Cartan-Eilenberg Categories and Descent Categories

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Part I. Cartan-Eilenberg categories.

- F.Guillén, V. Navarro, P.P., A. Roig: *A Cartan-Eilenberg approach to homotopical algebra*. Journal of Pure and Appl. Algebra (2009), *in press*. (ArXiv 0707.3704)
- GNPR, *Moduli spaces and formal operads*. Duke Math. J. **129** (2005), 292-335.
- GNPR, *Monoidal functors, acyclic models and chain operads*. Canadian J. Math. **60** (2008), 348-378.

Part II. Simplicial descent categories. By Beatriz Rodríguez.

Contents

1. Introduction

- R: commutative ring
- $\mathbf{C}_+(R)$: positive chain complexes of R-modules
- $\sim$: homotopy equivalence relation between morphisms of complexes
- $[X, Y]$: homotopy classes of morphisms between complexes
- $\mathbf{C}_+(Proj(R)) \subset \mathbf{C}_+(R)$: subcategory of complexes which are projective in each degree
- $\mathcal{W}$: quasi-isomorphisms

Theorem (1)

$$\left.\begin{array}{c} w : X \longrightarrow Y \in \mathcal{W} \\ P \in \mathcal{C}_{proj} \end{array}\right\} \Longrightarrow w_* : [P, X] \xrightarrow{\cong} [P, Y]$$

That is, given f as in the diagram, there exists g, unique up to homotopy, with $wg \sim f$.

$$\begin{tikzcd} & X \arrow[d, "w"] \\ P \arrow[ur, dotted, "g"] \arrow[r, "f"] & Y \end{tikzcd}$$

Theorem (2)

For any chain complex X there is a projective resolution, that is,

$$\exists \varepsilon : P_X \longrightarrow X \in \mathcal{W}, \textit{such that } \ P_X \in Ob\ \mathbf{C}_+(Proj(R)).$$

Take $\mathbf{K}_+(R) = \mathbf{C}_+(R)/\sim$, $\mathbf{D}_+(R) = \mathbf{C}_+(R)[\mathcal{W}^{-1}]$, etc.

Corollary

The natural functor $\mathbf{K}_+(Proj(R)) \longrightarrow \mathbf{D}_+(R)$, *induces an equivalence of categories*

$$\mathbf{K}_+(Proj(R)) \cong \mathbf{D}_+(R).$$

Moreover, the composition of the inverse of this equivalence with the inclusion $\mathbf{K}_+(Proj(R)) \longrightarrow \mathbf{K}_+(R)$,

$$\lambda : \mathbf{D}_+(R) \longrightarrow \mathbf{K}_+(R)$$

is a left adjoint functor of the localization functor $\gamma : \mathbf{K}_+(R) \longrightarrow \mathbf{D}_+(R)$.

Corollary

Given an additive functor $F : \mathbf{C}_+(R) \longrightarrow \mathcal{D}$, there exists its left derived functor

$$\begin{array}{rcl} \mathbb{L}F : \mathbf{D}_+(R) & \longrightarrow & \mathcal{D} \\ X & \mapsto & F(\lambda(X)) \end{array}$$

Main features of our approach:

- The initial data are two classes of morphisms $\mathcal{S} \subseteq \mathcal{W}$ of $\mathcal{C}$.
- From $\mathcal{S}, \mathcal{W}$ we define the cofibrant objects, which are "homotopy invariant".
- We work out a relative version which allows to include minimal models in our picture.
- We give an interpretation of derived functors in terms of the cofibrant model of the functor.
- We include cotriple cohomology in our setting.

1. Remarks on localization of categories.

Data: a category $\mathcal{C}$ and a class of morphisms $\mathcal{W}$.
The *localization of $\mathcal{C}$ with respect to $\mathcal{W}$* is

$$\gamma : \mathcal{C} \longrightarrow \mathcal{C}\left[\mathcal{W}^{-1}\right]$$

such that:

(i) For all $w \in \mathcal{W}$, $\gamma(w)$ is an isomorphism.

(ii) For any category $\mathcal{D}$ and any functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ that transforms morphisms $w \in \mathcal{W}$ into isomorphisms, there exists a unique functor $F' : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}$ such that $F' \circ \gamma = F$.

Example

Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a functor and take

$$\mathcal{W} = \{f \mid F(f) \text{is an isomorphism}\}.$$

These classes are saturated and satisfy the $3 out of 2$ property.

For any class $\mathcal{W}$, the *saturation* of $\mathcal{W}$ is the class of morphisms:

$$\overline{\mathcal{W}} = \gamma^{-1}(Iso_{\mathcal{C}[\mathcal{W}^{-1}]}).$$

A $\mathcal{W}$*-zigzag* f from X to Y is a finite sequence of morphisms of $\mathcal{C}$, going in either direction, between X and Y,

$$\begin{tikzcd} f : X \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \cdots \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & Y , \end{tikzcd}$$

where the morphisms going from right to left are in $\mathcal{W}$.
A morphism from a $\mathcal{W}$-zigzag f to a $\mathcal{W}$-zigzag g of the same type is a commutative diagram in $\mathcal{C}$,

$$\begin{tikzcd}
 & \bullet \arrow[r, dash] \arrow[dd] & \bullet \arrow[r, dash] \arrow[dd] & \cdots f \cdots \arrow[r, dash] & \bullet \arrow[dd] \arrow[dr, dash] & \\
X \arrow[ur, dash] \arrow[dr, dash] & & & & & Y . \\
 & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \cdots g \cdots \arrow[r, dash] & \bullet \arrow[ur, dash] &
\end{tikzcd}$$

A *hammock* is a commutative diagram H in $\mathcal{C}$

$$
\begin{tikzcd}
 & X_{11} \arrow[r, dash] \arrow[d, dash] & X_{12} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & X_{1p} \arrow[d, dash] \arrow[ddr, dash] & \\
 & X_{21} \arrow[r, dash] \arrow[d, dash] & X_{22} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & X_{2p} \arrow[d, dash] \arrow[dr, dash] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] \arrow[ddr, dash] & \vdots \arrow[d, dash] & \vdots \arrow[d, dash] & & \vdots \arrow[d, dash] & Y \\
 & X_{n-1,1} \arrow[r, dash] \arrow[d, dash] & X_{n-1,2} \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] & X_{n-1,p} \arrow[d, dash] \arrow[ur, dash] & \\
 & X_{n1} \arrow[r, dash] & X_{n2} \arrow[r, dash] & \cdots \arrow[r, dash] & X_{np} \arrow[uur, dash] &
\end{tikzcd}
$$

(i) in each column of arrows, all (horizontal) maps go in the same direction, and any row is a $\mathcal{W}$-zigzag,

(ii) in each row of arrows, all (vertical) maps go in the same direction, and they are arbitrary maps in $\mathcal{C}$.

