Title: Study of the Singularities in the flat Friedmann-Lemaître-Robertson-Walker Universe

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Study of the Singularities in the flat Friedmann-Lemaitre-Robertson-Walker Universe

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To my family
I would like to remember all the people that I have met during this way. I need to do it, because all of them are part of the person that I am today.

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A tots aquells que em mantenien enganxada a la realitat sense oblidar els meus somnis.
There is a definition saying that a theory is something that do some logical arguments and derives in a prediction. And the success of this theory is that this prediction occurs.\[1\]

But, here, there is a controversial point. A concatenation of cause and effect, governed by a series of logical arguments, and its prediction, need not have truth value. For example, today we have no doubt in general of the heliocentric model. But, the anthropocentric model could predict the position of the planets in the sky.

This relation between cause and effect will be true according the way it follows the laws of logic, but not true or false in the world nature. For this reason, in he continuation of my analysis we will differentiate between two key points: theories and facts.

The theories have no true value. They are beliefs, based in evidences. The facts, can be true or false.

In my view, the germ of scientific theories is subjective, and it is influenced by all the knowledge and stimuli that a person has had throughout his life. The Bohr atom, certainly reminds us a little solar system. With the uncertainty principle and the Schrödinger equation then changes our picture of the atom approaching a position closer to the experiments. Then, if scientific ideas in the beginning are subjective, how science can explain objectively what happens in a given phenomenon? Remember the words of Einstein in which he told us that a single experiment could disassemble a theory.

Physics, which is the science in which I focus, are not just theories. But, theories are one of its pillars. How some theories become facts and how other theories remain in a paradigm of a particular time is a matter of study.

So while we do not know how to explain it, we see that science gives us knowledge. This knowledge is far from a cumulative way. Scientific knowledge is not something that has accumulated throughout history. Things we take for granted today, in most cases refers to those beliefs, not dogmatic, but attempts, which are theories, and glimpse, within a paradigm, what is the truth now and what is a belief is not a simple task.

This work wants to study some aspects of Cosmology. As we have seen before, one of the main important aspects of the science, is that it gives a true knowledge about the nature. In this task, the cosmology is one of the most difficult areas, so it studies the Universe as a whole, its structure, its origin and its evolution.\[2\]

To ensure some knowledge about the theories of Cosmology we can see that we have to deal with a different issue respect other specialities:

- We can not do experiments with the universe as a whole. So we can not verify our theories from the experimental point of view. (We can do some experiments that verify some predictions or questions for some partial features of this theories, but not for the universe as a whole).

So, the most important way to verify the theories of the cosmology is the observation.

The ancient greeks though that the best way to learn about the nature is to observe it. In this sense the cosmology is not so far away from this way of thinking.
In the ancient Greece was born a different way of thinking. Those philosophers began a logical way of thinking. They were the first in try to explain the nature without a mythological point of view. In cosmology, we do a logical reasoning based on the scientific knowledge of the moment and the observations. Maybe we are not so far away from this ancient Greeks.

It is important to see that it is a logical reasoning. This means, it is the fact that distinguishes the Cosmology from the Cosmogonies. The cosmogonies are based in mythological explanations. As the ancient Greeks we want to show an explanation of the universe outside this. But, maybe, we can not be outside, in the sense that there are multiple Cosmogonies...

Some scientist and philosophers think that outside the current paradigm of the cosmology, "the imaginative scientists propose new speculative theories without verification and usually incompatible between them"[3]. And maybe in some aspects we can agree with them. But, maybe, if we can find multiple explanations of the Universe, its origin and its evolution, it means that at least at this moment we can not know the origin of the universe or if there exist, and we can not ensure nothing about the fate of the universe.

