CANONICAL HOMOTOPY OPERATORS FOR $\bar\partial$ IN THE BALL WITH RESPECT TO THE BERGMAN METRIC

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ABSTRACT. We notice that some well-known homotopy operators due to Skoda et. al. for the $\bar\partial$-complex in the ball actually give the boundary values of the canonical homotopy operators with respect to certain weighted Bergman metrics. We provide explicit formulas even for the interior values of these operators. The construction is based on a technique of representing a $\bar\partial$-equation as a $\bar\partial_b$-equation on the boundary of the ball in a higher dimension. The kernel corresponding to the operator that is canonical with respect to the Euclidean metric was previously found by Harvey and Polking. Contrary to the Euclidean case, any form which is smooth up to the boundary belongs to the domain of the corresponding operator $\bar\partial^*$, with respect to the metrics we consider. We also discuss the corresponding $\bar\Box$-operator and its canonical solution operator.

Moreover, our homotopy operators satisfy a certain commutation rule with the Lie derivative with respect to the vector fields $\partial/\partial\zeta_k$, which makes it possible to construct homotopy formulas even for the $\partial\bar\partial$-operator.

0. Introduction.

Let D be the unit disc. For any $\alpha > 0$,

$$P_\alpha u(z) = \frac{\alpha}{\pi}\int_D \frac{(1-|\zeta|^2)^{\alpha-1}u(\zeta)dm(\zeta)}{(1-\bar\zeta z)^{\alpha+1}}$$

is the orthogonal projection of $L^2_\alpha = L^2((1-|\zeta|^2)^{\alpha-1}dm(\zeta))$ onto the subspace of holomorphic functions, and the solution to $\bar\partial u = f$ with minimal norm in L^2_α is provided by the formula

$$K_\alpha f(z) = \frac{i}{2\pi}\int_D \left(\frac{1-|\zeta|^2}{1-\bar\zeta z}\right)^\alpha \frac{f(\zeta)\wedge d\zeta}{\zeta - z}.$$

Both of these claims follow from the easily checked formula

$$K_\alpha(\bar\partial u) = u - P_\alpha u. \tag{0.1}$$

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When α tends to zero, $K_\alpha f$ tends to the usual Cauchy integral, which gives the solution that has minimal norm in L^2 over the boundary.

The several variable situation is much more complicated. There is a similar wellknown formula for the orthogonal holomorphic projection P_α in the ball $\mathbb{B}$ in $\mathbb{C}^n$,

$$P_\alpha u(z) = \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}\int_{\mathbb{B}}\frac{(1-|\zeta|^2)^{\alpha-1}u(\zeta)dm(\zeta)}{(1-\bar\zeta\cdot z)^{\alpha+n}},$$

but there is no unique operator K_α such that (0.1) holds since this equation just determines the action of K_α on $\bar\partial$-closed $(0,1)$-forms.

The simplest way to obtain an explicit solution to the $\bar\partial$-equation in the ball is by means of the Cauchy-Fantappiè-Leray formula. With an appropriate choice of the section (e.g. $s(\zeta,z)=\bar\zeta$ for $|\zeta|=1$) one obtains the solution that is minimal in L^2 over the boundary (but even in this simple construction the resulting operator K can be chosen in a variety of ways). For some important estimates however, this solution is not adequate. The following well-known formula for (the complex tangential part of) the boundary values of a solution to $\bar\partial u = f$ for $\bar\partial$-closed f ($\beta = i\partial\bar\partial|\zeta|^2$, $\beta_k=\beta^k/k!$ and $\gamma = i\partial|\zeta|^2\wedge\bar\partial|\zeta|^2$),

$$\begin{aligned}K^b_\alpha f(z) = \sum_{q=0}^{n-1}&\frac{\Gamma(\alpha+n-q-1)}{(2\pi)^n\Gamma(\alpha)}\\ &\int_{|\zeta|<1}\frac{(1-|\zeta|^2)^{\alpha-1}f(\zeta)\wedge i^{q+1}\bar z\cdot d\zeta\wedge(d\bar z\cdot d\zeta)^q}{(1-\bar\zeta\cdot z)^{\alpha-1+n-q}(1-\zeta\cdot\bar z)^{q+1}}\\ &\wedge\left[(1-|\zeta|^2)\beta_{n-1-q}+\gamma\wedge\beta_{n-q-2}\right],\end{aligned} \tag{0.2}$$

was found by Skoda, [Sk], for $\alpha=1$. The general case, $\alpha>0$, follows for instance from Example 1 iii) in [BeAn]; for integer values of α, see also [Be]. This operator satisfies the homotopy relation

$$\bar\partial_b K^b_\alpha + K^b_\alpha\bar\partial = I - P_\alpha, \tag{0.3}$$

interpreted in the sense of complex tangential boundary values, and therefore in particular, $K^b_\alpha f$ is the boundary values of the minimal solution to $\bar\partial u = f$ if f is a $\bar\partial$-closed $(0,1)$-form. The limit case $\alpha=0$ corresponds to $L^2(\partial\mathbb{B})$.

In [Ch], Charpentier found formulas for the values in the interior of the solution (for (0,1)-forms) in the ball that are minimal in L^2_α for any $\alpha>0$. However, the boundary values of these formulas do not coincide with (0.2) for an arbitrary non-closed f.

In view of the L^2-theory for the $\bar\partial$ and $\bar\partial$-Neumann equation developed by Hörmander, Kohn et. al. in the 1960's, the question arose to find an explicit expression for the operator K acting on forms f, giving the minimal solution when f is $\bar\partial$-closed, and vanishing if f is orthogonal, with respect to the Euclidean metric, to the $\bar\partial$-closed forms. This was accomplished by Harvey and Polking in [HaPo] and by Range in [Ra].

In this paper we first define weighted Bergman norms, see (1.1) below, such that (0.2) gives the boundary values of the corresponding canonical homotopy operator. Expressed in these inner products

$$K_\alpha^b f(z) = (f, \overline{k_\alpha^b(\cdot, z)})_\alpha$$

where the kernel $k_\alpha^b(\zeta, z)$ has the simple expression

$$k_\alpha^b(\zeta, z) = \sum_{q=0}^{n-1} \frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)} \frac{\bar z \cdot d\zeta \wedge (d\bar z \cdot d\zeta)^q}{(1-\bar\zeta\cdot z)^{\alpha-1+n-q}(1-\zeta\cdot\bar z)^{q+1}}. \tag{0.4}$$

Since $\partial_\zeta k_\alpha^b(\zeta, z) = 0$ it follows that $K_\alpha^b f = 0$ if f is orthogonal to $\operatorname{Ker}\bar\partial$. Moreover, if $\bar\partial f = 0$ and u is the minimal solution to $\bar\partial u = f$, i.e. the one that is orthogonal to $\operatorname{Ker}\bar\partial$, then by (0.3) (and assuming for the moment that u is smooth),

$$K_\alpha^b f = K_\alpha^b \bar\partial u = u - \bar\partial_b K_\alpha^b u = u,$$

and therefore $K_\alpha^b f$ provides the complex tangential part of the boundary values of the minimal solution u. Thus $K_\alpha^b f$ is the complex tangential part of the boundary values of the canonical homotopy operator K_α, i.e. the one that satisfies (0.3), vanishes on forms orthogonal to $\operatorname{Ker}\bar\partial$ and gives the minimal solution when f is $\bar\partial$-closed. In §5 we give explicit formulas even for the interior values of $K_\alpha f$. They can be expressed as rational functions in the quantities $1-|\zeta|^2$, $1-|z|^2$, $1-\bar\zeta\cdot z$ and some simple forms.

It turns out that these norms have some other nice features. The limit when α tends to 0 is the natural norm of complex tangential forms on the boundary. If f is smooth up to the boundary and $f = f_1 + f_2$ is its orthogonal decomposition in a $\bar\partial$-closed form and a form that is orthogonal to the $\bar\partial$-closed forms, then both f_1 and f_2 are smooth. Moreover, for any $\alpha > 0$, all $(0,q)$-forms that are smooth up to the boundary belong to the domain of the Hilbert space adjoint $\bar\partial_\alpha^*$ of $\bar\partial$. This fact makes it easy to formulate the corresponding $\bar\partial$-Neumann equation $\bar\Box_\alpha u = f$. In the limit case $\alpha \to 0$ one gets the well-known $\bar\Box_b$-equation. We also discuss the canonical solution operators for these equations.

Furthermore $\bar\partial_\alpha^*$ satisfies a certain commutation rule with the vector field $\partial/\partial\zeta_k$, acting on forms as a Lie derivative, which makes it possible to find homotopy formulas for $\partial\bar\partial$. Consider the equation $\bar\partial\partial u = \theta$ in $\mathbb{B}$, where θ is a closed (q,q)-form. The most naive attempt to solve this equation consists in first taking some solution w to $\bar\partial w = \theta$ and then trying to solve $\partial u = w$. In general this breaks down as w is not necessarily ∂-closed. However, we will prove that

$$\partial K_\alpha f/\partial z_j = K_{\alpha+1}(\partial f/\partial\zeta_j). \tag{0.5}$$

Therefore (letting K_α just act on the barred part of θ) $w = K_\alpha\theta$ actually is ∂ closed if $d\theta = 0$, and thus $\bar K_\ell K_\alpha\theta$ is a solution to $\partial\bar\partial$. (The equation (0.5) is quite elementary when f is a $\bar\partial$-closed (0,1)-form, see the proof of Corollary 2.3.) Actually we obtain a sort of homotopy formulas for the $\partial\bar\partial$-complex. We further indicate that these solution operators for $\partial\bar\partial$ admit some expected estimates, but leave further investigation to another occasion.

The disposition is as follows. In §1 we introduce the weighted Bergman norms and state some basic results about canonical operators for the $\bar\partial$-complex and for the corresponding $\bar\partial$-Neumann equation(s). In §2 we compute an expression for $\bar\partial^*_\alpha$. The main result is Theorem 2.1, from which the above-mentioned result about the domain of $\bar\partial^*_\alpha$ follows. Moreover, it implies the the proposed commutation rule with $\partial/\partial z_j$.

The formulas for K_α are derived by a technique of representing a $\bar\partial$-equation as a $\bar\partial_b$-equation in a higher dimensional ball. In §3 we study the $\bar\partial_b$-complex and canonical formulas for $\bar\partial_b$ and $\bar\Box_b$. The technique of going up and down in the dimension is exploited in §4. It leads to a proof of the regularity result Theorem 1.3 and to a method to compute explicit formulas for the operators K_α. These computations are carried out in §5 and we also compare these formulas to some previously known homotopy formulas. In §6 we briefly discuss the corresponding operator $\bar\Box_\alpha$.

Finally, in §7 we present the idea to get homotopy formulas for $\partial\bar\partial$ and indicate some estimates.

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1. Weighted Bergman norms and some basic results.

The simplest norm of a $(0,q)$-form $f = \sum' f_I d\bar z^I$ in the ball $\mathbb{B}$ in $\mathbb{C}^n$ is the Euclidean norm

$$|f|^2 = \langle f, f\rangle = \sum{}' |f_I|^2,$$

if we adopt the convention that $d\bar\zeta_k$ has Euclidean norm one rather than $\sqrt{2}$. This corresponds to the metric form $\beta = i\partial\bar\partial|\zeta|^2$, and thus $\beta_n = \beta^n/n! = 2^n dm$, where dm denotes Lebesgue measure. However, this norm does not reflect the wellknown fact that the $\bar\partial$-operator near the boundary of a strictly pseudoconvex domain behaves differently in the normal and complex tangential directions. However the norm

$$\|f\|^2 = \langle\langle f, f\rangle\rangle = (1-|\zeta|^2)\,\langle f, f\rangle + \big\langle \bar\partial|\zeta|^2 \wedge f, \bar\partial|\zeta|^2 \wedge f\big\rangle$$

in fact does. Expressed in this norm for example the well-known Henkin-Skoda theorem says that there is a solution to $\bar\partial u = f$ whose complex tangential part $u|_b$ on the boundary is in $L^1(\partial\mathbb{B})$ if $(1/\sqrt{1-|\zeta|^2})\|f\|$ is in $L^1(\mathbb{B})$. We claim that

$$\|f\|^2 = |f|^2 - |\mathcal{L} f|^2$$

if

$$\mathcal{L} = \sum_j \bar\zeta_j \frac{\partial}{\partial \bar\zeta_j} \neg,$$

where $\neg$ denotes inner multiplication of vectors on forms. In fact, if $Tg = \bar\partial|\zeta|^2 \wedge g$, then $\langle \mathcal{L}f, g\rangle = \langle f, Tg\rangle$, and since inner multiplication is an anti-derivation and $\mathcal{L}\bar\partial|\zeta|^2 = |\zeta|^2$ we have that $\mathcal{L}T + T\mathcal{L} = |\zeta|^2 I$ and now the claim follows. In particular, this new norm coincides with the Euclidean norm on functions. Let

$$\|f\|^2_\alpha = (f, f)_\alpha = \frac{\Gamma(n+\alpha)}{\pi^n \Gamma(\alpha)} \int_{\mathbb{B}} (1 - |\zeta|^2)^{\alpha-1} \|f\|^2 dm. \tag{1.1}$$

When $\alpha \to 0$ we get

$$\|f\|^2_b = (f, f)_b = \frac{(n-1)!}{2\pi^n} \int_{\partial\mathbb{B}} \langle \bar\partial|\zeta|^2 \wedge f, \bar\partial|\zeta|^2 \wedge f\rangle \, d\sigma,$$

which is the usual normalized norm of the complex tangential part $f|_b$ of f on $\partial\mathbb{B}$. Moreover, if f is a $(0,q)$-form, α an integer and $\tilde f$ denotes the corresponding form in the ball in $\mathbb{C}^{n+\alpha}$ that is independent of the α last variables, then the norm $\|f\|_\alpha$ is equal to the tangential norm of $\tilde f|_b$, see §4. In particular, the function 1 has norm 1.

