HOW TO SIMPLIFY RETURN-MAPPING ALGORITHMS IN COMPUTATIONAL PLASTICITY: PART 1 - MAIN IDEA

MARTIN CERMAK∗† AND STANISLAV SYSALA†

∗ VSB-TU Ostrava (VSB-TU)
IT4Innovations National Supercomputing Center (IT4I)
Studentska 6231/1b, 708 00 Ostrava, Czech Republic
e-mail: martin.cermak@vsb.cz - web page: http://www.it4i.cz

† Institute of Geonics AS CR (IGN)
Department of IT4Innovations (IT4I)
Studentska 1768, 708 00 Ostrava, Czech Republic
e-mail: stanislav.sysala@ugn.cas.cz - web page: http://www.ugn.cas.cz/~sysala

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Abstract. An improved implicit return-mapping scheme for nonsmooth yield surfaces is suggested in this paper. It is based on a subdifferential formulation of the flow rule that does not require special treatment at singular points (apices, edges, etc.). Although the flow direction is multivalued in such a case, it is shown that the suggested scheme leads to solving a unique system of nonlinear equations similarly as for smooth yield surfaces. Improved return-mapping schemes of several elastoplastic models containing nonassociative plastic flow rules and nonlinear isotropic hardening laws are introduced together with the corresponding consistent tangent operators.

1 INTRODUCTION

The paper is devoted to numerical solution of small-strain quasi-static elastoplastic problems. Such a problem consists of the constitutive initial value problem (CIVP) and the balance equation representing the principle of virtual work. CIVP usually satisfies thermodynamical laws and can contain internal variables like hardening. There are several methods how to discretize and solve the problem. An overview of the methods can be found in the book [4, Chapter 7] and the references introduced therein.

We focus on the frequently used implicit Euler discretization. Then the incremental constitutive problem is solved by the elastic predictor / plastic corrector method. The plastic correction is usually called the return-mapping scheme. The scheme is relatively straightforward for a smooth yield surface and leads to solving a system of nonlinear
equations. In the presence of non-smooth corners on the yield surfaces, it is necessary to distinguish whether an updated stress tensor lies on smooth or nonsmooth portions of the surface [4, Chapter 8]. However, it leads to a "blind guessing" in the corresponding algorithm since the stress tensor is unknown.

To suppress this evident drawback, one can re-formulate the incremental constitutive problem e.g. using: a) generalized closed-point projection in associative plasticity [8]; b) theory of bipotentials in nonassociative plasticity [6].

In this paper, we present another approach based on the following subdifferential definition of the plastic flow rule:

\[ \dot{\varepsilon}^p \in \lambda \partial_\sigma g(\sigma, A), \]

where \( \dot{\varepsilon}^p, \lambda, \sigma, A, \) and \( g = g(\sigma, A) \) denote the plastic strain rate, the plastic multiplier rate, the stress tensor, the hardening thermodynamical forces and the plastic potential, respectively. Further, \( \partial_\sigma g(\sigma, A) \) denotes the subdifferential of the plastic potential \( g \) at \((\sigma, A)\) with respect to the stress variable. The definition (1) is known in literature (see, e.g., [4, Section 6.3.9]). On the first sight, it seems that (1) is not convenient for numerical treatment due to the presence of the multivalued flow direction.

The main goal of this paper is to show that the return-mapping scheme based on (1) directly simplifies the standard scheme at least for a wide class of isotropic models in sense that a unique system of nonlinear equations is solved without any "blind guessing". The improved schemes together with the corresponding consistent tangent operators are introduced for three elastoplastic models containing nonassociative plastic flow rules and nonlinear isotropic hardening laws: the Drucker-Prager model (Section 3), the general model in the Haigh-Westergaard coordinates (Section 4) and the Mohr-Coulomb model (Section 5).

It is surprising that for the Drucker-Prager and Mohr-Coulomb models, one can a priori decide whether the unknown stress tensor will lie on the smooth portion or not. The new technique also simplifies forms of the consistent tangent operators, mainly for the Mohr-Coulomb model. For detail derivation, we refer [1, 2].

An implementation of the "whole" initial boundary value elastoplastic problem containing the Drucker-Prager model is described in detail and illustrated on numerical examples within the related paper "PART II" [3].

2 THE CONSTITUTIVE INITIAL VALUE PROBLEM AND THE IMPLICIT EULER DISCRETIZATION

Consider the following constitutive initial value elastoplastic problem: Given the history of the strain tensor \( \varepsilon = \varepsilon(t), t \in [t_0, t_{\text{max}}] \), and the initial values

\[ \varepsilon^p(t_0) = \varepsilon^p, \ \varepsilon^p(t_0) = \varepsilon_0^p. \]
Find the generalized stress \((\mathbf{\sigma}(t), \kappa(t))\) and the generalized strain \((\mathbf{\varepsilon}^p(t), \hat{\mathbf{\varepsilon}}^p(t))\) such that

\[
\begin{align*}
\mathbf{\varepsilon} &= \mathbf{\varepsilon}^e + \mathbf{\varepsilon}^p, \\
\mathbf{\sigma} &= D_e : \mathbf{\varepsilon}^e, \quad \kappa = H(\hat{\mathbf{\varepsilon}}^p), \\
\hat{\mathbf{\varepsilon}}^p &= \hat{\lambda} \partial_\sigma g(\mathbf{\sigma}, \kappa), \\
\hat{\mathbf{\varepsilon}}^p &= \hat{\lambda} \ell(\mathbf{\sigma}, \kappa), \\
\hat{\lambda} &\geq 0, \quad f(\mathbf{\sigma}, \kappa) \leq 0, \quad \hat{\lambda} f(\mathbf{\sigma}, \kappa) = 0.
\end{align*}
\]

