

# Manipulation in games with multiple levels of output

Josep Freixas<sup>a</sup>, Cameron Parker<sup>b</sup>

<sup>a</sup> *Departament de Matemàtica Aplicada 3 i Escola Politècnica Superior d'Enginyeria de Manresa,  
Universitat Politècnica de Catalunya.*

*josep.freixas@upc.edu*

<sup>b</sup> *Department of Mathematics and Computer Science  
University of San Diego*

*San Diego, California, 92110-2492 USA.*

*cparker@sandiego.edu*

---

## Abstract

In  $(j, k)$ -games each player chooses amongst  $j$  ordered options and there are  $k$  possible outcomes. In this paper, we consider the case where players are assumed to prefer some outcomes to others, and note that when  $k > 2$  the players have an incentive to vote strategically. In doing so, we combine the theory of cooperative game theory with social choice theory, especially the theory of single-peaked preferences. We define the concept of a  $(j, k)$ -game with preferences and what it means for it to be manipulable by a player. We also consider Nash equilibriums with pure strategies for these games and find conditions that guarantee their existence.

*Keywords:* Cooperative Games, Insincere Voting, Nash Equilibrium,  $(j, k)$ -games, single-peaked preferences

*JEL:* C71, D72

---

## 1. Introduction

Simple games, in which a set of voters have two possible votes and there are two possible outputs, have been studied in the voting context since von Neumann and Morgenstern (1944). More recently in Fishburn (1973), Rubinstein (1980), and Felsenthal and Machover (1997) these games are extended to allow voters three ordered possible options and two outputs. In Freixas and Zwicker (2003) this idea is generalized to include many ordered levels of approval for the voter and many ordered levels of output. Games with multiple levels of input and output, and in particular how relative power can be determined amongst the players, have also been studied in Parker (2012) and Pongou et al. (2011). So far, the study of these games has always assumed that the players are voting in a genuine fashion and accept the final outcome. However, when the number of outputs is greater than two and a player prefers one of the middle outcomes, then she may wish to alter her vote from the one that most closely represents her honest input to another simply to have better chances to reach an outcome being as close as possible to her most preferred one. Or to put it in a way that is more charitable to the voter, the game itself fails to provide the voter a vote that accurately represents her preferences in all cases. In subsection 1.1, we give natural examples of voting situations in the areas of: law, academics, appropriations, and politics where such situations arise.

There are of course many results in the field of social choice theory dealing with the situation where there is an election with three or more candidates where manipulation by

the voters is possible, of which Arrow (1950) is the most famous, see for example Taylor and Pacelli (2008). However, this situation is different from the most general cases in social choice theory since the outcomes are assumed to have a natural ordering and this ordering is agreed upon by all voters. Thus, although a voter may prefer any outcome as their most desired outcome, their other preferences must be ordered in a way that is consistent with the overall ordering. This notion is made rigorous in section 2. In addition, this problem is distinguished from many social choice situations (e.g., social choice procedures or social welfare functions) since in our context the input to the voting system is a single vote, which is not necessarily an element of the set of output alternatives, rather than an ordered list of output alternatives.

The paper is organized as follows. Some motivating examples are given in the rest of section 1. Section 2 begins with the formalizations and terminologies of the notions of  $(j, k)$ -games with preferences and that of manipulation for this type of games. Section 3 begins with a comparison of how the results in Gibbard (1973) would apply if either we allowed players to have arbitrary preferences or we do not. The section also includes some basic results on manipulation for  $(j, k)$ -games with preferences. In section 4 we will turn our attention to looking at Nash equilibrium with pure strategies. Our starting point is the consideration of some subclasses of games with preferences having Nash equilibrium. We continue by showing the existence of games with preferences with as few as three players without Nash equilibrium. The central question in section 5 is whether Nash equilibrium with pure strategies exists for anonymous games with preferences. Although we leave the general question as an open conjecture, we partially prove it for some particular subcases. In section 6 we conclude the paper.

### 1.1. Examples

These examples have a common thread: The individual voters have a finite and naturally ordered set of voting options to choose from, and the body as a whole has a finite and naturally ordered set of outcomes. Thus can easily be modeled as a  $(j, k)$ -game from Freixas and Zwicker (2003). Similar examples have been explored in continuous framework of spatial and directional see Enelow and Hinich (1990) and Rabinowitz and Macdonald (1989) respectively. But also, these examples have in common that the system is set up under the assumption that each individual is not motivated by hoping for a particular outcome, but rather voting in a way that is their best assessment of what the best option is and trusting the voting system to produce the best outcome. However, if the voter is more interested in the final outcome than in the honesty of his or her actual vote, then as these examples show, the voter may not have a vote that will correspond to giving the best chance that his or her top preference will be the final outcome. Instead, the voter must guess or investigate how others will vote in order to decide on his or her vote. From these examples we can see many of the problems from the literature of social choice theory will also occur in the theory of  $(j, k)$ -games.

**Example 1** A juror has two choices—convict or acquit—but the outcome of the jury as a whole has a third option: a hung jury. For purposes of the example, we will suppose there are 12 jurors and each will vote for either conviction or acquittal and the outcome of the vote will be the unanimous decision of the jury, with a hung jury (and hence a potential retrial) if unanimity is not achieved. In reality, this vote might be taken many times in the process, with time for the jurors to persuade others between votes. We will assume we are just looking at the last vote. Hence if unanimity fails the jury will be hung.

The system is certainly set up under the assumption that each juror is supposed to give their honest assessment of the evidence. However, it is certainly plausible that a juror will prefer a hung jury, believing the proceedings were unfair and hence there should be a retrial. Although it is the job of the judges and not jury to decide the fairness of the trial, we must realize that individual motivations are often different from what the system assumes.

In this case, the juror will have to vote based on her assumption of what the other votes will be. She has no vote available to her that is the “best” in any objective sense.

**Example 2** At a certain school each student is given a grade in each class of A, B, C, D or F. A student graduates if he passes every course (with a D or better) and has a grade point average of a C average or better. Each of his teachers can then be thought of as voters deciding if he should graduate. A teacher might think that he deserves a D in her class but also wants him to graduate. She would like to give him the lowest grade possible without costing him his graduation.

**Example 3** Suppose each member of a budget committee is asked to vote to allocate either 5, 10, 15 or 20 percent of the budget to fund a project. The final allotment will be the average of the members’ suggestions. Again, if a committee member is more interested in the outcome of the averaging than her recommendation corresponding to her honest assessment, then she may wish to adjust her recommendation based on what she anticipates will be the recommendation of the others. In the cases where her honest assessment is that either 5 or 20 percent of the budget should be spent on the project, then she has no decision to make. Her best strategy and her honest assessment are identical and she does not have to take into account how others might vote. Only when she wants a middle value, does she have to consider strategizing.

A variation of this example would be the case in which the final allotment will be the median of the members’ suggestions. Again, if a committee member is more interested in the outcome, she can radicalize her vote to the extreme option, either 5 or 20 percent of the budget. However, the chance to manipulate, especially if the number of voters is not small, is here lower than for the mean.