Summarizing, these, could be theories, a kind of beliefs that makes more or less coherent and comprehensive the cosmovision of our universe in our paradigm, but those theories have no true value about the conclusions that we arrive, only the fact that we can not know the true about the origin and the fate of the Universe.
ABSTRACT

In this project we have performed an study of the singularities in Classical Cosmology, considering a flat Friedmann-Lemaître-Robertson-Walker for a linear equation of state and for a non-linear one. We start with the Introduction to this project. In Chapter 2, we review the concepts of Cosmology that concerns to our study and we develop the constituent equations of our model which are: The Friedmann Equation, the Raychaudhuri Equation, the Conservation Equation and the Equation of State (EoS), being this last one a relation between pressure $P$ and energy density $\rho$. Following the literature [4] we can find a definition for the future singularities that allow us to define analogously, in Chapter 3, all the different classes of singularities that we have obtained in the present study. The constituent equations found in Chapter 2 led us to compute different results: linear EoS (Chapter 4) and non-linear EoS (Chapter 5). We have computed analitically all the possible cases, obtaining as a result a general classification for the singularities for a model $P = -\rho + A\rho^\alpha$. In the conclusions we compare our classification with literature [5] and as result, the concrete cases of [5] concides with the corresponding ones of our general classification.
1. INTRODUCTION

As we have argued before, the most important way to ensure something in Cosmology is the observation. One of the most important observations is the redshift of distant Galaxies, their spectra are shifted toward longer wavelength. The shift increases with the distance, so if one galaxy is further from us, the shift will be larger. This implies that the distance between them and us is increasing. As the space expands, the wavelength of the light travelling through space also expands. We describe this expansion by a time-dependent scale factor $a(t)$. We define the present expansion rate as

$$H = \frac{\dot{a}}{a},$$

(1.1)

named Hubble Parameter.

Over large scales the Universe expands at the same rate. We can derive using general relativity $a(t)$ as a function of time. To do it we begin from the present expansion rate. Doing this calculus we find our first singularity, the one called Big Bang.

The Big Bang singularity satisfies that $a(t)$ goes to zero about 14 billion years ago ($10^{9}$), where we take the origin of the Universe in Big Bang theory. In the Big Bang Theory the Universe is expanding, so if we go backwards in time, the Universe will contract, and the result is that when $a(t)$ goes to zero, the energy density $\rho$ diverges. Here we are implicitly assuming that the Universe expands adiabatically (the total entropy of the Universe is constant), and also, that the dynamical laws of the Universe are the same at low than at high energy densities.
2. REVIEW OF COSMOLOGY

In both Special and General Relativity, spacetime is depicted by 4-dimensional manifold. In General Relativity, gravity is manifested as a curvature of this spacetime. The main idea is that a massive body curves the spacetime, and this curvature affects the motion of other bodies. "Matter tells spacetime how to curve, spacetime tells matter how to move" [2]. We can think on this idea in the following way: if there is no force acting on a body (and gravity is not a force in general relativity), the body will be a freely falling body, that will move in the curved spacetime following a geodesic.

General Relativity is based in two fundamental principles [5]
- The Principle of Equivalence. Free falling observers within a gravitational field are locally equivalent to inertial observers. These situation cannot be discriminated using local experiments.
- The Principle of General Covariance. The laws of physics must have the same form in all frames of reference. Therefore, they must change in a covariant way under general changes of coordinates. Mathematical speaking, physical laws must be formulated on terms of some geometric objects associated the 4-dimensional manifold.

So, gravity is a manifestation of the spacetime curvature. In Newtonian physics the source of a potential gravity is the mass. Using that in General Relativity the mass and the energy are equivalent, we can try to use the stress-energy tensor, named $T_{\mu\nu}$, which contains the information about the mass and the energy of the system, to take into account the action of the matter in the curvature of the spacetime. So, for first we can try something like:

$$G_{\mu\nu} = kT_{\mu\nu}, \quad (2.1)$$

where $k$ is a constant of proporcionality between the stress-energy tensor and a geometric tensor called $G_{\mu\nu}$.

There exists several constraints to be satisfied by the tensor $G_{\mu\nu}$: [5]
- It must be symmetric in both indexes
- It must depends on the metric and its derivatives only.
- It must cancel for flat spacetime.
- $\nabla_\mu G^{\mu\nu} = 0$.
- In order to be a dynamical theory that reproduce the Poisson equation it must contain a second derivative of the metric.
It can be shown that the only tensor that satisfies all the conditions is the so called Einstein tensor, given by:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$  \hspace{1cm} (2.2)

Then, the Einstein equation can be written as:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu},$$  \hspace{1cm} (2.3)

being $G$ the Newtonian gravitational constant.