hold for each instant \(t \in [t_0, t_{\text{max}}]\). Here, \(\mathbf{\varepsilon}^e, \mathbf{\varepsilon}^p, \hat{\mathbf{\varepsilon}}^p, \mathbf{\sigma}, \kappa, \hat{\lambda} f\) and \(g\) denote elastic and plastic parts of the strain tensor, the isotropic hardening variable, the stress tensor, the thermodynamical isotropic hardening force, the plastic multiplier, the yield function, and the plastic potential, respectively. Further, it is assumed that: \(H : \mathbb{R}_+ \to \mathbb{R}_+\) is a nondecreasing, continuous and piecewise smooth function satisfying \(H(0) = 0\); \(f, g : \mathbb{R}^{3 \times 3} \times \mathbb{R}_+ \to \mathbb{R}\) are convex at least in vicinity of the yield surface; \(\ell : \mathbb{R}^{3 \times 3} \times \mathbb{R}_+ \to \mathbb{R}_+\) is a positive value function. Finally, the fourth order tensor \(D_e\) represents a linear isotropic elastic response. One can write

\[
\mathbf{\sigma} = D_e : \mathbf{\varepsilon}^e = K(\mathbf{I} : \mathbf{\varepsilon}^e) + 2G\mathbb{I}_{\text{dev}} : \mathbf{\varepsilon}^e, \quad D_e = K\mathbf{I} \otimes \mathbf{I} + 2G\mathbb{I}_{\text{dev}},
\]

(2)

where \(K, G > 0\) denotes the bulk, and shear moduli, respectively. Further, \(\mathbf{I}\) is the identity second order tensor and \(\mathbb{I}_{\text{dev}}\) is the fourth order tensor representing the deviatoric part of a tensor, i.e.

\[
\mathbb{I}_{\text{dev}} : \mathbf{\varepsilon}^e = \mathbf{\varepsilon}^e - \frac{1}{3}(\mathbf{I} : \mathbf{\varepsilon}^e)\mathbf{I}.
\]

Consider the following partition:

\[t_0 < t_1 < \ldots < t_{k} < \ldots < t_m = t_{\text{max}}.\]

From now on, fix \(k = 1, \ldots, m\) and denote \(\mathbf{\sigma} := \mathbf{\sigma}(t_k), \mathbf{\varepsilon} := \mathbf{\varepsilon}(t_k), \mathbf{\varepsilon}^{e,tr} := \mathbf{\varepsilon}(t_k) - \mathbf{\varepsilon}^p(t_{k-1}), \hat{\mathbf{\varepsilon}}^{p,tr} := \hat{\mathbf{\varepsilon}}^p(t_{k-1})\) and \(\mathbf{\sigma}^{tr} := D_e : \mathbf{\varepsilon}^{e,tr}\). Then the corresponding incremental constitutive problem for the \(k\)-step received by the implicit Euler method reads as follows: Given \(\mathbf{\varepsilon}^{e,tr}\) and \(\hat{\mathbf{\varepsilon}}^{p,tr}\). Find \(\mathbf{\sigma}^{tr}, \hat{\mathbf{\varepsilon}}^{p,tr}\) and \(\Delta\lambda\) satisfying:

\[
\begin{align*}
\mathbf{\sigma} &= \mathbf{\sigma}^{tr} - \Delta\lambda D_e : \mathbf{N}, \quad \mathbf{N} \in \partial_\sigma g(\mathbf{\sigma}, H(\hat{\mathbf{\varepsilon}}^p)), \\
\hat{\mathbf{\varepsilon}}^p &= \hat{\mathbf{\varepsilon}}^{p,tr} + \Delta\lambda \ell(\mathbf{\sigma}, H(\hat{\mathbf{\varepsilon}}^p)), \\
\Delta\lambda &\geq 0, \quad f(\mathbf{\sigma}, H(\hat{\mathbf{\varepsilon}}^p)) \leq 0, \quad \Delta\lambda f(\mathbf{\sigma}, H(\hat{\mathbf{\varepsilon}}^p)) = 0.
\end{align*}
\]

If this problem has a solution then the remaining input parameter for the next step has the form \(\mathbf{\varepsilon}^p(t_k) = \mathbf{\varepsilon}(t_k) - D_e^{-1} : \mathbf{\sigma}(t_k)\). To solve the incremental problem we use the standard elastic predictor/plastic corrector method.

The elastic predictor. First, we verify whether the trial generalized stress \((\mathbf{\sigma}^{tr}, \hat{\mathbf{\varepsilon}}^{p,tr})\) is admissible:

\[
f(\mathbf{\sigma}^{tr}, H(\hat{\mathbf{\varepsilon}}^{p,tr})) \leq 0.
\]

(3)
If this inequality holds then we set
\[ \sigma = \sigma^{tr}, \quad \bar{\varepsilon} = \bar{\varepsilon}^{p,tr}, \quad \triangle \lambda = 0. \]

It is readily seen that the triplet \((\sigma, \bar{\varepsilon}, \triangle \lambda)\) solves the incremental problem and the corresponding consistent tangent operator has the form
\[ \frac{\partial \sigma}{\partial \varepsilon} = D_e. \]