Let

$$\omega = i\partial\bar\partial \log\left(\frac{1}{1 - |\zeta|^2}\right) = \frac{\beta}{1 - |\zeta|^2} + \frac{\gamma}{(1 - |\zeta|^2)^2}$$

be the Bergman metric form and let $|f|_B$ be the corresponding pointwise norm of forms, so that

$$|f|^2_B \omega^n / n! = \pm i^q \bar f \wedge f \wedge \omega^{n-q}/(n-q)!$$

(the sign is such that the norm is non-negative). Then one can verify that

$$\|f\|^2_\alpha = c_{n,\alpha,q} \int_{\mathbb{B}} (1 - |\zeta|^2)^{\alpha+n-q} |f|^2_B \omega^n / n!, \tag{1.2}$$

and therefore the $\| \ \|_\alpha$-norm corresponds to the L^2-norm with respect to the Bergman metric on the manifold $\mathbb{B}$ and the metric $(1 - |\zeta|^2)^{\alpha+n-q}$ on the fiber (on the trivial line bundle over $\mathbb{B}$). Thus from this point of view, for a fixed α, the fiber metric varies with q.

Let $\mathcal{E}_{0,q}$ denote the space of $(0,q)$-forms in $\mathbb{B}$ that are smooth up to the boundary and let $\mathcal{H}_{0,q} = \text{Ker}\,\bar\partial \cap \mathcal{E}_{0,q}$ and $\mathcal{H}^{\perp_\alpha}_{0,q}$ be the orthogonal complement of $\mathcal{H}_{0,q}$ in $\mathcal{E}_{0,q}$ with respect to $(\ ,\ )_\alpha$. In particular, $\mathcal{H}_{0,0}$ is just the space of holomorphic functions that are smooth up to the boundary. We will also use the notation $\mathcal{E} = \sum \mathcal{E}_{0,q}$, $\mathcal{H} = \sum \mathcal{H}_{0,q}$ etc. Let $\bar\partial^*_\alpha$ denote the formal adjoint of $\bar\partial$ with respect to the inner product $(\ ,\ )_\alpha$. Contrary to the Euclidean case we have

Proposition 1.1. *Let $\alpha > 0$. For any forms $\phi, \psi \in \mathcal{E}$, the formal adjoint $\bar\partial^*_\alpha$ maps $\mathcal{E}_{0,*+1} \to \mathcal{E}_{0,*}$ and we have*

$$(\bar\partial^*_\alpha \phi, \psi)_\alpha = (\phi, \bar\partial\psi)_\alpha. \tag{1.3}$$

In particular, any $\psi \in \mathcal{E}$ is in the domain of the Hilbert space adjoint of $\bar\partial$.

This proposition is proved in §2.

Let

$$\bar\Box_\alpha = \bar\partial\bar\partial^*_\alpha + \bar\partial^*_\alpha\bar\partial \colon \mathcal{E}_{0,*} \to \mathcal{E}_{0,*}$$

be the corresponding $\bar\partial$-Neumann operator.

Theorem 1.2. *Let $\alpha > 0$. The kernel of $\bar\Box_\alpha$ is $\mathcal{H}_{0,0}$. The kernel of $\bar\partial^*_\alpha$ is $\mathcal{H}^{\perp_\alpha}$ for $q > 0$, and its image is $\mathcal{H}^{\perp_\alpha}$. We have the orthogonal decompositions*

$$\mathcal{E}_{0,q} = \mathcal{H}_{0,q} \oplus \mathcal{H}^{\perp}_{0,q} = \bar\partial\mathcal{E}_{0,q-1} \oplus \bar\partial^*_\alpha\mathcal{E}_{0,q+1} = \bar\Box_\alpha\mathcal{E}_{0,q},$$

for $1 \le q \le n$ and

$$\mathcal{E}_{0,0} = \mathcal{H}_{0,0} \oplus \mathcal{H}^{\perp_\alpha}_{0,0} = \mathcal{H}_{0,0} \oplus \bar\partial^*_\alpha\mathcal{E}_{0,1} = \mathcal{H}_{0,0} \oplus \bar\Box_\alpha\mathcal{E}_{0,0}.$$

This theorem is proved in §4. In particular, as in the Euclidean case, any smooth form f can be decomposed as $f = f_1 + f_2$ where $\bar\partial f_1 = 0$ and $f_2 \in \mathcal{H}^{\perp_\alpha}$. We can now define the canonical homotopy operators for $\bar\partial$.

Definition. For a fixed $\alpha > 0$, $K_\alpha : \mathcal{E}_{0,*+1} \to \mathcal{E}_{0,*}$ is the operator which vanishes on $\mathcal{H}^{\perp_\alpha}$ and provides the minimal solution (with respect to $\| \, \|_\alpha$) to $\bar\partial u = f$ when applied to a $\bar\partial$-closed form.

We claim that K_α and P_α satisfy the homotopy relation

$$\bar\partial K_\alpha + K_\alpha\bar\partial = I - P_\alpha, \tag{1.4}$$

if $P_\alpha : \mathcal{E}_{0,0} \to \mathcal{H}_{0,0}$ is the orthogonal projection of functions onto the subspace of holomorphic ones. If K^q_α denotes the restriction of K_α to $\mathcal{E}_{0,q+1}$, (1.4) thus means that

$$K^0_\alpha\bar\partial f = f - P_\alpha f$$

for functions f, and

$$\bar\partial K^q_\alpha f + K^{q+1}_\alpha\bar\partial f = f$$

for $(0, q+1)$-forms f.

In fact, if $\bar\partial$-closed ϕ, the equation (1.4) simply says that $\phi = P_\alpha\phi$ if ϕ is a function and $\bar\partial K_\alpha\phi = \phi$ if ϕ is a form of higher degree. Both these cases are true by definition. If $\phi \in \mathcal{H}^{\perp_\alpha}$, then (1.4) states that $K_\alpha\bar\partial\phi = \phi$ which is also true since in that case ϕ is the minimal solution to $\bar\partial u = \bar\partial\phi$.

Notice that

$$\bar\partial K^q_\alpha : \mathcal{E}_{0,q+1} \to \mathcal{H}_{0,q+1}$$

is the orthogonal projection, and thus $\bar\partial K_\alpha$ is self-adjoint.

It follows from Theorem 1.2 that there is also an operator

$$E_\alpha : \mathcal{E}_{0,*} \to \mathcal{E}_{0,*}$$

that provides the unique solution to the $\bar\Box_\alpha$-equation for $q > 0$, vanishes on $\mathcal{H}_{0,0}$ and gives the minimal solution, i.e. the one in $\mathcal{H}_{0,0}^{\perp_\alpha}$, when applied to a function in $\mathcal{H}_{0,0}^{\perp_\alpha}$. It also follows that the adjoint K_α^* of K_α maps $\mathcal{E} \to \mathcal{E}$, more precisely, K_α^* vanishes on $\mathcal{H}$ and $K_\alpha^*\phi$ is the solution to $\bar\partial_\alpha^* u = \phi$ if $\phi \in \mathcal{H}^{\perp_\alpha}$. It is easy to verify that

$$\bar\Box_\alpha E_\alpha = E_\alpha \bar\Box_\alpha = I - P_\alpha \tag{1.5}$$

and

$$E_\alpha^* = E_\alpha, \quad \bar\partial E_\alpha = E_\alpha \bar\partial, \quad \bar\partial_\alpha^* E_\alpha = E_\alpha \bar\partial_\alpha^*. \tag{1.6}$$

(However, (1.5) and (1.6) do not determine E_α completely, since if E_α satisfies these two then e.g. also $\tilde E_\alpha = E_\alpha + P_\alpha$ will do.) The interrelation between E_α and K_α is provided by

Proposition 1.3. *If E_α and K_α are the canonical homotopy operators for $\bar\partial$ and $\bar\Box_\alpha$, then*

$$E_\alpha = K_\alpha K_\alpha^* + K_\alpha^* K_\alpha, \tag{1.7}$$

and

$$K_\alpha = \bar\partial_\alpha^* E_\alpha. \tag{1.8}$$

Proof. To prove (1.7) it is enough to verify that $\bar\Box_\alpha E_\alpha = I - P_\alpha$ and that E_α is orthogonal to $\mathcal{H}_{0,0}$ and vanishes also on $\mathcal{H}_{0,0}$. This last requirement is obvious from the definition of K_α as $E_\alpha = K_\alpha K_\alpha^*$ on functions. By the definition of K_α and (1.4) we have that

$$\bar\partial_\alpha^* K_\alpha = 0, \quad K_\alpha \bar\partial_\alpha^* = 0, \quad K_\alpha^* \bar\partial_\alpha^* = \bar\partial K_\alpha, \quad \bar\partial K_\alpha \bar\partial = \bar\partial.$$

Thus

$$\begin{aligned}\bar\Box_\alpha E_\alpha &= \bar\partial\bar\partial_\alpha^* K_\alpha^* K_\alpha + \bar\partial_\alpha^*\bar\partial K_\alpha K_\alpha^* = \bar\partial K_\alpha \bar\partial K_\alpha + \bar\partial_\alpha^* K_\alpha^* \bar\partial_\alpha^* K_\alpha^* = \\ &= \bar\partial K_\alpha + \bar\partial_\alpha^* K_\alpha^* = \bar\partial K_\alpha + K_\alpha \bar\partial = I - P_\alpha\end{aligned}$$

as desired.

It is readily verified that $\bar\partial_\alpha^* E_\alpha$ satisfies the defining properties of K_α so that (1.8) holds, but it can can also easily be obtained from (1.7). Namely,

$$\bar\partial_\alpha^* E_\alpha = \bar\partial_\alpha^* K_\alpha^* K_\alpha = K_\alpha \bar\partial K_\alpha = K_\alpha.$$

■

Remarks. There are analogues of Theorem 1.2 and Proposition 1.3 for the boundary operator $\bar\partial_b$ which we describe in §3. There are also analogues for the Euclidean metric. Proposition 1.3 is as simple as in our case and it is e.g. used in [Ra]. The analogue of Theorem 1.2 were first proved by Kohn, see [FoKo]. However, the formulation is more involved since $\mathcal{E}$ is not contained in the domains of $\bar\partial^*_\alpha$ and $\bar\Box_\alpha$.

One could also consider the homotopy operators T_ℓ for $\bar\partial$ that are canonical with respect to the weighted Bergman norms

$$\|f\|^2_{\ell,B} = \int_{\mathbb{B}} (1-|\zeta|^2)^{n+\ell} |f|^2_B \omega^n/n!,$$

for $\ell > 0$. Because of (1.2) it follows that for $(0,q)$-forms f,

$$T_\ell f = K_{\ell+q-1}\bar\partial K_{\ell+q} f,$$

and so they can be expressed in terms of the operators K_α. ■

2. A formula for $\bar\partial^*_\alpha$.

In this section we compute an expression for the formal adjoint $\bar\partial^*_\alpha$ of $\bar\partial$ with respect to the inner product $(\ ,\ )_\alpha$. Thus we look for the operator such that $(\bar\partial^*_\alpha f, g)_\alpha = (f, \bar\partial g)_\alpha$ for compactly supported forms f and g of degrees $(0,q-1)$ and $(0,q)$ respectively. For forms f and g let $f \lrcorner g$ denote inner multiplication with respect to the Euclidean metric, i.e. $\langle f \lrcorner g, \phi\rangle = \langle g, \bar f \wedge \phi\rangle$ for all forms ϕ. Then we have that $\mathcal{L} = \partial|\zeta|^2 \lrcorner$. The formal adjoint of $\bar\partial$ on $(0,q)$-forms with respect to the Euclidean metric is

$$\vartheta = -\sum_j \frac{\partial}{\partial\zeta_k}\frac{\partial}{\partial\bar\zeta_k} \neg = -i\beta \lrcorner \partial,$$

i.e. $\int \langle \vartheta f, g\rangle = \int \langle f, \bar\partial g\rangle$. We can now state the main result of this section.

Theorem 2.1. *If f is a $(0,q)$-form then*

$$\bar\partial^*_\alpha f = \vartheta f + \mathcal{L}\bar{\mathcal{L}}\partial f + (\alpha + n - q)\mathcal{L} f.$$

Corollary 2.2. *If $\partial/\partial\zeta_j$ acts as a Lie derivative on forms, then*

$$(\partial/\partial\zeta_j)\bar\partial^*_\alpha = \bar\partial^*_{\alpha+1}\partial/\partial\zeta_j.$$

Proof. To begin with, $\partial/\partial\zeta_j$ commutes with ϑ and $\mathcal{L}$. Since $\bar{\mathcal{L}}\partial$ is the Lie derivative with respect to the vector field $\sum \zeta_j(\partial/\partial\zeta_j)$, we have that

$$(\partial/\partial\zeta_j)\bar{\mathcal{L}}\partial = \bar{\mathcal{L}}\partial(\partial/\partial\zeta_j) + \partial/\partial\zeta_j.$$

The corollary then follows from Theorem 2.1. ■

Corollary 2.3. *For $\alpha > 0$, let K_α and P_α be the operators defined in Section 1. Then*

$$\partial P_\alpha f/\partial z_j = P_{\alpha+1}(\partial f/\partial \zeta_j), \tag{2.1}$$

for functions $f \in \mathcal{E}_{0,0}$ and

$$\partial K_\alpha f/\partial z_j = K_{\alpha+1}(\partial f/\partial \zeta_j), \tag{2.2}$$

for forms $f \in \mathcal{E}_{0,*}$.

