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# Modular forms with large coefficient fields via congruences

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## Abstract

We use the theory of congruences between modular forms to prove the existence of newforms with square-free level having a fixed number of prime factors such that the degree of their coefficient fields is arbitrarily large. We also prove a similar result for certain almost square-free levels.

## 1 Background

In this paper we will exploit the theory of congruences between modular forms to deduce the existence of newforms (in particular, cuspidal Hecke eigenforms) with levels of certain specific types having arbitrarily large coefficient fields. This property was already studied by Serre in [16] with other techniques. We will only consider newforms of weight 2 and trivial nebentypus. If the level is allowed to be divisible by a large power  $n$  of a fixed prime, or by the cube of a large prime  $p$ , then the coefficient fields of **all** newforms of this level will grow with  $n$  (with  $p$ , respectively) due to results of Hiroshi Saito (cf. [15], Corollary 3.4; see also [1]) showing that the maximal real subfield of certain cyclotomic field whose degree grows with  $n$  (with  $p$ , respectively) will be contained in these fields of coefficients. Thus, it is natural to deal with the question when the levels are square-free or almost-square-free, i.e., square-free except for the fact that they are divisible by a fixed power of a small prime.

In the square-free case, for any given number  $t$ , we will prove that in levels which are the product of exactly  $t$  primes there are newforms with arbitrarily large coefficient fields. We will recall results of Mazur on reducible primes for newforms of prime level that give the case of  $t = 1$ . Then, a generalization of these results will allow us to deduce the case  $t = 2$ . For  $t \geq 3$  we follow a completely different approach, namely, we exploit congruences involving certain elliptic curves whose construction is on the one hand related to Chen's celebrated results on (a partial answer to) Goldbach's conjecture (cf. [2]) and on the other hand inspired by Frey curves as in the proof of Fermat's Last Theorem (the diophantine problem that we will consider will be a sort of Fermat-Goldbach mixed problem). It is via the level lowering results in [14] that the desired congruence will be guaranteed. The precise statement of our first main result is the following:

**Theorem 1.** *Let  $B$  and  $t$  be two given positive integers. Then, there exist  $t$  different primes  $p_1, p_2, \dots, p_t$  such that if we call  $N$  their product, in the space of cuspforms of weight*

2, level  $N$  and trivial nebentypus there exists a newform  $f$  whose field of coefficients  $\mathbb{Q}_f$  satisfies:

$$[\mathbb{Q}_f : \mathbb{Q}] > B.$$

In the almost-square-free case, we will consider levels  $N = 2^k p_1 \dots p_t$  which are square-free except that they are divisible by a small power of 2. We study these particular levels because newforms of these levels can be related to prime values of certain binomials already studied in the literature (for a more thorough discussion, see Remark 1 on section 5). We will prove that for any fixed  $t$ , among newforms with such levels the fields of coefficients have unbounded degree. We will use congruences with certain  $Q$ -curves constructed from solutions to the problem of finding prime values attained by the expression  $(x^4 + y^2) / c$ . Again, these  $Q$ -curves will also have some features inspired by Frey curves, and the existence of the desired congruences will be a consequence of level lowering. For  $c = 1$  it is a celebrated result of Friedlander and Iwaniec that infinitely many primes are of the form  $x^4 + y^2$  (cf. [8]). Here we give the following generalization.

**Theorem 2.** *Let  $B > 0$  fixed, and  $\Lambda$  the usual Von Mangoldt function. Then, we have uniformly in  $c \leq (\log x)^B$*

$$\sum_{c|a^2+b^4 \leq cx} \Lambda((a^2 + b^4) / c) = K(c)x^{3/4} + o(x^{3/4}) \tag{1}$$

where  $a, b$  run over positive integers, and  $K$  is completely explicit in terms of  $c$ .

In Theorem 5 in section 6 we give the precise value of  $K(c)$ . For our application to congruences between modular forms we only need a mild version of the particular case with fixed  $c$  of the form  $c = 5^\ell$ . The following is a direct consequence of the previous theorem.

**Corollary 3.** *Let  $c$  be a positive odd integer. Then there are infinitely many primes of the form  $(x^4 + y^2) / c$  if and only if  $c$  can be written as the sum of two squares.*

The precise statement of our second main result, the one covering the almost-square-free level case, is the following:

**Theorem 4.** *Let  $B$  and  $t$  be two given positive integers. Then, there exist  $\alpha \in \{5, 8\}$  and  $t$  different odd primes  $p_1, p_2, \dots, p_t$  such that if we call  $N$  the product of these  $t$  primes, in the space of cuspforms of weight 2, level  $2^\alpha N$  and trivial nebentypus there exists a newform  $f$  with field of coefficients  $\mathbb{Q}_f$  satisfying:*

$$[\mathbb{Q}_f : \mathbb{Q}] > B.$$

Let us stress that the results on prime values of  $(x^4 + y^2) / c$ , in Theorem 2 and its corollary, are interesting in its own right, independently of the application to finding newforms with large coefficient fields. On the other hand it is also important to clarify that the proof of Theorem 2 is a generalization of Theorem 1.1 in [8], and does not contain

any new conceptual ingredient not present in [8]. It is however the case that some of the computations done in [8] do not apply to this case in a straightforward manner and, hence, they must be done now with the required level of generality in the variable  $c$ .

