George Boole's walk on the logical side of chance

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Despite my many trips to Ireland, I have never yet been to Cork. Therefore, I have never visited its university, formerly called Queen's College. But surfing the web I came across a description of the Aula Maxima of Queen's College. On the east-facing window of this hall there is the Boole Memorial Window, made up of ten glass panels in two rows of five panels each. Each panel represents a branch of knowledge. For instance, the second panel in the bottom row represents Mathematics, depicting Bacon, Napier and Newton, and the last panel in the top row represents Engineering and Architecture, in the figures of Archimedes and Phidias. The central panel in the bottom row represents Logic, and there are three figures: Aristotle and Euclid standing behind George Boole (1815-1864), seated at a desk, writing (Fig. 1). Born in Lincoln (Lincolnshire) in 1815, Boole was appointed professor of mathematics at Queen's College Cork in 1849, thanks to his papers and his reputation. Being a self-educated man, with no university training, this was a huge achievement, to say the least.[1] The Boole Memorial Window was built in 1866 in memory of Boole, the first professor of mathematics at Queen's College Cork.

Boole is chiefly remembered for his work on mathematical logic, especially his *An Investigation of the Laws of Thought on which are Founded the Mathematical Theories of Logic and Probabilities* (1854) [henceforth, *Laws of Thought*], a milestone in the historical development of mathematical logic. In the introduction, Boole explained that the design of his work was:

> ...to investigate the fundamental laws of those operations of the mind by which reasoning is performed; to give expression to them in the symbolical language of Calculus, and upon this foundation to establish the science of Logic and construct its method; to make that method itself the basis of a general method for the application of the mathematical doctrine of Probabilities, and, finally, to collect from the various elements of truth brought to view in the course of these inquiries some probable intimations concerning the nature and constitution of the human mind (Boole, 1854, I, §1).

From the very first lines, Boole stressed the novelty of the conception, method and results shown in his book, the more so since 'in its ancient and scholastic form, indeed, the subject of Logic stands almost exclusively associated with the great name of Aristotle' (Boole, 1854, I, §2).

Much has been written about Boole and his logic, Boolean algebra, or Boole's influence on circuit theory and computer programming.[2] This is clearly the reason why he appears in the panel of Logic in the Memorial Window in Queen's College Cork. However, although his most outstanding contribution is the application of mathematics to the domain of pure thought, that is to say, mathematical or formal logic, Boole did research into other branches of mathematics as well. Hence, his papers on analytical transformations and the theory of linear transformations seem to have been the seeds of the theory of invariants, later developed by Cayley and Sylvester. When it comes to differential equations, his research focused on criteria for distinguishing between singular solutions and particular solutions, integrating factors, symbolical methods and also partial differential equations. His textbooks on differential equations and on finite differences showed his teaching methods. As can be inferred from the complete title of the *Laws of Thought*, Boole was also interested in the theory of probability and, in

[1] For a thorough account of Boole's life and works, see MacHale (1985, reprinted in 2014).

[2] Boole's work on logic and its subsequent influence on the development of this subject are discussed in Kneale and Kneale (1962, chapter VI). See also the first part of Hailperin (1986).

particular, the application of his logical system to the calculus of events.[3] The *Laws of Thought* was not exactly a republication of Boole's former work on logic, *The Mathematical Analysis of Logic* (1847). Although the first part of the former was mainly devoted to the same subject as in the latter, the same system of fundamental laws being established in both, the methods shown in the *Laws of Thought* were more general and the range of applications wider (Boole, 1854, Preface). In particular, the second part (chapters XVI-XXI) focused on the theory of probability, in an attempt to construct an elaborate mathematical system regarding this subject.

From my experience in teaching statistics at various engineering schools, I am well aware that probability represents a rather overwhelming obstacle for students, due to the conceptual difficulties inherent in the topic. And, more often than not, students complain about the subjectiveness of probability.

What were Boole's views on the nature of the theory of probabilities? On the one hand, Boole considered the theory and method of probabilities to be based upon the general doctrine and method of logic because:

> Before we can determine the mode in which the expected frequency of occurrence of a particular event is dependent upon the known frequency of occurrence of any other events, we must be acquainted with the mutual dependence of the events themselves. Speaking technically, we must be able to express the event whose probability is sought, as a function of the events whose probabilities are given (Boole, 1854, I, §12).

This is why logic has to be studied prior to the theory of probabilities.