Define a category $\mathcal{C}_{\mathcal{W}}$ by:

- $Ob\ \mathcal{C}_{\mathcal{W}} = Ob\ \mathcal{C}$
- $Hom_{\mathcal{C}_{\mathcal{W}}}(X, Y) =$ equivalence classes of $\mathcal{W}$-zigzags from X to Y defined by hammocks

Theorem (Dwyer-Hirschhorn-Kan-Smith, 2004)

The category $\mathcal{C}_{\mathcal{W}}$, together with the obvious functor $\mathcal{C} \longrightarrow \mathcal{C}_{\mathcal{W}}$, is a solution to the universal problem of the category of fractions $\mathcal{C}\left[\mathcal{W}^{-1}\right]$, i.e.

$$\mathcal{C}_{\mathcal{W}} \cong \mathcal{C}\left[\mathcal{W}^{-1}\right].$$

Application to congruences

Let $\mathcal{C}$ be a category with a congruence $\sim$ and consider the functor

$$\pi : \mathcal{C} \longrightarrow \mathcal{C}/\sim$$

Say that a morphism $f : X \longrightarrow Y$ in $\mathcal{C}$ is a homotopy equivalence if there is a $g : Y \longrightarrow X$ such that

$$gf \sim 1_X, \; fg \sim 1_Y.$$

Let $\mathcal{S}$ be the class of homotopy equivalences and $\delta : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$.

Proposition

Suppose that $f \sim g$ implies $\delta f = \delta g$, then the categories $\mathcal{C}/\sim$ and $\mathcal{C}[\mathcal{S}^{-1}]$ are canonically isomorphic.

Definition

A *cylinder object over* X is an object $Cyl(X)$ in $\mathcal{C}$ together with morphisms $i_0, i_1 : X \longrightarrow \mathrm{Cyl}(X)$ and $p : Cyl(X) \longrightarrow X$ such that $p \in \mathcal{S}$ and $p \circ i_0 = \mathrm{id}_X = p \circ i_1$.

Corollary

If $\sim$ is defined by a cylinder object, then $\mathcal{C}/{\sim} = \mathcal{C}[\mathcal{S}^{-1}]$.

Proof:

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}"'] \arrow[d, "i_0"] \arrow[dr, "f_0"] & \\
X & Cyl(X) \arrow[l, "p"'] \arrow[r, "H"] & Y \\
 & X \arrow[ul, "\mathrm{id}"] \arrow[u, "i_1"'] \arrow[ur, "f_1"'] &
\end{tikzcd}$$

Examples

- $\mathcal{C} = \mathbf{Top}$: $Cyl(X) = X \times I$.
- $\mathcal{C} = \mathbf{C}_*(R)$: $Cyl(X) = X \oplus X[-1] \oplus X$ with differential

$$D = \begin{pmatrix} d & -id & 0 \\ 0 & -d & 0 \\ 0 & id & d \end{pmatrix}$$

- $\mathcal{C}$ a Quillen model category, a cylinder object is given by a factorization
$$X \sqcup X \rightarrowtail Cyl(X) \overset{\sim}{\twoheadrightarrow} X.$$

Relative localization

Definition

Let $(\mathcal{C}, \mathcal{E})$ be a category with weak equivalences and $\mathcal{M}$ a full subcategory. The *relative localization of the subcategory* $\mathcal{M}$ of $\mathcal{C}$ with respect to $\mathcal{E}$ is

$$\mathcal{M}[\mathcal{E}^{-1}, \mathcal{C}] = \text{full subcategory of } \mathcal{C}[\mathcal{E}^{-1}] \text{ generated by } \mathcal{M}.$$

Examples

1. Take $\mathcal{C} = \mathbf{Adgc(k)}$ and $\mathcal{S}$ the homotopy equivalences. If $\mathcal{M}$ is the class of minimal algebras,

$$\mathcal{M}[\mathcal{S}^{-1}] = \mathcal{M}, \quad \text{while} \quad \mathcal{M}[\mathcal{S}^{-1}, \mathcal{C}] = \mathcal{M}/\sim.$$

Examples

2. In some examples, $\mathcal{M}[\mathcal{E}^{-1}, \mathcal{C}] = \mathcal{M}[\mathcal{E}^{-1}]$:

   Suppose the following conditions on $\mathcal{E}$:

   (a) $\mathcal{E}$ has a right calculus of fractions,
   (b) for every morphism $w : X \longrightarrow M$ in $\overline{\mathcal{E}}$, with $M \in Ob\,\mathcal{M}$, there exists a morphism $N \longrightarrow X$ in $\overline{\mathcal{E}}$, where $N \in Ob\,\mathcal{M}$,

   then, $\bar{i} : \mathcal{M}[\mathcal{E}^{-1}] \cong \mathcal{M}[\mathcal{E}^{-1}, \mathcal{C}]$.

3. Cofibrant objects

Data: a category $\mathcal{C}$ with two classes of morphisms $\mathcal{S} \subseteq \mathcal{W}$

- $\mathcal{S}$ are the *strong (or global) equivalences*; we assume $\mathit{Iso}(\mathcal{C}) \subset \mathcal{S}$.
- $\mathcal{W}$ are the *weak (or local) equivalences*.

Examples

1. Given an abelian category $\mathcal{A}$, take

$$\begin{array}{rcl} \mathcal{C} & = & \mathbf{C}_*(\mathcal{A}), \\ \mathcal{W} & = & \text{quasi-isomorphisms}, \\ \mathcal{S} & = & \text{homotopy equivalences}. \end{array}$$

Examples

2. Topological spaces:

$$\begin{aligned} \mathcal{C} &= \textbf{Top}, \\ \mathcal{W} &= \text{weak homotopy equivalences}, \\ \mathcal{S} &= \text{homotopy equivalences}. \end{aligned}$$

3. Let X be a topological space,

$$\begin{aligned} \mathcal{C} &= \mathbf{Sh}(X, \mathbf{C}_+(\mathbb{Z})), \text{ sheaves of complexes of abelian groups.} \\ \mathcal{W} &= \{f : F \longrightarrow G, f_x : F_x \longrightarrow G_x \text{ qis}, \forall x \in X\}. \\ \mathcal{S} &= \{f : F \longrightarrow G, f(U) : F(U) \longrightarrow G(U), \text{qis}, \forall \text{ open} U \subseteq X\}. \end{aligned}$$

We fix the following notation for the localizing categories and functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\gamma"] \arrow[dr, "\delta"'] & & \mathcal{C}[\mathcal{W}^{-1}] \cong \mathcal{C}[\mathcal{S}^{-1}][\delta(\mathcal{W})^{-1}]\,. \\ & \mathcal{C}[\mathcal{S}^{-1}] \arrow[ur, "\gamma'"'] & \end{tikzcd}$$

Remark

For homotopy invariant notions we may restrict attention to γ'.

Definition

An object M of $\mathcal{C}$ is $(\mathcal{S}, \mathcal{W})$-*cofibrant* if for any diagram in $\mathcal{C}[\mathcal{S}^{-1}]$,

$$\begin{tikzcd} & \delta(X) \arrow[d, "\delta(w)"] \\ \delta(M) \arrow[ur, dotted, "g"] \arrow[r, "f"'] & \delta(Y) \end{tikzcd}$$

with $w \in \mathcal{W}$, there exists a *unique* morphism g of $\mathcal{C}[\mathcal{S}^{-1}]$ making the triangle commutative. That is,

$$\mathcal{C}[\mathcal{S}^{-1}](M, X) = \mathcal{C}[\mathcal{S}^{-1}](M, Y).$$

Remarks

1. Diagrams are in $\mathcal{C}[\mathcal{S}^{-1}]$.
2. M cofibrant $\Rightarrow \mathcal{C}[\mathcal{S}^{-1}](M, -)$ is an "exact functor".
3. Being cofibrant is "homotopy invariant".