Proof. To begin with we claim that $\partial v/\partial\zeta_j$ is orthogonal to the holomorphic functions with respect to $dm_{\alpha+1} = (1-|\zeta|^2)^\alpha dm$ if v is orthogonal to the holomorphic functions with respect to dm_α. In fact, for holomorphic h,

$$\int (1-|\zeta|^2)^\alpha \frac{\partial v}{\partial \zeta_j}\bar h = \alpha \int (1-|\zeta|^2)^{\alpha-1} v \bar\zeta_j \bar h = 0$$

by the assumption on v. Since $\partial/\partial\zeta_j$ commutes with $\bar\partial$, this means that if $v = K_\alpha f$ is the minimal solution to $\bar\partial u = f$ with respect to dm_α then $\partial v/\partial\zeta_j$ is the minimal solution to $\bar\partial u = \partial f/\partial\zeta_j$ with respect to $dm_{\alpha=1}$. Thus (2.2) holds for $\bar\partial$-closed (0,1)-forms f. Now (2.1) follows since $P_\alpha f = f - K_\alpha\bar\partial f$.

We are now going to prove (2.2) for an arbitrary form $f \in \mathcal{E}_{0,q}$. In view of Theorem 1.2 we may assume that either $f \in \mathcal{H}_{0,q}^{\perp_\alpha}$ or $f \in \mathcal{H}_{0,q}$. If $f \in \mathcal{H}_{0,q}^{\perp_\alpha}$ then $\bar\partial^*_\alpha f = 0$ and due to Corollary 2.2 it follows that $\bar\partial^*_{\alpha+1}\partial f/\partial\zeta_j = 0$. Thus, $\partial f/\partial\zeta_j \in \mathcal{H}_{0,q}^{\perp_{\alpha+1}}$. Because of the definition of K hence $K_\alpha f = 0 = K_{\alpha+1}(\partial f/\partial\zeta_j)$. Now let us consider $f \in \mathcal{H}_{0,q}$. Since $\partial/\partial\zeta_j$ commutes with $\bar\partial$, $\bar\partial(\partial K_\alpha f/\partial\zeta_j) = \partial f/\partial\zeta_j$ and therefore $\partial K_\alpha f/\partial\zeta_j$ is a solution to the equation $\bar\partial w = \partial f/\partial\zeta_j$. Finally we must ensure that it is the minimal solution, i.e. $\partial K_\alpha f/\partial\zeta_j \in \mathcal{H}_{0,q+1}^{\perp_{\alpha+1}}$. However, this is equalivalent to $\bar\partial^*_{\alpha+1}\partial K_\alpha f/\partial\zeta_j = 0$, which in turn follows from Corollary 2.2. ■

The formula (2.1) is well known; see e.g. [Br].

Proof of Proposition 1.1. To begin with, one notes that (1.3) holds for all complex α with $\operatorname{Re}\alpha > 1$. In view of Theorem 2.1 the general case follows by analytic continuation. ■

If we try the same trick in the Euclidean case, since the Euclidean adjoint with respect to the weight $(1-|\zeta|^2)^{\alpha-1}$ is

$$\vartheta - \frac{\alpha-1}{1-|\zeta|^2}\mathcal{L},$$

we get the familiar relation

$$\int_{\mathbb{B}} \langle g, \bar\partial f\rangle = \int_{\mathbb{B}} \vartheta g f + \int_{\partial\mathbb{B}} \langle \mathcal{L} g, f\rangle$$

when $\alpha \to 1$. This shows that g must satisfy the extra boundary condition $\mathcal{L}g|_{\partial\mathbb{B}} = 0$, in order to belong to the domain of the Euclidean L^2-adjoint of $\bar\partial$.

Proof of Theorem 2.1. If we for the moment forget about the normalizing constant in (1.1), we can define the formal adjoint $\bar\partial^*_\alpha$ for any real (or complex) α. If $\bar\partial^*_\omega$ denotes the formal adjoint with respect to the Bergman metric ω, then it follows from formula (1.2) that

$$(1-|\zeta|^2)\bar\partial^*_{-(n-q)} = \bar\partial^*_\omega. \tag{2.3}$$

However, as ω is a Kähler metric one has that $\bar\partial^*_\omega = -i[\omega\,\llcorner_\omega, \partial]$ (see e.g. [GrHa]), which is simply $-i\omega\,\llcorner_\omega\partial$ on $(0,q)$ forms, where $\llcorner_\omega$ denotes inner multiplication with respect to ω. We want to express $\omega\,\llcorner_\omega$ in terms of the Euclidean metric β. To this end we choose an ON-basis $e_1,\dots,e_n$ with respect to β for the space of $(1,0)$-forms, such that $|\zeta|e_n = \partial|\zeta|^2$. Then $\beta = i\sum_1^n e_k\wedge\bar e_k$ and

$$\omega = \frac{i\sum_1^n e_k\wedge\bar e_k}{1-|\zeta|^2} + \frac{|\zeta|^2 ie_n\wedge\bar e_n}{(1-|\zeta|^2)^2} = \frac{i\sum_1^{n-1} e_k\wedge\bar e_k}{1-|\zeta|^2} + \frac{ie_n\wedge\bar e_n}{(1-|\zeta|^2)^2}.$$

Thus, if $f_k = e_k/(1-|\zeta|^2)^{1/2}$ for $k<n$ and $f_n = e_n/(1-|\zeta|^2)$ then $f_1,\dots,f_n$ is an ON-basis with respect to ω. Since $f_k\,\llcorner_\omega\bar f_j = \delta_{jk}$ it follows that $e_k\,\llcorner_\omega\bar e_j$ vanishes for $j\neq k$, equals $(1-|\zeta|^2)$ for $j=k<n$ and $(1-|\zeta|^2)^2$ if $j=k=n$. Hence

$$\omega\,\llcorner_\omega = (1-|\zeta|^2)i\sum_1^{n-1} e_k\wedge\bar e_k\,\llcorner + (1-|\zeta|^2)^2 ie_n\wedge\bar e_n\,\llcorner = (1-|\zeta|^2)\left(\beta\,\llcorner - i\partial|\zeta|^2\wedge\bar\partial|\zeta|^2\,\llcorner\right).$$

Thus $\bar\partial^*_\omega = (1-|\zeta|^2)\left(\vartheta - \partial|\zeta|^2\wedge\bar\partial|\zeta|^2\,\llcorner\partial\right)$ and as $-\partial|\zeta|^2\wedge\bar\partial|\zeta|^2\,\llcorner = \partial|\zeta|^2\,\llcorner\bar\partial|\zeta|^2\,\llcorner$ the theorem follows from (2.3) in the case $\alpha = -(n-q)$. Since

$$\bar\partial^*_\omega\left((1-|\zeta|^2)f\right) = (1-|\zeta|^2)\bar\partial^*_\omega f - \partial|\zeta|^2\,\llcorner_\omega f$$

and $e_n\,\llcorner_\omega = (1-|\zeta|^2)^2 e_n\,\llcorner$, the general case follows (keeping (2.3) in mind)

$$\bar\partial^*_\alpha f = (1-|\zeta|^2)^{-(\alpha+n-q)}\bar\partial^*_{-(n-q)}\left((1-|\zeta|^2)^{\alpha+n-q}f\right).$$

■

3. The boundary complex and canonical operators on $\partial\mathbb{B}$.

For any $(0,q)$-form F over $\partial\mathbb{B}$ we denote its restriction to the complex tangential vectors by $f = F|_b$ and we refer to it as a tangential $(0,q)$-form. Thus f is determined by $\bar\partial|\zeta|^2\wedge F$ over $\partial\mathbb{B}$ and f is smooth if and only if $\bar\partial|\zeta|^2\wedge F$ is. The space of smooth tangential $(0,q)$-forms over $\partial\mathbb{B}$ is denoted by $\mathcal{E}^b_{0,q}$, and $\mathcal{O}^b$ denotes the space of smooth functions on $\partial\mathbb{B}$ which have holomorphic extensions to $\mathbb{B}$. The natural inner product is

$$(\phi,\psi)_b = \frac{(n-1)!}{2\pi^n}\int_{\partial\mathbb{B}}\langle\bar\partial|\zeta|^2\wedge\phi, \bar\partial|\zeta|^2\wedge\psi\rangle\, d\sigma. \tag{3.1}$$

Define $\bar\partial_b : \mathcal{E}^b_{0,q} \to \mathcal{E}^b_{0,q+1}$ by $\bar\partial_b u = f$ if $f = \bar\partial U|_b$ and U is any smooth extension of u to $\bar D$. Weakly it can be formulated as

$$\int_{\partial\mathbb{B}} u \wedge \bar\partial\xi = \int_{\partial\mathbb{B}} f \wedge \xi, \quad \xi \in \mathcal{E}_{n,n-q-2}(\bar D),$$

for $q \leq n-2$. This gives rise to the boundary complex

$$0 \to \mathcal{O}^b \to \mathcal{E}^b_{0,0} \to \mathcal{E}^b_{0,1} \to \ldots \to \mathcal{E}^b_{0,n-1} \to 0.$$

Since we noticed already in §1 that (3.1) is the limit of the inner product $(\ ,\ )_\alpha$ when α tends to 0, it follows from Theorem 2.1 that

$$\bar\partial_b^* f = \vartheta f + \mathcal{L}\bar{\mathcal{L}}\partial f + (n-q)\mathcal{L} f, \tag{3.2}$$

for $(0,q)$-forms f. It is well known that the orthogonal projection P of functions onto (boundary values of) holomorphic ones, the Szegö projection, preserves regularity (this follows easily from its explicit expression, see below) and hence maps $\mathcal{E}^b_{0,0} \to \mathcal{O}^b$.

Let us now consider tangential $(0,n-1)$-forms. If

$$\delta = \sum_{j=1}^n (-1)^{j+1} \zeta_j \widehat{d\zeta_j},$$

then $\langle \bar\partial|\zeta|^2 \wedge \bar\delta, \bar\partial|\zeta|^2 \wedge \bar\delta \rangle = 1$ and so any $f \in \mathcal{E}^b_{0,n-1}$ is $f = F\bar\delta$ for a unique function function F on $\partial\mathbb{B}$ and

$$\langle \bar\partial|\zeta|^2 \wedge f, \bar\partial|\zeta|^2 \wedge g \rangle = F\bar G.$$

Since $\mathcal{L}\bar\delta = 0$ it follows from (3.2) that $\bar\partial_b^* f = 0$ if and only if $\bar\partial_b \bar F = 0$. Let $\overline{\delta\mathcal{O}^b}$ be the subspace consisting of all f of this kind. It is clearly a closed subspace and we let S denote the orthogonal projection of $L^{2,b}_{0,n-1}$ onto $\overline{\delta\mathcal{O}^b}$. For $q < n-1$ the subspace of $\mathcal{E}^b_{0,q}$ consisting of $\bar\partial_b$-closed forms will be denoted by $\mathcal{H}^b_{0,q}$. Let

$$\bar\Box_b = \bar\partial_b \bar\partial_b^* + \bar\partial_b^* \bar\partial_b.$$

Theorem 3.1. *The kernel of* $\bar\partial_b^*$ *is* $\mathcal{H}^{b,\perp}$, $q \geq 1$, *and the kernel of* $\bar\Box_b$ *is* $\mathcal{O}^b \oplus \overline{\delta\mathcal{O}^b}$. *There are smooth orthogonal decompositions*

$$\mathcal{E}^b_{0,q} = \mathcal{H}^b_{0,q} \oplus \mathcal{H}^{b,\perp}_{0,q} = \bar\partial_b \mathcal{E}^{b,\perp}_{0,q-1} \oplus \bar\partial_b^* \mathcal{E}^b_{0,q+1} = \bar\Box_b \mathcal{E}^b_{0,q},$$

for $1 \leq q \leq n-2$,

$$\mathcal{E}^b_{0,0} = \mathcal{O}^b \oplus \mathcal{O}^{b,\perp} = \mathcal{O}^b \oplus \bar\partial_b^* \mathcal{E}^b_{0,1} = \mathcal{O}^b \oplus \bar\Box_b \mathcal{E}^b_{0,0}$$

and

$$\mathcal{E}^b_{0,n-1} = \bar\partial_b \mathcal{E}^b_{0,n-2} \oplus \overline{\delta\mathcal{O}^b} = \bar\Box_b \mathcal{E}^b_{0,n-1} \oplus \overline{\delta\mathcal{O}^b}.$$

This theorem is well-known, see [FoKo]; in the case of the ball it is quite simple and follows from the considerations about the integral formulas below. We now define

$$K^q : \mathcal{E}^b_{0,q+1} \to \mathcal{E}^b_{0,q}$$

as zero on $\mathcal{H}^{b,\perp}_{0,q}$ and as the unique operator $\mathcal{H}^b_{0,q+1} \to \mathcal{H}^{b,\perp}_{0,q}$ that inverts $\bar\partial_b$, for $q \le n-2$ whereas K^{n-2} is the one that vanishes on $\overline{\delta\mathcal{O}^b}$ and supply the minimal solution when applied to $f \in \overline{\delta\mathcal{O}^b}^\perp$. One easily verifies that

$$\bar\partial_b K + K\bar\partial_b = I - P - S \tag{3.3}$$

if $K = \sum K^q$. Thus $\bar\partial_b K$ is the orthogonal projection $\bar\partial_b K : \mathcal{E}^b_{0,q} \to \mathcal{H}^b_{0,q}$. The canonical fundamental solution to $\bar\Box_b$ is the operator $E : \mathcal{E}_{0,*} \to \mathcal{E}_{0,*}$ such that Ef is the unique solution to $\bar\Box_b u = f$ if $f \in \mathcal{E}^b_{0,q}$, $1 \le q \le n-2$, the minimal solution if $f \in \mathcal{O}^{b,\perp}$ or in $\overline{\delta\mathcal{O}^b}^\perp$ and zero if $f \in \mathcal{O}^b$ and $f \in \overline{\delta\mathcal{O}^b}^\perp$. We have the relations

$$E\bar\Box_b = \bar\Box_b E = I - P - S, \quad E^* = E$$

and

$$\bar\partial_b E = E\bar\partial_b, \quad \bar\partial_b^* E = E\bar\partial_b^*.$$

An explicit expression for E is found in [DaHa]. Analogously to Proposition 1.3 one can verify that

$$E = KK^* + K^*K \quad \text{and} \quad K = \bar\partial_b^* E. \tag{3.4}$$

We are now going to consider explicit expressions for these operators. We will use the convention that the kernel $k(\zeta, z)$ corresponds to the operator

$$Kf(z) = (f, \overline{k(\cdot, z)})_b,$$

where the inner product is computed after moving all differentials of z to the right.