On the other hand, probability admits of numerical measurement. Therefore, it belongs both to arithmetic (the science of number) and to logic:

> … there exists a definite relation between the laws by which the probabilities of events are expressed as algebraic functions of the probabilities of other events upon which they depend, and the laws by which the logical connexion of the events is itself expressed (Boole, 1854, I, §15)

It was the acknowledgement of the dual nature of the theory of probability, numerical and logical, that made the *Laws of Thought* different from all previous treatises. Accordingly, once Boole had established the object of the theory of probabilities:

> Given the probabilities of any events, of whatever kind [simple or compound], to find the probability of some other event connected with them [simple or compound] (Boole, 1854, XVI, §§4-5).

the solution to which had not been successfully investigated yet, he then introduced a new way of tackling it, which consists in:

> … substituting for *events* the propositions which assert that those events have occurred, or will occur; and viewing the element of numerical probability as having reference to the *truth* of those *propositions*, not to the *occurrence* of the *events* concerning which they make assertion (Boole, 1854, XVI, §6).

Hence, for instance, the numerical fraction p would no longer be the expression of the probability of the occurrence of an event E, but rather it would represent the probability of the truth of the proposition X, 'that the event E will occur'. Like this, the theory of probability could be studied as an application of the theory of propositions discussed in the first part of the *Laws of Thought*. By means of the method of the calculus of logic, it was possible to express the event whose probability was sought as a logical function of the events whose probabilities were given. This would result in a logical equation, which in turn would lead to a series of algebraic equations, implicitly involving the solution of the problem proposed (Boole, 1854, I, §15).

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[3] There is a thorough study about Boole's work on probability in the second part of Hailperin (1986).

In Chapter XVI Boole defined probability as follows:

> The probability of an event is the reason we have to believe that it has taken place, or that it will take place.
>
> The measure of the probability of an event is the ratio of the number of cases favourable to that event, to the total number of cases favourable or contrary, and all equally possible (equally likely to happen) (Boole, 1854, XVI, §2).

As he pointed out in a footnote, he had adopted Poisson's definition of probability (Poisson, 1837, I, art. 1 and 2). However, while Boole wrote 'equally likely to happen', in the original text Poisson referred to 'tous également possibles, ou qui ont tous une même chance' (Poisson, 1837, I, art. 2). According to Poisson (1837, I, art. 1), 'chance' and 'probability' were nearly synonyms in everyday life. Yet, in a work on probabilities, there was a difference in meaning between them: the word 'chance' was connected with the events themselves, regardless of the knowledge we have about them, whereas the word 'probability' referred to the above definition.

For the study of the chapters related to the theory of probabilities, in the preface of the *Laws of Thought* Boole recommended the reading of a treatise on probability written by Sir John Lubbock. Boole was here clearly alluding to *On Probability*, an undated and anonymous tract, no doubt written by John W. Lubbock (1803-1865) and John E. Drinkwater Bethune (1801-1851) and published by the Society for the Diffusion of Useful Knowledge around 1830. Lubbock and Drinkwater Bethune reflected on the use of the word 'chance':

> 1. In considering any future event, we are generally unable to determine whether or not it will happen; yet, we can often conjecture the number of cases which are possible, and of these how many favour the production of the event in question. In our uncertainty, we say that there is a **chance** it will happen; and thus our idea of **chance** arises from our wanting data which might enable us to decide whether or not the event will take place.
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> 5. Simpson has defined the probability of an event to be the ratio of the chances by which the event in question may happen to all the chances by which it may happen or fail. In this definition the word **chance** must be understood a way of happening; we, however, frequently say, "I left such a thing to chance", or, "such a thing is entirely chance"; these expressions, which are in some measure sanctioned by common use, are intended to signify that we are ignorant of the causes which produce the event in question, or that we do not influence its occurrence (Lubbock and Drinkwater Bethune, 1830, §§1 and 5, my emphasis).

From the above reflections, it is clear that there were two distinct uses of the word 'chance': one concerned with the happening of an event (that is, probability, in its mathematical acceptation), and another one connected, in ordinary language, with our ignorance of the causes that produce an event. Despite referring to the works of Poisson and Lubbock and Drinkwater Bethune, Boole seemed to avoid the word 'chance'. Did he do this on purpose, to stress the algebraic-logical nature of probability, and its independence from one's own mind?