**2 Theorem 1 for the case of prime level: Mazur’s argument**

Suppose that the level  $N$  is prime and that  $\ell > 3$  is a prime that divides  $N - 1$ . Then it is proved in [11] that the prime is Eisenstein, meaning that there is a newform  $f$  of weight 2 and level  $N$  such that if we call  $K_f$  its field of coefficients there is a prime  $\lambda$  dividing  $\ell$  in the ring of integers of  $K_f$  for which we have  $a_p \equiv 1 + p \pmod{\lambda}$  for all primes  $p$ . The residual mod  $\lambda$  Galois representation attached to  $f$  is reducible. In particular, we have

$$a_2 \equiv 3 \pmod{\lambda} \tag{1}$$

The coefficients of the modular form  $f$  and those of any Galois conjugate  $f^\sigma$  all satisfy the bound  $|a_p| \leq 2\sqrt{p}$ , in particular  $a_2$  and all its Galois conjugates have absolute value bounded above by  $2\sqrt{2} < 3$ . Then,  $a_2 - 3$  is a non-zero algebraic integer, whose norm is divisible by  $\ell$  because of congruence (1) and with absolute value at most  $(3 + 2\sqrt{2})^{\deg K_f}$ . Hence

$$\ell \leq (3 + 2\sqrt{2})^{\deg K_f}$$

Thus,  $\deg K_f$  is bigger than a fixed constant times  $\log \ell$ . Taking  $\ell$  big and using Dirichlet’s theorem to find an  $N \equiv 1 \pmod{\ell}$ , we can make  $\deg K_f$  as big as we like. This proves the case of prime level ( $t = 1$ ) of Theorem 1.

**3 A Frey curve adapted to Chen results, and the case  $t \geq 3$  of Theorem 1**

Let  $\ell$  be a (large) prime number, and assuming for the moment the truth of Goldbach’s conjecture let us write the even number  $2^{\ell+4}$  as the sum of two prime numbers:  $2^{\ell+4} = p + q$ . Since  $p$  and  $q$  are clearly non-congruent modulo 4, we assume without loss of generality that  $p \equiv 3 \pmod{4}$ . Let  $F$  be the semistable Frey curve associated to the triple  $p, q, 2^{\ell+4}$ :

$$y^2 = x(x - p) (x + 2^{\ell+4})$$

Its conductor is  $2pq$ , while its minimal discriminant is  $\Delta = (2^{\ell+4}pq)^2 / 2^8 = (2^\ell pq)^2$ .

The modularity of all semistable elliptic curves, proved by Wiles in [17], implies that there is a newform  $f$  of weight 2 and level  $2pq$  corresponding to  $F$ .

The mod  $\ell$  Galois representation  $F[\ell]$  of  $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  is irreducible by results of Mazur, and unramified at 2 because the 2-adic valuation of the discriminant is divisible by  $\ell$  (as in the original Frey curves related to solutions of Fermat’s Last Theorem). Although it comes initially from a newform  $f$  of level  $2pq$ , by level-lowering (cf. [14]) it arises also from a newform  $f'$  of level  $pq$ .

The trace of the action of  $\text{Frob } 2$  on  $F[\ell]$  is  $\pm(1 + 2)$ , because this is the well-known necessary condition for level-raising, i.e., for the existence of an  $\ell$ -adic Galois representation with semistable ramification at 2 providing a lift of  $F[\ell]$ , and we have such a lift by construction: it is given by the Galois action on the full  $\ell$ -adic Tate module  $T_\ell(F)$  of the curve  $F$ . So if we call  $\{a_p\}$  the coefficients of  $f'$  we get

$$a_2 \equiv \pm 3 \pmod{\lambda} \tag{2}$$

for a prime  $\lambda$  dividing  $\ell$  in the field of coefficients of  $f'$ . From this congruence we can argue as we did in the previous section using congruence (1) and conclude easily that just by choosing the prime  $\ell$  sufficiently large we can construct newforms of weight 2 and level  $N = pq$  a product of two different primes with arbitrarily large field of coefficients. Since Goldbach's conjecture remains open, in order to get an unconditional result we need to move to the case of three primes in the level. Using the results of Chen on Goldbach's problem (cf. [2]) we know that for  $\ell$  sufficiently large  $2^{\ell+4}$  can be written as the sum of a prime and a "pseudo-prime", i.e., a number that is either a prime or the product of two different primes. Then, in particular, one of the following is true for infinitely many  $\ell$ :  $2^{\ell+4}$  can be written as the sum of two primes  $p$  and  $q$ , or  $2^{\ell+4}$  can be written as the sum of a prime  $p$  and the product of two primes  $qr$ . If the first is true, as we have just seen, this will prove Theorem 1 for levels which are the product of two primes, and if the second is true a similar argument with the triple  $p, qr, 2^{\ell+4}$  shows that the theorem is true for levels which are the product of three primes.

Thus, to finish the proof of the case  $t = 3$  of Theorem 1, it remains to show that if the result holds for  $t = 2$  then it also holds for  $t = 3$ . But this is just an application of raising the level (cf. [13]), because whenever we have a modular form of level  $pq$  and an irreducible mod  $\ell$  Galois representation attached to it, the same residual representation is also realized in some newform of level  $pqr$ , as long as the prime  $r$  satisfies the required condition for level raising (and it is well known that there are infinitely many primes  $r$  that satisfy this condition, cf. [13]). Thus, whenever we have found a newform  $f'$  as in the previous argument, of level  $pq$  and satisfying (2), there are also newforms with levels of the form  $pqr$  also satisfying (2) and from this the proof of the theorem for the case  $t = 3$  follows exactly as explained above.

