Cycles in the cycle prefix digraph*

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Abstract

Cycle prefix digraphs are a class of Cayley coset graphs with many remarkable properties such as symmetry, large number of nodes for a given degree and diameter, simple shortest path routing, Hamiltonicity, optimal connectivity, and others. In this paper we show that the cycle prefix digraphs, like the Kautz digraphs, contain cycles of all lengths l, with l between two and N, the order of the digraph, except for $N-1$.

1 Introduction

Several families of digraphs have been proposed as a model for directed interconnection networks for parallel architectures, distributed computing and communication networks [6, 2]. One of these is the family of Kautz digraphs that have been extensively studied in the literature. In particular, Villar (see [10]) showed that Kautz digraphs are almost pancyclic and contain cycles of any length between two and N, the order of the digraph, except for $N-1$. Cycle prefix digraphs constitute a family of Cayley coset digraphs which is particularly attractive. For diameter two, the cycle prefix digraphs and the Kautz digraphs are isomorphic. In general they have many properties that make them an interesting alternative to the Kautz digraphs. Firstly, the cycle prefix digraphs are vertex symmetric, or arc transitive, and therefore in the associated network each node is able to execute the same communication software. These digraphs are also specially

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significant for the construction of networks with order as large as possible for a given maximum degree and diameter. They form, together with other families constructed from them [2], most of the entries of the table of largest known vertex symmetric digraphs[1]. Other remarkable properties are: Hamiltonicity [8], simple shortest path routing, and optimal connectivity [9]. The wide diameter of a cycle prefix digraph with diameter D is $D+2$ [1] (the wide diameter of a graph is considered an important measure of communication efficiency and reliability, see [7]). For $D \geq 3$, the digraphs are $D-$reachable and a message sent from any vertex may reach all other vertices (including the originator) in exactly D steps [2]. Other aspects of interest are their modularity and simple definition. A broadcasting algorithm based on their hierarchical structure was given in [3].

In this paper we are interested in the existence of cycles of all lenghts in these digraphs. This knowledge will help in designing good communication strategies for the corresponding networks. In the next section we will introduce the notation and give some known results and in Section 3, given a cycle prefix digraph of order $|V|$, we obtain cycles of any length l, for $l = 2 \dots |V|$, except when $l = |V| - 1$.

2 Cycle prefix digraphs. Notation and previous results.

$\Gamma_\Delta(D)$ will denote a *cycle prefix digraph* of degree Δ and diameter D. These digraphs were introduced as Cayley coset digraphs by Faber and Moore in 1988 [4, 5] and they may also be defined on an alphabet of $\Delta + 1$ symbols $0, 1, \dots, \Delta$ as follows: Each vertex $x_1 x_2 \cdots x_D$ is a sequence of distinct symbols from the alphabet. The adjacencies are given by

$$x_1 x_2 \cdots x_D \quad \rightarrow \quad \begin{cases} x_2 x_3 x_4 \cdots x_D x_{D+1}, & x_{D+1} \neq x_1, x_2, \dots, x_D \\ x_2 x_3 x_4 \cdots x_D x_1 & \\ x_1 x_2 \cdots x_{k-1} x_{k+1} \cdots x_D x_k, & 2 \leq k \leq D-1 \end{cases}$$

The first kind of adjacency, that introduces a new symbol, will be called a *shift*. The other adjacencies will be called *rotations*: r_k is the adjacency rotating the symbol in position k to the end of the word. Through this paper $\Rightarrow$ will denote the adjacency r_1.

[1] An up-to-date table of the largest known vertex symmetric digraphs is maintained at `http://www-mat.upc.es/grup_de_grafs/table_vsd.html`

$\Gamma_\Delta(D)$ is a vertex symmetric digraph that has order $(\Delta+1)_D = \frac{(\Delta+1)!}{(\Delta+1-D)!}$, diameter D and is Δ-regular ($\Delta \geq D$). The digraph has a hierarchical structure and may be decomposed into $\binom{\Delta+1}{D}$ subdigraphs isomorphic to $\Gamma_{D-1}(D-1)$, with $D \geq 3$, see [3]. Shift adjacencies connect different subdigraphs whereas rotations join vertices in the same subdigraph.