The concept of probability depended, nevertheless, on partial knowledge (Boole, 1854, XVI, §2). Consequently, the probability of an event may change, depending on our knowledge about the circumstances under which the event occurs. In general, this knowledge could be obtained from the particular constitution of the piece involved in the occurrence of the event (e.g. the constitution of a die) or from the long-continued observation of the success and failure of events. Likewise, Lubbock and Drinkwater Bethune (1830, §9) asserted: 'Probability does not exist in the abstract, but always refers to the knowledge possessed by some particular individual'. An assertion that, in a way, is in keeping with one of the meanings of the word 'subjective': dependent on an individual's perception for its existence. If we adopt this sense of the word, after all, we could agree with our students that the concept of probability is a rather subjective topic!

The information we possess about the occurrence of a certain event turns out to be fundamental when computing probabilities. We all know that the probability of rolling a '6' with a fair die is 1/6. But what if we learn beforehand that an even number or a multiple of '3' came up? What is then the probability of rolling a '6'? However simple, this example illustrates the main idea of conditional probability, that is, the probability of an event *A*, given another event *B*, $P(A|B)$. Boole wrote a good deal on conditional probability, or, as he called it, 'the connexion of causes and effects':

> **From the probabilities of causes** assigned *à priori* [*sic*], or given by experience, and their respective probabilities of association with an effect contemplated, it may be required **to determine the probability of that effect** (…). On the other hand, it may be required **to determine the probability of a particular cause**, or of some particular connexion among a system of causes, **from observed effects**, and the known tendencies of the said causes, singly or in connexion, to the production of such effects (Boole, 1854, XX, §1, my emphasis).

This kind of question was also known as 'inverse probability' at the time.[4] One of the first to use such an expression was Augustus De Morgan (1806-1871) in his *An Essay on Probabilities* (1838), a work recommended by Boole in his preface, as preliminary reading for the study of the chapters related to the theory of probabilities. De Morgan referred to 'inverse questions': 'Where we know the event which has happened, and require the probability which results therefrom to any particular set of circumstances under which it might have happened' (De Morgan, 1838, II, 31-32), as opposed to 'direct questions': 'Where we know the previous circumstances and require the probability of an event' (De Morgan, 1838, II, 31). According to De Morgan, inverse questions originated with Thomas Bayes (1702-1761) and his *An Essay towards Solving a Problem in the Doctrine of Chances* (published posthumously in the *Philosophical Transactions of the Royal Society* in 1763), a rather forgotten fact at the time (De Morgan, 1838, preface, vii).

Boole believed that social problems could be studied by means of the theory of probability, since 'phaenomena, in the production of which large masses of men are concerned, do actually exhibit a very remarkable degree of regularity' (Boole, 1854, I, §15). In this context, probabilities could be regarded as founded upon continued observation. Thus, for instance, chapter XXI of the *Laws of Thought* dealt with a practical application in the social field, namely, the question of the probability of judgments (connected with decisions of judges, juries and so on), a very popular question in the nineteenth century. Here again, Boole distinguished between direct and inverse questions:

> The **direct questions** of probability are those in which the probability of correct decision for each member of the tribunal, or of guilt for the accused party, are supposed to be known *à priori* [*sic*], and in which the probability of a decision of a particular kind, or with a definite majority, is sought. **Inverse problems** are those in which, from the data furnished by experience, it is required to determine some element which, though it stand to those data in the relation of causes to effect, cannot directly be made the subject of observation; as when from the records of the decisions of courts it is required to determine the probability that a member of a court will judge correctly. To this species of problems, the most difficult and the most important of the whole series, attention will chiefly be directed here (Boole, 1854, XXI, §2, my emphasis).

A detailed discussion of the question of the probability of judgments is beyond the scope of this paper.[5] But seeing as Boole was so interested in it, it is nevertheless worth providing a simple example, concerning unanimous decisions from witnesses and inverse probability:

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[4] For a full history of the inverse probability, see Dale (1999).
[5] See, for instance, Hailperin (1986, 376-389).

> Example 3. The probability that a witness *A* speaks the truth is *p*, the probability that another witness *B* speaks the truth is *q*, and the probability that they disagree in a statement is *r*. What is the probability that if they agree, their statement is true? (Boole 1854, XVIII, §4).

In De Morgan (1838) I found a much simpler version of this example, rather rhetoric, considering $p = q = 3/4$ in Boole's example:

> EXAMPLE II: Two witnesses, on each of whom it is 3 to 1 that he speaks the truth, unite in affirming that an event did happen, which of itself is equally likely to have happened or not to have happened. What is the probability that the event did happen?
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> The fact observed is the agreement of the two witnesses in asserting the event: the two possible antecedents (equally likely) are: 1. The event did happen. 2. The event did not happen. If it did happen, the probability that both witnesses should state its happening is that of their both telling the truth, which is ¾ x ¾, or 9/16. If it did not happen, then the probability that both witnesses should assert its happening is that of their both speaking falsely, which is ¼ x ¼, or 1/16. Consequently, the probability that the event did happen is the (9 +1)th part of 9, or 9/10; that is, it is 9 to 1 in favour of the event having happened (De Morgan, 1838, III, 56).