So far in all of the examples the voting has been symmetric amongst all voters, that is, each player’s vote is treated the same. In such a framework, names need not be attached to the votes, so these games are called anonymous. The framework of  $(j, k)$ -games is certainly flexible to allow for the voters to have different roles. We can modify the previous example to allow for asymmetric voting.

**Example 4** Suppose in the above example there is a budgetary supervisor who determines the maximum amount that can be spent on a project. Hence, he also votes 5, 10, 15 or 20 percent, but his vote represents the maximum amount that can be spent on the project. Thus the outcome of the voting will be the smaller of the average of all the non-supervisors’ votes and the vote of the supervisor. We can see that all of the non-supervisors could have an incentive to vote strategically, but it is not clear the supervisor does.

These examples include the possibility that a player may vote in a way to increase her chance to get her favorite outcome, and to do so she might have to guess what other voters are going to do. That is, she may not have a vote that best represents her preferences independent of the actions of all other players. In the paper we will call such a game “manipulable”. This term does have a negative connotation and seems to imply that a

player changing her vote based on trying to get her preferred outcome is somehow doing something unethical. It has been suggested that a better term for this kind of voting would be either “strategic”, “tactical” (more positive) or “insincere” (more negative) voting and reserve the term “manipulation” for more untoward activities such as bribing voters or destroying ballots. We do not mean to imply any moral judgement upon a player using her best strategy. If the term manipulable is pejorative it is against the game not providing the player a vote that best represents her preferences.

## 2. Definitions and Notation

The following two definitions are from Freixas and Zwicker (2003) while adopting some of the notation from Felsenthal and Machover (1997):

**Definition 1** An ordered  $j$ -partition of a finite set  $N$  of players or voters is a sequence  $A = (A_1, A_2, \dots, A_j)$  of disjoint, possibly empty sets whose union is  $N$ . If  $a \in A_i$  we say that  $a$  approves at level  $i$  or votes for the  $i$ th option. If  $i_1 < i_2$  we say those voting at approval level  $i_1$  are voting at a higher level of approval than those approving at level  $i_2$ . We denote the set of all  $j$ -partitions of  $N$  by  $j^N$ . For every ordered  $j$ -partition  $A$  we define  $T_A : N \rightarrow \{1, 2, \dots, j\}$  by  $T_A(a) = i$  if  $a \in A_i$ . For two ordered  $j$ -partitions  $A$  and  $B$ , we write  $A^j \subseteq B$  if  $T_A(a) \geq T_B(a)$  for all  $a \in N$ . We say the ordered  $j$ -partitions,  $A$  and  $B$  *agree outside of player  $a$*  if  $T_A(x) = T_B(x)$  for all  $x \neq a$ .

Notice that  $X^j \subseteq Y$  means that each player in  $N$  has a higher or equal level of approval in  $Y$  than they do in  $X$ .

**Definition 2** A  $(j, k)$ -(*simple*)game consists of a finite set  $N$ , a set of voting outcomes  $\{v_1, v_2, \dots, v_k\}$  ordered by  $v_1 > v_2 > \dots > v_k$  and a value function  $V : j^N \rightarrow \{v_1, v_2, \dots, v_k\}$  that is monotonic that is, if for two ordered  $j$ -partitions  $X, Y$ , if  $X^j \subseteq Y$  then  $V(X) \leq V(Y)$ .

We will assume voters have at least two choices to vote for,  $j > 1$ , and the collective decision has at least two choices  $k > 1$  as well.

**Definition 3** A  $(j, k)$ -game  $V$  is:

- *exhaustive* if  $V$  is itself a surjective function;
- *smooth* if for all  $j$ -partition  $A$  with  $V(A) = v_h$  for some  $h > 1$  and  $T_A(a) = i$  for some  $i > 1$ ; it yields  $V(B) = v_h$  or  $v_{h-1}$  if  $A$  and  $B$  agree outside of  $a$  and  $T_B(a) = i - 1$ ;
- *strongly smooth* if whenever  $A$  and  $B$  are  $j$ -partitions that agree outside of  $a$  and  $T_A(a) = 1$  and  $T_B(a) = j$  then either  $V(A) = V(B)$  or  $V(A)$  and  $V(B)$  are consecutive outputs.
- *anonymous* if  $V(A) = V(\pi(A))$  for all permutation  $\pi : N \rightarrow N$ .

Since we normally include only achievable outcomes in the game, we will assume throughout the paper that all games are exhaustive. Smoothness is a strong assumption but is natural in many situations. In fact examples 1, 2, and 3 are all smooth. This condition, however, can be lost with minor variations. For instance, in example 3 the game is only smooth since the possible allocation options are equally spaced, if the option 10 is removed or an option of 30 is added the game ceases to be smooth. Strongly smooth games may occur in practice when

the number of voters is large and all voters play an equivalent or a quite similar role within the game as it occurs for example in anonymous  $(j, k)$ -games. Examples of  $(3, 3)$ -games of this type are simple majority rule and absolute majority rule as defined, e.g., Freixas and Zwicker (2009), see also Dougherty and Edward (2010) in which abstention or absence is considered as intermediate input and the outcome may pass, be postponed or defeated.

We define special types of voters of a  $(j, k)$ -game:

**Definition 4** In a  $(j, k)$ -game  $V$ , player  $a$  is:

- *null for the input level  $i$  ( $i < j$ )* if  $V(A) = V(B)$  for all  $A$  and  $B$  that agree outside of  $a$  with  $T_A(a) = i$  and  $T_B(a) = i + 1$ ;
- *null* if  $V(A) = V(B)$  for all  $A, B$  that agree outside of  $a$ ; i.e. if  $a$  is null for all input levels  $i < j$ ;
- *$v_h$ -passer* ( $h \leq k$ ) at input level  $i$  ( $i \leq j$ ) if  $V(A) = v_h$  for all  $A$  with  $T_A(a) = i$ ;
- *passer* if  $a$  is  $v_h$ -passer at some input level for all  $h \leq k$ .
- *dictator* if  $a$  is a passer and the rest of players are null.

Some comments on these definitions are the following:

- a null is a voter who is never able to affect the outcome of a division and therefore does not play an essential role in the  $(j, k)$ -game. Observe that any input level is a universally optimal vote for a null voter. Thus, the presence of null voters in a game never affects if this is manipulable or not.
- a dictator is the most radical form of being a passer since other voters are null, while this requirement does not necessarily apply for passers.
- if a game has a passer then  $j \geq k$ . If  $j = k$  and the game has a passer then the passer is also a dictator. If the game is the dictatorship of some player and  $j > k$ , the dictator is null for  $j - k$  input levels.
- two players cannot be a passer for different outputs, thus a game can have at most a single passer.

We now define a preference function for each player that gives their ranking outcomes with 1 being the outcome which they most prefer and  $k$  being their least favorite outcome. Each player's preference must correspond to the ordering of the outcomes in the following way: if  $k \geq 5$  and if the player's most favorite outcome is  $v_3$  then she must prefer  $v_2$  to  $v_1$  and  $v_4$  to  $v_5$ . However, there is no assumption on whether she will prefer  $v_5$  to  $v_2$ .