Examples

1. Projective modules: $\mathbf{C}_+(\mathit{Proj}(R))$ are cofibrant in $\mathbf{C}_+(R)$. Remark that there are cofibrant objects not in $\mathbf{C}_+(\mathit{Proj}(R))$: any contractible complex is cofibrant.
2. $\mathcal{C} = \mathbf{Top}$ and $\mathcal{W}$ the weak homotopy equivalences. Then the CW complexes are cofibrant.

Proposition

M is $(\mathcal{S},\mathcal{W})$-*cofibrant* if and only if for any $X \in Ob\ \mathcal{C}$,

$$\mathcal{C}[\mathcal{S}^{-1}](M,X) = \mathcal{C}[\mathcal{W}^{-1}](M,X).$$

Corollary

Let $(\mathcal{C},\mathcal{S},\mathcal{W})$ be a category with strong and weak equivalences and $\mathcal{M}$ be a full subcategory of $\mathcal{C}_{cof}$. The functor

$$j : \mathcal{M}[\mathcal{S}^{-1},\mathcal{C}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$$

is fully faithful.

In particular, the functor j reflects isomorphisms, that is, if $X, Y \in Ob\ \mathcal{M}$, then

$$f \in \mathcal{W} \Longrightarrow f \in \mathcal{S}.$$

For $\mathcal{C} = \mathbf{Top}$, the example of topological spaces with homotopy equivalences and weak homotopy equivalences, and $\mathcal{M} = \mathbf{CW}$ this is the classical Whitehead's theorem.

3. Cartan-Eilenberg categories

Definition

$(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a *left Cartan-Eilenberg* category if there are sufficiently many cofibrant objects:

$$\forall X \in Ob\,\mathcal{C},\ \exists M \in Ob\,\mathcal{C}_{cof},\ f : \delta(M) \longrightarrow \delta(X) \in \overline{\delta(\mathcal{W})}.$$

Theorem

$(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left CE category iff j induces an equivalence of categories

$$j : \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \cong \mathcal{C}[\mathcal{W}^{-1}]$$

Examples

1. Any catgeory $\mathcal{C}$ with classes $\mathcal{S} = \mathcal{W}$ is a left CE category.
2. If $\mathcal{A}$ is an abelian category with sufficiently many projectives and $\mathcal{W} = qis$, there is an equivalence of categories

$$K_+(Proj(\mathcal{A})) \cong D_+(\mathcal{A}).$$

3. Every topological space is weakly equivalent to a CW-complex, so **Top** is a left CE.

Examples

4. More generally, if $\mathcal{C}$ is a Quillen model category and $\mathcal{W}$ is the class of weak equivalences, there is an equivalence of categories
$$\pi\mathcal{C}_{\mathsf{cf}} \cong \mathcal{C}_{\mathsf{f}}[\mathcal{W}^{-1}], \tag{1}$$
hence $\mathcal{C}_{\mathsf{f}}$ with the weak equivalences and the right/left homtopies is a left CE. E.g.
$\mathbf{C}_*(\mathcal{A}), \mathbf{Top}, \mathit{PreSh}(X, \mathbf{SSets}), \mathbf{Adgc}(\mathbf{k}), \ldots$
5. There are variations on the model axioms proposed by J. Baues, K. Brown, D. Cisinski, R. Thomason, etc. leading to an equivalence as (1) which also give left CE structures.
6. There is the dual notion of right CE ...

Corollary (Cofibrant model functor)

Let $(\mathcal{C},\mathcal{S},\mathcal{W})$ be a left CE category. There exist

$$r:\mathcal{C}[\mathcal{S}^{-1}]\longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1},\mathcal{C}],\quad \varepsilon: ir\Rightarrow 1$$

such that:

(1) $\varepsilon_X : ir(X)\longrightarrow X$ *is a cofibrant left model of* X.

(2) $r(\delta(\mathcal{W}))\subset Iso_{\mathcal{C}_{cof}[\mathcal{S}^{-1},\mathcal{C}]}$ *and induces an equivalence of categories*

$$\bar{r}:\mathcal{C}[\mathcal{W}^{-1}]\longrightarrow \mathcal{C}_{cof}[\mathcal{S}^{-1},\mathcal{C}]$$

quasi-inverse to j.

(3) M *cofibrant* $\Longrightarrow \varepsilon_M : r(M)\longrightarrow M$ *is an isomorphism in* $\mathcal{C}[\mathcal{S}^{-1}]$.

Consider the composition

$$\lambda : \mathcal{C}[\mathcal{W}^{-1}] \xrightarrow{\bar{r}} \mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \xrightarrow{i} \mathcal{C}[\mathcal{S}^{-1}].$$

One can prove that λ is left adjoint to γ'.
More precisely, we have

Proposition (Bousfield localization)

$(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left CE iff the localization $\gamma' : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ admits a left adjoint

$$\lambda : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}].$$

In such case, $\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}]$ is the essential image of λ.

For a left CE category we have a diagram

$$\begin{tikzcd}
& \mathcal{C}[\mathcal{S}^{-1}] \arrow[dl, shift left, "r"] \arrow[dr, "\gamma'"] & \\
\mathcal{C}_{cof}[\mathcal{S}^{-1}, \mathcal{C}] \arrow[ur, shift left, "i"] \arrow[rr, shift left, "j"] & & \mathcal{C}[\mathcal{W}^{-1}] \arrow[ul, dotted, "\lambda"] \arrow[ll, shift left, "\overline{r}"]
\end{tikzcd}$$

- i is the inclusion functor,
- $j, \overline{r}$ are inverse equivalences,
- $r = \overline{r}\gamma'$ is a coreflection, $i \dashv r$
- $\lambda = i\overline{r} : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{C}[\mathcal{S}^{-1}]$. It follows that $\lambda \dashv \gamma'$.

Theorem

Let $\mathcal{M}$ be a full subcategory of $\mathcal{C}$, $w : Y \longrightarrow X \in \mathcal{W}$ and $M \in Ob\,\mathcal{M}$. Suppose that

(i) *for any w, M, and any $f \in \mathcal{C}(M, X)$, there exists a morphism $g \in \mathcal{C}[\mathcal{S}^{-1}](M, Y)$ such that $w \circ g = f$ in $\mathcal{C}[\mathcal{S}^{-1}]$;*

(ii) *for any w and M, the map*

$$w_* : \mathcal{C}[\mathcal{S}^{-1}](M, Y) \longrightarrow \mathcal{C}[\mathcal{S}^{-1}](M, X)$$

is injective; and

(iii) *for each object X of $\mathcal{C}$ there exists a morphism $\varepsilon : M \longrightarrow X$ in $\mathcal{C}$ such that $\varepsilon \in \mathcal{W}$ and $M \in Ob\,\mathcal{M}$;*

Then,

(1) *every object in $\mathcal{M}$ is cofibrant;*

(2) *$(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category; and*

(3) *the functor $\mathcal{M}[\mathcal{S}^{-1}, \mathcal{C}] \longrightarrow \mathcal{C}\left[\mathcal{W}^{-1}\right]$ is an equivalence of categories.*