Proposition 3.2. *The canonical operator K is given by $Kf(z) = (f, \overline{k(\cdot, z)})_b$ where*

$$k(\zeta, z) = \sum_{q=0}^{n-2} k^q(\zeta, z) = \sum_{q=0}^{n-2} \frac{(n-q-2)!}{(n-1)!} \frac{\zeta \cdot (\partial/\partial\zeta) \neg [\bar\zeta \cdot d\zeta \wedge \bar z \cdot d\zeta \wedge (d\bar z \cdot d\zeta)^q]}{(1-\bar\zeta \cdot z)^{n-q-1}(1-\zeta\cdot\bar z)^{q+1}}.$$

Moreover, Pf and Sf are given by the kernels

$$p(\zeta, z) = \frac{1}{(1-\bar\zeta\cdot z)^n} \qquad \text{and} \qquad s(\zeta, z) = \frac{\overline{\delta(z)} \wedge \delta(\zeta)}{(1-\zeta\cdot\bar z)^n},$$

in the sense that $Pf(z)$ and $Sf(z)$ are the the boundary values of the holomorphic function $(f, \overline{p(\cdot, z)})_b$ and the anti-holomorphic form $(f, \overline{s(\cdot, z)})_b$, respectively.

Proof. Notice that $k^q(\zeta, z)$ is equivalent to

$$k^q(\zeta, z) = \frac{(n-q-2)!}{(n-1)!} \frac{\bar z \cdot d\zeta \wedge (d\bar z \cdot d\zeta)^q}{(1-\bar\zeta\cdot z)^{n-q-1}(1-\zeta\cdot\bar z)^{q+1}}$$

as a tangential form. If K, P and S are defined as in the proposition, then the homotopy formula (3.3) holds, see [He2] or [Sk], (this also follows from (0.3) when $\alpha \to 0$, noting that a tangential $(0, n-1)$-form f is in $\overline{\delta\mathcal{O}}^{\perp}$ if and only if it has a $\bar\partial$-closed extension to $\mathbb{B}$). It is equally well-known (and obvious from the self-adjointness) that P, and hence S, are the orthogonal projections.

Now assume that $q \geq 1$ and let $H^q f = (f, \overline{h^q})_b$ where

$$h^q(\zeta, z) = \frac{(n-q-2)!}{(n-1)!q} \frac{(d\bar z \cdot d\zeta)^q}{(1 - \bar\zeta \cdot z)^{n-q-1}(1 - \zeta \cdot \bar z)^q}.$$

To begin with, H preserves regularity. In fact, any integral of the form

$$Tu(z) = \int_{\partial\mathbb{B}} \frac{w(z,\zeta) d\sigma(\zeta)}{(1 - \bar\zeta \cdot z)^{n-q-1}(1 - \zeta \cdot \bar z)^q},$$

where w is smooth, has smooth boundary values. If $q = 0$ it occurs a logarithm but the same argument works anyway.

Since $\bar\partial_{b,\zeta} \overline{h(\zeta, z)} = \overline{k(\zeta, z)}$ it follows that $K = H\bar\partial_b^*$ and hence also K (and K^*) preserves regularity, (it is well-known that P and S do). It is clear that H is self-adjoint since $h(\zeta, z) = (-1)^q \overline{h(z, \zeta)}$ and hence K is the canonical operator, since $K = H\bar\partial_b^*$ vanishes on $\mathcal{H}^{b,\perp}$ and $\bar\partial_b K = \bar\partial H \bar\partial_b^*$ is self-adjoint. Thus Proposition 3.2 is proved. ■

In the same way the operator $K^* = \bar\partial_b H$ preserves regularity, and by (3.4) therefore E does, and hence Theorem 3.1 follows.

Using (3.4) we get

Proposition 3.3. *The operator E is given by a kernel $e(\zeta, z)$ such that*

$$e(\zeta, z) = \mathcal{O}\left(\frac{1}{|1 - \bar\zeta \cdot z|^{n-1}}\right).$$

A more explicit formula for $e(\zeta, z)$ is computed in [DaHa].

Proof. Let $e^q(\zeta, z)$ be the kernel for the restriction of E to $\mathcal{E}^b_{0,q}$. Since $k^q(\zeta, z) = \mathcal{O}(|\zeta - z|/|\zeta - z|^n)$, $k(\zeta, z)$ and $k^*(\zeta, z)$ are integrable, and hence it follows from (3.4) and Fubini's theorem that

$$e^q(\zeta, z) = (k^q(\cdot, \zeta), \overline{k^q(\cdot, z)})_b + (k^{q-1}(z, \cdot), \overline{k^{q-1}(\zeta, \cdot)})_b.$$

Thus

$$e^q(\zeta, z) = \int_{|w|=1} \mathcal{O}\left(\frac{1}{|1 - \bar\zeta \cdot w|^{n-1/2}|1 - \bar w \cdot z|^{n-1/2}}\right),$$

and the proposed estimate then follows in a standard way (see e.g. Lemma 5.2 in [AnCa]).

■

4. Going up and down in dimension, canonical operators in the ball.

In this section we exploit the technique of going up and down in the dimension. We have defined our norm $(\, , \,)_\alpha$ for an integer α so that it is the projection of the boundary norm of the ball in $\tilde{\mathbb{B}}$ in $\mathbb{C}^{n+\alpha}$ (see formula (4.2) below). Our basic idea is to use this fact to obtain results and expressions for the interior values of the kernel for the operator K_α in $\mathbb{B}$ from the corresponding one on $\partial\tilde{\mathbb{B}}$. This kind of technique has been used many times before in the case of functions, see for instance, [Am], [An2], [Be], [Br], [Ru] and others. We extend this technique to the case of higher degree forms. It turns out that it actually is enough to go up just one dimension, and this makes it possible to get results for non-integer values of α as well. Therefore, our first result is restricted to the case when $\tilde{\mathbb{B}}$ is the ball in $\mathbb{C}^{n+1}$. The crucial point is part c), which is less obvious than it might look like at first sight.

Let $\mathbb{B}$ denote the ball in $\mathbb{C}^n$ and $\tilde{\mathbb{B}}$ the ball in $\mathbb{C}^{n+1}$ and let (z, w) be coordinates in $\mathbb{C}^n \times \mathbb{C}$. If $f(z)$ is a form in $\mathbb{B}$ then $\tilde{f}(z, w) = f(z)$ is a form in $\tilde{\mathbb{B}}$. Moreover, as before $\tilde{f}|_b$ denotes its restriction to the complex tangent space over $\partial\tilde{\mathbb{B}}$. We say that the tangential form ϕ over $\partial\tilde{\mathbb{B}}$ is invariant if it is rotation invariant in the last variable, i.e. if $\tau^*\phi = \phi$ for all $\tau(z, w) = (z, e^{i\theta}w)$.

Proposition 4.1 (Going up one dimension). *Let $\alpha > 1$.*

a) For any forms g, f in $\mathbb{B}$, $f = g$ if and only if $\tilde{f}|_b = \tilde{g}|_b$ on $\partial\tilde{\mathbb{B}}$.

b) For any forms g, f in $\mathbb{B}$,

$$(f, g)_\alpha = (\tilde{f}, \tilde{g})_{\alpha-1}.$$

c) If $f \in \mathcal{E}_{0,q}(\mathbb{B})$, then $\tilde{f}|_b$ is a smooth invariant form on $\partial\tilde{\mathbb{B}}$. Conversely, to any smooth invariant $(0, q)$-form on $\partial\tilde{\mathbb{B}}$ there is a unique form $f \in \mathcal{E}_{0,q}(\mathbb{B})$ such that $\tilde{f}|_b = \phi$.

d) For any $f \in \mathcal{E}_{0,q}(\mathbb{B})$, $(\bar\partial f)\tilde{} = \bar\partial\tilde{f}$ and $\bar\partial u = f$ if and only if $\bar\partial_b \tilde{u}|_b = \tilde{f}|_b$.

e) For any $f \in \mathcal{E}_{0,q}(\mathbb{B})$, $(\bar\partial^*_\alpha f)\tilde{} = \bar\partial^*_{\alpha-1}\tilde{f}$ and $(\bar\Box_\alpha f)\tilde{} = \bar\Box_{\alpha-1}\tilde{f}$.

f) If $f = f_1 + f_2$ is the smooth orthogonal decomposition of $f \in \mathcal{E}_{0,q}$ in $\mathbb{B}$ with respect to $(\, , \,)_\alpha$ then $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$ is the orthogonal decomposition of $\tilde{f}$ in $\tilde{\mathbb{B}}$ with respect to $(\, , \,)_{\alpha-1}$.

g) Any invariant $(0, n)$-form ϕ on $\partial\tilde{\mathbb{B}}$ is orthogonal to $\operatorname{Ker}\bar\partial_b^*$, so $\bar\partial_b u = f$ is solvable.

Remark. Parts c) and d) together mean that one can solve $\bar\partial u = f$ in $\mathbb{B}$ by taking an invariant solution to $\bar\partial_b\phi = \tilde{f}|_b$ on $\partial\tilde{\mathbb{B}}$ and then find the unique u such that $\tilde{u}|_b = \phi$. If ϕ is smooth then u is smooth up to the boundary, but as it will be clear from the proof, in general one loses one half unit of regularity on the "complex normal" component of u. For instance, if ϕ is bounded, then $u = u_1 + \bar\partial|z|^2 \wedge u_2/\sqrt{1 - |z|^2}$ where u_j are bounded.

Proposition 4.1 will be proved without any reference to the regularity result Theorem 1.2 except for part f). However, this part will not be used in the proof of Theorem 1.2. In the case of functions we already know that the decomposition is smooth since the projection P_α preserves regularity. ∎

Proof.

a) $\tilde f|_b = 0$ if and only if $(\bar\partial|z|^2 + \bar\partial|w|^2)\wedge f(z) = 0$ for $|z|^2+|w|^2=1$ and this is true if and only if $f(z)=0$ for $|z|<1$.

b) Since

$$(1-|z|^2-|w|^2)\left\langle \tilde f, \tilde g\right\rangle + \left\langle \bar\partial(|z|^2+|w|^2)\wedge \tilde f, \bar\partial(|z|^2+|w|^2)\wedge \tilde g\right\rangle = \\ = (1-|z|^2)\langle f,g\rangle + \langle \bar\partial|z|^2\wedge f, \bar\partial|z|^2\wedge g\rangle\,),$$

the desired equality is obtained by integrating the expression for $(\tilde f,\tilde g)_{\alpha-1}$ in the last variable w.