**Definition 5** A *preference function* for a player  $a$  in a  $(j, k)$ -game,  $V$ , is a bijective function  $P_a : \{v_1, v_2, \dots, v_k\} \rightarrow \{1, 2, \dots, k\}$  such that if  $v_{i_1}$  and  $v_{i_3}$  are such that  $P_a(v_{i_1}) < P_a(v_{i_3})$  and  $v_{i_2}$  is between  $v_{i_1}$  and  $v_{i_3}$  then  $P_a(v_{i_2}) < P_a(v_{i_3})$ .  $P$  is said to be a universal preference function if  $P(a)$  (written  $P_a$ ) is a preference function for each  $a \in N$ . A  $(j, k)$ -game with preferences is a  $(j, k)$ -game,  $V$  together with a universal preference function  $P$ . It is denoted  $V_P$  or just  $V$  when the preference function is clear. For each  $a \in N$  we let  $v_a^* = P_a^{-1}(1)$  be  $a$ 's most preferred outcome.

This definition implies that preferences of the voters satisfy single-peaked preferences as it is known in the social choice literature, see for example Black (1948), Sen (1966), Moulin (1988), and Moulin (1980). The assumption of single peakedness for players is implicit in the definition of  $(j, k)$ -game where the output levels are ordered from the highest level of approval ( $v_1$ ) to the lowest one ( $v_k$ ), so intermediate levels for the output represent partial level of support. Is it in that context reasonable to assume that a player who prefers  $v_1$  to  $v_2$ , will prefer  $v_k$  to  $v_2$ ? Of course this is plausible, but the assumption of single peakedness for players avoids this counterintuitive way of thinking and in some sense it may be considered as a rational requisite we assume for the behavior of players.

The fact that only single-peaked preferences are considered distinguishes the problem from those discussed in Gibbard (1973) and Satterthwaite (1975) as will be seen in more detail in section 3.1.

**Definition 6** In a  $(j, k)$ -game with preferences a player  $a$  is said to have a *universally optimal vote* of level  $i$  if whenever ordered  $j$ -partitions  $A$  and  $B$  agree outside of  $a$  and  $T_A(a) = i$  then  $P_a(V(A)) \leq P_a(V(B))$ . The game is said to be *manipulable by  $a$*  if  $a$  has no universally optimal vote.

So if level  $i$  is player  $a$ 's universally optimal vote in a  $(j, k)$ -game then the player can not improve his satisfaction with the outcome by voting differently from level  $i$ . Conversely, if the game is manipulable by  $a$  then no matter how he votes there is always a situation where he could have improved the outcome from his point of view by voting differently. In other words, in deciding his vote he could improve his outcome by correctly anticipating how others will vote.

**Definition 7** A  $(j, k)$ -game with preferences is said to be *manipulable* if it is manipulable by some player.

We can look back at the examples in section 1.1 and see which are manipulable in the sense of definition 6. We can see right away by proposition 2 that example 2 does not involve manipulation in the sense that we are defining it, since there are only two possible outcomes. The reason this example does not work is illuminating. Notice the teacher's motives for selecting the student's grade are partially, but not entirely, outcome-driven. If her grade was entirely driven by outcomes, then she would simply assign an A if she wished the student to pass and an F if she wished him to fail. However, her motives are to give a grade that most reflects the student's performance in her class while achieving her desired outcome. This motivation cannot be captured in our model as we only look at manipulability where the motives are entirely outcome-based. To contrast this, let's modify the example to one that is manipulable by definition 6, by providing this example with a third possible outcome.

**Example 5** Now suppose that every student has the possibility to graduate with honors if they receive all A's and B's with at most two B's. Now there are three outcomes for each student: graduate with honors, graduate without honors, and fail. It certainly seems reasonable that any teacher's preferences would align with definition 5 as it would seem unreasonable that a teacher's first choice would be that the student graduates with honors, second preference that the student fails and last preference is that the student graduates without honors. If a teacher's first preference is that the student graduates without honors then they might grade strategically. They would want to give the grade that is high enough for the student to graduate but not so high that the student graduates with honors. There

would be no grade they could give that would achieve this goal always and would have to take into account how others vote. Thus this game would be manipulable.

### 3. Manipulability

#### 3.1. Gibbard-like Theorem

In this section, we look at how the results in Gibbard (1973) would apply if we allowed players to have arbitrary preferences, that is a player can prefer the outcomes in any order. However, we will see by an example that a Gibbard-like Theorem would not apply if we assumed that every player has single-peaked preferences as defined in definition 5. First we define an unrestricted preference function that is also called an agenda.

**Definition 8** Let  $V$  be a  $(j, k)$ -game.  $P$  is said to be an unrestricted universal preference function (or an agenda) if for each  $a \in N$ ,  $P(a)$  (written  $P_a$ ) is a bijective function  $P_a : \{v_1, v_2, \dots, v_k\} \rightarrow \{1, 2, \dots, k\}$ . A  $(j, k)$ -game with unrestricted preferences is a  $(j, k)$ -game,  $V$  together with a universal unrestricted preference function  $P$ . It is written  $V_P$  or just  $V$  when  $P$  is clear.

The following theorem follows from Gibbard (1973):

**Theorem 1** Suppose  $k > 2$  and let  $V$  be a  $(j, k)$ -game that is exhaustive and does not contain a passer. Then there exists a universal unrestricted preference function  $P$  so that  $V_P$  is manipulable.

PROOF: Suppose to the contrary that for any  $P$ ,  $V_P$  is not manipulable. Define the voting scheme,  $\nu$  (see Gibbard (1973)) by for each  $P$ ,  $\nu(P) = V(A)$  where  $A$  is the  $j$ -partition where each player votes their optimal vote (or one of their optimal votes) with respect to  $P$ . Let  $v_h$  be an outcome and let  $P$  be such that every player's top choice is  $v_h$ . It is easy to see that the result of  $P$  must be  $v_h$ . This is because  $V$  is exhaustive so there must be a  $j$ -partition  $A$  so that  $V(A) = v_h$ . Each player's optimal vote must then lead to  $v_h$  since otherwise there would be a situation (namely all the other players vote as in  $A$ ) where they had a better vote. Thus  $\nu$  has at least 2 outcomes and by Gibbard's theorem there must exist a dictator (in the voting scheme) call it  $a$ .

We now show  $a$  is also a passer for  $V$  (but not necessarily a dictator by definition 4). Let  $h \leq k$ , and let  $P$  be an agenda such that  $P_a(v_h) = 1$  and  $P_x(v_h) = k$  for  $x \neq a$ . Let  $i$  be an optimal vote of  $a$  with respect to  $P_a$  and  $A$  be the  $j$ -partition where each player votes their optimal vote with respect to  $P$  (thus  $T_A(a) = i$ ). Then  $V(A) = \nu(P) = v_h$  (since  $a$  is the dictator). Also if  $B$  is any other  $j$ -partition such that  $T_B(a) = i$ , then  $\nu(B) = v_h$  also. This is because if it wasn't  $v_h$  then all the other players would do better than they did with  $A$  contradicting the fact that their vote in  $A$  was their optimal vote. This is a contradiction and so  $V_P$  is manipulable for some  $P$ .

△

This result does not hold when only single-peaked preferences as defined in definition 5 are considered. This is seen in this simple example (see also Moulin (1980)).