Example (Complexes of exact categories)

An exact category $\mathcal{A}$ is an additive category with a class $\mathcal{E}$ of exact sequences

$$0 \longrightarrow A' \rightarrowtail A \twoheadrightarrow A'' \longrightarrow 0$$

satisfying:

- E0 $\forall A \in \mathcal{A}, 1_A$ is an admissible mono and epi.
- E1 admissible monos and epis are closed under composition.
- E2

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d, dotted] \\ A' \arrow[r, tail, dotted] & B' \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} A' \arrow[r, two heads, dotted] \arrow[d, dotted] \arrow[dr, phantom, "PB"] & B' \arrow[d] \\ A \arrow[r, two heads] & B \end{tikzcd}$$

Examples

1. Any abelian category $\mathcal{A}$ with $\mathcal{E}$ the class of all exact sequences.
2. Let be $\mathcal{A}$ an abelian category and $F(\mathcal{A})$ the category of filtered objects. Take $\mathcal{E}$ the sequences which are exact at any stage of the filtration. . Then $(F(\mathcal{A}), \mathcal{E})$ is an exact category.

Remark

In an abelian category we can decompose a morphism

$$
\begin{tikzcd}
& A \arrow[rrr] \arrow[dr, two heads, "f"] & & & B \arrow[dr, two heads] & \\
\ker f \arrow[ur, tail] & & \operatorname{coim} f \arrow[r, "\overline{f}"] & \operatorname{im} f \arrow[ur, tail] & & \operatorname{coker} f
\end{tikzcd}
$$

with $\overline{f}$ an isomorphism. If it is possible to get such a decomposition in an exact category, $\overline{f}$ is not necessarily an isomorphism.

One can do some homological algebra on an exact category $(\mathcal{A}, \mathcal{E})$:

1. A chain complex A_* is *acyclic* if the differential factorizes as

$$\begin{tikzcd} A_{n+1} \arrow[rr, "d"] \arrow[dr, two heads] & & A_n \arrow[rr, "d"] \arrow[dr, two heads] & & A_{n-1} \\ & Z_n \arrow[ur, tail] & & Z_{n-1} \arrow[ur, tail] & \end{tikzcd}$$

   with $Z_n \rightarrowtail A_n \twoheadrightarrow Z_{n-1}$ an exact sequence (of $\mathcal{E}$).
2. A morphism $w : A \longrightarrow B$ in an exact category is a quasi-isomorphism if the cone $c(w)$ is an acyclic complex. If $\mathcal{A}$ is *idempotent complete*, then the contractible complexes are acyclic.
3. Projective objects are defined with respect to admissible epis.

Theorem

Let $(\mathcal{A}, \mathcal{E})$ be an idempotent complete exact category with enough projective objects. With the usual structure of homotopy equivalences and quasi-isomorphisms, $(\mathbf{C}_+(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a left CE category.

Example

- $\mathcal{A}$ an abelian category and $\mathbf{FC}_+(\mathcal{A})$ the category of positive filtered chain complexes (X, W) of $\mathcal{A}$ (we assume the filtration $W_p = 0$ for $p < 0$).
- $\mathcal{S}$ the class of filtered homotopies
- $\mathcal{W}$ the class of filtered quasi-ismomorphisms (that is, the w with $Gr_p^W w$ are quasi-isomorphism).

Then $(\mathbf{FC}_+(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a left CE category.

The filtered complexes (P, W) such that $Gr_p^W(P)$ is projective in each degree are a sufficient class of cofibrant objects.

Example (A non left CE category: Freyd's example, CN)

- I: class of all ordinals
- $R = \mathbb{Z}[I]$ polynomial ring "freely generated" by I
- $\mathcal{A}$ abelian category of *small* R-modules

Freyd (1966) observed that $Ext^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z}) = \mathbf{D}(\mathcal{A})(\mathbb{Z}, \mathbb{Z}[1])$ is a proper class, so the derived category $\mathbf{D}_+(\mathcal{A})$ is not locally small.

As a consequence $\mathbf{C}_+(\mathcal{A})$ with the homotopy equivalences and quasi-isomorphisms *is not a left (nor right) CE category*.

Example (Another non-example)

- $\mathbb{P}^1$: projective space over $\mathbb{C}$
- $QCoh(\mathbb{P}^1)$: abelian category of quasicoherent sheaves on $\mathbb{P}^1$

The category $\mathbf{C}_+(Coh(\mathbb{P}^1))$, with the classes of homotopy equivalences and quasi-isomorphisms, is not a left CE category.

Resolvent functors

Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences.

Definition

A *resolvent functor (or cofibrant replacement functor)* is a functor $R : \mathcal{C} \longrightarrow \mathcal{C}$ together with a natural transformation $\varepsilon R \Rightarrow 1_{\mathcal{C}}$, such that

(i) $R(X) \in \mathcal{C}_{cof}$.

(ii) $\varepsilon_X : R(X) \longrightarrow X$ is in $\overline{\mathcal{W}}$.

Remark

If R is a resolvent functor, then $\overline{\mathcal{W}} = R^{-1}(\overline{\mathcal{S}})$.

Proposition

Let (R, ε) be a left resolvent functor on $\mathcal{C}$. Then,

1. $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a left Cartan-Eilenberg category;
2. the canonical functor $\alpha : \mathcal{C}_{cof}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is an equivalence of categories; and
3. an object X of $\mathcal{C}$ is cofibrant if and only if $\varepsilon_X : RX \longrightarrow X$ is an isomorphism in $\mathcal{C}[\mathcal{S}^{-1}]$.

Proposition

Let $\mathcal{C}$ be a category, $\mathcal{S}$ a class of morphisms and $R : \mathcal{C} \longrightarrow \mathcal{C}$ a functor with an augmentation $\varepsilon : R \Rightarrow 1_{\mathcal{C}}$. Take $\mathcal{W} = R^{-1}(\overline{\mathcal{S}})$. Then,

$$\left.\begin{array}{l} R(\mathcal{S}) \subseteq \mathcal{S} \\ R\varepsilon_X, \varepsilon_{R(X)} \in \mathcal{S} \end{array}\right\} \Longrightarrow \begin{array}{l} R \text{ is cofibrant replacement} \\ \quad \text{for } (\mathcal{C}, \mathcal{S}, \mathcal{W}) \end{array}$$

Example

Free R-module generated by an R-module.

5. Minimal models: Sullivan categories

Definition

Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a category with strong and weak equivalences. We say that a *cofibrant* object M of $\mathcal{C}$ is *minimal* if if any weak equivalence $w : M \longrightarrow M$ of $\mathcal{C}$ is an isomorphism, that is,

$$End_{\mathcal{C}}(M) \cap \mathcal{W} = Aut_{\mathcal{C}}(M).$$

Examples

- In an abelian category $\mathcal{A}$ with sufficiently many projectives, a complex of projectives with zero differential is minimal.
- A discrete topological space is minimal in **Top**.