To treat the case of more than three primes, we modify the argument above by further raising the level. Starting with the irreducible mod  $\ell$  representation afforded by  $F[\ell]$  of conductor either  $pq$  or  $pqr$ , and for any given  $t \geq 4$ , we can find forms giving the same residual representations in levels equal to the product of  $t$  primes by just raising the level  $t - 2$  ( $t - 3$ , respectively) times. For this, we have to take care not to lose spurious primes as we add on new ones. The required analysis is carried out in [4].

We conclude that Theorem 1 is true for any  $t \geq 3$ . Since the case of prime level was dealt with in the previous section, at this point only the case of  $t = 2$  remains unsolved (and a proof of Goldbach's conjecture would be enough to handle it).

#### 4 The case $t = 2$ of Theorem 1 via a result of Ogg

One way to treat Theorem 1 for the case where  $N = pq$  (without proving Goldbach's conjecture) is to appeal to the results of Ogg in [12]. If  $p$  and  $q$  are distinct primes, Ogg finds a degree 0 cuspidal divisor on  $X_0(pq)$  whose image on  $J_0(pq)$  has order equal to the numerator of the fraction  $(p - 1)(q + 1)/24$ . Take  $\ell > 3$ . If  $\ell$  divides  $(p - 1)(q + 1)$ , using exactly the same arguments applied in (cf. [11]) in the case of prime level we find an eigenform  $f$  at level  $pq$  that is Eisenstein mod  $\ell$ , therefore giving a reducible residual mod  $\ell$  representation. In particular, this means that the coefficient  $a_2$  satisfies again congruence (1) as in section 2, for some prime  $\lambda$  dividing  $\ell$  in its field of coefficients  $K_f$ , and we deduce as before that the degree of  $K_f$  is large (it grows with  $\ell$ ). We need to ensure that  $f$  is genuinely a newform, i.e., that its eigenvalues do not arise at level  $p$  or at level  $q$ . We begin as before by taking  $\ell$  large. Then we find  $q \equiv -1 \pmod{\ell}$  and pick  $p$  to be a random

prime that is not congruent to 1 mod  $\ell$ . Since the Eisenstein primes at prime level  $N$  are divisors of  $N - 1$ , we see that  $\ell$  is not an Eisenstein prime at either level  $p$  or level  $q$  while it is an Eisenstein prime at level  $pq$ , thus the form  $f$  must be a newform of level  $pq$ . This completes the proof of the case  $t = 2$  of Theorem 1. Thus, putting together the results of the last three sections, we conclude that Theorem 1 for any positive value of  $t$ .

## 5 The proof of Theorem 4

To prove Theorem 4 we follow a strategy similar to the one explained in section 3, except that now we will start from a diophantine equation such that the elliptic curve corresponding to any solution is a  $Q$ -curve defined over  $\mathbb{Q}(i)$ . For the Fermat-type equation  $x^4 + y^2 = z^p$  an attached  $Q$ -curve was proposed by Darmon in [3] and in the work of Ellenberg [6] it was shown using the modularity of this curve that the diophantine equation does not have non-trivial solutions for large  $p$ . We will consider instead the diophantine problem:

$$x^4 + y^2 = 5^\ell p$$

The result that we will prove in the next sections (see Theorem 5 in section 6, specialized to the case  $c = 5^\ell$ ), which is a generalization of the case  $\ell = 0$  solved by Friedlander-Iwaniec in [8], implies that for any prime exponent  $\ell$  there exist infinitely many primes  $p$  such that there are integer solutions  $A, B$  to this equation. Thus, if  $\ell$  is a given prime and  $A, B, p$  satisfy

$$A^4 + B^2 = 5^\ell p$$

with  $p$  prime, we consider, as in the work of Darmon and Ellenberg, the elliptic curve  $E$ :

$$y^2 = x^3 + 4Ax^2 + 2(A^2 + iB)x$$

For simplicity, and since we have infinitely many primes  $p$  satisfying the equation, we assume  $p \neq \ell$ .

The following properties of this curve are known (cf. [6]):

$E$  is 2-isogenous to its Galois conjugate, in particular it is a  $Q$ -curve. If we call  $W$  its Weil restriction defined over  $\mathbb{Q}$ , it is a  $GL_2$ -type abelian surface and thus it has a compatible family of 2-dimensional Galois representations of  $G_{\mathbb{Q}}$  attached. These representations have coefficients in  $\mathbb{Q}(\sqrt{2})$ . This abelian surface is semistable outside 2 and it is modular. Computing the conductor of  $W$  it follows that the modular form  $f$  attached to  $W$  has level  $2^\alpha 5p$ , with  $\alpha = 5$  or 8. It has weight 2 and trivial nebentypus. This newform has coefficients in  $\mathbb{Q}(\sqrt{2})$  and it has an inner twist.

We now consider for the prime  $\ell$  we started with and  $\lambda \mid \ell$  in  $\mathbb{Q}(\sqrt{2})$  the residual mod  $\lambda$  representation  $\bar{\rho}_{W,\lambda}$  attached to  $W$ . Assuming that  $\ell > 13$  it is known that this representation is irreducible (cf. [6]). Since the discriminant of  $E$  is  $512(A^2 + iB)5^\ell p$  we can, as in [6], apply the Frey trick at the semistable prime 5 (observe that 5 is unramified in  $\mathbb{Q}(i)/\mathbb{Q}$ ): locally at 5 the valuation of the discriminant is divisible by  $\ell$  (on the other hand, this does not happen locally at the prime  $p$ ). Thus, we conclude that  $\bar{\rho}_{W,\lambda}$  is unramified at 5: more precisely it has conductor  $2^\alpha p$  with  $\alpha \in \{5, 8\}$ . If we apply lowering the level (cf. [14]) we see that there is a newform  $f'$  of level  $2^\alpha p$ , weight 2 and trivial nebentypus such that some of its corresponding residual Galois representations in characteristic  $\ell$  is  $\bar{\rho}_{W,\lambda}$ .