**Example 6** Consider the  $(3, 3)$ -game,  $V$ , where  $V(A) = v_1$  if  $A_1 = N$  and  $V(A) = v_3$  if  $A_3 = N$  and  $V(A) = v_2$  otherwise. Clearly this game does not have a passer if  $|N| > 1$ .

Then each voter  $a$  has an optimal vote of  $h$  where  $v_a^* = v_h$ . This is clearly optimal if  $h = 2$  since  $a$  is a  $v_2$ -passer at level 2. If  $h \neq 2$  then supporting at level  $h$  will guarantee her first choice when it is possible and her second choice otherwise. Notice this would not be an optimal voting strategy if the player had been allowed to prefer  $v_1$  first and  $v_3$  second, since if  $A = (\{a\}, \emptyset, N \setminus \{a\})$ ,  $a$  would be better by decreasing her level of support to 3.

### 3.2. Games with Preferences

A simple well-known observation that can be drawn from definition 5 is that while a player's most desirable outcome can be any outcome, her least desirable outcome must be one of the extreme choices.

**Proposition 1** *If  $P_a$  is a preference function for voter  $a$  on a  $(j, k)$ -game, then  $P_a^{-1}(k) \in \{v_1, v_k\}$ .*

If in addition a player's most desired outcome is also one of the extreme choices, then her preferences must be in order or reverse order of the order on the outcomes and this player has no need to manipulate since she always has a universally optimal vote. This is seen in the next proposition.

**Proposition 2** *Consider a  $(j, k)$ -game. If  $a \in N$  is such that  $v_a^* = v_1$  then  $P_a(v_{i_1}) < P_a(v_{i_2})$  if and only if  $i_1 < i_2$  and furthermore voter  $a$  has a universally optimal vote of level 1. Similarly, if  $v_a^* = v_k$  then  $P_a(v_{i_1}) < P_a(v_{i_2})$  if and only if  $i_1 > i_2$  and voter  $a$  has a universally optimal vote of level  $k$ . In either case the game is not manipulable by  $a$ .*

PROOF: Suppose  $P_a(v_1) = 1$ , if  $1 < i_1 < i_2 \leq k$  then  $v_{i_1}$  is between  $v^* = v_1$  and  $v_{i_2}$  so  $P_a(v_{i_1}) < P_a(v_{i_2})$ . Next suppose  $A$  and  $B$  agree outside of  $a$  with  $T_A(a) = 1$ . Then  $T_A \leq T_B$  pointwise and hence by monotonicity  $V(B) \leq V(A)$  and hence  $P_a(V(A)) \leq P_a(V(B))$ . A similar argument holds with  $P_a(v_k) = 1$ .

△

**Proposition 3** *If player  $a$  has a universally optimal vote of level  $i$  then there exists a  $j$ -partition  $A$  with  $V(A) = v_a^*$  and  $T_A(a) = i$ .*

PROOF: Assume the contrary. As  $V$  is exhaustive there exists a  $j$ -partition  $B$  with  $T_B(a) \neq i$  and  $V(B) = v_a^*$ . Let  $A$  be the  $j$ -partition that agrees  $B$  outside of  $a$  and  $T_A(a) = i$ , as  $V(A) \neq v_a^* = P_a^{-1}(1)$ , player  $a$  cannot have a universally optimal vote of level  $i$ , a contradiction.

△

The next corollary is now immediate.

**Corollary 1** *If  $V(A) \neq v_a^*$  for all  $j$ -partition  $A$  with  $T_A(a) = i$  then player  $a$  does not have a universally optimal vote at level  $i$ .*

Recall that passers, and dictators as a particular case, for exhaustive  $(j, k)$ -games are only possible if  $j \geq k$  and if they exist they are unique.

**Proposition 4** *If  $a$  is a  $v_a^*$ -passer at input level  $i$  then player  $a$  has a universally optimal vote at level  $i$ .*



PROOF: Clearly,  $1 = P_a(v_a^*) = P_a(V(A)) \leq P_a(V(B))$  for all  $B$  that agrees with  $A$  outside of  $i$ .

△

For this we get the following corollary which gives us a converse of theorem 1.

**Corollary 2** *A  $(j, k)$ -game with preferences is not manipulable if it has a dictator.*

PROOF: Every voter other than the dictator is a null voter, and as was noted earlier, any input level  $i$  is a universally optimal vote for them. In addition, the dictator is a  $v_a^*$ -passer, so has a universally optimal vote.

△

The next result is a simple test for deciding if a given player has a universally optimal vote.

**Proposition 5** *Level input  $i$  is not a universally optimal vote for  $a$  if there exist  $j$ -partitions  $A$  and  $B$  that agree outside of  $a$  with  $T_A(a) = i$  and  $P_a(V(B)) < P_a(V(A))$  and a  $j$ -partition  $C$  such that  $V(C) = v_a^*$  with  $T_C(a) = i$ .*

PROOF:  $V(B)$  is preferable to  $V(A)$  for  $a$ , thus  $i$  is not a universally optimal vote for  $a$ .

△

The precise technical test for checking if a player has a universally optimal vote follows.

**Proposition 6** *Player  $a$  has a universally optimal vote at input level  $i$  if and only if the following conditions hold:*

- $V(A) = v_a^*$  for some  $A$  with  $T_A(a) = i$ ,
- for all  $A'$  with  $T_{A'}(a) = i$  and  $V(A') < v_a^*$  it yields  $V(B') = V(A')$  or  $V(B') > v_a^*$  with  $P_a(V(A')) \leq P_a(V(B'))$  for all  $B'$  that agrees  $A'$  outside of  $i$  and  $T_{B'}(a) < i$ , and
- for all  $A''$  with  $T_{A''}(a) = i$  and  $V(A'') > v_a^*$  it yields  $V(B'') = V(A'')$  or  $V(B'') < v_a^*$  with  $P_a(V(A'')) \leq P_a(V(B''))$  for all  $B''$  that agrees  $A''$  outside of  $i$  and  $T_{B''}(a) > i$ .

The second and third parts of the previous result essentially say that: if  $a$  changes her vote to the immediate higher input level, then  $v_a$  must be on the right of her ideal point  $v_a^*$  and either there is no gain for  $V$  or the gain for  $V$  is big enough to reach the left part of her ideal point where the outcomes are less preferred to the initial one. Analogous comment follows if  $a$  changes her vote to the immediate lower input level.

PROOF: ( $\Rightarrow$ ) The first part follows from proposition 3. If  $a$  has a universally optimal vote at input level  $i$  and only she changes her vote in the partition to another consecutive level of approval, then the image for  $V$  of the new partition cannot be better than the previous value, which does not happen if the  $V$  has the same value over the two partitions or the new value for  $V$  is a worse outcome for  $a$ . By monotonicity of  $V$  only vote changes for  $a$  in the consecutive inputs of  $i$  need to be considered.

( $\Leftarrow$ ) Just observe that  $V(A')$  is at least as preferable as  $V(B')$  for  $a$  and  $V(A'')$  is at least as preferable as  $V(B'')$  for  $a$ . Thus  $a$  has a universally optimal vote for  $a$ . The reasoning for the second part is analogous.

△

The next result is a consequence of previous result for smooth  $(j, k)$ -games.