Cycle prefix digraphs are Hamiltonian, see [8]. Another result that we will use from the paper of Jiang and Ruskey [8] is the existence in $\Gamma_D(D)$ of a Hamiltonian path in each subdigraph isomorphic to $\Gamma_{D-1}(D-1)$ that contains D adjacencies of type r_1 such that first symbols are different from one to another.

Finally, we will also use in this paper the fact that an adjacency r_1, connects two vertices such that the first symbol of the first vertex is equal to the last symbol of the second vertex, i.e. $x_1 \dots x_D \Rightarrow x_2 \dots x_D x_1$.

3 Cycles in $\Gamma_\Delta(D)$

In this section we obtain for the cycle prefix digraph of degree Δ and diameter D ($\Delta \geq D$) $\Gamma_\Delta(D) = G(V,E)$ cycles of any length $|V|$, for $N = 2 \dots |V|$, except when $N = |V|-1$. We will distinguish between short cycles (with length less than or equal to $D+1$) and long cycles.

3.1 Short cycles.

We show in the following theorem that each vertex $\mathbf{x}$ in $\Gamma_\Delta(D)$, $\Delta \geq D$, belongs to a cycle $C_k(x)$ of length k, $2 \leq k \leq D$. Cycles $C_k(x)$ are vertex–disjoint, except for $\mathbf{x}$.

Theorem 1 *Let $\Gamma_\Delta(D)$, $\Delta \geq D$, be the cycle prefix digraph of degree Δ and diameter D and let $\mathbf{x} = x_1 x_2 .. x_D$ be a vertex in $\Gamma_\Delta(D)$. Then $\mathbf{x}$ belongs to a family of cycles $C_k(\mathbf{x})$, where $C_k(\mathbf{x})$ has length k, $2 \leq k \leq D$, and* $\bigcap_{k=2}^{D} C_k(\mathbf{x}) = \{\mathbf{x}\}$.

Proof. The cycle $C_k(\mathbf{x})$ is obtained by successively applying the rotation $r_{D-(k-1)}$ to $\mathbf{x}$.

Remark 1 *All D adjacencies of the cycle C_D are of type r_1.*

Example 1 *In* $\Gamma_7(5)$*, the cycles* $C_k(\mathbf{x})$ *for a vertex* $\mathbf{x} = 63412$ *are:*

$$\begin{array}{llllllll} C_2(\mathbf{x}): & \mathbf{63412} & \to 63421 & \to \mathbf{63412} & & & \\ C_3(\mathbf{x}): & \mathbf{63412} & \to 63124 & \to 63241 & \to \mathbf{63412} & & \\ C_4(\mathbf{x}): & \mathbf{63412} & \to 64123 & \to 61234 & \to 62341 & \to \mathbf{63412} & \\ C_5(\mathbf{x}): & \mathbf{63412} & \to 34126 & \to 41263 & \to 12634 & \to 26341 & \to \mathbf{63412} \end{array}$$

The next theorem allows the construction of cycles of length $D+1$. We show that each vertex x in $\Gamma_\Delta(D)$, $\Delta \geq D$, may be included in $\Delta + 1 - D$ different cycles of this length where the adjacencies are all of type shift. We will call this cycles *shift cycles*.

Theorem 2 *Given* $\Gamma_\Delta(D)$*,* $\Delta \geq D$ *and a vertex* $\mathbf{x} = x_1 x_2 .. x_D$ *then* $\mathbf{x}$ *is in* $\Delta + 1 - D$ *cycles,* $C_y(\mathbf{x})$*, of length* $D+1$ *and* $\bigcap_{y \neq x_1,\ldots,x_D} C_y(\mathbf{x}) = \{\mathbf{x}\}$*.*

Proof.

$$C_y : \mathbf{x} = x_1 x_2 .. x_D \to x_2 .. x_D y \to x_3 .. x_D y x_1 \ldots \to y x_1 .. x_{D-1} \to x_1 x_2 .. x_D$$

Since in each cycle C_y, the symbol y is different, all the cycles have only $\mathbf{x}$ as a common vertex. The number of cycles of length $D+1$ is the number of choices of $y \neq x_1, \ldots, x_D$ from the alphabet of $\Delta + 1$ symbols: $\Delta + 1 - D$.

3.2 Long cycles.

We begin the construction of cycles of length k, $k > D + 1$ by studying the cases $D = 2$ and $D = 3$. We use then the recursive structure of the cycle prefix digraphs to generalize the results to the case $\Gamma_\Delta(D)$, $\Delta \geq D$.