Remarks

- For an object M, being minimal is not a homotopy invariant property.
- The inclusion functor

$$\mathcal{C}_{min}[\mathcal{S}^{-1}] \longrightarrow \mathcal{C}_{min}[\mathcal{S}^{-1}, \mathcal{C}]$$

is not, generically, an equivalence of categories.

Definition

We say that $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ is a *left Sullivan category* if there are enough minimal left models.

Example (A trivial one)

Let $\mathbf{k}$ be a field. Then $\mathbf{C}_+(\mathbf{k})$, with homotopy equivalences and quasi-isomorphisms, is a Sullivan category.

Example (Sullivan's original example)

- $\mathbf{k}$ a field of characteristic zero
- $\mathbf{Adgc}(\mathbf{k})_1$ the category of connected and simply connected commutative differential graded $\mathbf{k}$-algebras
- $\mathcal{S}$ be the class of homotopy equivalences: defined after the path object $Path(A) := A \otimes \mathbf{k}[t, dt]$
- $\mathcal{W}$ quasi-isomorphisms
- $\mathcal{M}_S$ subcategory of Sullivan's minimal algebras

Theorem (Sullivan 1977, Griffiths-Morgan 1981)

$(\mathbf{Adgc}(\mathbf{k})_1, \mathcal{S}, \mathcal{W})$ *is a left Sullivan category and* $\mathcal{M}_S$ *is the subcategory of minimal objects of* $\mathbf{Adgc}(\mathbf{k})_1$*.*

Example (Filtered algebras, Cirici 2009)

- **k** a field of characteristic zero
- $\mathbf{FAdgc}(\mathbf{k})_1$ category of filtered *dgc* 1-connected algebras.
- filter the path object $Path(A) = A[t, dt]$ in such a way that t and dt have weight 0
- $\mathcal{S}$ be the class of homotopy equivalences: defined after the filtered path object $Path(A)$
- $\mathcal{W}$ quasi-isomorphisms
- $\mathcal{M}_S$ subcategory of Sullivan's minimal algebras

Theorem

$(\mathbf{FAdgc}(\mathbf{k})_1, \mathcal{S}, \mathcal{W})$ *is a left Sullivan category.*

Application to Hodge Theory

Let (A, W) be a filtered *dgc* $\mathbf{k}$-algebra.
Recall that the differential $d : A \longrightarrow A$ is strict with respect to W if

$$d(W_p A) = W_p A \cap d(A).$$

Lemma (Deligne)

d is strict with respect to W iff the spectral sequence $E_r^{pq}(W)$ degenerates in the E_1-term.

Theorem

Let A be a filtered dgc *algebra such that d is strict and Λ the filtered minimal model. Then Λ is a minimal model of A in* $\mathbf{Adgc}(\mathbf{k})_1$.

Corollary

Let X be a (simply connected) compact Kähler manifold, $A(X)$ be the algebra of C^∞-differential forms over X, then the Hodge filtration of $A(X)$ passes to the minimal model of $A(X)$.

Corollary (Morgan 1978)

Let X be a compact algebraic variety, $D \subseteq X$ a divisor with normal crossings and $U = X \setminus D$. Let $A_X(logD)$ be the algebra of differential forms with logarithmic singularities along D. The weight filtration of $A_X(logD)$ induces a filtration on its minimal model.

Example (Differential graded operads)

Definition

Let $\mathcal{C}$ be a symmetric monoidal category.
An *operad* of $\mathcal{C}$ is a sequence of objects $\{P(n)\}_{n\geq 1}$ together with the following data:

1. a unit morphism $\eta : \mathbf{1} \longrightarrow P(1)$,
2. an action of the symmetric group Σ_n on $P(n)$, $n \geq 1$,
3. product morphisms

   $$\gamma_{\ell;m_1,\ldots,m_\ell} : P(l) \otimes P(m_1) \otimes \cdots \otimes P(M_\ell) \longrightarrow P(m),$$

   for all $\ell, m_i \geq 1$, where $m = m_1 + \cdots + m_\ell$, satisfying certain compatibility conditions.

Examples

1. Dg operads

   Let $\mathbf{k}$ be a field and consider $\mathbf{C}_+(\mathbf{k})$ as a symmetric monoidal category with the tensor product $\otimes_{\mathbf{k}}$. An operad of $\mathbf{C}_+(\mathbf{k})$ is called a *dg operad*; denote $\mathbf{Op}(\mathbf{k})$ the category of dg operads. Some examples:

   - Com operad : $Com(n) = \mathbf{k}$, $n \geq 1$
   - Ass operad : $Ass(n) = \mathbf{k}[\Sigma_n]$, $n \geq 1$
   - Endomorphism operad: given $V \in Ob\ \mathbf{C}_+(\mathbf{k})$, the operad of endomorphisms of V, End_V, is

   $$End_V(n) = \underline{Hom}_{\mathbf{C}_+(\mathbf{k})}(V^{\otimes n}, V),\ n \geq 1$$

Examples

2. Moduli space of curves of genus 0 with ℓ labelled points, $\overline{\mathcal{M}}_0$

$$\mathcal{M}_{0,\ell} \quad = \quad \text{moduli space of } \ell \text{ labeled points on } \mathbb{P}^1 \text{ },\ell \geq 3,$$

(1)
$\overline{\mathcal{M}}_{0,\ell} =$ Grothendieck-Knudsen compactification
of $\mathcal{M}_{0,\ell}$

Take

$$\overline{\mathcal{M}}_0(1) = *, \quad \overline{\mathcal{M}}_0(\ell) = \overline{\mathcal{M}}_{0,\ell+1}$$

Geometric operations on $\overline{\mathcal{M}}_0$:

- Action of symmetric groups: permutation of labeled points
- Composition: There are algebraic maps

$$\circ_i : \overline{\mathcal{M}}_{0,\ell} \times \overline{\mathcal{M}}_{0,m} \longrightarrow \overline{\mathcal{M}}_{0,\ell+m-2},\ 0 \leq i \leq \ell,$$

given by

$$(C; x_0, \ldots, x_\ell) \circ_i (D; y_0, \ldots, y_m) =$$
$$((C \sqcup D)/x_i \sim y_0; x_0, \ldots, x_{i-1}, y_1, \ldots, y_m, x_{i+1}, \ldots, x_\ell)$$

- $\mathbf{k}$ field of characteristic zero
- take the path object $Path(P) = P \otimes \mathbf{k}[t, dt]$ and the induced homotopy relation of operads
- A quasi-isomorphism is a morphism of operads $w : P \longrightarrow Q$ such that $w(n); P(n) \longrightarrow Q(n)$ is is quasi-isomorphism, $n \geq 1$.
- it is possible adapt Sullivan's defintion of minimal algebra to $\mathbf{Op}(\mathbf{k})$, so there is a class of minimal operads $\mathcal{M}$.

Theorem (Markl 1996, Guilllén-Navarro-Pascual-Roig 2005)

Let $\mathbf{Op}_1(\mathbf{k})$ *be the full subcategory of operads* P *such that* $H_*(P(1)) = 0$*. Then* $(\mathbf{Op}_1(\mathbf{k}), \mathcal{S}, \mathcal{W})$ *is a left Sullivan category and* $\mathcal{M}$ *is the subcategory of minimal objects.*

We associate to $\overline{\mathcal{M}}_0$ two dg operads:

- operad of rational singular chains, $S_*(\overline{\mathcal{M}}_0; \mathbb{Q})$
- operad of rational homology $H_*(\overline{\mathcal{M}}_0; \mathbb{Q})$

Definition

A dg operad P is *formal* if P and HP are weakly equivalent.