Now we can conclude as in section 3: we focus on the coefficient  $a_5$  of  $f'$  and we want to use the fact that there is a mod  $\lambda$  congruence between  $f'$  and a newform  $f$  corresponding to an abelian variety which is semistable at 5 (namely, the abelian surface  $W$ ). Then, the necessary condition for level-raising must be satisfied and this translates into the following congruence:

$$a_5 \equiv \pm 6 \pmod{\lambda}$$

From this congruence we see, as in sections 2 and 3, that the minimal field of definition of  $a_5$ , and a fortiori the field of coefficients of  $f'$ , has a degree that grows with  $\ell$ . This solves the case of almost-square-free level  $2^\alpha p$  with  $\alpha = 5$  or  $8$ , i.e., the case  $t = 1$ , of Theorem 4. The case of Theorem 4 for levels of the form  $2^\alpha p_1 \cdot \dots \cdot p_t$ , with  $t > 1$  fixed and the  $p_i$  odd, different primes, can be deduced from this by  $t - 1$  applications of raising the level as explained in section 3.

**Remark 1.** The referee asks whether or not we can extend Theorem 4 to cover other almost-square-free levels by choosing other Frey curves or  $Q$ -curves. This seems plausible, but will require extra work because of the following reason. On one hand, several Frey curves or  $Q$ -curves have been proposed in the literature to attack diophantine equations of Fermat type, and for all of them the resulting modular forms have almost-square-free level, for example the equations  $x^4 + 2y^2 = z^p$  and  $x^4 + 3y^2 = z^p$  are solved in [5] using Frey  $Q$ -curves. On the other hand, the problem of prime values for the kind of binomials appearing on the left hand side of these equations is solved in the literature only for two cases, namely, the case we have considered and the case of irreducible cubic forms (cf. [9] and [10]). Consider for example  $x^3 + 2y^3$ . We can apply to it the standard trick of factorizing the form and construct a Frey elliptic curve, but this curve will be defined over  $\mathbb{Q}(\sqrt[3]{-2})$ . There is no reason for this curve to be a  $Q$ -curve, so probably there is no way to relate this equation to classical modular forms.

### 6 Prime values of $(x^4 + y^2)/c$

We now introduce some notation which will be used from now on. For any given prime  $p$ , and any integer  $d$  we denote  $v_p(d)$  the highest power of  $p$  dividing  $d$ . Moreover, we will write  $d$  as  $d = d_1 d_2^2 = d_1 d_3^2 d_4^4$ , where  $d_1, d_3$  are square-free. We will consider  $\Lambda(r)$  the usual Von Mangoldt function, extended as zero over non integer numbers.

Then, the main result of this section is the following theorem.

**Theorem 5.** *Let  $B > 0$  fixed. We have uniformly in  $c \leq (\log x)^B$*

$$\sum_{(a^2+b^4)/c \leq x} \sum \Lambda((a^2 + b^4)/c) = 4\pi^{-1} \kappa G(c) (cx)^{3/4} + o(x^{3/4}) \tag{2}$$

where  $a, b$  run over positive integers,  $G$  is a multiplicative function, and

$$\kappa = \int_0^1 (1 - t^4)^{1/2} dt = \Gamma\left(\frac{1}{4}\right)^2 / 6\sqrt{2\pi}. \tag{3}$$

**Remark 2.** It is important to emphasize here that the constant  $G(c)$ , explicitly described in Lemma 8, can take the zero value, and it does precisely in those  $c$  which are non representable as the sum of two squares, or such that  $v_2(c) \equiv 3 \pmod{4}$ . For trivial reasons there is at most one prime in the sequence  $(a^2 + b^4)/c$  in these cases, since none of the elements is in fact coprime with  $c$  if  $c$  is non representable as the sum of two squares, and if  $v_2(c) \equiv 3 \pmod{4}$ , every integer of the sequence has to be even. Hence, the proof that follows restricts to those values of  $c$  such that this constant is non zero since, in any other case, the result is trivial.

The proof of Theorem 5 relies in the verification of the hypothesis needed to apply the Asymptotic Sieve due to Friedlander and Iwaniec in [7], but now for the sequence  $a(c)_n = 0$  for any  $(n, c) > 1$ , and for  $n$  coprime to  $c$  given by

$$a(c)_n = \sum_{(a^2+b^2)/c=n} \mathfrak{z}(b), \tag{4}$$

where  $a, b$  are integers non necessarily positive, and  $\mathfrak{z}$  is the function with value  $\mathfrak{z}(m^2) = 2$ , for any integer  $m \neq 0$ ,  $\mathfrak{z}(0) = 1$ , and  $\mathfrak{z}(b) = 0$  in any other case. From now on we will only consider integers  $n$  coprime to  $c$  and, then, we have  $a(c)_n = a_{cn}^{old}$  where  $a_n^{old}$  is the sequence related with Theorem 1 of [8]. We now include for reading convenience the hypotheses and main result of the Asymptotic Sieve. The following, with the exception of (13), is basically a copy of Section 2 in [8]. We explain the difference between (13) and (2.8) in [8] at the end of this section.