Since digraphs $\Gamma_\Delta(D)$ are vertex symmetric, without loss of generality we may consider the identity vertex $I = 12 \ldots D$ as the starting vertex of a cycle.

$\mathbf{\Gamma_2(2)}$

Cycles containing $I = 12$ are:

C_2	12	21				
C_3	12	23	31			
C_4	12	21	13	31		
C_6	12	21	13	32	23	31

The cycle C_5 does not exist.

$\mathbf{\Gamma_3(3)}$

Remark 2 *Any arc in $\Gamma_\Delta(3)$ of type r_1, say, $xyz \Rightarrow yzx$, can be replaced by a path with 3, 4 or 6 vertices . These vertices have symbols $\{y,z,t\}$ with $t \neq x$, therefore they are in the same subdigraph and the paths are isomorphic, respectively, to C_3, C_4 and C_6 in the former case, see Figure 1. For example, the arc $123 \Rightarrow 231$ can be replaced with vertices that contain the symbols $\{2,3,4\}$:*

$$123 \Rightarrow 231: \quad 123 \left\{ \begin{array}{cccccc} 234 & 342 & 423 & & & \\ 234 & 243 & 432 & 423 & & \\ 234 & 243 & 432 & 324 & 342 & 423 \end{array} \right\} 231$$

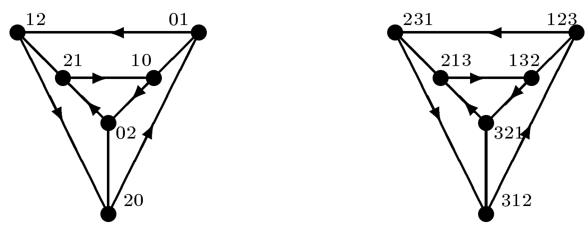

Figure 1: The digraph $\Gamma_2(2)$ and one subdigraph of $\Gamma_\Delta(3)$ isomorphic to it.

- Cycles of length 5. It is not possible to obtain a cycle of length 5 with all the vertices in the same subdigraph (isomorphic to $\Gamma_2(2)$), i.e. vertices with symbols $\{1,2,3\}$. We show here several possible cycles of length 5. Notice that in all of them there is always an adjacency of type r_1:

$$\begin{array}{c} 123 \rightarrow 234 \Rightarrow 342 \rightarrow 421 \rightarrow 412 \\ 123 \rightarrow 234 \rightarrow 341 \Rightarrow 413 \rightarrow 132 \\ 123 \Rightarrow 231 \rightarrow 314 \rightarrow 341 \rightarrow 412 \end{array}$$

- Cycles of length 6. The Hamiltonian cycles of each subdigraph $\Gamma_2(2)$ have length 6.

- Cycles of length 7. They may be constructed from any cycle of length 3, C_3, by replacing one arc with a path of four vertices. This substitution is always possible because all the adjacencies are of type r_1. From $C_3(I): 123 \Rightarrow 231 \Rightarrow 312$ we obtain

$$123 \rightarrow 234 \rightarrow 243 \Rightarrow 432 \rightarrow 423 \rightarrow 231 \Rightarrow 312$$

- Cycles of length 8. Notice that there is always an adjacency of type r_1 in the cycles of length 5 listed above which can be replaced by a path of three vertices to obtain a cycle of length 8:

$$123 \to 234 \to 341 \Rightarrow 413 \Rightarrow 134 \to 342 \to 421 \to 412$$

- Cycle of length 11. It may be constructed from any cycle of length 3 by replacing two arcs with paths of four vertices and one arc with a path of three vertices.

- Cycles of length $9, 10, 12, \ldots 22, 24$. $\Gamma_3(3)$ contains four subdigraphs isomorphic to $\Gamma_2(2)$. In each $\Gamma_2(2)$, as was stated in Section 2, there is a Hamiltonian cycle containing three adjacencies of type r_1. Following the substitution process of Remark 2, we have cycles of lengths $9, 10, 12 \ldots 22$ and 24 vertices. see Figure 2.