Theorem (Guillén-Navarro-Pascual-Roig 2005)

$S_*(\overline{\mathcal{M}}; \mathbb{Q})$ *is a formal dg operad.*

Example (Deformation functors)

- $\mathcal{C} = \mathbf{Art}(\mathbb{C})$ category of local artinian $\mathbb{C}$-algebras, with residue field $\mathbb{C}$
- $\hat{\mathcal{C}}$ category of complete local noetherian $\mathbb{C}$-algebras, with residue field $\mathbb{C}$
- $\mathbf{Cat}(\mathcal{C}, \mathbf{Sets})$ covariant functors $F : \mathcal{C} \longrightarrow \mathbf{Sets}$ with $F(\mathbb{C}) = \{*\}$
- there is a natural functor

$$\hat{\mathcal{C}} \longrightarrow \mathbf{Cat}(\mathcal{C}, \mathbf{Sets})$$

  Its image is the subcategory of *prorepresentable* functors.
- tangent space $t_F = F(\mathbb{C}[\varepsilon]), \varepsilon^2 = 0$

Definition

A morphism $u : F \longrightarrow G$ is

- unramified if t_u is injective,
- smooth if for any surjection $A \longrightarrow B$ in $\mathcal{C}$, the map

$$\eta : F(A) \longrightarrow G(A) \times_{G(B)} F(B)$$

  is surjective
- étale if it is unramified and smooth ($\Rightarrow t_u$ bijective)

Given a functor F and morphisms of $\mathcal{C}$ $A' \longrightarrow A \longleftarrow A''$, we consider

$$\beta : F(A' \times_A A'') \longrightarrow F(A') \times_{F(A)} F(A'').$$

Define the following properties:

- H1 β is surjective for any simple surjection $A'' \longrightarrow A$,
- H2 β is bijective for $A = \mathbb{C}$, $A'' = \mathbb{C}[\varepsilon]$,
- H3 $\dim_{\mathbb{C}} t_F < \infty$.

Definition

Say that F has a *hull* if there is an object $C \in Ob\ \hat{C}$ and an étale morphism $h_C \longrightarrow F$.

Theorem (Schlesinger 1968)

Any deformation functor satisfying properties $H1 - H3$ has a hull.

Fact: an étale morphism $h_R \longrightarrow h_{R'}$ is an isomorphism.

So we can interpret Schlessinger theorem in the following form:

Corollary

Let **Def** *be the category of deformation functors satisfying* $H1 - H3$*. Take* $\mathcal{S} = \mathcal{W}$ *the class of étale morphisms of functors. Then,* $(\mathbf{Def}, \mathcal{W})$ *is a left Sullivan category and its minimal models are the prorepresentable functors.*

6. Functor categories: models of functors and derived functors

Consider $(\mathcal{C}, \mathcal{W})$ a category with weak equivalences. Given a functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ we look for an approximation

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{D} \\ \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, dotted] & \end{tikzcd}$$

Obviously, if $F(\mathcal{W}) \subset Iso_{\mathcal{D}}$, then F induces a functor

$$F' : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D}.$$

If $F : \mathcal{C} \longrightarrow \mathcal{D}$ is a functor, a *right Kan extension* of F along $\gamma : \mathcal{C} \longrightarrow \mathcal{C}[\mathcal{W}^{-1}]$ is a functor

$$\mathrm{Ran}_{\gamma} F : \mathcal{C}[\mathcal{W}^{-1}] \longrightarrow \mathcal{D},$$

together with a natural transformation

$$\theta_F = \theta_{\gamma,F} : (\mathrm{Ran}_{\gamma} F)\gamma \Rightarrow F,$$

satisfying a universal property.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{D} \\
\mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, "\mathrm{Ran}_{\gamma} F"'] &
\end{tikzcd}$$

Definition

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called *left derivable* if it exists the right Kan extension of F along γ. The functor

$$\mathbb{L}_{\mathcal{W}} F := (\mathrm{Ran}_{\gamma} F)\gamma$$

is called a *left derived functor of F* with respect to $\mathcal{W}$.

There is a natural transformation

$$\theta_F : \mathbb{L}_{\mathcal{W}} F \Rightarrow F.$$

Notation:

- $\mathbf{Cat}'((\mathcal{C},\mathcal{W}),\mathcal{D})$: category of left derivable functors from $(\mathcal{C},\mathcal{W})$ to $\mathcal{D}$.
- $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C},\mathcal{D})$ full subcategory of $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C},\mathcal{D})$ of functors F such that $F(\mathcal{W}) \subseteq Iso_{\mathcal{D}}$; which is isomorphic to $\mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}],\mathcal{D})$.

We have

$$\mathbf{Cat}_{\mathcal{W}}(\mathcal{C},\mathcal{D}) \subset \mathbf{Cat}'((\mathcal{C},\mathcal{W}),\mathcal{D})$$

and

$$\mathbb{L}_{\mathcal{W}} : \mathbf{Cat}'((\mathcal{C},\mathcal{W}),\mathcal{D}) \longrightarrow \mathbf{Cat}_{\mathcal{W}}(\mathcal{C},\mathcal{D})$$

Let

$$\widetilde{\mathcal{W}} = \text{preimage of isomorphisms by } \mathbb{L}_{\mathcal{W}} : \mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D}) \longrightarrow \mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D}).$$

Proposition

1. $\mathbb{L}_{\mathcal{W}}$ and the natural transformation θ induce a resolvent functor $\mathbb{L}_{\mathcal{W}} : \mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D}) \longrightarrow \mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$,
2. $(\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D}), \widetilde{\mathcal{W}})$ is a left CE category,
3. $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ is the subcategory of cofibrant objects.

Theorem (Derivability criterion)

Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ be a left CE category. For any category $\mathcal{D}$,

1. $\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$ *is a full subcategory of* $\mathbf{Cat}'((\mathcal{C}, \mathcal{W}), \mathcal{D})$*;*
2. *if $F \in Ob\ \mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, then*

$$\mathbb{L}_{\mathcal{W}} F = F' \lambda \gamma,$$

   where $F' : \mathcal{C}[\mathcal{S}^{-1}] \longrightarrow \mathcal{D}$ is induced by F;

3. *$\theta_F : \mathbb{L}_{\mathcal{W}} F \longrightarrow F$ is defined by $\theta_F = F' * \varepsilon' * \delta$, i.e.*

$$(\theta_F)_X = F'(\varepsilon'_{\delta X}), \quad X \in Ob\ \mathcal{C}.$$

Example

Let $\mathcal{A}, \mathcal{B}$ be abelian categories, and assume $\mathcal{A}$ has enough projectives.

With the ususal structure, $(\mathbf{C}_+(\mathcal{A}), \mathcal{S}, \mathcal{W})$ is a left CE category.

In this case the criterion above reduces a well known fact:
An additive functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ induces an additive functor $F : \mathbf{C}_+(\mathcal{A}) \longrightarrow \mathbf{K}_+(\mathcal{B})$ which, by additivity, sends homotopy equivalences to isomorphisms. Hence it is left derivable.