Consider a sequence of real, nonnegative, numbers  $\mathcal{A} = (a_n)_{n \geq 1}$ , and  $x$  a positive number. We want to obtain an asymptotic formula for

$$S(x) = \sum_{p \leq x} a_p \log p,$$

where the sum runs over prime numbers, in terms of  $\mathcal{A}(x) = \sum_{n \leq x} a_n$ . We suppose

$$\mathcal{A}(x) \gg \mathcal{A}(\sqrt{x}) (\log x)^2, \tag{5}$$

$$\mathcal{A}(x) \gg x^{1/3} \left( \sum_{n \leq x} a_n^2 \right)^{1/2}. \tag{6}$$

As usual in sieve theory, we will assume that for any integer  $d > 1$

$$\mathcal{A}_d(x) = \sum_{\substack{n \leq x \\ d|n}} a_n = g(d)\mathcal{A}(x) + r_d(x),$$

where  $g$  is a multiplicative function, and  $r_d(x)$  is regarded as an error term. For the function  $g$  we assume the following hypotheses

$$0 \leq g(p^2) \leq g(p) < 1, \tag{7}$$

$$g(p) \ll p^{-1}, \tag{8}$$

$$g(p^2) \ll p^{-2}, \quad \text{and} \tag{9}$$

$$\sum_{p \leq y} g(p) = \log \log y + e + O\left((\log y)^{-10}\right), \tag{10}$$

for every  $y$  and some  $e$  depending only on  $g$ . For the error term we will assume

$$\sum_{d \leq DL^2}^3 |r_d(t)| \leq A(x)L^{-2}, \tag{11}$$

uniformly in  $t \leq x$ , for some  $D$  in the range

$$x^{2/3} < D < x. \tag{12}$$

The superscript 3 in (11) restrict the sum to cube free moduli and  $L = (\log x)^{2^{24}}$ . We also require

$$\mathcal{A}_d(x) \ll d^{-1} \tau(d)^8 \mathcal{A}(x) \log x \text{ uniformly in } d \leq x^{1/3}, \tag{13}$$

and finally an estimate in bilinear forms like

$$\sum_m \left| \sum_{\substack{N < n \leq 2N \\ mn \leq x \\ (n, m\Pi) = 1}} \beta(n) \mu(mn) a_{mn} \right| \leq \mathcal{A}(x)L^{-4}, \tag{14}$$

where the coefficients are given by

$$\beta(n) = \beta(n, K) = \sum_{k|n, k \leq K} \mu(k), \tag{15}$$

for any  $K$  in the range

$$1 \leq K \leq xD^{-1}, \tag{16}$$

$N$  verifying

$$\Delta^{-1} \sqrt{D} < N < \delta^{-1} \sqrt{x} \tag{17}$$

for some  $\Delta \geq \delta \geq 2$ , and  $\Pi$  is the product of all primes  $p < P$  for some  $P$  which can be chosen conveniently in the range

$$2 \leq P \leq \Delta^{1/2^{35}} \log \log x. \tag{18}$$

In this conditions we have

**Proposition 6.** *Let  $\mathcal{A}$  be a sequence verifying the above hypotheses. Then,*

$$S(x) = H\mathcal{A}(x) \left\{ 1 + O\left(\frac{\log \delta}{\log \Delta}\right) \right\} \tag{19}$$

where  $H$  is the positive constant given by the convergent product

$$H = \prod_p (1 - g(p)) \left(1 - \frac{1}{p}\right)^{-1}, \tag{20}$$

and the implied constant depends only on the function  $g$ .

Normally  $\delta$  is a large power of  $\log x$  and  $\Delta$  a small power of  $x$ .

**Remark 3.** It is important to note that (13) is not the original assumption (1.6) in [7], but a slightly weaker. However, as the authors mention in that paper, (1.6) is only required to reduce the hypotheses

$$\sum_m \tau_5(m) \left| \sum_{\substack{N < n \leq 2N \\ mn \leq x \\ (n, m\Pi) = 1}} \beta(n) \mu(mn) a_{mn} \right| \leq \mathcal{A}(x)(\log x)^{-3}, \tag{21}$$

and

$$\sum_{d \leq D} \mu^2(d) \tau_5(d) |r_d(t)| \leq A(x) (\log x)^{-3}, \tag{22}$$

to (11) and (14). We just have to follow the reasoning in Section 2, p. 1047 of [7] to see that this reduction is also possible with our hypothesis (13).

### 7 Proof of Theorem 5

To prove Theorem 5 we will use Proposition 6 for the sequence given in (4). Hence, we have to check that the sequence verifies hypotheses (5) through (14). Given an integer  $d \geq 1$ , we denote  $\mathcal{A}_d(x; c) = \sum_{n \leq x, n \equiv 0 \pmod{d}} a(c)_n$ . The first thing that needs to be done is to find a good approximation of  $\mathcal{A}_d(x; c)$  in terms of a multiplicative function. Now,  $\mathcal{A}_d(x; c) = 0$  for  $(d, c) > 1$  and for  $(d, c) = 1$  we have

$$\mathcal{A}_d(x; c) = \sum_{k|c} \mu(k) \mathcal{A}_{ckd}^{old}(cx), \tag{23}$$

and we know by [8] that

$$\mathcal{A}_d^{old}(x) = g(d) \mathcal{A}^{old}(x) + r_d^{old}(x),$$

where the functions  $g, r^{old}$  satisfy conditions (5) through (14). Note that for any integer  $d$  the definition of  $\mathcal{A}_d^{old}(x)$  is implicit in (23) for  $c = 1$ , and observe that  $g, r^{old}$  are precisely the functions  $g, r$  appearing in [8]. Hence, to approximate  $\mathcal{A}_d(x; c)$  we are tempted to use the approximation of  $\mathcal{A}^{old}(x)$  given in Lemma 3.4 of [8]. However, this lemma only works for cube-free integers  $d$  which do not cover completely our case, since  $c$  will be any number  $c \leq (\log x)^B$ . Because of this, an extra technical difficulty appears. Indeed, any cube-free integer can be factorized as  $d = d_1 d_2^2$  with  $d_1, d_2$  square-free and coprime, conditions that are highly convenient when dealing with multiplicative functions. When the integers are not cube-free as it is our case, we need to control the contribution that comes from common divisors (c.f. Lemma 8), and we have to keep track of it throughout the argument.