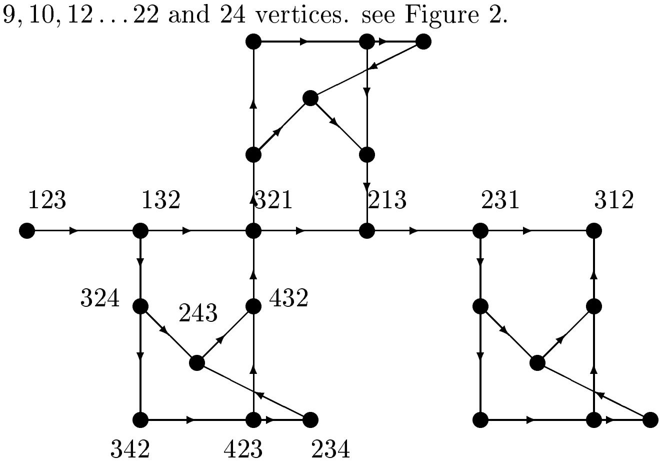

Figure 2: An Hamiltonian cycle in $\Gamma_2(2)$ and the substitution of each arc r_1 by paths of three, four or six vertices.

In general, to obtain a cycle of length k ($k \neq 5, 7, 8, 11$) we write $k = 6(1+q)+r$ with $0 \leq q \leq 3$ and $0 \leq r < 6$. We consider a Hamiltonian cycle in one of the subdigraphs isomorphic to $\Gamma_2(2)$. This cycle has three arcs of type r_1 that may be replaced by a path with three, four or six vertices, then:

- If $r = 0, 3, 4$. Replace q adjacencies r_1 by q Hamiltonian paths and a different adjacency r_1 by a path with r vertices.

- If $r = 1, 2$ then $1 \leq q < 3$. Replace $q-1$ adjacencies r_1 with Hamiltonian paths, use the remaining adjacencies r_1 to complete $6+r$ vertices.

- If $r = 5$ then $1 \leq q < 3$. Replace $q-1$ adjacencies r_1 with Hamiltonian paths. We need three more adjacencies to complete the 11 remaining vertices with two paths of length 4 and one path of length 3. This process is only possible if $q \neq 2$ and, consequently, $k \neq 23$.

$\Gamma_\Delta(3)$, $\Delta > 3$

We recall that $\Gamma_\Delta(3)$ contains $\binom{\Delta+1}{3}$ subdigraphs isomorphic to $\Gamma_2(2)$, and that the order of the digraph is $|\Gamma_\Delta(3)| = (\Delta+1)_3 = 6\binom{\Delta+1}{3} = |\Gamma_2(2)|\binom{\Delta+1}{3}$.

Next we introduce the *substitution tree* $\mathcal{T}$ *of* $\Gamma_\Delta(D)$, see Figure 3. This tree will be used to select pairs of cycles in different subdigraphs with vertices differing only in one symbol. The cycles may be joined to obtain new longer cycles.

Each node of this tree is a subset of D symbols used in $\Gamma_\Delta(D)$ and the tree is made as follows. We choose the vertex $I = 12\ldots D$ as root of $\mathcal{T}$. At the first level we replace each one of the symbols $1, 2\ldots D$ by $D+1$, at the second level we replace each symbol in a node of the first level by $D+2$ and so on. Boldface numbers in Figure 3 indicate the replaced symbol. After this process, we obtain a tree with depth $\Delta + 1 - D$, maximum degree D and such that a new symbol is introduced at each level. Notice that the branches of $\mathcal{T}$ are not directly related to the arcs of $\Gamma_\Delta(D)$.

To obtain a cycle of length k, $k \geq 23$, we start by writing the cycle with symbols $\{1, 2, 3\}$ (associated to the root of $\mathcal{T}$), which is isomorphic to the Hamiltonian path in $\Gamma_2(2)$. We can replace the three adjacencies r_1 of this cycle by paths using the symbols $\{2, 3, 4\}$, $\{1, 3, 4\}$ and $\{1, 2, 4\}$ respectively and obtain in this way, cycles of lengths $k = 9, \ldots 22$ and 24. Next we will replace the three adjacencies r_1 in the Hamiltonian path with symbols $\{2, 3, 4\}$ by paths using $\{3, 4, 5\}$, $\{2, 4, 5\}$ and $\{2, 3, 5\}$. Then we will replace two adjacencies r_1 in the Hamiltonian path which has symbols $\{1, 3, 4\}$ by paths on $\{1, 4, 5\}$ and $\{1, 3, 5\}$. Finally we replace one adjacency r_1 in the Hamiltonian path with symbols $\{1, 2, 4\}$ by paths using $\{1, 2, 5\}$. A new symbol, 6, is introduced here following the scheme given by $\mathcal{T}$.