The plastic corrector (the return-mapping scheme). Let (3) do not hold and assume that the incremental constitutive problem has a solution. Then clearly \(\triangle \gamma > 0\) and the problem reduces into the following form: Given \(\varepsilon^{e,tr}\) and \(\bar{\varepsilon}^{p,tr}\) such that \(f(\sigma^{tr}, H(\bar{\varepsilon}^{p,tr})) > 0\). Find \(\sigma, \bar{\varepsilon}^{p}\) and \(\triangle \lambda > 0\) satisfying:
\[ \begin{align*}
\sigma &= \sigma^{tr} - \triangle \lambda D_e : N, \quad N \in \partial g(\sigma, H(\bar{\varepsilon}^{p})), \\
\bar{\varepsilon}^{p} &= \bar{\varepsilon}^{p,tr} + \triangle \lambda(\sigma, H(\bar{\varepsilon}^{p})), \\
f(\sigma, H(\bar{\varepsilon}^{p})) &= 0.
\end{align*} \tag{4} \]

If the plastic potential \(g\) is differentiable on the yield surface then the flow direction \(N\) is always singlevalued and the return-mapping scheme leads to solving a system of nonlinear equation. In the rest of the paper, we demonstrate on three models that the plastic correction leads to solving a unique system of nonlinear equations even if \(g\) is nonsmooth everywhere on the yield surface.

3 THE DRUCKER-PRAGER MODEL

The yield function and the plastic potential are given as follows:
\[ g(\sigma, \kappa) = \gamma(p(\sigma), \varrho(\sigma)) = \sqrt{\frac{1}{2} \varrho + \eta p - \xi (c_0 + \kappa)}, \]
\[ f(\sigma, \kappa) = \hat{f}(p(\sigma), \varrho(\sigma), \kappa) = \sqrt{\frac{1}{2} \varrho + \eta p}, \]
respectively, where the parameters \(\eta, \xi > 0\) are usually calculated from the friction angle using a sufficient approximation of the Mohr-Coulomb yield surface, the parameter \(\bar{\eta}\) depends on the dilatancy angle and \(c_0 > 0\) denotes the initial cohesion. Further, \(p(\sigma) = I : \sigma / 3, \quad s(\sigma) = I_{\text{dev}} : \sigma\) and \(\varrho(\sigma) = \sqrt{s : s} = \|s\|\) define the hydrostatic stress, the deviatoric stress, and its norm, respectively, related to \(\sigma\). Notice that \(\varrho^2 / 2 = J_2(s)\). Finally, we let the function \(H\) in an abstract form and choose the associative hardening law, i.e., set \(\ell(\sigma, \kappa) = \xi\). Define \(\rho^{tr} = I : \sigma^{tr} / 3, \quad s^{tr} = I_{\text{dev}} : \sigma^{tr}, \quad \rho^{tr} = \|s^{tr}\|\) and \(n^{tr} = s^{tr} / \rho^{tr}\) for \(\rho^{tr} \neq 0\). The following result has been proved in [1].
Theorem 1. Let \((\sigma, \bar{\varepsilon}^p, \Delta \lambda)\) be a solution to problem (4). Then \((p, \varrho, \bar{\varepsilon}^p, \Delta \lambda)\) is a solution to the following system:

\[
\begin{aligned}
p &= p^{tr} - \Delta \lambda K \bar{\eta}, \\
\varrho &= (\varrho^{tr} - \Delta \lambda G \sqrt{2})^+, \\
\bar{\varepsilon}^p &= \bar{\varepsilon}^{p,tr} + \Delta \lambda \xi, \\
\hat{f}(p, \varrho, H(\bar{\varepsilon}^p)) &= 0, \\
\end{aligned}
\]

where \((\cdot)^+\) denotes a positive part of a function.

Conversely, if \((p, \varrho, \bar{\varepsilon}^p, \Delta \lambda)\) is a solution to (5) then \((\sigma, \bar{\varepsilon}^p, \Delta \lambda)\) is the solution to (4), where

\[
\sigma = \begin{cases} 
\sigma^{tr} - \Delta \lambda \left(G \sqrt{2} n^{tr} + K \bar{\eta} I\right) & \text{if } \varrho^{tr} > \Delta \lambda G \sqrt{2}, \\
(p^{tr} - \Delta \lambda K \bar{\eta}) I & \text{if } \varrho^{tr} \leq \Delta \lambda G \sqrt{2}.
\end{cases}
\]

So the return-mapping scheme reduces into the system (5) of nonlinear equations regardless the plastic potential is differentiable at the unknown stress tensor \(\sigma\) or not. Further, one can simply insert (5)\(_1\)–\(_3\) into (5)\(_4\) leading to the nonlinear equation \(q(\Delta \lambda) = 0\), where

\[
q(\gamma) := \sqrt{\frac{1}{2}} \left(\varrho^{tr} - \gamma G \sqrt{2}\right)^+ + \eta \left(p^{tr} - \gamma K \bar{\eta}\right) - \xi \left(c_0 + H(\bar{\varepsilon}^{p,tr} + \gamma \xi)\right), \quad \gamma \in \mathbb{R}_+.
\]

We summarize the return-mapping scheme and the related consistent tangent operator introduced in [1]. For the sake of simplicity, assume that \(H\) is differentiable at \(\bar{\varepsilon}^{p,tr} + \Delta \lambda \xi\) and set \(H_1 = H'(\bar{\varepsilon}^{p,tr} + \Delta \lambda \xi)\).