Hence, our next objective is to generalize Lemma 3.4 of [8] to any integer  $d$ . As in [8], we start approximating  $\mathcal{A}_d(x; c)$  by

$$M_d(x; c) = \sum_{k|c} \mu(k) \frac{1}{ckd} \sum_{0 < (a^2 + b^2) \leq cx} \mathfrak{z}(b) \rho(b; ckd),$$

for any  $d$  coprime to  $c$ , where  $\rho(b, d)$  denotes the number of solutions  $\alpha \pmod{d}$  to the congruence  $\alpha^2 + b^2 \equiv 0 \pmod{d}$ , and  $M_d(x; c) = 0$  otherwise. The following is a trivial consequence of Lemma 3.1 of [8].

**Lemma 7.** *Let  $B > 0$ . For any  $c \leq (\log x)^B$  we have*

$$\sum_{d \leq D} |\mathcal{A}_d(x; c) - M_d(x; c)| \ll D^{1/4} x^{9/16 + \varepsilon}$$

for any  $D \geq 1$  and  $\varepsilon > 0$  and the implied constant depending only on  $\varepsilon$ .

Now, we need to find out the main term of  $M_d(x; c)$ , as we mentioned, by generalizing Lemma 3.4 of [8].

**Lemma 8.** *Let  $B > 0$ . We have uniformly for any  $c \leq (\log x)^B$*

$$M_d(x; c) = g_c(d) (4\kappa c^{3/4} G(c)) x^{3/4} + O(h(d)H(c)x^{1/2}),$$

where  $\kappa$  is given in (3),  $g_c(d) = 0$  for any  $(c, d) > 1$ , and  $g_c(d) = g(d)$  otherwise where  $g$  and  $h$  are the multiplicative functions given by

$$g(d) = \frac{1}{d} \sum_{v_4|d_4} v_4^2 \sum_{\substack{v_3|d_3 \\ \left(\frac{d_4}{v_4}\right)=1}} v_3(v_3, d_1^*) \sum_{\substack{v_1|\frac{d_1^*}{(v_3, d_1^*)} \\ \left(\frac{d_3 d_4}{v_3 v_4}\right)=1}} \rho(\delta(v_1, v_3, v_4)^2) \frac{\varphi(\delta(v_1, v_3, v_4))}{d_1^* d_3 d_4}, \tag{24}$$

and

$$h(d) = \frac{1}{d} \sum_{v_4|d_4} v_4^2 \sum_{\substack{v_3|d_3 \\ \left(\frac{d_4}{v_4}\right)=1}} v_3 \sum_{\substack{v_1|\frac{d_1^*}{(v_3, d_1^*)} \\ \left(\frac{d_3 d_4}{v_3 v_4}\right)=1}} \rho(\delta(v_1, v_3, v_4)^2) \tau(\delta(v_1, v_3, v_4)). \tag{25}$$

Here  $d_1^* = d_1/(d_1, 2)$  and  $\delta(v_1, v_3, v_4) = \frac{d_1^* d_3 d_4}{(d_1^*, v_3) v_1 v_3 v_4}$ . Finally  $G(c) = \sum_{k|c} \mu(k)g(ck)$  and  $H(c) = c^{1/2} \sum_{k|c} h(ck)$ .

*Proof.* We restrict only to integers  $d$  coprime to  $c$  since the result is trivial otherwise. Given  $d = d_1 d_2^2 = d_1 d_3^2 d_4^4$ , with  $d_1, d_3$  square-free, and an integer  $b$ , let us call  $b_2 = (b, d_2)$ ,  $b_1 = (b/b_2, d_1^*)$ . Then, it is fairly straightforward to prove that

$$\rho(b; d) = b_2 \rho((d_1^* d_2 / b_1 b_2)^2)$$

where  $\rho$  is the multiplicative function given by

$$\rho(p^\alpha) = 1 + \chi_4(p),$$

whit  $\chi_4$  the character of conductor 4, except  $\rho(d) = 0$  if  $4|d$ . Now, by definition, we have

$$M_d(x; c) = \sum_{k|c} \mu(k) M_{ckd}^{old}(cx) \tag{26}$$

where

$$\begin{aligned} M_d^{old}(x) &= \frac{1}{d} \sum_{0 < (a^2 + b^2) \leq x} \mathfrak{z}(b) \rho(b; d) = \frac{2}{d} \sum_{|r| \leq x^{1/4}} \rho(r^2, d) \left\{ (x - r^4)^{1/2} + O(1) \right\} \\ &= T_d + E_d, \end{aligned}$$