c) Since $\tau^*\tilde f = \tilde f$, we have that $\tau^*\tilde f|_b = \tilde f|_b$ and since $\tilde f$ is smooth up to the boundary of $\tilde{\mathbb{B}}$ if f is smooth up to the boundary of $\mathbb{B}$, the first half of c) follows. Assume therefore that ϕ is smooth and invariant. Let Φ denote some smooth $(0,q)$-form on $\partial\tilde{\mathbb{B}}$ that represents ϕ. Then there are forms a and b such that

$$\Phi = wd\bar w\wedge a(z,w) + b(z,w),$$

where a and b are forms with no differential $d\bar w$. The invariance condition means that $a(z,e^{it}w) = a(z,w)$ and $b(z,e^{it}w)=b(z,w)$. Thus actually $a(z,w)=a(z)$ and $b(z,w)=b(z)$. Moreover, $wd\bar w = -\sum z_j d\bar z_j$ as tangential forms so

$$f(z) = -\sum z_j d\bar z_j\wedge a(z) + b(z)$$

is the desired form in $\mathbb{B}$ such that $\tilde f|_b = \phi$. The uniqueness follows from part a). It remains to see that actually f is smooth if ϕ is. If $a_I(z)$ is any of the coefficients in the form $a(z)$ then we know that the function $wa_I(z)$ is smooth on $\partial\tilde{\mathbb{B}}$ and we want to prove that a_I is smooth up to the boundary of $\mathbb{B}$. For simplicity we assume that $n=1$ and $a_I = a$. The possible problem is when w is near 0. Here $(w,t)\mapsto (e^{it}\sqrt{1-|w|^2}, w)$ are coordinates on $\partial\tilde{\mathbb{B}}$ and so $(w,t)\mapsto wa(e^{it}\sqrt{1-|w|^2})$ is a C^∞-function. In particular for real x, $(x,t)\mapsto xa(e^{it}\sqrt{1-x^2})$ is smooth and odd in x. Therefore there is a smooth function u such that $xa(e^{it}\sqrt{1-x^2}) = xu(x^2,t)$. Thus $a(re^{it}) = u(1-r^2,t)$ and hence it is smooth up to the boundary.

d) The first claim is obvious. Since $\tilde u$ is a (smooth) representative of $\tilde u|_b$ it follows that $(\bar\partial u)\tilde{}$ represents $\bar\partial_b\tilde u|_b$ and hence $\bar\partial_b\tilde u|_b = \tilde f|_b$ if and only if $(\bar\partial\tilde u)|_b = \tilde f|_b$, i.e. if and only if $(\bar\partial u)\tilde{}|_b = \tilde f|_b$, and by part a) this holds if and only if $\bar\partial u = f$.

e) The claim about $\bar\partial^*_\alpha$ follows from Theorem 2.1 and the second claim is then immediate.

f) In view of Theorem 1.2, a smooth $(0,q)$-form f is in $\mathcal{H}^{\perp_\alpha}$ if (and only if) $\bar\partial^*_\alpha f = 0$. Hence the statement follows immediately from parts d) and e). For functions, it follows from the fact that $(\tilde f, h)_{\alpha-1} = 0$ for all holomorphic h in $\tilde{\mathbb{B}}$ if $(f,g)_\alpha = 0$ for all holomorphic g in $\mathbb{B}$.

g) This immediately follows from part c) and the solvability of $\bar\partial$ in $\mathbb{B}$. However, we prefer a direct argument. Assume that $f = F\bar\delta$ is invariant. If $\tau(\zeta,\zeta_{n+1}) =$

$(\zeta, \lambda\zeta_{n+1})$ then $\tau^*\bar\delta = \bar\lambda\bar\delta$ and hence $F(\zeta, \lambda\zeta_{n+1}) = \lambda F(\zeta, \zeta_{n+1})$. If $h = H\delta$ and $H \in \mathcal{O}^b$ then

$$(f, \bar h) = \int_{\partial\tilde B} F(\zeta, \zeta_{n+1}) H(\zeta, \zeta_{n+1}) d\sigma(\zeta, \zeta_{n+1}) =$$
$$= \int_{\partial\tilde B} F(\zeta, \lambda\zeta_{n+1}) H(\zeta, \lambda\zeta_{n+1}) d\sigma(\zeta, \lambda\zeta_{n+1}) =$$
$$= \int_{\partial\tilde B} F(\zeta, \zeta_{n+1}) \lambda H(\zeta, \lambda\zeta_{n+1}) d\sigma(\zeta, \zeta_{n+1}),$$

and integrating over $|\lambda| = 1$ we get that $(f, \bar h) = 0$ as the holomorphic function $\lambda \mapsto \lambda H(\zeta, \lambda\zeta_{n+1})$ vanishes at the origin.

■

We want to find formulas for the canonical operator $K_\alpha f$. In order to motivate the idea, let us for the moment assume the regularity for the orthogonal decomposition (Theorem 1.2). ¿From part f) we have that $K_{\alpha-1}\tilde f = (K_\alpha f)\tilde{}$. However, already in §0 we gave an argument that showed that the boundary values of the canonical operator K_α were provided by the explicitly given operator $K^b_\alpha : \mathcal{E}_{0,*}(\tilde{\mathbb{B}}) \to \mathcal{E}^b_{0,*-1}(\partial\tilde{\mathbb{B}})$, cf. (0.2) ($K^b_\alpha$ preserves regularity for the same reason as K do, see §3). Summing up, we must have

$$(K_\alpha f)\tilde{}|_b = K^b_{\alpha-1}\tilde f. \tag{4.1}$$

However, we want to establish (4.1) without assuming Theorem 1.2. This can be done with a limit procedure but we will supply a more direct approach.

It is easily verified by inspection of the explicit formula that $K^b_{\alpha-1}\tilde f$ is rotation invariant in the last variable since $\tilde f$ is. In virtue of Proposition 4.1 we can then take (4.1) as the *definition* of an operator $K_\alpha : \mathcal{E}_{0,*}(\mathbb{B}) \to \mathcal{E}_{0,*-1}(\mathbb{B})$ in $\mathbb{B}$ and we are going to show that it actually coincides with the canonical one, and at the same time we will conclude the regularity of the orthogonal decomposition.

¿From formula (0.4) it is clear that $k_\alpha(\zeta, z)$ is ∂_ζ-exact, and hence there is an operator (preserving regularity for the same reason as above) $H^b_{\alpha-1} : \mathcal{E}_{0,q}(\tilde{\mathbb{B}}) \to \mathcal{E}^b_{0,q}(\partial\tilde{\mathbb{B}})$ such that $K^b_{\alpha-1}\phi = H^b_{\alpha-1}\bar\partial^*_{\alpha-1}\phi$. It has also the property that $H^b_{\alpha-1}\phi$ is invariant if ϕ is, and therefore we can define the operator

$$H_\alpha : \mathcal{E}_{0,q}(\mathbb{B}) \to \mathcal{E}_{0,q}(\mathbb{B})$$

by the formula

$$(H_\alpha f)\tilde{}|_b = H^b_{\alpha-1}\tilde f$$

for $\alpha > 1$.

By part f) of Proposition 4.1 (for functions), $(P_\alpha u)\tilde{} = P_{\alpha-1}\tilde u$. This can preferably be verified also by direct inspection of the kernels. We now claim that our operator

K_α satisfies the homotopy relation (1.4). In fact, by the definition of $K_\alpha f$, (0.3) and Proposition 4.1 we have

$$(\bar\partial K_\alpha f)\tilde{}|_b = \bar\partial_b (K_\alpha f)\tilde{}|_b = \bar\partial_b K^b_{\alpha-1}\tilde f = \tilde f|_b - (P_{\alpha-1}\tilde f)|_b - K^b_{\alpha-1}(\bar\partial \tilde f) =$$
$$= \tilde f|_b - (P_\alpha f)\tilde{}|_b - K^b_{\alpha-1}((\bar\partial f)\tilde{}) = \tilde f|_b - (P_\alpha f)\tilde{}|_b - (K^b_\alpha(\bar\partial f))\tilde{}|_b.$$

By definition this means (1.4).

Since $K^b_{\alpha-1}\phi = H^b_{\alpha-1}\bar\partial^*_{\alpha-1}\phi$ it follows from Proposition 4.1 that $K_\alpha = H_\alpha\bar\partial^*_\alpha$. It will be established in §5 that H_α is self-adjoint, by computing its kernel. Thus $\bar\partial K_\alpha = \bar\partial H_\alpha \bar\partial^*_\alpha$ is self-adjoint and therefore $f = \bar\partial K_\alpha f + K_\alpha\bar\partial f$ is the orthogonal decomposition of a smooth form f, and in particular this decomposition is smooth (the case with functions f is already dealt with). Moreover, since $K_\alpha = H_\alpha\bar\partial^*_\alpha$ vanishes on $\mathcal{H}^{\perp_\alpha}$, it follows that K_α is the canonical homotopy operator.

We have thus proved

Theorem 4.2. *The operator $K_\alpha : \mathcal{E}_{0,*}(\mathbb{B}) \to \mathcal{E}_{0,*-1}(\mathbb{B})$, defined by (4.1), is the canonical homotopy operator with respect to $(\ ,\ )_\alpha$. Moreover, $K_\alpha = H_\alpha\bar\partial^*_\alpha$ and $H_\alpha : \mathcal{E}_{0,*} \to \mathcal{E}_{0,*}$ is self-adjoint. In particular, $K^*_\alpha = \bar\partial H_\alpha$ maps $\mathcal{E}_{0,*}$ into $\mathcal{E}_{0,*+1}$ and hence $E_\alpha = K^*_\alpha K_\alpha + K_\alpha K^*_\alpha$ also maps $\mathcal{E}_{0,*}$ into $\mathcal{E}_{0,*}$.*

Proof of Theorem 1.2. The desired smooth decomposition is thus provided by $f - P_\alpha f = \bar\Box E_\alpha f = \bar\partial\bar\partial^*_\alpha\bar\Box_\alpha f + \bar\partial^*_\alpha\bar\partial\bar\Box_\alpha f$, cf. § 1. ∎

In the arguments above it is tacitly understood that $\alpha > 1$. However, Proposition 4.1 as well as the succeeding discussion works just as well for $\alpha = 1$ if $(\ ,\ )_{\alpha-1}$ is interpreted as the norm $(\ ,\ )_b$ over $\partial\tilde{\mathbb{B}}$. By repeated use of Proposition 4.1 one can therefore express K_α for integers α in terms of the operator K on the boundary of the ball in $\mathbb{C}^{n+\alpha}$. However, for completeness, we formulate a result about such a multi-dimensional step ending at the boundary of a higher dimensional ball.

Let therefore m be an integer and let $\tilde{\mathbb{B}}$ denote the ball in $\mathbb{C}^{n+m}$, where (z, w) are coordinates. We say that a tangential form on $\partial\tilde{\mathbb{B}}$ is invariant if $\tau^* f = f$ for each unitary mapping τ in the w variable. If f is any form in $\mathbb{B}$ we let $\tilde f(z, w) = f(z)$. Finally $(\ ,)_b$ denotes the inner product for tangential forms on $\partial\tilde{\mathbb{B}}$.

Proposition 4.3 (Going up several dimensions).

a) For any forms g, f in $\mathbb{B}$, $f = g$ if and only if $\tilde f|_b = \tilde g|_b$ on $\tilde{\mathbb{B}}$.

b) For any forms g, f in $\mathbb{B}$,

$$(f, g)_m = (\tilde f, \tilde g)_b. \tag{4.2}$$

c) If $f \in \mathcal{E}(\mathbb{B})_{0,q}$, then $\tilde f|_b$ is a smooth invariant form on $\partial\tilde{\mathbb{B}}$. Conversely, for any smooth invariant $(0, q)$-form ϕ on $\partial\tilde{\mathbb{B}}$ there is a unique form $f \in \mathcal{E}_{0,q}(\mathbb{B})$ such that $\tilde f|_b = \phi$. In particular, $\phi \equiv 0$ if $q > n$.

d) For any $f \in \mathcal{E}_{0,q}(\mathbb{B})$, $(\bar\partial f)\tilde{} = \bar\partial\tilde f$ and $\bar\partial u = f$ if and only if $\bar\partial_b\tilde u|_b = \tilde f|_b$.

e) For any $f \in \mathcal{E}_{0,q}(\mathbb{B})$, $(\bar\partial^*_\alpha f)\tilde{} = \bar\partial^*_{\alpha-m}\tilde f$ and $(\bar\Box_m f)\tilde{}|_b = \bar\Box_b\tilde f|_b$.

f) If ϕ is invariant and $\phi = \phi_1 + \phi_2$ is its orthogonal decomposition in $\mathcal{H}^b_{0,q} \oplus \mathcal{H}^{b,\perp}_{0,q}$ then ϕ_j are also invariant. If ϕ is invariant and $\bar\partial_b$-closed, then the minimal solution to $\bar\partial_b u = \phi$ is also invariant.

Proof. Parts a), d) and e) follow exactly as the corresponding statements in Proposition 4.1.

b) Repeated use of Proposition 4.1 b) yields that $(f, g)_{m+\epsilon} = (\tilde f, \tilde g)_\epsilon$. Then let $\epsilon \to 0$.

c) Let Φ denote a representing $(0, q)$-form over $\partial\tilde{\mathbb{B}}$. For each fixed z, Φ is a form in $d\bar w_j$ over the sphere $|w| = \sqrt{1 - |z|^2}$ which is rotation invariant. By the lemma below therefore

$$\Phi = \sum w_j d\bar w_j \wedge a(z, w) + b(z, w),$$

where a and b have degrees at most $(0, m - 1)$ in $d\bar w_j$. Taking this for granted, we can replace $\sum w_j d\bar w_j$ by $-\sum z_j d\bar z_j$, and therefore we obtain a representative for ϕ which is at most of degree $m - 1$ in $d\bar w_j$. Iterating we finally obtain a representing form with no occurrences of $d\bar w_j$, and thus we have obtained our f. The smoothness of f follows from part c) of Proposition 4.1 applied m times.