Return to the smooth portion of the yield surface. It happens if and only if

\[
q(0) > 0 \quad \text{and} \quad q\left(\frac{\varrho^{tr}}{G \sqrt{2}}\right) < 0.
\]

1. Find \(\Delta \lambda \in (0, \varrho^{tr}/G \sqrt{2})\) satisfying

\[
q(\Delta \lambda) = \sqrt{\frac{1}{2}} \left(\varrho^{tr} - \Delta \lambda G \sqrt{2}\right)^+ + \eta \left(p^{tr} - \Delta \lambda K \bar{\eta}\right) - \xi \left(c_0 + H(\bar{\varepsilon}^{p,tr} + \Delta \lambda \xi)\right) = 0.
\]

2. Compute

\[
\sigma = \sigma^{tr} - \Delta \lambda \left(G \sqrt{2} n^{tr} + K \bar{\eta} I\right), \quad \bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \Delta \lambda \xi.
\]

3. Set

\[
\frac{\partial \sigma}{\partial \varepsilon} = D_e - \Delta \lambda \frac{2 G^2 \sqrt{2}}{\varrho^{tr}} \left(\mathbb{I}_{dev} - n^{tr} \otimes n^{tr}\right) - (G \sqrt{2} n^{tr} + K \bar{\eta} I) \otimes \frac{G \sqrt{2} n^{tr} + \eta K I}{G + K \bar{\eta} + \xi^2 H_1}.
\]
Return to the apex of the yield surface. It happens if and only if
\[ q \left( \frac{\varrho^{tr}}{G\sqrt{2}} \right) \geq 0. \]

1. **Find** \( \triangle \lambda \geq \varrho^{tr}/G\sqrt{2} \) satisfying
\[ q(\triangle \lambda) = \eta (p^{tr} - \triangle \lambda K\bar{\eta}) - \xi (c_0 + H(\bar{\varepsilon}^{p,tr} + \triangle \lambda \xi)) = 0. \]

2. **Compute**
\[ \sigma = (p^{tr} - \triangle \lambda K\bar{\eta}) I, \quad \bar{\varepsilon}^p = \bar{\varepsilon}^{p,tr} + \triangle \lambda \xi. \]

3. **Set**
\[ \frac{\partial \sigma}{\partial \varepsilon} = \frac{\xi^2 KH_1}{K\eta\bar{\eta} + \xi^2 H_1} I \otimes I \]
if \( q(\varrho^{tr}/G\sqrt{2}) > 0. \)

If \( H \) is a linear function then \( \triangle \lambda \) can be found in closed form. In paper "PART II" [3], we use this return-mapping scheme to solve the corresponding initial boundary value elastoplastic problem.

4 **THE GENERAL MODEL IN THE HAIGH-WESTERGAARD COORDINATES**

In this section, we demonstrate that the treatment introduced in Section 3 can be generalized on a wide class of models given by the Haigh-Westergaard coordinates. For the sake of brevity, we consider the general model inspired by the plastic part of the Jirasek-Grassl damage-plastic model, see [5].

Beside the notation introduced in Section 2 and 3 we use the Lode angle,
\[ \theta := \theta(\sigma) = \frac{1}{3} \arccos \left( \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}} \right), \quad J_2 := \frac{1}{2} s : s = \frac{1}{2} \varrho^2, \quad J_3 = \frac{1}{3} s^3 : I \]
and the functions
\[ \tilde{\varrho} := \tilde{\varrho}(\sigma) = \varrho(\sigma) \tilde{r}(\cos \theta(\sigma)), \quad \hat{\varrho} := \hat{\varrho}(\sigma) = \varrho(\sigma) \hat{r}(\cos \theta(\sigma)). \]

Here, \( \tilde{r}, \hat{r} \) are nonnegative smooth functions and the mapping \( \sigma \mapsto \tilde{r}(\cos \theta(\sigma)) \) is smooth. Consider the following forms of the functions \( f, g \) and \( \ell \):
\[ f(\sigma, \kappa) = \hat{f}(p(\sigma), \varrho(\sigma), \hat{\varrho}(\sigma), \kappa), \]
\[ g(\sigma, \kappa) = \hat{g}(p(\sigma), \varrho(\sigma)), \]
\[ \ell(\sigma, \kappa) = \hat{\ell}(p(\sigma), \varrho(\sigma), \hat{\varrho}(\sigma)). \]
respectively. We assume that: \( \hat{f} \) is increasing with respect to \( \varrho \) and \( \tilde{\varrho} \), convex and continuously differentiable at least in vicinity of the yield surface; \( \hat{g} \) is an increasing function with respect to \( \varrho \), convex and twice continuously differentiable at least in vicinity of the yield surface; \( \hat{\ell} \) is a nonnegative smooth function. Finally, denote \( \hat{g}_V(p, \varrho) := \frac{\partial \hat{g}}{\partial p}, \hat{g}_\varrho(p, \varrho) := \frac{\partial \hat{g}}{\partial \varrho} \) and \( \theta^{tr} := \theta(\sigma^{tr}) \). We have the following result, see [1].

**Theorem 2.** Let \((\sigma, \varepsilon^{p}, \Delta \lambda)\) be a solution to problem (4). Then \((p, \varrho, \bar{\varepsilon}^{p}, \triangle \lambda)\) is a solution to the following system:

\[
\begin{align*}
\label{eq:6}
p - p^{tr} + \Delta \lambda K \hat{g}_V(p, \varrho) &= 0, \\
\varrho - [\varrho^{tr} - \Delta \lambda 2G \hat{g}_\varrho(p, \varrho)]^{+} &= 0, \\
\varepsilon^{p} - \varepsilon^{p, tr} - \Delta \lambda \hat{\ell}(p, \varrho, \varrho^{tr} \cos(\theta^{tr})) &= 0, \\
\hat{f}(p, \varrho, \varrho^{tr}, H(\varepsilon^{p})) &= 0.
\end{align*}
\]

Conversely, if \((p, \varrho, \varepsilon^{p}, \Delta \lambda)\) is a solution to (6) then \((\sigma, \varepsilon^{p}, \Delta \lambda)\) is the solution to (4), where

\[
\sigma = \begin{cases} 
\sigma^{tr} - \Delta \lambda [2G \hat{g}_\varrho(p, \varrho)n^{tr} + K \hat{g}_V(p, \varrho)I] & \text{if } \varrho > 0, \\
[p^{tr} - \Delta \lambda K \hat{g}_V(p, 0)]I & \text{if } \varrho = 0.
\end{cases}
\]

These system can be solved by the semismooth Newton method introduced in [7]. We see that two different cases can happen: if \( \varrho > 0 \) then the plastic corrector returns the stress to the smooth portion of the yield surface, otherwise to the apices.