**Proposition 7** *A smooth game with preferences is manipulable for player  $a$  if and only if*

- *there exists an input level  $i$  such that  $T_A(a) = i$  and  $V(A) = v_a^* \notin \{v_1, v_k\}$  for some  $A$ , and*
- *there exists either  $A'$  with  $V(A') < v_a^*$ ,  $T_{A'}(a) = i$  that agrees  $B'$  outside of  $A'$  with  $T_{B'}(a) = h < i$  with  $V(B') > V(A')$  being consecutive output values, or  $A''$  with  $V(A'') > v_a^*$ ,  $T_{A''}(a) = i$  that agrees  $B''$  outside of  $A''$  with  $T_{B''}(a) = h > i$  with  $V(A'') > V(B'')$  being consecutive output values.*

PROOF:  $(\Leftarrow)$   $V(A) = v_a^* \notin \{v_1, v_k\}$  for some  $A$  otherwise the game would not be manipulable for  $a$  (proposition 2). In the first case of the second item the condition  $V(A') < v_a^*$  combined with smoothness implies  $v_a^* \geq V(B') > V(A')$ . Thus  $P_a(V(B')) < P_a(V(A'))$  and therefore input level  $i$  is not a universally optimal vote for  $a$ . In the second case of the second item the condition  $V(A'') > v_a^*$  combined with smoothness implies  $V(A'') > V(B'') \geq v_a^*$ . Thus  $P_a(V(B'')) < P_a(V(A''))$  and therefore input level  $i$  is not a universally optimal vote for  $a$ .

$(\Rightarrow)$  As the game is manipulable for  $a$ ,  $v_a^* \notin \{v_1, v_k\}$  (Proposition 2). As  $V$  is exhaustive it exists  $A$  with  $V(A) = v_a^*$ . Let  $i$  be any input level such that  $T_A(a) = i$ . As the game is manipulable, it exists  $A'$  with  $T_{A'}(a) = i$  and  $V(A') \neq v_a^*$  (Proposition 4).

Assume first  $V(A') < v_a^*$ , as the game is smooth and manipulable, input  $i$  is not a universally optimal vote for  $a$ , thus it exists  $B'$  with  $V(B') > V(A')$  being consecutive and with  $B'$  and  $A'$  agree outside of  $a$  for some  $h = T_{B'}(a) < i$ .

The second case,  $V(A') > v_a^*$  is analogous.

△

The “or” condition above just expresses that voter  $a$  is not null at input levels  $i - 1$  or  $i$  over some partition  $A$  with  $T_A(a) = i$  and  $V(A) \neq v_a^*$ .

Let’s consider the *simplest* case for which manipulability is possible. This means that the  $(j, k)$ -games must have at least three outputs ( $k = 3$ , because of proposition 2) and the minimum possible choices we can select for inputs and number of voters is 2 ( $j = n = 2$ ).

A careful enumeration and the use of the above propositions for these three parameters leads to the following conclusion:

1. the number of  $(2, 3)$ -games with preferences for 2 voters is 80,
2. 52 of these games are not manipulable: 20 of them because the top choice preference for the two voters is not  $v_2$  (proposition 2), while the remaining 32 have  $v_2$  as the top choice preference for at least one of the voters,
3. the remaining 28 games are manipulable.

#### 4. The general case: existence of a Nash Equilibrium with pure strategies

An interesting aspect of the presence of single peaked preferences for players in a  $(j, k)$ -game with preferences is how it affects the existence of Nash equilibrium with pure strategies (see Yamamura and Kawasaki (2013) for this observation in a different setting). Throughout the paper we will use NE to stand for Nash equilibrium with pure strategies. To see this, consider the very simple case of an anonymous  $(2, 3)$ -game with just two voters:

**Example 7** Let  $V$  be a  $(2, 3)$ -game with  $N = \{a, b\}$  and suppose  $V(N, \emptyset) = v_1, V(\emptyset, N) = v_3, V(\{a\}, \{b\}) = V(\{b\}, \{a\}) = v_2$ . Then without the single-peaked requirement, this game could have no NE if  $a$ 's top preference is  $v_2$ , but  $b$  likes that outcome the least. However, under single peaked preferences,  $b$ 's preferences would not be permissible by proposition 1.

In fact as we shall see in this section and the next,  $(j, k)$ -games with preferences will always have an NE if either  $k = 3, n = 2$  or if  $j = 2$  and the game is anonymous. We will see that for games with  $n \geq 3$  it is possible to have no NE's.

The simplest case to see the existence of an NE is when  $k = 3$ .

**Theorem 2** *Every  $(j, 3)$ -game has an NE.*

PROOF: Let  $N_i = \{a \in N : P_a(v_i) = 1\}$ , so  $N_1, N_2$  and  $N_3$  form a partition of  $N$ . Players in  $N_i$  for  $i \in \{1, 3\}$  can do no better than voting approval level 1 or  $j$  respectively so their votes can be fixed. There are two cases. If there is a  $j$ -partition  $A$  such that  $N_1 \subseteq A_1$  and  $N_3 \subseteq A_j$  with  $V(A) = v_2$ , then this is an NE. This is because players in  $N_2$  have their first choice and the other players never have an incentive to change their votes.

Now suppose no such  $j$ -partition exists, then further subdivide  $N_2$  into  $N'_2 = \{a \in N_2 : P_a(v_1) = 2\}$  and  $N''_2 = \{a \in N_2 : P_a(v_3) = 2\}$ . Let  $A = (N_1 \cup N'_2, \emptyset, \dots, \emptyset, N_3 \cup N''_2)$ . This is an NE since no voter in  $N_2$  can achieve their first preferred outcome so they can do no better than voting as extreme as possible for their second preferred choice.

△

Next we show that in all two-player games, there is an NE. Consider a two-player  $(j, k)$ -game with preferences where the two players are denoted  $a$  and  $b$ . Let  $\{c_{h,i}\}_{1 \leq h \leq j, 1 \leq i \leq j}$  be the bidimensional matrix where  $h$  denotes the input level chosen by voter  $a$  and  $i$  denotes the input level chosen by voter  $b$  and  $c_{h,i}$  is the corresponding output.

The monotonicity demanded to  $V$  implies:

$$c_{h,i} \geq c_{h',i'} \quad \text{whenever} \quad h \leq h' \quad \text{and} \quad i \leq i'. \quad (1)$$

In particular, the greatest output is  $c_{1,1}$ , while the lowest output is  $c_{j,j}$ .

For each  $i = 1, \dots, j$  let  $c_{*,i}$  be such that  $P_a(c_{*,i}) \leq P_a(c_{h,i})$  for all  $1 \leq h \leq j$ . So  $c_{*,i}$  is the best possible outcome for player  $a$  when player  $b$  chooses input level  $i$ . Define  $a_i$  in the following way: if  $v_a^* \geq c_{*,i}$  then  $a_i = \min\{h \mid c_{h,i} = c_{*,i}\}$  and if  $v_a^* < c_{*,i}$  then  $a_i = \max\{h \mid c_{h,i} = c_{*,i}\}$ .

**Lemma 1** *If  $i < i'$  then  $a_i \geq a_{i'}$ .*

PROOF: Suppose to the contrary that  $a_i < a_{i'}$ .