Lemma 4.4. *If g is a $(0, q)$-form over $\partial\mathbb{B}$ which is rotation invariant, i.e. $\tau^* g = g$ for all unitary mappings τ, then either $q = 0$ or $g = \bar\partial|w|^2 \wedge h$ on $\partial\mathbb{B}$ for some $(0, q - 1)$-form h. In particular, it is zero considered as a tangential form.*

Proof. We first prove that a complex tangential rotation invariant $(0, q)$-form is vanishes if $q \geq 1$. First assume that g is a complex tangential $(0, n - 1)$-form. Then $g = G\bar\delta$ for some function G. Since $\tau^*\bar\delta = \overline{\det\tau}\bar\delta$, it follows that $G(\tau(w)) = (\det)\tau G(w)$ for all τ, which implies that $G = 0$ if $n > 1$. If now g is a $(0, q)$-form, $q \geq 1$, we take an arbitrary point on $\partial\mathbb{B}$ and choose q complex tangential vectors. There is a $(q + 1)$-dimensional complex subspace H of $\mathbb{C}^n$ through this point containing these vectors. Since the restriction of g to $\partial\mathbb{B} \cap H$ is rotation invariant it follows from above that its value on these vectors is zero. Hence $g = 0$ as a tangential form. If g denotes a representative, thererfore $\bar\partial|w|^2 \wedge g = 0$. This in turn means that $g = \bar\partial|w|^2 \wedge h$ for some $(0, q - 1)$-form h. (In fact, $T\mathcal{L} + \mathcal{L}T = |w|^2$ and therefore one can take $h = \mathcal{L}g/|w|^2$). ■

f) By c) there is an f such that $\tilde f|_b = \phi$, and if $f = f_1 + f_2$ is its orthogonal decomposition with respect to $(\ ,\ )_m$, then it follows from b), d) and e) that $\tilde f_j|_b = \phi_j$ and hence ϕ_j are invariant. However, this can easily be established directly without reference to the somewhat harder statement c). Take a form ϕ on the boundary that is invariant in the last variables and let ϕ_1 be the orthogonal projection onto the $\bar\partial_b$-closed ones. Then $\tau^*\phi_1$ is $\bar\partial_b$-closed for any unitary mapping on the last α variables and since ϕ is invariant, we have that $\|\tau^*\phi_1 - \phi\| = \|\phi_1 - \phi\|$ since the measure on the boundary is invariant under τ. Thus $\tau^*\phi_1 = \phi_1$ and therefore ϕ_1 is invariant. Obviously $\phi_2 = \phi - \phi_1$ is also invariant. The last statement of f) is shown along the same line. If ϕ is invariant and $\bar\partial_b u = \phi$, then also $\bar\partial_b \tau^* u = \phi$ and $\|u\| = \|\tau^* u\|$. Hence if u is the minimal solution, then $\tau^* u = u$ by uniqueness. ■

It follows from the preceding proposition that $(E_m f)\rceil_b = E\tilde f$, and this observation makes it possible to express the kernel of E_m in terms of the kernel $e(\zeta, z)$ for E on $\tilde{\mathbb{B}}$. However, in this paper we restrict ourselves to computing formulas for the operator K_α, relying on formula (4.1).

5. Explicit expressions for the canonical operators and comparison with some known homotopy formulas.

The main result in this section is the following.

Theorem 5.1. *Let $\alpha > 0$. The canonical operator K_α^q is given by $K_\alpha^q f = (f, \overline{k_\alpha(\cdot, z)})_\alpha$ for a $(0, q+1)$-form f, where*

$$\begin{aligned}
&k_\alpha^q(\zeta, z) = \\
&c_{n,\alpha,q}\frac{1}{(1-\bar\zeta\cdot z)^{n+\alpha-q}(1-\zeta\cdot\bar z)^{q+1}(1-|a|^2)^n} \\
&\Big[[(1-\bar\zeta\cdot z)P_{n-q-1}^{\alpha-1,-n}\bar z\cdot d\zeta - (1-|z|^2)P_{n-q-1}^{\alpha,-n}\bar\zeta\cdot d\zeta]\wedge(d\bar z\cdot d\zeta)^q \\
&+ qP_{n-q-1}^{\alpha,-n}\bar z\cdot d\zeta\wedge(d\bar z\cdot d\zeta)^{q-1}\wedge\bar\partial|z|^2\wedge\bar\zeta\cdot d\zeta\Big], \\
&c_{n,\alpha,q} = \frac{\Gamma(n+\alpha-q-1)}{\Gamma(n+\alpha)}\frac{\Gamma(\alpha)\Gamma(n-q)}{\Gamma(n+\alpha-q-1)}, \\
&1-|a|^2 = 1-\frac{(1-|\zeta|^2)(1-|z|^2)}{|1-\bar\zeta\cdot z|^2},
\end{aligned}$$

and $P_{n-q-1}^{\alpha,-n}$ and $P_{n-q-1}^{\alpha-1,-n}$ are polynomials in $|a|^2$ of degree $n-1-q$. More precisely,

$$P_m^{\alpha,\beta} = P_m^{\alpha,\beta}(1-2|a|^2)$$

where $P_m^{\alpha,\beta}(x)$ are the Jacobi polynomials

$$P_m^{\alpha,\beta}(x) = \frac{(-1)^m}{2^m m!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^m}{dx^m}\{(1-x)^{m+\alpha}(1+x)^{m+\beta}\}. \tag{5.1}$$

Moreover, $k_\alpha^q(\zeta, z) = \partial_\zeta h_\alpha^q(\zeta, z)$ where h_α^q corresponds to the operator H_α^q defined in §4. This operator is in fact self-adjoint.

Remarks. Since $P_{n-q-1}^{\alpha-1,-n}(1) = \Gamma(n+\alpha-q-1)/\Gamma(\alpha)\Gamma(n-q)$, $k_\alpha(\zeta, z)$ coincides with $k_\alpha^b(\zeta, z)$ as tangential forms when z is on the boundary. Notice that $1-|a|^2 = \mathcal{O}(|\zeta-z|^2)$ for (ζ, z) on compact subsets of $\mathbb{B}\times\mathbb{B}$. Since $P_{n-q-1}^{\alpha-1,-n}(-1) = P_{n-q-1}^{\alpha,-n}(-1)$ and $\bar z\cdot d\zeta\wedge\bar\zeta\cdot d\zeta = \mathcal{O}(|\zeta-z|)$ it follows that $k_\alpha^q(\zeta, z) = \mathcal{O}(|\zeta-z|^{-2n+1})$ as expected. If $n=1$ (and $q=0$) then $1-|a|^2 = |\zeta-z|^2/|1-\bar\zeta z|^2$ and therefore the kernel reduces to

$$k_\alpha(\zeta, z) = \frac{1}{\alpha}\frac{\bar z d\zeta}{(1-\bar\zeta z)^\alpha(\zeta-z)},$$

which corresponds to the weighted Cauchy integral mentioned in the introduction. If

$$\sigma = |1-\bar\zeta\cdot z|^2 - (1-|\zeta|^2)(1-|z|^2).$$

then

$$\frac{P_{n-q-1}^{\alpha-1,-n}}{(1-|a|^2)^n} = \sum_{k=0}^{n-q-1} c_k \frac{|a|^{2k}}{(1-|a|^2)^{q+1+k}} =$$
$$= |1-\bar\zeta\cdot z|^{2(q+1)} \sum_{k=0}^{n-q-1} c_k \frac{(1-|\zeta|^2)^k(1-|z|^2)^k}{\sigma^{q+1+k}}$$

and similarly for $P_{n-q-1}^{\alpha,n}$, and therefore we can write the kernel as

$$k_\alpha^q(\zeta,z) = C_{n,q,\alpha}\frac{1}{(1-\bar\zeta\cdot z)^{n+\alpha-2q-1}}$$
$$\Bigg[\sum_{k=0}^{n-q-1}[c_k(1-\bar\zeta\cdot z)\bar z\cdot d\zeta - c_k'(1-|z|^2)\bar\zeta\cdot d\zeta]\frac{(1-|\zeta|^2)^k(1-|z|^2)^k}{\sigma^{q+1+k}}\wedge(d\bar z\cdot d\zeta)^q$$
$$+ q\bar z\cdot d\zeta\wedge(d\bar z\cdot d\bar\zeta)^{q-1}\wedge\bar\partial|z|^2\wedge\bar\zeta\cdot d\zeta\sum_{k=0}^{n-q-1} c_k'\frac{(1-|\zeta|^2)^k(1-|z|^2)^k}{\sigma^{q+1+k}}\Bigg],$$

where $c_{n-q-1} = c'_{n-q-1}$.

At the end of this section we compare this kernel to the kernels of some previously known homotopy operators.

Proof. By formula (4.1), $K_\alpha f(z)$ is obtained from the formula for $K^b_{\alpha-1}\tilde f(z,z_{n+1})$ by replacing z_{n+1} with $\sqrt{1-|z|^2}$ and all occurrences of $d\bar z_{n+1}$ with $\bar\partial|z|^2/\sqrt{1-|z|^2}$. It is enough to find the expression for the kernel for $\alpha>1$, since the general case then follows by analytic continuation. Let $\langle\langle\ ,\ \rangle\rangle\tilde{}$ denote the corresponding inner product in $\tilde B$. If f, a and b are $(0,*)$-forms in $\mathbb{B}$ then it is easily checked (cf. the proof of Theorem 4.1 b)) that

$$\langle\langle f, a+d\bar w\wedge b\rangle\rangle\tilde{} = \frac{w}{1-|\zeta|^2}\langle\langle f,\bar\partial|\zeta|^2\wedge b\rangle\rangle\,. \tag{5.2}$$

¿From (0.4) we have that

$$k^{b,q}_{\alpha-1}(\zeta,w;z,z_{n+1}) =$$
$$= \Omega\bar z\cdot d\zeta\wedge(d\bar z\cdot d\zeta)^q + \big(\bar z_{n+1}\wedge(d\bar z\cdot d\zeta)^q + q(d\bar z\cdot d\zeta)^{q-1}\wedge d\bar z_{n+1}\big)\wedge dw,$$

where

$$\Omega = \frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)}\frac{(1-|\zeta|^2-|w|^2)^{\alpha-2}}{(1-\bar\zeta\cdot z-\bar w z_{n+1})^{\alpha+n-q-1}(1-\zeta\cdot\bar z-w\bar z_{n+1})^{q+1}}.$$

Since

$$K^{b,q}_{\alpha-1}f(z,z_{n+1}) =$$
$$\frac{\Gamma(n+\alpha)}{\pi^{n+1}\Gamma(\alpha-1)}\int_{\tilde{\mathbb{B}}}(1-|\zeta|^2-|w|^2)^{\alpha-2}\Big\langle\Big\langle f,\overline{k^{b,q}_{\alpha-1}(\cdot;z,z_{n+1})}\Big\rangle\Big\rangle\tilde{}\,dm(\zeta,w),$$

we get by (5.2) that

$$K^q_\alpha f(z) = \frac{\Gamma(n+\alpha)}{\pi^n\Gamma(\alpha)}\int_{\mathbb{B}}(1-|\zeta|^2)^{\alpha-1}\left\langle\left\langle f, \overline{k^q_\alpha(\cdot,z)}\right\rangle\right\rangle dm(\zeta),$$

where

$$k^q_\alpha(\zeta,z) = (1-|\zeta|^2)^{-(\alpha-1)}\int_{|w|<\sqrt{1-|\zeta|^2}}\Omega\Bigg\{\bar z\cdot d\zeta\wedge(d\bar z\cdot d\zeta)^q +$$

$$-\frac{1}{1-|\zeta|^2}\left(\sqrt{1-|z|^2}(d\bar z\cdot d\zeta)^q - \bar z\cdot d\zeta\wedge q(d\bar z\cdot d\zeta)^{q-1}\wedge\frac{\bar\partial|z|^2}{\sqrt{1-|z|^2}}\right)\wedge$$

$$\wedge\,\partial|\zeta|^2\wedge w\Bigg\}\frac{(\alpha-1)dm(w)}{\pi}.$$

If we make the change of variables $\sqrt{1-|\zeta|^2}\tau = w$ in this integral we get

$$k^q_\alpha(\zeta,z) = \frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)}\frac{1}{(1-\bar\zeta\cdot z)^{\alpha+n-q-1}(1-\zeta\cdot\bar z)^{q+1}}$$

$$\Bigg\{\bar z\wedge(d\bar z\cdot d\zeta)^q m_{\alpha-2,\alpha+n-q-1,q+1} -$$

$$-\frac{1}{\sqrt{1-|\zeta|^2}\sqrt{1-|z|^2}}\Big((1-|z|^2)(d\bar z\cdot d\zeta)^q - \bar z\cdot d\zeta\wedge q(d\bar z\cdot d\zeta)^{q-1}\wedge$$

$$\bar\partial|z|^2\Big)\wedge\partial|\zeta|^2\wedge m'_{\alpha-2,\alpha+n-q-1,q+1}\Bigg\},$$

where

$$m_{\alpha-2,j,k} = \frac{\alpha-1}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}dm(\tau)}{(1-a\bar\tau)^j(1-\bar a\tau)^k},$$

$$m'_{\alpha-2,j,k} = \frac{\alpha-1}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}\tau\, dm(\tau)}{(1-a\bar\tau)^j(1-\bar a\tau)^k}$$

and

$$a = \frac{\sqrt{1-|\zeta|^2}\sqrt{1-|z|^2}}{1-\bar\zeta\cdot z}.$$

An integration by parts in the expression for m' reveals that

$$m'_{\alpha-2,j,k} = \frac{aj}{\alpha}m_{\alpha,j+1,k}$$

and hence

$$k^q_\alpha(\zeta,z) =$$

$$\frac{\Gamma(\alpha+n-q-1)}{\Gamma(n+\alpha)}\Bigg[\frac{m_{\alpha-2,\alpha+n-q-1,q+1}}{(1-\bar\zeta\cdot z)^{\alpha+n-q-1}(1-\zeta\cdot\bar z)^{q+1}}\bar z d\zeta\wedge(d\bar z\cdot d\zeta)^q -$$

$$-\frac{\alpha+n-q-1}{\alpha}\frac{m_{\alpha-1,\alpha+n-q,q+1}}{(1-\bar\zeta\cdot z)^{\alpha+n-q}(1-\zeta\cdot\bar z)^{q+1}} \tag{5.3}$$

$$\Big((1-|z|^2)(d\bar z\cdot d\zeta)^q - \bar z\cdot d\zeta\wedge q(d\bar z\cdot d\zeta)^{q-1}\wedge\bar\partial|z|^2\Big)\wedge\partial|\zeta|^2\Bigg].$$