For derivation of the consistent tangent operator, we use these derivatives:

\[
\begin{align*}
\frac{\partial p^{tr}}{\partial \varepsilon} &= K I, \\
\frac{\partial s^{tr}}{\partial \varepsilon} &= 2G I_{\text{dev}}, \\
\frac{\partial \varrho^{tr}}{\partial \varepsilon} &= 2G n^{tr}, \text{ if } s^{tr} \neq 0, \\
\frac{\partial n^{tr}}{\partial \varepsilon} &= \frac{2G}{\varrho^{tr}} (I_{\text{dev}} - n^{tr} \otimes n^{tr}), \text{ if } s^{tr} \neq 0, \\
\frac{\partial \theta^{tr}}{\partial \varepsilon} &= \frac{2G \sqrt{6}}{\varrho^{tr} \sin(3\theta^{tr})} \left[(n^{tr} \otimes (n^{tr})^3)I - I_{\text{dev}}(n^{tr})^2\right], \text{ if } s^{tr} \neq 0, \varrho^{tr} \neq 0, \frac{2\pi}{3}.
\end{align*}
\]

Return to the apices. Assume that \( \varrho^{tr} < \Delta \lambda 2G \hat{g}_\varrho(p, \varrho) \). Then \( \varrho = 0 \) and the system (6) reduces into

\[
\begin{align*}
p + \Delta \lambda K \hat{g}_V(p, 0) &= p^{tr}, \\
\varepsilon^{p} - \Delta \lambda \hat{\ell}(p, 0, 0) &= \varepsilon^{p, tr}, \\
\hat{f}(p, 0, 0, H(\varepsilon^{p})) &= 0.
\end{align*}
\]
The derivative of the reduced system yields

\[
\left(\begin{array}{c}
\frac{\partial p}{\partial \varepsilon} \\
\frac{\partial \theta}{\partial \varepsilon} \\
\frac{\partial \varepsilon}{\partial \varepsilon} \\
\frac{\partial \lambda}{\partial \varepsilon}
\end{array}\right) = \left(\begin{array}{cccc}
1 + \triangle \lambda K \hat{g}_{VV}(p,0) & \triangle \lambda K \hat{g}_{V,s} & 0 & K \hat{g}_{V,V,s} \\
-\triangle \lambda \hat{\lambda}_{V}(p,0,0) & 1 + \triangle \lambda 2G \hat{g}_{\theta V,s} & 0 & 2G \hat{g}_{\theta V,s} \\
\hat{f}_{V}(p,0,0,H(\varepsilon^p)) & -\triangle \lambda \hat{\lambda}_{V,s} & 1 & -\hat{\lambda}_s \\
\hat{f}_{s}(p,0,0,H(\varepsilon^p)) & \hat{f}_{V,s} & \hat{f}_{V,s} & 0
\end{array}\right)^{-1} \left(\begin{array}{c}
KI \\
0 \\
0 \\
0
\end{array}\right).
\]

Here \( \hat{f}_{V} = \partial \hat{f}/\partial p, \hat{f}_{s} = \partial \hat{f}/\partial \varepsilon, \hat{\lambda}_{V} = \partial \hat{\lambda}/\partial p \) and \( \hat{g}_{VV} = \partial \hat{g}_{V}/\partial p \). Using (7) we obtain

\[
\frac{\partial \sigma}{\partial \varepsilon} = KI \otimes \left( I - \triangle \lambda \hat{g}_{VV}(p,0) \frac{\partial p}{\partial \varepsilon} - \hat{g}_{V}(p,0) \frac{\partial \triangle \lambda}{\partial \varepsilon} \right).
\]

Return to the smooth portion. Assume that \( q^{tr} > \triangle \lambda 2G \hat{g}_{\theta}(p,\theta) \) and \( \theta^{tr} \neq 0,2\pi/3 \). Then \( \theta, q^{tr} > 0 \) and the system (6) reduces into

\[
\begin{align*}
p + \triangle \lambda K \hat{g}_{V}(p,\theta) &= p^{tr}, \\
\theta + \triangle \lambda 2G \hat{g}_{\theta}(p,\theta) &= q^{tr}, \\
\varepsilon^{p} - \triangle \lambda \hat{\lambda}(p,\theta, \hat{r}(\cos \theta^{tr})) &= \varepsilon^{p, tr}, \\
\hat{f}(p,\theta, \hat{r}(\cos \theta^{tr}), H(\varepsilon^{p})) &= 0.
\end{align*}
\]