Theorem

Let $(\mathcal{C}, \mathcal{S}, \mathcal{W})$ *be a left CE category and* $\mathcal{D}$ *any category. In* $\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D})$, *take*

$$\widetilde{\mathcal{W}} = \{\phi : F \to G; \phi_M \text{ is } \cong, \forall M \in \mathcal{C}_{cof}\}.$$

The functor

$$\mathbb{L}_{\mathcal{W}} : \mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}), \quad \mathbb{L}_{\mathcal{W}} F := F' \lambda \gamma,$$

and the natural transformation $\theta : \mathbb{L}_{\mathcal{W}} F \Rightarrow F$ *defined by* $(\theta_F)_X = F'(\varepsilon'_{\delta(X)})$, *for each object* X *of* $\mathcal{C}$, *satisfy*

1. $(\mathbb{L}_{\mathcal{W}}, \theta)$ *is a left resolvent functor on* $(\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{W}})$;
2. $(\mathbf{Cat}_{\mathcal{S}}(\mathcal{C}, \mathcal{D}), \widetilde{\mathcal{W}})$ *is a left CE category; and*
3. $\mathbf{Cat}_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ *is the subcategory of its cofibrant objects.*

In order to get total derived functors for $(\mathcal{C}, \mathcal{W}_{\mathcal{C}}) \longrightarrow (\mathcal{D}, \mathcal{W}_{\mathcal{D}})$, we make the following assumption:

- $(\mathcal{C}, \mathcal{S}, \mathcal{W}_{\mathcal{C}})$ is a left CE category with a resolvent functor R

Define

- $\mathbf{Cat}_{\mathcal{S},\mathcal{W}}(\mathcal{C}, \mathcal{D}) = \{F : \mathcal{C} \longrightarrow \mathcal{D} \mid F(\mathcal{S}) \subseteq \mathcal{W}_{\mathcal{D}}\}$.
- $\widetilde{\mathcal{W}} = \{\varphi : F \to G \mid \varphi_M \in \mathcal{W}_{\mathcal{D}}, \quad M \in Ob\ \mathcal{C}_{cof}\}$,
- $\widetilde{\mathcal{S}} = \{\varphi : F \to G \mid \varphi_X \in \mathcal{W}_{\mathcal{D}}, \quad X \in Ob\ \mathcal{C}\}$.

Theorem

1. $(\mathbf{Cat}_{\mathcal{S},\mathcal{W}}(\mathcal{C},\mathcal{D}),\widetilde{\mathcal{S}},\widetilde{\mathcal{W}})$ *is a left CE category*
2. $R^*(F) = F \circ R$, $\varepsilon^*_F = F \circ \varepsilon$, *is a left resolvent functor*
3. *Moreover,*
$$F\,\textit{cofibrant} \Leftrightarrow F(\mathcal{W}_{\mathcal{C}}) \subseteq \mathcal{W}_{\mathcal{D}}.$$

Example

Let A be a commutative ring and $R : \mathbf{C}_+(A) \longrightarrow \mathbf{C}_+(A)$ the resolvent functor defined by the free functorial resolution. Any additive functor $F : \mathbf{Mod}(A) \longrightarrow \mathcal{B}$, $\mathcal{B}$ abelian category, induces an additive functor

$$F : \mathbf{C}_+(A) \longrightarrow \mathbf{C}_+(\mathcal{B})$$

with $F(\mathcal{S}) \subset \mathcal{W}$, therefore $FR \Rightarrow F$ is a left cofibrant model of F in $\mathbf{Cat}_{\mathcal{S},\mathcal{W}}(\mathbf{C}_+(A), \mathbf{C}_+(\mathcal{B}))$.

Corollary

$F\varepsilon : F \circ R \Rightarrow F$ is a cofibrant model, hence the left derived functor $\mathbb{L}F$ of $\gamma_{\mathcal{D}} \circ F$ is induced by $\gamma_{\mathcal{D}} \circ F \circ R$:

$$
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F \circ R} & \mathcal{D} \\
{\scriptstyle \gamma_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \gamma_{\mathcal{D}}} \\
\mathcal{C}[\mathcal{W}^{-1}] & \xrightarrow{\mathbb{L}F} & \mathcal{D}[\mathcal{W}^{-1}] \,.
\end{array}
$$

7. CE structres defined by a cotriple

Definition

Let $\mathcal{X}$ be a category. A cotriple $\mathbf{G} = (G, \varepsilon, \delta)$ in $\mathcal{X}$ is given by

1. a functor $G : \mathcal{X} \longrightarrow \mathcal{X}$
2. a natural transformation $\varepsilon : G \Rightarrow id$
3. a natural transformation $\delta : G \Rightarrow G^2$

satisfying

$$\begin{aligned} \delta G \cdot \delta = G\delta \cdot \delta \quad &: \quad G \Rightarrow G^3, \\ \varepsilon G \cdot \delta = 1_G = G\varepsilon \cdot \delta \quad &: \quad G \Rightarrow G. \end{aligned}$$

Examples

1. Let

$$U : \mathcal{X} \leftrightarrows \mathcal{Y} : F$$

be a pair of adjoint functors, $F \dashv U$. They define a cotriple in $\mathcal{X}$ with $G = FU$.
For example, if $R \longrightarrow S$ is a ring homomorphism, take

$$U : \mathbf{Mod}(S) \leftrightarrows \mathbf{Mos}(R) : F$$

with U the forgetful functor and $F = \otimes_R S$, to obtain a cotriple in $\mathbf{Mod}(S)$.

Examples

2. Cotriples from models: let $\mathcal{X}$ be a category with arbitrary coproducts and $\mathcal{M}$ a set of objects of $\mathcal{X}$ (the models). Define $G : \mathcal{X} \longrightarrow \mathcal{X}$ by

$$G(X) = \coprod_{f:M\to X, M\in\mathcal{M}} M_f$$

with $M_f = M$. One can easily define ε, δ to obtain a cotriple.

For example, take $\mathcal{X} = \mathbf{Top}$ and $\mathcal{M} = \{\Delta^n, n \geq 0\}$.

The standard construction associated to a cotriple

Let $\mathbf{G}$ be a cotriple in $\mathcal{X}$. The *standard construction associated to* $\mathbf{G}$ is the augmented simplicial functor $B_\bullet G$ defined by

$$\begin{aligned} B_n G &= G^{n+1}, \\ \partial_i &= G^i \varepsilon G^{n-i} : G^{n+1} \Rightarrow G^n, \quad 0 \le i \le n, \\ s_i &= G^i \delta G^{n-i} : G^{n+1} \Rightarrow G^{n+2}, \quad 0 \le i \le n. \end{aligned}$$

The natural transformation ε defines an augmentation

$$\varepsilon : B_\bullet G \Rightarrow 1.$$

Let $\mathcal{A}$ be an additive category with an additive cotriple $\mathbf{G}$. Denote also by G the induced additive cotriple on $\mathbf{C}_+(\mathcal{A})$.