When the spacetime is flat, its geometry is the Minkowski’s one:

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2,$$  \hspace{1cm} (2.4)

in cartesian coordinates $(t, x, y, z)$. At sufficiently large scales we can consider an homogeneous (all places look the same) and isotropic (all directions look the same) Universe. If we consider an homogeneous spacetime in space but not in time, the hypersurfaces related to this $t = \text{constant}$ are homogeneous, and this $t$ is called the cosmic time. Taking into account that the model must consider a homogeneous and isotropic Universe, the curvature of the spacetime must be the same everywhere and into every direction (but in time is not homogeneous, it may change in time). Considering a curvature that only depends on time it can be shown that the metric (using a suitable choice of coordinates) is the Robertson-Walker (RW) metric:

$$ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1-kr^2} + r^2d\theta^2 + r^2\sin^2(\theta)d\phi^2 \right].$$  \hspace{1cm} (2.5)

Where $k$ is a constant, related to the curvature of space and $a(t)$ is a function of time, associated with the expansion (or possible contraction) of the Universe. We can define the curvature radius of space (at time $t$) as:

$$r_{\text{curv}} = \frac{a(t)}{\sqrt{|k|}}.$$  \hspace{1cm} (2.6)

The RW metric (2.5) is given in spherical coordinates. If we take $t = r = \text{constant}$ in (2.5) the 2D surfaces have the metric of a sphere with radius $r$. The factor $a(t)$ is called the scale factor.

If in (2.6) $k \neq 0$: with this equation we can rescale $r$ to make $k = \pm 1$, in such a way that we can always rescale the possible values of $k$ to $k = -1, 0, 1$ in (2.5). We are going to consider one by one this possible cases.

1. $k = 0$. The space part ($t = \text{constant}$) of the RW metric is flat. The spacetime however is curved. Usually, people say that the Universe is flat in this case (note that flat Universe is different from flat spacetime, where we use the Minkowski metric instead of the RW one).
2. $k > 0$. The coordinate system has a singularity in this case at $r = \frac{1}{\sqrt{k}}$. If we choose a coordinate transformation $r = k^{-1/2} \sin \chi$ the metric becomes

$$ds^2 = -dt^2 + a^2(t)k^{-1} \left[ d\chi^2 + \sin^2(\chi)d\theta^2 + \sin^2(\chi)\sin^2(\theta)d\phi^2 \right].$$

The space part has a metric of a hipersphere, a sphere with one extra dimension. $\chi$ is a new angular coordinate whose values range over $0 - \pi$, just like $\theta$. The singularity at $r = \frac{1}{\sqrt{k}}$ disappears in this coordinate system transformation, it is a coordinate singularity, not a singularity of the spacetime. The original coodinates covers only the half of the hypersurface, as the coordinate singularity $r = \frac{1}{\sqrt{k}}$ divides the hipersurface in two halves. The case $k > 0$ corresponds to a closed universe whose spatial curvature is positive. This is a finite Universe, with circumference $\frac{2\pi a}{\sqrt{k}} = 2\pi r_{\text{curv}}$, and we can think of $r_{\text{curv}}$ as the radius of the hypersphere.

3. $k < 0$. We have not a coordinate singularity and $r$ can range from 0 to $\infty$. The substitution $r = k^{1/2} \sinh \chi$ is often useful in calculations. The case $k < 0$ corresponds to an open Universe, whose (spatial) curvature is negative. The metric is then:

$$ds^2 = -dt^2 + a^2(t)k^{-1} \left[ d\chi^2 + \sin^2(\chi)d\theta^2 + \sin^2(\chi)\sin^2(\theta)d\phi^2 \right].$$

This Universe is infinite, just like the case $k = 0$.

The RW metric at a given time has two associated length scales: the curvature radius, $r_{\text{curv}} = \frac{a}{\sqrt{k}}$, and the Hubble time, $t_H = H^{-1}$, where $H = \frac{\dot{a}}{a}$ is the Hubble parameter. Since we take $c = 1$, the Hubble time multiplied by the speed of light gives the Hubble length, $l_H = ct_H = H^{-1}$.