First suppose that  $c_{*,i} \leq v_a^*$ . Then  $c_{*,i'} = c_{a_{i'},i'} \leq c_{a_i,i'} \leq c_{a_i,i} \leq v_a^*$  so  $P_a(c_{a_i,i'}) \leq P_a(c_{a_{i'},i'}) = P_a(c_{*,i'})$ . It must be that,  $c_{a_i,i'} = c_{*,i'}$ . But this contradicts the construction of  $a_{i'} = \min\{h \mid c_{h,i'} = c_{*,i'}\}$ .

Now suppose that  $v_a^* < c_{*,i}$ . Then by construction  $c_{a_{i'},i} \neq c_{*,i}$  and  $P_a(c_{a_i,i}) < P_a(c_{a_{i'},i})$ . Thus  $v_a^* > c_{a_{i'},i} \geq c_{a_{i'},i'} = c_{*,i'}$ . By construction of  $a_{i'}$ ,  $c_{a_{i'},i'} < c_{a_i,i'}$  and  $P_a(c_{a_{i'},i'}) < P_a(c_{a_i,i'})$ . This implies that  $c_{a_{i'},i'} \leq c_{a_{i'},i} < v_a^* < c_{a_i,i'} \leq c_{a_i,i}$ . So  $P_a(c_{a_{i'},i}) \leq P_a(c_{a_{i'},i'}) < P_a(c_{a_i,i'}) \leq P(c_{a_i,i})$  which contradicts the fact that  $P_a(c_{a_{i'},i}) \geq P(c_{a_i,i})$ .

△

Thus if  $a$  always votes  $a_i$  in response to  $b$ 's vote of  $i$ , then any change in  $b$ 's vote will result in either  $a$  not changing his vote or changing his vote in opposite direction as  $b$ . Similarly we can for each  $h$ , define  $b_h$  with  $b_h \leq b_{h'}$  whenever  $h < h'$ . It can now be shown that an NE  $(h, i)$  is achieved if  $a_{b_h} = h$  and  $b_{a_i} = i$ .

**Theorem 3** *Every two-player  $(j, k)$ -game with preferences has an NE.*

PROOF: We will create a sequence of votes that leads in a finite number of steps to an NE. Start with  $i_1 = 1$   $h_1 = a_{i_1}$ . If  $b_{h_1} = i_1$  then an NE has been achieved. Otherwise  $b$  will change her vote to  $i_2 = b_{h_1} > i_1$ . If  $a_{i_2} = h_1$  then this is an NE, otherwise  $a$  will change his vote to  $h_2 = a_{i_2} < h_1$  which is either an NE or the process continues. Since  $h_m$  and  $i_m$  are monotone sequences, this process cannot go on indefinitely and at some point an NE must be achieved.

△

Notice that not only does every two-player  $(j, k)$ -game with preferences have an NE, but an equilibrium can be achieved by successive unilateral improvements by each player. This result does not hold in general when  $n \geq 3$  as the following example demonstrates:

**Example 8** A  $(3, 27)$ -game with 3 voters without an NE.

Voters' preferences:

$$\begin{aligned} P_1 : & \quad 16 > \dots > 12 > 17 > \dots > 27 > 11 > \dots > 1. \\ P_2 : & \quad 12 > 11 > 10 > 13 > \dots > 27 > 9 > \dots > 1. \\ P_3 : & \quad 15 > \dots > 1 > 16 > \dots > 27. \end{aligned}$$

whenever dots are used it means that the preferences follow the consecutive ordering. The matrix  $A$  with  $3^3 = 27$  entries:  $a_{i,h,l}$  with  $1 \leq i \leq 3$ ,  $1 \leq h \leq 3$  and  $1 \leq l \leq 3$ , provides the output for the game for all tripartition. The next three matrices describe the tridimensional matrix  $A$ .

$a_{i,h,1}$	27	26	16
	25	15	10
	24	13	5
$a_{i,h,2}$	23	22	9
	21	14	7
	20	6	4
$a_{i,h,3}$	19	17	8
	18	12	3
	11	2	1

To check that, in fact, this game does not have an NE, we are going to list the value of the game for the best response for each player to the given actions of the two others.

The best response for player 1:

$a_{.,1,1}$	$a_{.,1,2}$	$a_{.,1,3}$	$a_{.,2,1}$	$a_{.,2,2}$	$a_{.,2,3}$	$a_{.,3,1}$	$a_{.,3,2}$	$a_{.,3,3}$
24	20	18	15	14	12	16	9	8

The best response for player 2:

$a_{1,\cdot,1}$	$a_{1,\cdot,2}$	$a_{1,\cdot,3}$	$a_{2,\cdot,1}$	$a_{2,\cdot,2}$	$a_{2,\cdot,3}$	$a_{3,\cdot,1}$	$a_{3,\cdot,2}$	$a_{3,\cdot,3}$
16	22	17	10	14	12	13	20	11

The best response for player 3:

$a_{1,1,\cdot}$	$a_{1,2,\cdot}$	$a_{1,3,\cdot}$	$a_{2,1,\cdot}$	$a_{2,2,\cdot}$	$a_{2,3,\cdot}$	$a_{3,1,\cdot}$	$a_{3,2,\cdot}$	$a_{3,3,\cdot}$
19	17	9	18	15	10	11	13	5

As there is no number present in the three previous lists we deduce that the game does not have an NE.

Notice the above example is quite convoluted and it is not easy to see an application of it in political theory. It does seem possible that large classes of more realistic games may all contain an NE. In the next section we will determine some special subclasses that have an NE.

## 5. Some particular classes with a Nash Equilibrium with pure strategies

We will first show that all anonymous  $(2, k)$ -games with preferences must have an NE. Anonymous  $(j, k)$ -games are covered in detail in Freixas and Zwicker (2009).

To do this let  $V$  be an anonymous  $n$  player  $(2, k)$ -game with preferences. For simplicity when  $l$  is between 0 and  $n$  denote  $V(l)$  be the outcome of any 2-partition where  $l$  voters vote 1. (Since  $V$  is anonymous it does not matter which voters vote 1.) It is clear by monotonicity that  $V(l_1) \leq V(l_2)$  if and only if  $l_1 \leq l_2$ .

**Proposition 8** *Every anonymous  $(2, k)$ -game with preferences has an NE.*

PROOF: Let  $V$  be as above. Let  $l_0$  be such that  $V(l_0)$  is the lowest outcome that is the top choice for one of the voters. We will prove by induction on  $n$  that  $V$  has an NE whose outcome is at the level of  $V(l_0)$  or higher.

Clearly the theorem holds for  $n = 1$  so suppose  $n > 1$ .

First suppose that  $V(l_0) = V(n)$ . Thus every player's top choice is  $V(n)$ . In this case every voter voting 1 is an NE whose outcome is at the level of  $V(l_0)$  or higher.