$T_d$  being the main term of  $M_d^{old}(x)$ , and

$$E_d \ll \frac{1}{d} \sum_{|r| \leq x^{1/4}} \rho(r^2, d) \ll x^{1/4}.$$

Now suppose  $(r, d_4) = v_4$ . Then  $(r^2, d_3 d_4^2) = v_4^2 (r/v_4, d_3)$ , and splitting the sum over the divisors of  $d_4$  we get, after some calculations,

$$T_d(x) = \frac{2}{d} \sum_{v_4|d_4} v_4^4 \sum_{\substack{|r| \leq x^{1/4} \\ \left(\frac{d_4}{v_4}\right)=1}} (r, d_3) \rho \left( \left( \frac{d_1^*}{(r, d_1^*)} \frac{d_3}{(r, d_3)} \frac{d_4}{v_4} \right)^2 \right) (x - r^4)^{1/2},$$

where  $z = z(v_4) = x/v_4^4$ . Similarly, splitting the inner sum over the divisors of  $d_3$  and  $d_1^*$ , it is easy to get

$$T_d(x) = \frac{2}{d} \sum_{v_4|d_4} v_4^4 \sum_{\substack{v_3|d_3 \\ (v_3, \frac{d_4}{v_4})=1}} v_3^3 \sum_{\substack{v_1|\frac{d_1^*}{(v_3, d_1^*)} \\ (v_1, \frac{d_3 d_4}{v_3 v_4})=1}} v_1^2 \rho(\delta(v_1, v_3, v_4)^2) \sum_{\substack{|r| \leq u^{1/4} \\ (r, \delta(v_1, v_3, v_4))=1}} (u - r^4)^{1/2},$$

where  $\delta(v_1, v_3, v_4) = \frac{d_1^* d_3 d_4}{(v_3, d_1^*) v_1 v_3 v_4}$ , and  $u = u(v_1, v_3, v_4) = z/(v_3 v_1)^4$ . Estimating the inner sum, (also done in Lemma 3.4 of [8]), we get

$$T_d(x) = \frac{2}{d} \sum_{v_4|d_4} v_4^4 \sum_{\substack{v_3|d_3 \\ (v_3, \frac{d_4}{v_4})=1}} v_3^3 \sum_{\substack{v_1|\frac{d_1^*}{(v_3, d_1^*)} \\ (v_1, \frac{d_3 d_4}{v_3 v_4})=1}} v_1^2 \rho(\delta(v_1, v_3, v_4)^2) \left\{ \frac{\varphi(\delta(v_1, v_3, v_4))}{\delta(v_1, v_3, v_4)} 2\kappa u^{3/4} + O(\tau(\delta(v_1, v_3, v_4))u^{1/2}) \right\}.$$

Substituting the value of  $u$  in the previous formula we get, for any integer  $d \geq 1$  not necessarily coprime with  $c$ ,

$$M_d^{old}(x) = g(d)4\kappa x^{3/4} + O(h(d)x^{1/2}),$$

where  $g$  and  $h$  are the multiplicative functions given by (24) and (25) respectively. We just have to plug this into (26) to get the result with  $g_c = g$  if  $(d, c) = 1$ , and 0 otherwise. Note that

$$G(c) = \sum_{k|c} \mu(k)g(ck) = \prod_{p|c} \left( g(p^{v_p(c)}) - g(p^{v_p(c)+1}) \right),$$

defines a multiplicative function. Also, observe that the multiplicativity of both  $g$  and  $h$  is a direct consequence of the definition in each case. It is straightforward to see that the value at prime powers is given by  $g(2^{4\alpha+r}) = 1/2^{3\alpha+r}$ , and for  $p$  odd,

$$g(p^{4\alpha+r}) = \frac{1}{p^{4\alpha+r}} (1 + \chi_4(p)) \left( 1 - \frac{1}{p} \right) \frac{p^\alpha - 1}{p - 1} + \frac{1}{p^{3\alpha+r}} g_r(p) \tag{27}$$

where

$$g_r(p) = \begin{cases} 1 & \text{if } r = 0 \\ 1 + \chi_4(p) \left( 1 - \frac{1}{p} \right) & \text{if } r = 1 \\ 1 + (1 + \chi_4(p)) \left( 1 - \frac{1}{p} \right) & \text{if } r = 2, 3. \end{cases} \tag{28}$$

For  $h$  we will only need its value for cube-free integers which comes from the following

$$h(p)p = 1 + 2\rho(p), \quad h(p^2)p^2 = p + 2\rho(p), \tag{29}$$

already gotten in Lemma 3.4 of [8].

We are now in position to verify hypotheses (5) through (14) for the approximation

$$\mathcal{A}_d(x; c) = g_c(d)\mathcal{A}(x; c) + r_d(x).$$

First of all we note that, by Lemmas 7 and 8, we have

$$\mathcal{A}_d(x; c) = g(d)4\kappa G(c)(cx)^{3/4} + O(d^{1/4}x^{9/16+\varepsilon}) + O(H(c)h(d)x^{1/2}), \tag{30}$$

meanwhile, trivially, we have  $a_c(n) \ll \tau(n)$ . From here, (5) and (6) follow immediately by noting that  $H(c) \ll c^\varepsilon \ll \log x$ , which is an easy consequence of the definition of  $h$ . Also