Observe that $m_{\alpha,j,k} = 1$ when $a = 0$. It is not hard to see that $m_{\gamma,j,k}$ only depends on $|a|^2$. More precisely, see [An2],

$$m_{\gamma,j,k} = F(j, k, \gamma + 2, |a|^2) \tag{5.4}$$

where F is the hypergeometric function,

$$\begin{aligned} F(m, b, c, z) = \\ &\frac{(c-1)!}{(b-1)!(c-b-1)!}\int_0^1 t^{b-1}(1-t)^{c-b-1}\frac{dt}{(1-tz)^m} = \\ &= 1 + \frac{mb}{c\cdot 1}z + \frac{m(m+1)b(b+1)}{c(c+1)\cdot 1\cdot 2}z^2 + \dots. \end{aligned}$$

The hypergeometric functions that appear in the kernel are of the forms

$$F(\alpha + n - q - 1, q + 1, \alpha, |a|^2) \qquad \text{and} \qquad F(\alpha + n - q, q + 1, \alpha + 1, |a|^2)$$

and therefore, if we apply the well-known formula (see for instance [GrRy]),

$$F(m, b, c, z) = (1 - z)^{c-m-b}F(c - m, c - b, c, z),$$

the functions that appear are

$$\frac{F(-n + q + 1, \alpha - q - 1, \alpha, |a|^2)}{(1 - |a|^2)^n} \qquad \text{and} \qquad \frac{F(-n + q + 1, \alpha - q, \alpha + 1, |a|^2)}{(1 - |a|^2)^n}.$$

If m is a non-positive integer then actually $F(m, b, c, z)$ is polynomial in z of degree $-m$. More precisely,

$$F(-m, \alpha + 1 + \beta + m, \alpha + 1, |a|^2) = \frac{\Gamma(\alpha + 1)\Gamma(m + 1)}{\Gamma(\alpha + 1 + m)}P_m^{\alpha,\beta}(1 - 2|a|^2),$$

where $P_m^{\alpha,\beta}(x)$ are the Jacobi polynomials (5.1). Thus the hypergeometric functions that appear in our expression for the kernel are in fact rational functions. If we replace F by its rational expression in (5.4) and plug it into (5.3) we obtain the stated formula for $k_\alpha^q(\zeta, z)$.

The boundary values of the operator H_α^q defined in §4 are given by the kernel

$$h_\alpha^b(\zeta, z) = c_{q,\alpha,n}\frac{(d\bar{z}\cdot d\zeta)^q}{(1 - \bar{\zeta}\cdot z)^{\alpha-1+n-q}(1 - \zeta\cdot\bar{z})^q}$$

if $q > 0$ and

$$h_\alpha^{b,0}(\zeta, z) = c_{0,\alpha,n}\frac{\log(1 - \zeta\cdot\bar{z})}{(1 - \bar{\zeta}\cdot z)^{\alpha-1+n}}.$$

The operator $H^q_\alpha f(z)$ is obtained in the same manner from $H^{b,q}_{\alpha-1}(z, z_{n+1})$ and a similar computation as for k^q_α yields that

$$h^q_\alpha(\zeta, z) = c_{n,q,\alpha}\Bigg(\frac{(d\bar z\cdot d\zeta)^q}{(1-\bar\zeta\cdot z)^{\alpha+n-q-1}(1-\zeta\cdot\bar z)^q} m_{\alpha-2,\alpha+n-q-1,q} - \\ - c\frac{q(d\bar z\cdot d\zeta)^{q-1}\wedge\bar\partial|z|^2\wedge\partial|\zeta|^2}{(1-\bar\zeta\cdot z)^{\alpha+n-q}(1-\zeta\cdot\bar z)^q} m_{\alpha-1,\alpha+n-q,q}\Bigg)$$

for $q > 0$ and

$$h_\alpha(\zeta, z) = \\ \frac{c_{n,0,\alpha}}{(1-\bar\zeta\cdot z)^{\alpha+n-1}}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}\big[\log(1-\zeta\cdot\bar z)+\log(1-\bar a\tau)\big]dm(\tau)}{(1-a\bar\tau)^{\alpha+n-1}}.$$

for some real constants $c_{n,q,\alpha}$ and c and therefore these kernels satisfy $h^q_\alpha(z,\zeta) = (-1)^q\overline{h^q_\alpha(\zeta, z)}$ and hence they correspond to self-adjoint operators. This completes the proof of Theorem 5.1. ■

Remark. For $q > 0$, h^q_α can be expressed in terms of Jacobi polynomials in the same way as k^q_α. In the expression above for the function h^0_α, the first term is independent of a (because of rotation invariance) and hence equals $\log(1-\zeta\cdot\bar z)$. The function

$$\frac{\alpha-1}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-2}\log(1-\bar a\tau)dm(\tau)}{(1-a\bar\tau)^{\alpha+n-1}}$$

is equal to $\Phi(|a|^2)$ where $\Phi(0) = 0$ and (by a simple computation)

$$\Phi'(|a|^2) = \frac{\alpha+n}{\pi}\int_{|\tau|<1}\frac{(1-|\tau|^2)^{\alpha-1}dm(\tau)}{(1-\bar\tau a)^{\alpha+n}(1-\tau\bar a)}.$$

The last term is a rational function in $|a|^2$ as before and Φ will involve a logarithm.

An alternative way to compute k^q_α is to first compute h^q_α and then use that $k^q_\alpha = \partial_\zeta h^q_\alpha$. In that computation it is worth to notice that

$$\partial m_{\alpha,j,k} = \frac{jk}{\alpha} m_{\alpha+1,j+1,k+1}\partial|a|^2.$$

■

We will now compare our homotopy operator with some other previously known ones. One method to construct weighted homotopy operators is described in Example 1 in [BeAn]. Let

$$S = (1-\bar\zeta\cdot z)\bar z - (1-|z|^2)\bar\zeta \tag{5.5}$$

or

$$S = -(1-\bar z\cdot\zeta)\bar\zeta + (1-|\zeta|^2)\bar z. \tag{5.6}$$

With either choice we have

$$S\cdot(z-\zeta)=|1-\zeta\cdot z|^2-(1-|\zeta|^2)(1-|z|^2)=\sigma.$$

Moreover, if $s=S\cdot d(\zeta-z)$ and

$$B_\alpha f(z)=\int_{\mathbb{B}}\sum_{k=0}^{n-1}c_{n,\alpha,k}$$

(5.7)
$$\frac{(1-|\zeta|^2)^{\alpha-1}i^{k+1}s\wedge(\bar\partial s)^k\wedge\Big((1-|\zeta|^2)\beta_{n-k-1}+\gamma\wedge\beta_{n-k-2}\Big)\wedge f}{(1-\zeta\cdot z)^{\alpha+n-k-1}(S\cdot(z-\zeta))^{k+1}},$$

with

$$c_{n,\alpha,k}=\frac{1}{(2\pi)^n}\frac{\Gamma(\alpha+n-k-1)}{\Gamma(\alpha)},$$

we have that

$$\bar\partial B_\alpha+B_\alpha\bar\partial=I-P_\alpha$$

where P_α is the Euclidean orthogonal projection with respect to α. In particular, with either choice of S, $B_\alpha f$ is the minimal solution to $\bar\partial u=f$ in L^2_α whenever f is a $(0,1)$-closed form. By the choice (5.6) of S one obtains the kernel first found by Charpentier in [Ch]. It differs from our canonical one since they even do not coincide when $z\in\partial\mathbb{B}$.

Now let us consider the choice (5.5). The kernel in (5.7) is homogeneous in S and therefore if z is on the boundary, then S is equivalent to $\bar z$. Therefore, if f is a $(0,q+1)$-form only the term for $k=q$ occurs in (5.7) when $z\in\partial\mathbb{B}$, and thus, cf (0.2), the boundary values of $B_\alpha f$ is the boundary values of the canonical operator. However we claim that they do not coincide in the interior. For simplicity we restrict to the case $q=0$. We will compare the behaviour of the kernels $B_\alpha(\zeta,z)$ and

$$K_\alpha=c_\alpha k^0_\alpha\wedge(1-|\zeta|^2)^{\alpha-1}((1-|\zeta|^2)\beta_{n-1}+\gamma\wedge\beta_{n-2})$$

when $|\zeta|\to1$. If we take $(1-|\zeta|^2)^{-(\alpha-1)}K_\alpha(\zeta,z)$ and let $|\zeta|\to1$ we get

(5.8)
$$c\frac{i\bar z\cdot d\zeta\wedge\gamma\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{\alpha+n-1}(1-\zeta\cdot\bar z)}.$$

Since $q=0$, only differentials of ζ occur in s and $\bar\partial s$ and so

$$s=(1-\bar\zeta\cdot z)\bar z\cdot d\zeta-(1-|z|^2)\bar\zeta\cdot d\zeta\quad\text{and}\quad i\bar\partial s=i\bar z\cdot d\zeta\wedge z\cdot d\bar\zeta+(1-|z|^2)\beta.$$

If we further note that $S\cdot(z-\zeta)=|1-\bar\zeta\cdot z|^2$ and let $|\zeta|\to1$ in $(1-|\zeta|^2)^{-(\alpha-1)}B_\alpha(\zeta,z)$ we get

(5.9)
$$\sum_{k=0}^{n-1}c_{n,\alpha,k}\frac{i\bar z\cdot d\zeta\wedge(1-|z|^2)^k\gamma\wedge\beta_{n-2}}{(1-\bar\zeta\cdot z)^{\alpha+n+k-2}|1-\zeta\cdot z|^{2(k+1)}}.$$

Since we already know that they coincide when $|z|=1$, the constant c in (5.8) must equal $c_{n,\alpha,0}$. However, then it is obvious that (5.8) and (5.9) are not equal for $|z|<1$.

6. The $\bar\Box_\alpha$ operator.

In this section we briefly discuss the operator $\bar\Box_\alpha = \bar\partial_\alpha^*\bar\partial + \bar\partial\bar\partial_\alpha^*$ and relate it to the Euclidean and Bergman boxes $\bar\Box_E = \vartheta\bar\partial + \bar\partial\vartheta$ and $\bar\Box_B = \bar\partial_\omega^*\bar\partial + \bar\partial\bar\partial_\omega^*$. It is worth to emphasize again that the Euclidean metric used here, $i\partial\bar\partial|\zeta|^2$, differs from the usual one with a factor 1/2.

By direct application of Theorem 2.1 and the formula $\bar\partial_\omega^* = (1-|\zeta|^2)(\vartheta + \mathcal{L}\bar{\mathcal{L}}\partial)$ (cf. the proof of Theorem 2.1) we get that

$$\bar\Box_\alpha = \bar\Box_E + \mathcal{L}\bar{\mathcal{L}}\partial\bar\partial + \bar\partial\mathcal{L}\bar{\mathcal{L}}\partial + (\alpha+n-q)(\mathcal{L}\bar\partial + \bar\partial\mathcal{L}) - \mathcal{L}\bar\partial$$

on $(0,q)$-forms and

$$\bar\Box_B = (1-|\zeta|^2)\left(\bar\Box_E + \mathcal{L}\bar{\mathcal{L}}\partial\bar\partial + \bar\partial\mathcal{L}\bar{\mathcal{L}}\partial\right) - T(\vartheta + \mathcal{L}\bar{\mathcal{L}}\partial).$$

Modulo the factor $1-|\zeta|^2$ hence $\bar\Box_B$ and $\bar\Box_\alpha$ have the same principal part. In particular, on functions we have

$$\bar\Box_\alpha = \bar\Box_E + \mathcal{L}\bar{\mathcal{L}}\partial\bar\partial + (\alpha+n-q)\mathcal{L}\bar\partial \quad \text{and} \quad \bar\Box_B = (1-|\zeta|^2)\left(\bar\Box_E + \mathcal{L}\bar{\mathcal{L}}\partial\bar\partial\right).$$

As for any Kähler metric, the Euclidean $\bar\Box_E$ is equal to one half of the Euclidean Laplace Δ and $\bar\Box_B$ is one half of the invariant (Bergman) Laplacian Δ_B. Thus the homogenous formal solutions to $\bar\Box_E$ and $\bar\Box_B$ are the harmonic and the Bergman harmonic functions respectively. Anyway, the true solutions are holomorphic functions, in virtue of the extra boundary condition imposed on the domain of these operators. For the operator $\bar\Box_\alpha$ we know that all homogenous solutions that are smooth up to the boundary are holomorphic functions. In this case this depends on the extra first order term $(\alpha+n-q)\mathcal{L}\bar\partial$.