Notice that in $\Gamma_\Delta(3)$, because of the Hamiltonian path in $\Gamma_2(2)$, there

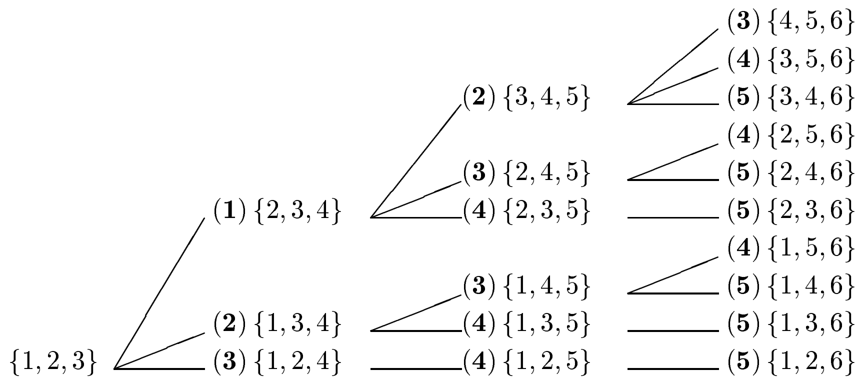

Figure 3: The substitution tree $\mathcal{T}$ of $\Gamma_5(3)$

are $\binom{\Delta+1}{3} - 1$ adjacencies of type r_1 that may be replaced.

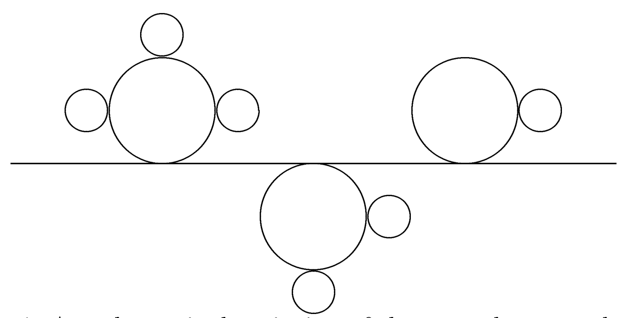

Figure 4: An schematic description of the procedure to substitute adjacencies r_1 in a Hamiltonian cycle of a subdigraph $\Gamma_2(2)$

To obtain a cycle of length k in $\Gamma_\Delta(3)$, we write $k = 6(1+q)+r$, $q \leq \binom{\Delta+1}{3} - 1$ and $r < 6$, and we distinguish the following cases:

- Case $r = 0, 3, 4$. We construct q Hamiltonian paths on the Hamiltonian cycle with symbols $\{1,2,3\}$, and we replace one of the remaining arcs r_1 by a path of length r.
- Case $r = 1, 2$. Notice that $q < \binom{\Delta+1}{3} - 1$. We construct $q-1$ Hamiltonian paths on the Hamiltonian cycle with symbols $\{1,2,3\}$,

and we replace two of the remaining arcs r_1 by two paths of total length $6+r$.

- Case $r=5$. We construct in this case $q-1$ Hamiltonian paths and we replace the three remaining arcs r_1 by two paths of total length $11 = 4+4+3$. This is always possible except when $(\binom{\Delta+1}{3}-1)-(q-1)=2$, that is, when $k=6\binom{\Delta+1}{3}-1=|\Gamma_\Delta(3)|-1$.

The main result of this paper is summarized by the following theorem:

Theorem 3 *The cycle prefix digraph* $\Gamma_\Delta(D)$, $\Delta \geq D$, *contains cycles of any length* k, $2 \leq k \leq |\Gamma_\Delta(D)|$, *except when* $k = |\Gamma_\Delta(D)|-1$.

Proof. Theorems 1 and 2 ensure the existence of cycles of length 2 to $D+1$ in $\Gamma_\Delta(D)$. To prove the existence of longer cycles we will proceed by induction over D. The assertion holds when $D=3$.

Suppose that there exists a cycle of length k, $2<k\leq|\Gamma_\Delta(D-1)|$, $k \neq |\Gamma_\Delta(D-1)|-1$ in $\Gamma_\Delta(D-1)$. We need to construct paths of length $k=|\Gamma_\Delta(D-1)|-1$ and $|\Gamma_\Delta(D-1)|<k\leq|\Gamma_\Delta(D)|$, $k\neq|\Gamma_\Delta(D)|-1$, in $\Gamma_\Delta(D)$.