(7),(8),(9) and (10) are easy consequences of (27), (28) and the Prime Number Theorem in the arithmetic progression modulo 4. Note also that these conditions were already verified in [8] since, for cube-free integers,  $g(d)$  is the same function as the one appearing in that reference. In order to get (11) we note that, by (30) used for any given  $d$  and for  $d = 1$ , we have

$$r_d(x) = O(g(d)d^{1/4}x^{9/16+\varepsilon}) + O(H(c)h(d)x^{1/2}).$$

Moreover

$$\sum_{d \leq x}^3 h(d) \ll \prod_{p \leq x} (1 + h(p)) (1 + h(p)^2) \ll (\log x)^6,$$

and

$$\sum_{d \leq x}^3 g(d) \ll \prod_{p \leq x} (1 + g(p)) (1 + g(p)^2) \ll (\log x)^2,$$

which gives

$$\sum_{d \leq D}^3 |r_d(t)| \ll D^{1/4}x^{9/16+\varepsilon},$$

and, in particular, implies (11). For the remainder conditions, (13) and (14), we want to use the analogous results obtained in [8] for the sequence  $a_n^{old}$ . It is then mandatory to obtain the relation between the size of  $\mathcal{A}_d(x; c)$  and  $\mathcal{A}_d^{old}(x)$ . Now, (3.18) of [8] is, with our notation

$$\mathcal{A}^{old}(x) = 4\kappa x^{3/4} + O(x^{1/2}). \tag{31}$$

□

Hence, in view of (30), with  $d = 1$ , to compare  $\mathcal{A}^{old}(x)$  with  $\mathcal{A}(x; c)$  we need to control  $G(c)$ , and  $G(c)^{-1}$  for any  $c \leq (\log x)^B$ . This is the content of the next lemma.

**Lemma 9.** *Let  $c$  be an integer. If  $c$  is not representable as the sum of two squares, or  $v_2(c) \equiv 3 \pmod{4}$ , then  $G(c) = 0$ . Otherwise we have*

$$\frac{1}{c} \leq G(c) \leq \frac{1}{c^{3/4}}.$$

**Remark 4.** The lemma is intended to show bounds which are enough for our purpose and by no means need to be optimal.

*Proof.* By making an easy case by case analysis in (27) and (28) depending on  $r$ , we see that whenever  $p^\alpha$  is representable as a sum of two squares, we have

$$G(p^\alpha) \geq \frac{1}{p^\alpha},$$

which in particular implies the lower bound. The upper bound is a direct consequence of (31) and (30) since, by definition,  $\mathcal{A}(x; c) \leq \mathcal{A}(x; 1) = \mathcal{A}^{old}(x)$ .

We now deal with (13), for  $(d, c) = 1$ . It is clear that in this case  $\mathcal{A}_d(x; c) \leq \mathcal{A}_{cd}^{old}(cx)$  and it is trivial to get

$$\mathcal{A}_d^{old}(x) \ll \frac{1}{d} \tau(d) \mathcal{A}^{old}(x),$$

uniformly in  $d \leq x^{1/2-\varepsilon}$ . Hence, we have

$$\mathcal{A}_d(x; c) \ll \frac{1}{cd} \tau(cd) \mathcal{A}^{old}(cx) \leq \frac{1}{d} \tau(d) \tau(c) \mathcal{A}(x; c) \ll \frac{1}{d} \tau(d) \mathcal{A}(x; c) \log x,$$

where we have used the lower bound in Lemma 9, together with (30) with  $d = 1$ , (31) and the bound  $c \leq (\log x)^B$ . Hence, we are left with the bilinear condition (14). We will get this bound from the similar one achieved in Proposition 4.1 of [8].  $\square$

**Proposition 10.** *Let  $\eta > 0$ ,  $A > 0$  and  $B > 0$ . Then for any  $c \leq (\log x)^B$  we have*

$$\sum_m \left| \sum_{\substack{N < n \leq 2N \\ mn \leq x \\ (n, m\pi) = 1}} \mu(mn) \beta(n) a(c)_{mn} \right| \leq \mathcal{A}(x; c) L^{4-A}, \tag{32}$$

for every  $N$  with

$$x^{1/4+\eta} < N < x^{1/2} (\log x)^{-U}, \tag{33}$$

where the coefficients  $\beta(n)$  are given by (15) for any  $1 \leq C \leq N^{1-\eta}$ ,

$$(\log \log x)^2 \leq \log P \leq (\log x) (\log \log x)^{-2}.$$

where  $\Pi = \prod_{p < P} p$ . Here  $U$  and the implied constant in (32) need to be taken sufficiently large in terms of  $\eta$  and  $A$ .

*Proof of Proposition 10:*

$$\begin{aligned} \sum_m \left| \sum_{\substack{N < n \leq 2N \\ mn \leq x \\ (n, m\pi) = 1}} \mu(mn) \beta(n) a(c)_{mn} \right| &= \sum_{(m,c)=1} \left| \sum_{\substack{N < n \leq 2N \\ mn \leq x \\ (n, m\pi) = 1}} \mu(mn) \beta(n) a_{cmn}^{old} \right| \\ &\leq \sum_{c|m} \left| \sum_{\substack{N < n \leq 2N \\ mn \leq cx \\ (n, m\pi) = 1}} \mu(mn) \beta(n) a_{mn}^{old} \right| \leq \sum_m \left| \sum_{\substack{N < n \leq 2N \\ mn \leq cx \\ (n, m\pi) = 1}} \mu(mn) \beta(n) a_{mn}^{old} \right| \\ &\leq \mathcal{A}^{old}(x) L^{4-A}, \end{aligned}$$

where we have used Proposition 4.1 of [8], and  $c \leq (\log x)^B$  together with the lower bound on  $P$ . We just have to use (30) with  $d = 1$ , (31), and the lower bound in Lemma 9 to get the result.

Theorem 5 is now a direct consequence of Proposition 6 and the upper bound in Lemma 9. Recall that  $H$  is given by (20), which in this case is

$$H = \prod_p (1 - \chi_4(p) p^{-1}) = L(1, \chi_4)^{-1} = \frac{4}{\pi},$$

and we are counting only positive integers  $a, b$  in Theorem 5.

**Competing interests**

The authors declare that they have no competing interests.

**Authors' contributions**

The three authors have contributed to this work in equal parts. All authors read and approved the final manuscript.

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