The derivative of the reduced system yields

\[
\left(\begin{array}{c}
\frac{\partial p}{\partial \varepsilon} \\
\frac{\partial \theta}{\partial \varepsilon} \\
\frac{\partial \varepsilon}{\partial \varepsilon} \\
\frac{\partial \lambda}{\partial \varepsilon}
\end{array}\right) = \left(\begin{array}{cccc}
1 + \triangle \lambda K \hat{g}_{VV,s} & \triangle \lambda K \hat{g}_{V,s} & 0 & K \hat{g}_{V,V,s} \\
\triangle \lambda 2G \hat{g}_{\theta V,s} & 1 + \triangle \lambda 2G \hat{g}_{\theta V,s} & 0 & 2G \hat{g}_{\theta V,s} \\
\hat{f}_{V,s} & -\triangle \lambda \hat{\lambda}_{V,s} & 1 & -\hat{\lambda}_s \\
\hat{f}_{s,s} & \hat{f}_{V,s} & \hat{f}_{V,s} & 0
\end{array}\right)^{-1} \left(\begin{array}{c}
KI \\
2Gn^{tr} \\
\triangle \lambda \hat{\lambda}_{\theta} \frac{\partial \theta^{tr}}{\partial \varepsilon} \\
-\hat{f}_{\theta} \frac{\partial \theta^{tr}}{\partial \varepsilon}
\end{array}\right).
\]

Here,

\[
\begin{align*}
\hat{f}_{\theta} &= -\frac{\partial \hat{f}(p,\theta, \hat{r}(\cos \theta^{tr}), H(\varepsilon^{p}))}{\partial \theta} \hat{r}''(\cos \theta^{tr}) \sin \theta^{tr}, \\
\hat{\lambda}_{\theta} &= -\frac{\partial \hat{\lambda}(p,\theta, \hat{r}(\cos \theta^{tr}))}{\partial \theta} \hat{r}''(\cos \theta^{tr}) \sin \theta^{tr}, \\
\hat{g}_{VV,s} &= \frac{\partial \hat{g}_{V}(p,\theta)}{\partial p}, \quad \hat{g}_{V,s} = \frac{\partial \hat{g}_{V}(p,\theta)}{\partial \theta}, \quad \hat{g}_{V,s} = \frac{\partial \hat{g}_{V}(p,\theta)}{\partial p} = \hat{g}_{V,s}, \\
\hat{g}_{\theta \theta,s} &= \frac{\partial \hat{g}_{\theta}(p,\theta)}{\partial \theta}, \quad \hat{g}_{\theta,s} = \hat{g}_{\theta}(p,\theta), \quad \hat{g}_{\theta,s} = \hat{g}_{\theta}(p,\theta), \\
\hat{\lambda}_{s} &= \hat{\lambda}(p,\theta, \hat{r}(\cos \theta^{tr})), \quad \hat{\lambda}_{V,s} = \frac{\partial \hat{\lambda}(p,\theta, \hat{r}(\cos \theta^{tr}))}{\partial p}, \\
\hat{\lambda}_{\theta,s} &= \frac{\partial \hat{\lambda}(p,\theta, \hat{r}(\cos \theta^{tr}))}{\partial \theta} + \frac{\partial \hat{\lambda}(p,\theta, \hat{r}(\cos \theta^{tr}))}{\partial \theta} \hat{r}(\cos \theta^{tr}),
\end{align*}
\]

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\[ \dot{f}_{v,s} = \frac{\partial \dot{f}(p, q, g\tilde{G}(\cos \theta r), H(\tilde{e}^p))}{\partial p}, \quad \dot{f}_{\kappa,s} = \frac{\partial \dot{f}(p, q, g\tilde{G}(\cos \theta r), H(\tilde{e}^p))}{\partial p} H'(\tilde{e}^p), \]

\[ \dot{f}_{\phi,s} = \frac{\partial \dot{f}(p, q, g\tilde{G}(\cos \theta r), H(\tilde{e}^p))}{\partial q} \quad \text{and} \quad \frac{\partial \dot{f}(p, q, g\tilde{G}(\cos \theta r), H(\tilde{e}^p))}{\partial \tilde{e}} \tilde{r}(\cos \theta r). \]

Using (7) we obtain

\[ \frac{\partial \sigma}{\partial e} = \mathbb{D}_e - [2G\tilde{g}_v(p, q)n^{tr} + K\tilde{g}_v(p, q)] \otimes \frac{\partial \Delta \lambda}{\partial e} - K\Delta \lambda \otimes \left( \tilde{g}_{vV,s} \frac{\partial p}{\partial e} + \tilde{g}_{v\phi,s} \frac{\partial q}{\partial e} \right), \]

\[ -2G\Delta \lambda \left( \tilde{g}_v(p, q) \frac{\partial n^{tr}}{\partial e} + n^{tr} \otimes \left( \tilde{g}_{vV,s} \frac{\partial p}{\partial e} + \tilde{g}_{v\phi,s} \frac{\partial q}{\partial e} \right) \right). \]

5 THE MOHR-COULOMB MODEL

The yield function and the plastic potential are given as follows:

\[ f(\sigma, \kappa) = \dot{f}(\sigma_1, \sigma_2, \sigma_3, \kappa) = (1 + \sin \phi)\sigma_1 - (1 - \sin \phi)\sigma_3 - 2(c_0 + \kappa) \cos \phi, \]

\[ g(\sigma, \kappa) = \dot{g}(\sigma_1, \sigma_2, \sigma_3) = (1 + \sin \psi)\sigma_1 - (1 - \sin \psi)\sigma_3, \]

where \( \sigma_1 \geq \sigma_2 \geq \sigma_3 \) denote the principal stresses, i.e. the eigenvalues of \( \sigma \). The parameters \( \phi, \psi, c_0 \) represent the friction angle, the dilatancy angle, and the initial cohesion, respectively. Finally, we let the function \( H \) in an abstract form and choose the associative hardening law, i.e., set \( \ell(\sigma, \kappa) = 2 \cos \phi \).