Now suppose that  $V(l_0)$  is lower than  $V(n)$ . Let  $a$  be a voter for which  $V(l_0)$  is his top outcome. Let  $V_1$  be the  $n - 1$  player game where voter  $a$ 's vote is fixed at 2. Thus  $V_1$  has the same voters as  $V$  minus  $a$ , and  $V_1(k) = V(k)$ . Let  $l_1$  be such that  $V(l_1)$  is the lowest level that is the top choice for one of the voters in  $V_1$ . Since the outcomes of  $V$  and  $V_1$  are the same with the possible exception that if  $V(n) \neq V(n - 1)$  then  $V(n)$  is not a possible outcome of  $V_1$ , it follows that  $V(l_0) \leq V(l_1)$ . By the inductive hypothesis, there exists a 2-partition  $A$  in  $V_1$  that is an NE at the level of  $V(l_1)$  or higher. We will call the number of 1 votes in  $A$ :  $\bar{l}$ . We claim that the 2-partition  $A'$  created from  $A$  by adding in  $a$  with a vote of 2 forms an NE for  $V$ . First note that none of the voters of  $V_1$  will benefit by changing their vote in  $A$  since then they would have benefited in  $V_1$ . Also  $a$  would not do better to increase his level of support to 1 since  $a$  prefers  $V(l_0)$  to  $V(\hat{l} + 1)$  and  $V(\bar{l})$  is between  $V(l_0)$  and  $V(\hat{l} + 1)$  thus by Definition 5  $a$  prefers  $V(\bar{l})$  to  $V(\bar{l} + 1)$  (or the two outcomes could be the same). Thus  $V$  has an NE whose outcome is at the level of  $V(l_0)$  or higher.

△

**Proposition 9** *An anonymous  $(j, k)$ -game with preferences has an NE if  $k < \frac{n+3}{2}$ .*

PROOF: For an anonymous game one needs to only look at the number of voters at each support level, so if  $(n_1, n_2, \dots, n_j)$  is such that  $\sum_{i=1}^j n_i = n$  is a decomposition of  $n$  with  $j$  parts, then it can be uniquely mapped to a single output. Consider the sequence  $(n, 0, \dots, 0, 0), (n-1, 0, \dots, 0, 1), \dots, (0, 0, \dots, 0, n)$ , if either three in a row or the first two or last two map to the same output (which must happen if  $k < \frac{n+3}{2}$ ) then the game must have an NE. To see this suppose that  $(n_1, 0, \dots, 0, n_j)$  is the middle of the three (or the first or last in the latter cases). Then no player can unilaterally change the outcome by changing his or her vote, thus it is an NE.

△

Because of the assumption of single peaked preference, if a player has a unilateral incentive to change his vote it must be in the direction of his top choice. To be clear, if  $A$  is a  $j$ -partition such that  $V(A) > v_a^*$  then  $a$  can only increase the preferability of the outcome by decreasing his level of support. In general decreasing his level of support (by even one level) can decrease the preferability of the outcome if it causes the outcome to leapfrog  $a$ 's top choice to a less preferable one. We now consider the case where this doesn't happen, i.e.  $a$  does no worse by decreasing his level of support by one level. Of course, a completely analogous situation holds when  $V(A) < v_a^*$ .

**Definition 9** In a  $(j, k)$ -game with preferences, a player  $a$  is said to *always have a simple strategy from above* if whenever  $A$  is a  $j$ -partition such that  $V(A) > v_a^*$ ,  $a$  can do no worse by decreasing his level of support by one level (if possible). Similarly,  $a$  is said to *always have a simple strategy from below* if whenever  $A$  is a  $j$ -partition such that  $V(A) < v_a^*$ ,  $a$  can do no worse by increasing his level of support by one level.

A player will always have a simple strategy from above in a couple of cases. One, the game itself does not allow leap-frogging, i.e. the game is smooth. Alternatively, in any game if the player prefers all the outcomes below her top choice to all the outcomes above her top choice. In such a situation when  $V(A) > v_a^*$  then she prefers all choices below  $V(A)$  to  $V(A)$  and so can do no worse by decreasing her level of support. Notice in the game in Example 8, voter  $p_1$  does not always have a simple strategy since  $V(\{p_2\}, \{p_1\}, \{p_3\}) > v_{p_1}^*$  but he does worse if he decreases his level of support by one. Similarly,  $p_2$  does not always have a simple strategy from below since  $V(\{p_3\}, \{p_1\}, \{p_2\}) < v_{p_2}^*$  but she does worse if she increases her level of support by one.

The following lemma gives one possible construction of an NE in a game where every player always has a simple strategy from above. Of course, a similar result holds for games where every player always has a simple strategy from below.

**Lemma 2** *Let  $V$  be a  $(j, k)$ -game with preferences, in which every player always has a simple strategy from above. Let  $A$  be a  $j$ -partition such that for all  $a \in N$ :*

1. *if  $V(A) \leq v_a^*$  then either  $T_A(a) = 1$  or if  $A^*$  agrees with  $A$  outside of  $a$  and  $T_{A^*}(a) < T_A(a)$ , then  $V(A^*) > v_a^*$ ,*
2. *if  $V(A) > v_a^*$  then  $T_A(a) = j$ .*

*Then  $A$  is an NE.*

PROOF: First suppose  $T_A(a) \neq 1$  let  $A^*$  agrees with  $A$  outside of  $a$  and  $T_{A^*}(a) = T_A(a) - 1$  then  $V(A^*) > v_a^*$ . Since  $a$  always has a simple strategy from above  $a$  would do no worse than decrease her level of support in  $A^*$  by one level and hence has no incentive to increase her level of support in  $A$ . Also if  $T_A(a) \neq j$  then  $V(A) \leq v_a^*$  so  $a$  has no incentive to decrease her level of support. Thus  $A$  is an NE.

△

**Proposition 10** *If in a  $(j, k)$ -game with preferences all the players always have a simple strategy from above or all the players always have a simple strategy from below, then the game has an NE. In particular, all smooth games have an NE.*

PROOF: We will show it for games where all the players always have a simple strategy from above, since an analogous argument holds for simple strategies from below. First, order the players  $a_1, a_2, \dots, a_n$  so that  $v_{a_{i_1}}^* \leq v_{a_{i_2}}^*$  whenever  $i_1 \leq i_2$ . Start with the  $j$ -partition  $A_{n+1} = (\emptyset, \dots, \emptyset, N)$ . If  $V(A_{n+1}) > v_{a_n}^*$  then no player has incentive to increase their level of support and  $A_{n+1}$  is an NE. Otherwise construct the  $j$ -partition  $A_n$ , which agrees with  $A_{n+1}$  outside of  $a_n$  and  $T_{A_n}(a_n) = i$ , is the highest level of support that  $a_n$  can give and still satisfy the condition:  $V(A_n) \leq v_{a_n}^*$ . If  $V(A_n) > v_{a_{n-1}}^*$ , then  $A_n$  satisfies the conditions of lemma 2 and is an NE. Otherwise, continue the construction by allowing  $a_{n-1}$  to increase his level of support up to the highest level  $i' < j$  such that the obtained  $j$ -partition  $A_{n-1}$ , which agrees with  $A_n$  outside of  $a_{n-1}$  and  $T_{A_{n-1}}(a_{n-1}) = i'$ , fulfills the condition:  $V(A_{n-1}) \leq v_{a_{n-1}}^*$ . Again if  $V(A_{n-1}) > v_{a_{n-2}}^*$  then by lemma 2,  $A_{n-1}$  is an NE. Otherwise the construction continues in the same manner. Either the process stops at an NE or  $A_1$  is constructed so that  $V(A_1) \leq v_{a_1}^*$  and hence  $V(A_1) \leq v_{a_k}^*$  for all  $k$ ,  $1 \leq k \leq n$  and hence by lemma 2 is an NE.