In general, the metric of spacetime can always be written as:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu,\nu=0}^3 g_{\mu\nu}dx^\mu dx^\nu, \quad (2.7)$$

we introduce the Einstein summation rule: there is a sum over repeated numbers. The objects $g_{\mu\nu}$ are the components of the metric tensor. The metric of Minkowski space has the components:

$$g_{\mu\nu} = \begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}, \quad (2.8)$$
in cartesian coordinates. The RW metric of (2.5) has components:

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & a^2 & 0 & 0 \\
0 & 0 & a^2 r^2 & 0 \\
0 & 0 & 0 & a^2 r^2 \sin^2 \theta
\end{pmatrix}.
\] (2.9)

In both cases the tensor matrix form a diagonal matrix, this is because the coordinate system is orthogonal.

The Einstein Equation (2.3) is a covariant equation. This implies that it will have the same solutions indepenently the coordinate system that we have choose. In the Appendix we have done a development in order to derive it, using two different coordinate systems: one has been developed in espherical coordinates (2.9); and the another one using a metric that is conformal to the Minkowski’s metric (2.8) (with \( k = -1, 0, 1 \)) that we have been able to derive with the help of a conformal time.

In the general case, (see Appendix A.2) we can write for every value of \( k \) the coefficients for the Einstein Tensor:

\[
G_{00} = \frac{3}{a^2} (\ddot{a}^2 + k),
\] (2.10)

\[
G_{11} = -(2 \dddot{a} + \dot{a}^2 + k) = G_{22} = G_{33}.
\] (2.11)

In our case, we are going to develop the calculus taking the special value of a flat Universe \( k = 0 \). First of all we have to consider the metric that we are using (RW with \( k = 0 \) and in cartesian coordinates):

\[
g_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & a^2(t) & 0 & 0 \\
0 & 0 & a^2(t) & 0 \\
0 & 0 & 0 & a^2(t)
\end{pmatrix},
\] (2.12)

or equivalently:

\[
g^{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & a^{-2}(t) & 0 & 0 \\
0 & 0 & a^{-2}(t) & 0 \\
0 & 0 & 0 & a^{-2}(t)
\end{pmatrix}.
\] (2.13)

If we do the partial derivatives of each component the only ones that are different from zero are the partial derivatives with respect to the time (or respect the coordinate 0), we can write them as:

\[
g_{11,0} = 2a\dot{a}, \quad g_{22,0} = 2a\dot{a}, \quad g_{33,0} = 2a\dot{a}.
\] (2.14)
The Christoffel symbols of first class:

\[
\Gamma_{ijk} = \frac{1}{2} (g_{jk,i} + g_{ki,j} - g_{ij,k}), \tag{2.15}
\]

and the Christoffel symbols of second class:

\[
\Gamma_{ij}{}^h = \Gamma_{ijk} g^{kh}. \tag{2.16}
\]

Applying this to our case we obtain the following:

\[
\Gamma_{110} = -a \dot{a}, \quad \Gamma_{220} = -a \dot{a}, \quad \Gamma_{330} = -a \dot{a}. \tag{2.17}
\]

\[
\Gamma_{101} = \Gamma_{011} = a \dot{a}, \quad \Gamma_{202} = \Gamma_{022} = a \dot{a}, \quad \Gamma_{303} = \Gamma_{033} = a \dot{a}. \tag{2.18}
\]

\[
\Gamma_{101} = \Gamma_{101} g^{11} = a \ddot{a} - 2 = \frac{\dot{a}}{a} = \Gamma_{011}, \tag{2.19}
\]

\[
\Gamma_{101} = \Gamma_{011} = \Gamma_{202} = \Gamma_{022} = \Gamma_{303} = \Gamma_{033}, \tag{2.20}
\]

With the Christoffel symbols we can calculate the Ricci tensor and the scalar curvature:

\[
R_{ab} = \Gamma^c_{ab,c} - \Gamma^c_{ac,b} + \Gamma^c_{ab} \Gamma^e_{ec} - \Gamma^c_{ae} \Gamma^e_{eb}, \tag{2.22}
\]

\[
R = g^{ab} R_{ab}. \tag{2.23}
\]

In our case:

\[
R_{00} = -\Gamma^1_{01,0} - \Gamma^2_{02,0} - \Gamma^3_{03,0} - \Gamma^1_{01} \Gamma^1_{10} - \Gamma^2_{02} \Gamma^2_{20} - \Gamma^3_{03} \Gamma^3_{30} = -3\dot{H} - 3H^2 = -3(\dot{H} + H^2), \tag{2.24}
\]

where \( H = \frac{\dot{a}}{a} \).