7. Homotopy formulas for $\partial\bar\partial$.

In [An 1,2] were found operators M_α acting on d-closed (1,1)-forms such that

$$M_\alpha\partial\bar\partial u = u - \Pi_\alpha u,$$

where Π_α is the orthogonal projection in L^2_α onto the pluriharmonic functions. Explicitly,

$$\Pi_\alpha = P_\alpha + \bar P_\alpha - \Pi_\alpha^0, \tag{7.1}$$

where, as before, P_α is the holomorphic projection and $\Pi_\alpha^0 = P_\alpha\bar P_\alpha$ is the (orthogonal) projection onto the constants. In particular, $M_\alpha\theta$ provides the L^2_α minimal solution to $\partial\bar\partial u = \theta$ if $d\theta = 0$. Sharp estimates of the kernels were given.

By abstract nonsense it follows that there is an operator R_α, acting on 3-forms such that

$$\partial\bar\partial M_\alpha\theta = \theta - R_\alpha d\theta,$$

for any (reasonable) $(1,1)$-form θ. We will now construct (semi-explicit) operators M_α and R_α with these properties and moreover we will extend to the case of (q,q)-forms. It should be emphasized that the operator M_α constructed below, and acting on $(1,1)$-forms (probably) *is not* the same as the operator in [An 1,2], but it anyway gives the minimal solutions, and admits the same estimates.

So far, the operator K_α is just defined on $(0,q)$-forms. We first extend the definition to an operator

$$K_\alpha : \mathcal{E}_{p,*} \to \mathcal{E}_{p,*-1}$$

by the formula

$$K_\alpha(a_{IJ}(\zeta)d\bar\zeta^J \wedge d\zeta^I)(z) = K_\alpha(a_{IJ}(\zeta)d\bar\zeta^J) \wedge dz^I.$$

Also the operator P_α is extended in the same way. By the apparent formula $\bar K_\alpha f = \overline{K_\alpha \bar f}$ we also get an operator $\bar K_\alpha : \mathcal{E}_{*,q} \to \mathcal{E}_{*-1,q}$. It is clear that the formula $\bar\partial K_\alpha + K_\alpha\bar\partial = I - P_\alpha$ still holds. The main observation now is

Proposition 7.1. *Let K_α be the canonical operator with respect to α, and P_α the corresponding orthogonal holomorphic projection. Then*

$$\partial K_\alpha = -K_{\alpha+1}\partial, \quad \textit{and} \quad \partial P_\alpha = P_{\alpha+1}\partial.$$

Proof. By the definition and Corollary 2.3 we have

$$\partial K_\alpha(a(\zeta)d\bar\zeta^J \wedge d\zeta^I) = \partial K_\alpha(a(\zeta)d\bar\zeta^J) \wedge dz^I =$$

$$= \sum(-1)^{|J|-1}\frac{\partial}{\partial z_k}K_\alpha(a d\bar\zeta^J) \wedge dz_k \wedge dz^I =$$

$$= \sum(-1)^{|J|-1}K_{\alpha+1}\big(\frac{\partial a}{\partial \zeta_k}d\bar\zeta^J\big) \wedge dz_k \wedge dz^I =$$

$$= K_{\alpha+1}\big(\sum(-1)^{|J|-1}\frac{\partial a}{\partial\zeta_k}d\bar\zeta^J \wedge d\zeta_k \wedge d\zeta^I\big) = -K(\partial a(\zeta) \wedge d\bar\zeta^J \wedge d\zeta^I).$$

The statement about P_α now follows:

$$\partial P_\alpha u = \partial(u - K_\alpha\bar\partial u) = \partial u - K_{\alpha+1}\bar\partial\partial u = P_{\alpha+1}\partial u.$$

■

For (q,q)-forms we define

$$M_\alpha = \frac{i}{2}(\bar K_\alpha K_{\alpha+1} - K_\alpha \bar K_{\alpha+1})$$

and

$$D_\alpha = \frac{1}{2}(\partial\bar K_\alpha\bar\partial K_\alpha + \bar\partial K_\alpha\partial\bar K_\alpha)$$

if $q \geq 1$. For $q=0$ we let $D_\alpha = P_\alpha\bar P_\alpha$. Finally,

$$\Pi_\alpha = \bar\partial K_\alpha + \bar\partial K_\alpha - D_\alpha.$$

Certainly all these operators map real forms onto real ones. Moreover we have

Theorem 7.2. *a) The operator $D_\alpha : \mathcal{E}_{q,q} \to \mathcal{E}_{q,q}$ is a projection onto the d-closed forms.*

b) The operator $\Pi_\alpha : \mathcal{E}_{q,q} \to \mathcal{E}_{q,q}$ is a projection onto the $\partial\bar\partial$-closed forms.

c) The operator M_α is a homotopy operator for $i\partial\bar\partial$ in the following sense,

$$M_\alpha(i\partial\bar\partial u) = u - \Pi_\alpha u \tag{7.2}$$

and

$$i\partial\bar\partial M_\alpha\theta = D_{\alpha+1}\theta = \theta - R_{\alpha+1}d\theta, \tag{7.3}$$

where

$$R_{\alpha+1}d = \bar K_{\alpha+1}\partial + K_{\alpha+1}\bar\partial - \frac{1}{2}\big(\bar K_{\alpha+1}\partial K_{\alpha+1}\bar\partial + K_{\alpha+1}\bar\partial\bar K_{\alpha+1}\partial\big).$$

In particular, $M_\alpha\theta$ is a solution to $i\partial\bar\partial u = \theta$ if $d\theta = 0$. Actually any operator $\bar K_\alpha K_\beta$ solves the $\partial\bar\partial$-equation. This is obvious from Proposition 7.1, since if $d\theta = 0$ then $K_\beta\theta$ is a ∂-closed solution to $\bar\partial f = \theta$ and thus $\bar K_\alpha f$ solves $\partial u = f$ which means that $\partial\bar\partial u = \theta$.

Recall that $\bar\partial K_\alpha$ is a projection of (q,q)-forms onto the $\bar\partial$-closed ones, and analogously for its conjugate, and therefore b) says that the projection Π of (q,q)-forms onto $\operatorname{Ker}\partial\bar\partial$ is the sum of one projections onto $\operatorname{Ker}\bar\partial$ and one onto $\operatorname{Ker}\partial$ minus one projection onto $\operatorname{Ker} d$. When $q = 0$ this is just the formula (7.1) above.

Proof. It is enough to prove part c) since a) and b) are immediate consequences of (7.2) and (7.3). To see (7.2), note that by Proposition 7.1,

$$\begin{aligned} &-\partial\bar\partial\bar K_\alpha K_{\alpha+1} = \partial\bar K_{\alpha+1}\bar\partial K_{\alpha+1} = (I - \bar K_{\alpha+1}\partial)(I - K_{\alpha+1}\bar\partial) = \\ &= I - \bar K_{\alpha+1}\partial - K_{\alpha+1}\bar\partial + \bar K_{\alpha+1}\partial K_{\alpha+1}\bar\partial \end{aligned}$$

if $q \geq 2$. From this, (7.2) follows. The case $q = 1$ is completely analogous, just replace the projections $\bar\partial K_\alpha$ by P_α and $\partial\bar K_\alpha$ by $\bar P_\alpha$. To see (7.3), again by Proposition 7.1 we have

$$\begin{aligned} &\bar K_\alpha K_{\alpha+1}\partial\bar\partial = -\bar K_\alpha\partial K_\alpha\bar\partial = (I - \partial\bar K_\alpha)(I - \bar\partial K_\alpha) = \\ &= I - \big(\partial\bar K_\alpha + \bar\partial K_\alpha + \partial\bar K_\alpha\bar\partial K_\alpha\big), \end{aligned}$$

from which (7.3) follows. ■

Instead of trying to give a complete description of all the operators involved, we concentrate on M_α. It is possible to give a semi-explicit expression as some real-analytic function of the quantities $1 - |\zeta|^2$, $1 - \bar\zeta\cdot z$ and some simple forms by going up and down in the dimensions, cf. Section 5 in [AnCa], but we restrict our ambition to indicate that it admits some expected L^1-estimates.

For $(0,q)$-forms let $\|f\|$ denote the pointwise norm

$$\|f\| = \sqrt{1-|\zeta|^2}|f| + |\bar\partial|\zeta|^2 \wedge f|,$$

and for (p, q)-forms we set

$$|\!|\!|\theta|\!|\!| = (1-|\zeta|^2)|\theta| + \sqrt{1-|\zeta|^2}\big(|\bar\partial|\zeta|^2\wedge\theta| + |\partial|\zeta|^2\wedge\theta|\big) + |\partial|\zeta|^2\wedge\bar\partial|\zeta|^2\wedge\theta|.$$

A straight forward estimation of expressions of $Kf = K_\alpha f$ yields that it admits the following classical estimates of Henkin and Skoda,

$$\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1}\|Kf\| \leq C\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1/2}\|f\|, \tag{7.4}$$

for any $\ell > 0$ and

$$\int_{\partial\mathbb{B}}\|Kf\| \leq C\int_{\mathbb{B}}(1-|\zeta|^2)^{-1/2}\|f\|, \tag{7.5}$$

if α is sufficiently large (depending on ℓ). This means that, roughly speaking, measured in our norms K regularizes one half unit. In fact, (7.4) follows immediately from (7.5), at least for integer values of ℓ by going up in the dimension, and (7.5) is immediate from standard estimates.

For the operators M_α we have the following expected result.

Proposition 7.3. *For (q, q)-forms θ and any $\ell > 0$ we have the estimates*

$$\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1}|\!|\!|M\theta|\!|\!| \leq C\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell}|\!|\!|\theta|\!|\!|, \tag{7.6}$$

and

$$\int_{\partial\mathbb{B}}|\!|\!|M\theta|\!|\!| \leq C\int_{\mathbb{B}}|\!|\!|\theta|\!|\!|. \tag{7.7}$$

if $M = M_\alpha$ and α is large enough (depending on ℓ).

For $q = 1$ this gives the Henkin-Skoda estimate of solutions to the $\partial\bar\partial$-equation, and therefore the statement may be thought of as a generalized version. It can certainly be proved by the usual method as well, but our purpose is to show that actually our operator M works.

Proof. Extend the norm $\|f\|$ on $(0, q)$-forms f to (p, q)-forms in the naive way, by setting $\|f_I dz^I\| = \|f_I\|$. Then one can readily verify that

$$|\!|\!|\theta|\!|\!| \sim \sqrt{1-|\zeta|^2}\|\theta\| + \|\partial|\zeta|^2\wedge\theta\|. \tag{7.8}$$

This follows from the observation that

$$\|\theta\| \sim \sqrt{1-|\zeta|^2}|\theta| + |\bar\partial|\zeta|^2\wedge\theta|.$$

Notice that the kernel for the commutator

$$f \mapsto Lf = \partial|z|^2\wedge Kf - K(\partial|\zeta|^2\wedge f)$$

is $\mathcal{O}(|\zeta-z|)$ times the kernel for K. Therefore, L will regularize one half unit better than K, and hence by (7.4) we have

$$(7.9)\quad \int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1}\|\partial|\zeta|^2\wedge Kf\| \lesssim \int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1/2}\|\partial|\zeta|^2\wedge f\| + \int_{\mathbb{B}}(1-|\zeta|^2)^{\ell}\|f\|.$$

Keep in mind that the norm $\|\ \|$ is non-isotropic just in the barred part of f here. If we now apply (7.4) and (7.9) to $f=\bar K\theta$, we get

$$\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1}\|K\bar K\theta\| \sim \int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1}\big(\sqrt{1-|\zeta|^2}\|K\bar K\theta\| + \|\partial|\zeta|^2\wedge K\bar K\theta\|\big) \lesssim$$
$$\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1/2}\big(\sqrt{1-|\zeta|^2}\|\bar K\theta\| + \|\partial\wedge\bar K\theta\| + \sqrt{1-|\zeta|^2}\|\bar K\theta\|\big).$$

Written in simple norms the right hand side is

$$\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1/2}\big((1-|\zeta|^2)|\bar K\theta| + \sqrt{1-|\zeta|^2}\big|\bar\partial|\zeta|^2\wedge\bar K\theta\big| +$$
$$+\sqrt{1-|\zeta|^2}\big|\partial|\zeta|^2\wedge\bar K\theta\big| + \big|\partial|\zeta|^2\wedge\bar\partial|\zeta|^2\wedge\bar K\theta\big|\big).$$

If we now let $\|\ \|$ denote the non-isotropic norm of the unbarred part instead this integral can be written

$$\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1/2}\big(\sqrt{1-|\zeta|^2}\|\bar K\theta\| + \|\bar\partial|\zeta|\wedge\bar K\theta\|\big),$$

and then, using the conjugates of (7.4) and (7.9), we finally get the estimate

$$\int_{\mathbb{B}}(1-|\zeta|^2)^{\ell-1}\|K\bar K\theta\| \lesssim \int_{\mathbb{B}}(1-|\zeta|^2)^{\ell}\big(\sqrt{1-|\zeta|^2}\|\theta\| + \|\bar\partial|\zeta|^2\wedge\theta\|\big),$$

which in view of (7.8) is the desired estimate (7.6). In the same way (7.7) is obtained. ■

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