the graphs.

Our result confirms both their calculations and the usefulness of the concise spectral methods described here.

Finally, the values for Kemeny's constant and the number of spanning trees of a scale-free pseudofractal graph, which is another particular case of n -triangulation graph, were obtained by one of the authors in [39]. We find the same expressions by using $N_0 = E_0 = 3$ in Eq. (25) and Eq. (26).

5. Conclusion

We have obtained a closed-form expression for the normalized Laplacian spectrum of a n -triangulation of any simple connected graph. This was possible through the analysis of the eigenvectors in relation to adjacent vertices at different iteration steps. Our method could be also applied to find the spectra of other graph families which are constructed iteratively. We have also studied the spectrum structure. The knowledge of this spectrum makes very easy, in relation to formerly known methods, the calculation of several invariants related with structural and dynamic characteristics of the iterated triangulations of a graph. As an example, we find the multiplicative degree-Kirchhoff index, Kemeny's constant and the number of spanning trees.

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