The standard construction induces a functor

$$\begin{array}{rcl} B : \mathbf{C}_+(\mathcal{A}) & \longrightarrow & \mathbf{C}_+(\mathcal{A}) \\ K_* & \mapsto & TotB_*K_* \end{array}$$

with a natural transformation $\varepsilon : B \Rightarrow 1$ induced by the augmentation of $B_\bullet G$.

Definition

A class of morphisms $\mathcal{S}$ of $\mathbf{C}_+(\mathcal{A})$ is a *class of summable morphisms* if:

1. $\mathcal{S}$ is saturated
2. $\mathcal{S}$ contains the homotopy equivalences
3. let $f : C_{**} \longrightarrow D_{**}$ be a morphism of first quadrant double complexes. Then

$$f_n : C_{*n} \longrightarrow D_{*n} \in \mathcal{S}, \quad n \geq 0 \Longrightarrow Tot\ f \in \mathcal{S}.$$

If $\mathcal{A}$ is abelian and the morphisms in $\mathcal{S}$ are quesi-isomorphism, we say that it is an *acyclic class of morphisms*.

Let $\mathcal{S}$ be a class of summable morphisms of $\mathbf{C}_+(\mathcal{A})$, and $\mathbf{G}$ an additive cotriple on $\mathcal{A}$.

Definition

$\mathbf{G}$ is *compatible* with $\mathcal{S}$ if $G(\mathcal{S}) \subset \mathcal{S}$.

Theorem

Let $\mathcal{A}$ be an additive category, $\mathbf{G}$ *an additive cotriple in $\mathcal{A}$ and $\mathcal{S}$ a class of summable morphisms in* $\mathbf{C}_+(\mathcal{A})$ *compatible with G. Take $\mathcal{W} = G^{-1}(\mathcal{S})$, then,*

1. $\mathcal{W} = B^{-1}(\mathcal{S})$
2. (B, ε) *is a resolvent functor for* $(\mathbf{C}_+(\mathcal{A}), \mathcal{S}, \mathcal{W})$*,*
3. *is a left CE category,*
4. $K \in Ob\ \mathbf{C}_+(\mathcal{A})$ *is cofibrant iff* $\varepsilon_K : BK \to K \ \in \mathcal{S}$*.*

Application to functor categories

Data:

- $\mathcal{X}$ a category with a cotriple $\mathbf{G}$
- $\mathcal{A}$ an abelian category
- denote also by $\mathbf{G}$ the induced cotriple in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$
- $\mathcal{S}$ a class of summable morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$, compatible with $\mathbf{G}$, as for example
  - the class of homotopy equivalences $\mathcal{S}_h$
  - the class of quasi-isomorphisms
  - the class of pointwise homotopy equivalences $\mathcal{S}_{pt}$

Corollary

Let $\mathcal{X}$ be a category and $\mathcal{A}$ an additive category. Let $\mathbf{G}$ *be an additive cotriple on* $\mathbf{Cat}(\mathcal{X}, \mathcal{A})$*, and $\mathcal{S}$ a class of summable morphisms in* $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$ *compatible with* $\mathbf{G}$*. Then,*

1. (B, ε) *is a left resolvent functor for* $(\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A})), \mathcal{S}, \mathcal{W})$
2. $(\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A})), \mathcal{S}, \mathcal{W})$ *is a left CE category*
3. $K \in Ob\ \mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$ *is cofibrant if* $\varepsilon_K : BK \longrightarrow K \in \mathcal{S}$.

Remark

The theorem applies for functor categories with values in a *simplicial descent category* with a compatible cotriple. For example, we can replace $\mathbf{C}_+(\mathcal{A})$ by the category **SSets**.

Example

- $\mathcal{X} = \mathbf{Top}$ be the category of topological spaces.
- $\mathcal{S}$ the class of homotopy equivalences in $\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_+(\mathbb{Z}))$.
- $\mathbf{G}$ defined on $\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_+(\mathbb{Z}))$ by

$$G(K)(X) = \bigoplus_{n,\ \sigma \in \mathbf{Top}(\Delta^n, X)} K(\Delta^n, \sigma).$$

Then, $(\mathbf{Cat}(\mathbf{Top}, \mathbf{C}_+(\mathbb{Z})), \mathcal{S}, G^{-1}(\mathcal{S}))$ is a left Cartan-Eilenberg category, in which the singular simplex functor

$$S_* : \mathbf{Top} \longrightarrow \mathbf{C}_+(\mathbb{Z}),$$

is a cofibrant model of $H_0(-, \mathbb{Z})$.

Theorem (Acyclic models theorem, Barr 2002)

Let $\mathcal{X}$ be a category with a cotriple $\mathbf{G}$, let $\mathcal{A}$ be an abelian category, and $\mathcal{S}$ a class of acyclic morphisms in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$ compatible with the cotriple induced by $\mathbf{G}$.

If K, L are objects of $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$ such that K is cofibrant and L is G-acyclic, then the map

$$H_0 : [K, L] \longrightarrow [H_0 K, H_0 L]$$

is bijective, that is, given a morphism $f : H_0 K \longrightarrow H_0 L$ there is a unique morphism $\hat{f} : K \longrightarrow L$ in $\mathbf{Cat}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))[\mathcal{S}^{-1}]$ such that, $H_0 \hat{f}$

Definition
L is G-acyclic if if $L \longrightarrow H_0 L$ is in $\mathcal{W}$, that is, $G(L) \longrightarrow G(H_0 L)$ is in $\mathcal{S}$.

For the proof, it suffices to consider the diagram

$$\begin{tikzcd} BK \arrow[rr, dotted, "\hat{f}"] \arrow[d] & & L \arrow[d] \\ K \arrow[r] & H_0 K \arrow[r, "f"] & H_0 L \end{tikzcd}$$

in which, by hypothesis the left vertical morphism is in $\mathcal{S}$ and the right vertical morphism is in $\mathcal{W}$. The existence of $\hat{f}$ follows since $B_* K$ is cofibrant.

Monoidal variation

We have also adapted all this machinery to the monoidal and the symmetric monoidal settings For example, we get a left CE structure on categories $\mathbf{SyMon}(\mathcal{X}, \mathbf{C}_+(\mathcal{A})$, and as a consequence an acyclic model theorem:

Theorem (Guillén-Navarro-Pascual-Roig 2007)

Let $\mathcal{X}$ be a monoidal category with a monoidal cotriple $\mathbf{G}$, let $\mathcal{A}$ be an abelian monoidal category, and $\mathcal{S}$ a class of acyclic morphisms in $\mathbf{SyMon}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$ compatible with the cotriple induced by $\mathbf{G}$.

If K, L are objects of $\mathbf{SyMon}(\mathcal{X}, \mathbf{C}_+(\mathcal{A}))$ such that K is cofibrant and L is G-acyclic, then the map

$$H_0 : [K, L] \longrightarrow [H_0K, H_0L]$$

is bijective.

Symmetric monoidal functors admit an extension to operad categories. As a consequence of the acyclic models theorem we obtain

Corollary (GNPR 2007)

The functors of singular chains and ordered cubical chains

$$S_*, C_*^{ord} : \mathbf{Op}_{\mathbf{Top}} \longrightarrow \mathbf{Op}_{\mathbf{C}_+(\mathbb{Z})}$$

are weakly equivalent.