Recall the notation \( \varepsilon^{e, tr} := \varepsilon(t_k) - \varepsilon^p(t_{k-1}), \varepsilon^{p, tr} := \varepsilon(t_{k-1}) \) and \( \sigma^{tr} := \mathbb{D}_e : \varepsilon^{e, tr} \) and denote \( \varepsilon_1^{e, tr} \geq \varepsilon_2^{e, tr} \geq \varepsilon_3^{e, tr} \) as the eigenvalues of \( \varepsilon^{e, tr} \). Then the eigenvalues \( \sigma_1^{tr} \geq \sigma_2^{tr} \geq \sigma_3^{tr} \) of \( \sigma^{tr} \) satisfies

\[ \sigma_i^{tr} = \frac{1}{3} (3K - 2G)(\varepsilon_1^{e, tr} + \varepsilon_2^{e, tr} + \varepsilon_3^{e, tr}) + 2G\varepsilon_i^{e, tr}, \quad i = 1, 2, 3. \]

Further, define the nonnegative values,

\[ \gamma_{s,l} = \frac{\sigma_1^{tr} - \sigma_2^{tr}}{2G(1 + \sin \psi)}, \quad \gamma_{s,r} = \frac{\sigma_2^{tr} - \sigma_3^{tr}}{2G(1 - \sin \psi)}, \quad \gamma_{l,a} = \frac{\sigma_1^{tr} + \sigma_2^{tr} - 2\sigma_3^{tr}}{2G(3 - \sin \psi)}, \quad \gamma_{r,a} = \frac{2\sigma_1^{tr} - \sigma_2^{tr} - \sigma_3^{tr}}{2G(3 + \sin \psi)} \]

and the functions,

\[ q_s(\gamma) = (1 + \sin \phi)\sigma_1^{tr} - (1 - \sin \phi)\sigma_3^{tr} - 2 \cos \phi \left[ c_0 + H (\varepsilon^{p, tr} + 2\gamma \cos \phi) \right] \]

\[ -\gamma \left[ \frac{4}{3} (3K - 2G) \sin \psi \sin \phi + 4G(1 + \sin \psi \sin \phi) \right], \]

\[ q_l(\gamma) = \frac{1}{2} (1 + \sin \phi)(\sigma_1^{tr} + \sigma_2^{tr}) - (1 - \sin \phi)\sigma_3^{tr} - 2 \cos \phi \left[ c_0 + H (\varepsilon^{p, tr} + 2\gamma \cos \phi) \right] \]

\[ -\gamma \left[ \frac{4}{3} (3K - 2G) \sin \psi \sin \phi + G(1 + \sin \psi)(1 + \sin \phi) + 2G(1 - \sin \psi)(1 - \sin \phi) \right], \]
$q_r(\gamma) = (1 + \sin \phi)\sigma_1^{tr} - \frac{1}{2}(1 - \sin \phi)(\sigma_2^{tr} + \sigma_3^{tr}) - 2 \cos \phi \left[c_0 + H (\bar{e}^{p,tr} + 2\gamma \cos \phi)\right]$

$-\gamma \left[\frac{4}{3} (3K - 2G) \sin \psi \sin \phi + 2G(1 + \sin \psi )(1 + \sin \phi) + G(1 - \sin \psi)(1 - \sin \phi)\right],$

$q_a(\gamma) = 2p^{tr} \sin \phi - 4K \gamma \sin \psi \sin \phi - 2 \cos \phi \left[c_0 + H (\bar{e}^{p,tr} + 2\gamma \cos \phi)\right].$

Using these auxiliary values and functions, one can a priori determine whether the unknown stress tensor $\sigma$ lies on the smooth portion, the left edge, the right edge or at the apex of the yield surface. The return-mapping algorithm can be written as follows [2].

Return to the smooth portion ($\sigma_1 > \sigma_2 > \sigma_3$). It happens if and only if $\min\{\gamma_{s,l}, \gamma_{s,r}\} > 0$ and $q_a(\min\{\gamma_{s,l}, \gamma_{s,r}\}) < 0$. Then the inequalities $\varepsilon_1^{e,tr} > \varepsilon_2^{e,tr} > \varepsilon_3^{e,tr}$ must hold.

1. Find $\Delta \lambda \in (0, \min\{\gamma_{s,l}, \gamma_{s,r}\})$ satisfying $q_a(\Delta \lambda) = 0$.

2. Compute $\bar{e}^p = \bar{e}^{p,tr} + 2\Delta \lambda \cos \phi$,

$\sigma_1 = \sigma_1^{tr} - \Delta \lambda \frac{2}{3} (3K - 2G) \sin \psi - \Delta \lambda 2G(1 + \sin \psi),$

$\sigma_2 = \sigma_2^{tr} - \Delta \lambda \frac{2}{3} (3K - 2G) \sin \psi,$

$\sigma_3 = \sigma_3^{tr} - \Delta \lambda \frac{2}{3} (3K - 2G) \sin \psi + \Delta \lambda 2G(1 - \sin \psi).$

3. Set

$\sigma = \sum_{i=1}^{3} \sigma_i E_i^{tr}, \quad E_i^{tr} = \frac{(e_i^{e,tr} - \varepsilon_j^{e,tr}) (e_i^{e,tr} - \varepsilon_k^{e,tr})}{(\varepsilon_j^{e,tr} - \varepsilon_j^{e,tr}) (\varepsilon_i^{e,tr} - \varepsilon_k^{e,tr})}, \quad i \neq j \neq k \neq i, \quad i = 1, 2, 3,$

i.e. $E_i^{tr}$ are the eigenprojections of $e^{e,tr}$, see [4].