△

Suppose that  $V$  is a game in which all players prefer all the outcomes above their top choice to all the outcomes below their top choice. It is worth noticing that sometimes an NE can be rather unsatisfactory for  $n - 1$  voters. Assume for example that  $A$  and  $B$  are  $j$ -partitions that agree outside of  $a_1$  with  $T_A(a_1) = j$ ,  $T_B(a_1) = j - 1$  and  $T_B(x) = 1$  for all  $x \neq a_1$ . Players  $a_2, \dots, a_n$  feel very unsatisfied, although none of them can unilaterally change the outcome for the better.

Notice since examples 1, 2, 3, and 5 are all smooth, they must all have an NE.

If we further assume that a game is strongly smooth then we can in fact easily find an NE as the following proposition shows:

**Proposition 11** *Strongly smooth  $(j, k)$ -games with preferences have an NE in which all voters vote either 1 or  $j$ .*

PROOF: Let  $a \in N$ , for any  $j$ -partition  $C$ , let  $C'$  and  $C''$  be  $j$ -partitions that agree with  $C$  outside of  $a$  with  $a \in C'_1$  and  $a \in C''_j$ . Then since  $V$  is strongly smooth either  $V(C) = V(C')$  or  $V(C) = V(C'')$ . Thus in all situations, a player can change her level of support to either 1 or  $j$  without changing the result.

Let  $V^*$  be the  $(2, k)$ -game defined from  $V$  as follows,  $V^*(A_1, A_j) = V(A)$  for any  $j$ -partition  $A$  with  $A_1 \cup A_j = N$ . The strongly smooth condition on  $V$  guarantees that  $V^*$  will be smooth and so by proposition 10 it has an NE. The corresponding  $j$ -partition is also an NE in  $V$ .

A notion of equilibrium outcome in other types of voting games with single-peaked preferences has been studied in Ortuño-Ortín (1997) and Yamamura and Kawasaki (2013). Although these models are different from ours, the typical property of NE in those contexts with single peaked preferences is that voters vote for a radical alternative. However, outside the class of strongly smooth  $(j, k)$ -games there is no a clear evidence that players wish to radicalize their votes. When there are several possible outcomes depending on a player's vote, it could perfectly happen that they prefer some intermediate outcome and therefore these players would not necessarily want to radicalize their votes.

## 6. Conclusion and Future Work

This paper looks at the situation where voters have multiple options that are naturally ordered, and there are more than two ordered outcomes. We look at these situations from the point of view of the voter, and assume that she has her own preferred outcomes and may not have a vote that accurately represents her preferences. In fact, if the voter's top choice is anything other than one of the extreme choices, it will almost certainly be the case that she will not have a universally optimal vote.

In section 1.1 we see many diverse situations where this could occur. We define what it means for a game to be manipulable by a player. This occurs when a voter's best vote depends on his belief about what the other voters are going to vote. This can only occur when the voter's top choice is not one of the extreme choices. Intuitively, if the voter's top choice is a middle choice, then she is very likely to have no universally optimal vote, unless she either has no power or all the power in the vote. This leads to the well-studied issues from social choice theory.

Finally, we looked at the existence of an NE in these games. We saw many situations that guarantee the existence of an NE. For example, an NE must occur when the game is smooth, when  $k = 3$  or  $n = 2$  or in anonymous games when  $j = 2$  or  $k$  is too small. However, we did see an example of a game with no NE. Future work would be to extend these conditions. In particular, we conjecture that all anonymous games must have an NE.

## Acknowledgments

The authors would like to thank Josep's son Marc Freixas for his contributions to construction of a game with preferences without an NE (Example 8). This research was partly funded by a Faculty Research Grant at the University of San Diego and by Grant MTM2012-34426/FEDER of "Ministerio de Economía y Competitividad".

We also would like to thank the editor for his careful monitoring during the review process, a referee whose reviews of the proofs were very accurate and helpful, and another referee who guided us to study the relationship of our model with the one by Gibbard (1973) for unrestricted preferences as well as for providing us some other motivations concerning social choice aspects of the paper.

## References

- Arrow, K. J., 1950. A difficulty in the concept of social welfare. *The Journal of Political Economy* 58 (4), 328–346.



- Black, D., 1948. On the rationale of group decision-making. *The Journal of Political Economy* 56, 23–34.
- Dougherty, K. L., Edward, J., 2010. The properties of simple vs. absolute majority rule: cases where absences and abstentions are important. *Journal of Theoretical Politics* 22 (1), 85–122.
- Enelow, J. M., Hinich, M. J., 1990. *Advances in the spatial theory of voting*. Cambridge University Press.
- Felsenthal, D. S., Machover, M., 1997. Ternary voting games. *Internat. J. Game Theory* 26 (3), 335–351.
- Fishburn, P. C., 1973. *The theory of social choice*. Princeton University Press, Princeton, N.J.
- Freixas, J., Zwicker, W. S., 2003. Weighted voting, abstention, and multiple levels of approval. *Soc. Choice Welf.* 21 (3), 399–431.
- Freixas, J., Zwicker, W. S., 2009. Anonymous yes-no voting with abstention and multiple levels of approval. *Games Econom. Behav.* 67 (2), 428–444.
- Gibbard, A., 1973. Manipulation of voting schemes: a general result. *Econometrica* 41, 587–601.
- Moulin, H., 1980. On strategy-proofness and single peakedness. *Public Choice* 54 (4), 437–455.
- Moulin, H., 1988. *Axioms of cooperative decision making*. Vol. 15 of *Econometric Society Monographs*. Cambridge University Press, Cambridge.
- Ortuño-Ortín, I., 1997. A spatial model of political competition and proportional representation. *Soc. Choice Welf.* 14 (3), 427–438.
- Parker, C., 2012. The influence relation for ternary voting games. *Games Econom. Behav.* 75 (2), 867–881.
- Pongou, R., Tchantcho, B., Difo Lambo, L., 2011. Political influence in multi-choice institutions: cyclicity, anonymity, and transitivity. *Theory and Decision* 70 (2), 157–178.
- Rabinowitz, G., Macdonald, S. E., 1989. A directional theory of issue voting. *The American Political Science Review*, 93–121.
- Rubinstein, A., 1980. Stability of decision systems under majority rule. *J. Econom. Theory* 23 (2), 150–159.
- Satterthwaite, M. A., 1975. Strategy-proofness and Arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *J. Econom. Theory* 10 (2), 187–217.
- Sen, A., 1966. A possibility theorem in majority decision. *Econometrica* 34, 491–496.

- Taylor, A. D., Pacelli, A. M., 2008. Mathematics and politics, 2nd Edition. Springer, New York, strategy, voting, power and proof.
- von Neumann, J., Morgenstern, O., 1944. Theory of Games and Economic Behavior. Princeton University Press, Princeton, New Jersey.
- Yamamura, H., Kawasaki, R., 2013. Generalized average rules as stable Nash mechanisms to implement generalized median rules. Soc. Choice Welf. 40 (3), 815–832.