In Lubbock and Drinkwater Bethune (1830) there is a more general account of this problem, with *n* witnesses, but again with the same probability of giving the right decision:

> Ex. 24. A jury consists of *n* individuals; let the probability of each separately giving a right decision be *p*, what is the probability that a unanimous decision is a correct one? Two hypotheses can be formed, namely, that the decision is a correct one, or the contrary; the event observed is a unanimous decision, and the *à priori* [*sic*] probability of this event on the first hypothesis is p^n, the *à priori* [*sic*] probability of the event on the second hypothesis is $(1-p)^n$, therefore the probability of the first hypothesis is $\frac{p^n}{p^n+(1-p)^n}$, which is greater than 1/2, only when $p > 1/2$. Therefore it is probable that a unanimous verdict is a correct one, only when it is probable that each juryman considered separately will give a correct decision (Lubbock and Drinkwater Bethune, 1830, §50).

Now, let us outline how Boole solved the problem by applying the doctrine and method of logic, explained in the first part of the *Laws of Thought*. If x represents the hypothesis that *A* speaks truth, and y represents the hypothesis that *B* speaks truth, then the hypothesis that *A* and *B* disagree in their statement can be represented by:

$$x(1-y) + y(1-x)$$

Consequently, the hypothesis that *A* and *B* agree can be represented by:

$$xy + (1-x)(1-y)$$

where xy symbolises the hypothesis that they agree in the truth. The data given in the example can be expressed as follows:

$$\text{Prob. } x = p, \text{Prob. } y = q, \text{Prob. } x(1-y) + y(1-x) = r$$

To determine the probability that if both witnesses agree, their statement is true, we have to compute the conditional probability:

$$\frac{\text{Prob. } xy}{\text{Prob. } xy + (1-x)(1-y)}$$

which is evidently the same as:

$$\frac{\text{Prob. } xy}{1 - r}$$

Let consider the following system:

$$\begin{cases} x(1-y) + y(1-x) = s \\ xy = w \end{cases}$$

Using the reduction of systems of propositions, studied in chapter VIII of the *Laws of Thought*, this system gives:

$$\{x(1-y)+y(1-x)\}(1-s)+s\{xy+(1-x)(1-y)\}+xy(1-w)+w(1-xy)=0$$

From this we can infer that:

$$w=\frac{x(1-y)(1-s)+y(1-x)(1-s)+sxy+s(1-x)(1-y)+xy}{2xy-1}$$

After developing this equation and applying Proposition IV (Boole, 1854, XVII, §14), Boole arrived at the following expression:

$$\text{Prob. } w=\frac{xy(1-s)}{xy(1-s)+x(1-y)s+(1-x)ys+(1-x)(1-y)(1-s)}=$$
$$=\frac{p+q-r}{2}$$

Finally, the probability sought is:

$$\frac{\text{Prob. } xy}{1-\ r}=\frac{p+q-r}{2(1-r)}$$

I chose this example because it reminded me of some of the exercises we do in class to illustrate the application of conditional probability and Bayes' theorem. We would say that the events 'the statement is true' and 'the statement is false' together form the sample space. Let 'unanimous yes' be an event from the same sample space, of which we know the joint probabilities with 'the statement is true' and with 'the statement is false', respectively. The goal is to compute the probability of the event 'the statement is true', given the event 'unanimous yes', that is, $P($*the statement is true*$|$*unanimous yes*$)$.

Boole was certainly concerned with the question of the probability of judgments. Not only was chapter XXI of the *Laws of Thought* devoted to this question. In 1857 Boole published the paper 'On the application of the theory of probabilities to the question of the combination of testimonies or judgments' in the *Transactions of the Royal Society of Edinburgh*, for which he had won the Keith Medal for the period 1855-1857, the highest prize awarded by the Royal Society of Edinburgh. This seems to point out that Boole's work on probability was highly regarded in his lifetime. Yet, as the Boole Memorial Window suggests, Boole is largely remembered for his work on mathematical logic. I have to acknowledge that that has been my vision for many years, too. It was precisely to make up for my unawareness that I decided to get a glimpse of Boole's contribution to the development of the theory of probabilities to celebrate the bicentenary of his birth.

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