- Case $k=|\Gamma_{D-1}(D-1)|-1$. The path

$$12\ldots D \to 23\ldots D(D+1) \Rightarrow 3\ldots D(D+1)2 \to$$

$$4\ldots D(D+1)21 \to 4\ldots D(D+1)12 \to \ldots \to (D+1)12\ldots(D-1)$$

  has $D+2$ vertices and one adjacency of type r_1 which can be replaced by a path with $|\Gamma_{D-1}(D-1)|-D-3$ vertices and symbols $\{1,3\ldots D,(D+1)\}$, to obtain a path with $|\Gamma_{D-1}(D-1)|-1$ vertices.

- Case $|\Gamma_{D-1}(D-1)|<k\leq|\Gamma_\Delta(D)|$ and $k\neq|\Gamma_\Delta(D)|-1$. We start with a Hamiltonian path in one of the subdigraphs isomorphic to $\Gamma_{D-1}(D-1)$

$$12\ldots D \to \ldots \to D1\ldots(D-1)$$

  The results obtained by Jiang and Ruskey [8] and quoted at the end od Section 2, ensure that this path has D adjacencies r_1 which can be replaced following the scheme given by the corresponding substitution tree $\mathcal{T}$. Overall there are $\binom{\Delta+1}{D}-1$ adjacencies r_1 to replace.

Moreover $|\Gamma_\Delta(D)| = |\Gamma_{D-1}(D-1)|\binom{\Delta+1}{D}$. To obtain the cycle of length k, we write $k = |\Gamma_{D-1}(D-1)|(q+1)+r$, with $q \le \binom{\Delta+1}{D} - 1$, $r < |\Gamma_{D-1}(D-1)|$, and we consider the following cases:

- Case $r = 0, D, D+1 \ldots |\Gamma_{D-1}(D-1)| - 2$. Replace q adjacencies r_1 by Hamiltonian paths in q different copies of $\Gamma_{D-1}(D-1)$ and use a remaining adjacency r_1 to replace it with r vertices.
- Case $r = 1, 2, \ldots D-1$. In this case $q < \binom{\Delta+1}{D} - 1$. Replace $q-1$ adjacencies r_1 by Hamiltonian paths in $q-1$ different copies of $\Gamma_{D-1}(D-1)$ and use two remaining adjacencies r_1 to replace them with $|\Gamma_{D-1}(D-1)| + r$ vertices. This can always be done because $|\Gamma_{D-1}(D-1)| + r = a + b$ with $D < a, b < |\Gamma_{D-1}(D-1)| - 1$ and there are always two arcs to replace: $\binom{\Delta+1}{D} - 1 - (q-1) \ge 2$.
- Case $r = |\Gamma_{D-1}(D-1)| - 1$. Replace $q-1$ adjacencies r_1 by Hamiltonian paths. We need to add $2|\Gamma_{D-1}(D-1)| - 1$ vertices. We must use three of the remaining adjacencies r_1 to ensure that each adjacency is replaced by less than $|\Gamma_{D-1}(D-1)| - 1$ vertices. This process can be done when $\binom{\Delta+1}{D} - 1 - (q-1) \ge 3$. When $\binom{\Delta+1}{D} - 1 - (q-1) = 2$ then $q = \binom{\Delta+1}{D} - 2$ and it is not possible to have a cycle of length

$$k = |\Gamma_{D-1}(D-1)|\binom{\Delta+1}{D} - 1 = |\Gamma_\Delta(D)| - 1$$

4 Conclusions

In this paper we have studied the cycle structure of cycle prefix digraphs. In particular, for the cycle prefix digraph of degree Δ and diameter D, $\Delta \ge D$, $\Gamma_\Delta(D) = G(V,E)$, we have obtained cycles of any length l, for $l = 2 \ldots |V|$, except when $l = |V| - 1$. The existence for $\Gamma_\Delta(D) = G(V,E)$ of cycles of length $|V| - 1$ is an open problem. We have checked that all cycle prefix digraphs with small order ($|V| \le 720$) do not contain a cycle of length $|V| - 1$ and we conjecture that, as it happens for the Kautz digraphs, these digraphs are almost pancyclic.

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