Return to the left edge ($\sigma_1 = \sigma_2 > \sigma_3$). It happens if and only if $0 < \gamma_{s,l} < \gamma_{l,a}$, $q_l(\gamma_{s,l}) \geq 0$ and $q_l(\gamma_{l,a}) < 0$. Then the inequalities $\varepsilon_1^{e,tr} \geq \varepsilon_2^{e,tr} > \varepsilon_3^{e,tr}$ must hold.

1. Find $\Delta \lambda \in [\gamma_{s,l}, \gamma_{l,a})$ satisfying $q_l(\Delta \lambda) = 0$.

2. Compute $\bar{e}^p = \bar{e}^{p,tr} + 2\Delta \lambda \cos \phi$,

$\sigma_1 = \frac{1}{2}(\sigma_1^{tr} + \sigma_2^{tr}) - \Delta \lambda \frac{2}{3} (3K - 2G) \sin \psi - \Delta \lambda G(1 + \sin \psi),$

$\sigma_3 = \sigma_3^{tr} - \Delta \lambda \frac{2}{3} (3K - 2G) \sin \psi + \Delta \lambda 2G(1 - \sin \psi).$

3. Set

$\sigma = \sigma_1 E_{12}^{tr} + \sigma_3 E_3^{tr}, \quad E_{12}^{tr} = I - E_3^{tr}, \quad E_3^{tr} = \frac{(e_i^{e,tr} - \varepsilon_1^{e,tr} I)(e_i^{e,tr} - \varepsilon_2^{e,tr} I)}{(\varepsilon_3^{e,tr} - \varepsilon_1^{e,tr})(\varepsilon_3^{e,tr} - \varepsilon_2^{e,tr})}. $
Return to the right edge ($\sigma_1 > \sigma_2 = \sigma_3$). It happens if and only if $0 < \gamma_{s,r} < \gamma_{r,a}$, $q_r(\gamma_{s,r}) \geq 0$ and $q_r(\gamma_{s,a}) < 0$. Then the inequalities $\varepsilon_{e,tr}^{e_1} > \varepsilon_{e,tr}^{e_2} \geq \varepsilon_{e,tr}^{e_3}$ must hold.

1. **Find** $\Delta \lambda \in [\gamma_{s,r}, \gamma_{r,a})$ satisfying $q_r(\Delta \lambda) = 0$.

2. **Compute** $\bar{\varepsilon} = \bar{\varepsilon}_{p, tr} + 2\Delta \lambda \cos \phi$,

   \[
   \sigma_1 = \sigma_{1, tr} - \Delta \lambda \frac{2}{3}(3K - 2G)\sin \psi - \Delta \lambda 2G(1 + \sin \psi),
   \]

   \[
   \sigma_3 = \frac{1}{2}(\sigma_{2, tr} + \sigma_{3, tr}) - \Delta \lambda \frac{2}{3}(3K - 2G)\sin \psi + \Delta \lambda G(1 - \sin \psi).
   \]

3. **Set**

   \[
   \sigma = \sigma_{1, tr}E_{23}^{tr} + \sigma_{3, tr}E_{23}^{tr}, \quad E_{23}^{tr} = I - E_1^{tr}, \quad E_1^{tr} = \frac{(\varepsilon_{e, tr}^{e_1} - \varepsilon_{e, tr}^{e_2})I)(\varepsilon_{e, tr}^{e_1} - \varepsilon_{e, tr}^{e_3})I}{(\varepsilon_{e, tr}^{e_1} - \varepsilon_{e, tr}^{e_2})(\varepsilon_{e, tr}^{e_1} - \varepsilon_{e, tr}^{e_3})}.
   \]

Return to the apex ($\sigma_1 = \sigma_2 = \sigma_3$). It happens if and only if $0 < \max\{\gamma_{l,a}, \gamma_{r,a}\}$ and $q_a(\max\{\gamma_{l,a}, \gamma_{r,a}\}) \geq 0$.

1. **Find** $\Delta \lambda \geq \max\{\gamma_{l,a}, \gamma_{r,a}\}$ satisfying $q_a(\Delta \lambda) = 0$.

2. **Compute** $\bar{\varepsilon} = \bar{\varepsilon}_{p, tr} + 2\Delta \lambda \cos \phi$, $p = p_{tr} - 2\Delta \lambda K \sin \psi$.

3. **Set** $\sigma = pI$.

For the sake of brevity, we do not introduce the consistent tangent operator. It can be found in [2]. We only mention that from the results summarized above, it follows that

\[
\sigma_1 > \sigma_j \implies \varepsilon_{e, tr}^{e_i} > \varepsilon_{e, tr}^{e_j}, \quad i < j, \quad i, j = 1, 2, 3.
\]

This implication significantly simplifies derivation of the consistent tangent operator in comparison to [4, Append. A]. For example, the derivatives $\partial \varepsilon_{e, tr}^{e_i} / \partial \varepsilon$ are correctly defined and equal to $E_{i, tr}^{tr}$, $i = 1, 2, 3$, for the return to the smooth portion.

6 CONCLUSIONS

In this paper, the subdifferential formulation of the flow rule was used for numerical purposes in computational plasticity. It was shown on several models that such a formulation directly improved the implicit return-mapping scheme for nonsmooth yield surfaces and consequently also simplifies derivation of the consistent tangent operator. It seems that the new technique would be universal and usable for a broad class of elastoplastic models.
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