\[
R_{11} = \Gamma^0_{11,0} + \Gamma^0_{01} \Gamma^1_{11} + \Gamma^0_{11} \Gamma^2_{02} + \Gamma^0_{11} \Gamma^3_{03} - \Gamma^0_{01} \Gamma^1_{01} - \Gamma^0_{02} \Gamma^2_{20} - \Gamma^0_{03} \Gamma^3_{30} = \dot{a}^2 + a\ddot{a} + 3\dot{a}^2 - 2\ddot{a}^2 = 2\ddot{a}^2 + a\ddot{a}. \tag{2.25}
\]

Using (2.13), (2.24) and (2.25) we can obtain the scalar curvature:

\[
R = g^{\mu\nu} R_{\mu\nu} = 3(\dot{H} + H^2) + 3a^{-2}(2\ddot{a}^2 + a\ddot{a}) = 6(\dot{H} + 2H^2). \tag{2.26}
\]
With this results we are able to calculate the components of Einstein’s tensor:

\[ G_{00} = R_{00} - \frac{1}{2} g_{00} R = -3(\dot{H} + H^2) + 3(\dot{H} + 2H^2) = \frac{3 \dot{a}^2}{a^2}, \quad (2.27) \]

\[ G_{11} = R_{11} - \frac{1}{2} g_{11} R = -\dot{a}^2 - 2\ddot{a}a = G_{22} = G_{33}. \quad (2.28) \]

Returning to the Einstein Equation we are going to assume the perfect fluid form for the stress-energy tensor:

\[ T_{\mu\nu} = (\rho + P) u_{\mu} u_{\nu} + P g_{\mu\nu} \quad (2.29) \]

We can use the perfect fluid form in a very large-scale such that the galaxies are "microscopic fluid particles". Isotropy implies that the fluid in the RW coordinates has \( u^\mu = (1, 0, 0, 0) \) (we must consider the metric: \( g_{\mu\nu} = diag(-1, a^2, a^2, a^2) \)). With these considerations our stress-energy tensor takes the form:

\[ T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & a^2P & 0 & 0 \\ 0 & 0 & a^2P & 0 \\ 0 & 0 & 0 & a^2P \end{bmatrix}. \quad (2.30) \]

Homogeneity implies that \( \rho = \rho(t), P = P(t) \).

Using (11) and (12), the Einstein equation, (4), becomes now:

\[ 3 \frac{\dot{a}^2}{a^2} (\dot{a}^2 + k) = 8\pi G \rho, \quad (2.31) \]

\[ -2 \frac{\ddot{a}}{a} - \left( \frac{\dot{a}}{a} \right)^2 - \frac{k}{a^2} = 8\pi G P. \quad (2.32) \]

We can rearrange this pair of equations in the following form:

\[ \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \rho, \quad (2.33) \]

\[ \frac{\ddot{a}}{a} = -4\frac{\pi G}{3} (\rho + 3P), \quad (2.34) \]

where (2.33) is named the Friedmann Equation and (2.34) is named the acceleration equation.

From the first law of thermodynamics:

\[ \Delta U = Q + W \quad (2.35) \]
where $U$ is the energy of the system, $W$ the work done by the system, taking into account that the FLRW Universe is thought as an adiabatic expansion of the Universe with null variation of entropy, $Q = 0$. We can imagine a volume $V$ proportional to $a^3$, containing a total energy $U = \rho a^3$, we can rewrite equation (2.35) as [6]:

$$d(\rho a^3) = -Pda^3$$  \hspace{1cm} (2.36)

That we can rewrite in the following form:

$$\dot{P}a^3 = \frac{d}{dt} [a^3 (\rho + P)]$$  \hspace{1cm} (2.37)

$$\dot{\rho} + 3(\rho + P) \frac{\dot{a}}{a} = 0$$  \hspace{1cm} (2.38)

And now notice that the Friedmann Equtation and the Acceleration equation are not independent, we can recover (2.34) from (2.33):

$$\dot{a}^2 = \frac{8\pi G}{3}\rho a^2$$

$$2\frac{\ddot{a}}{a^2} = \frac{8\pi G}{3} (\dot{\rho} + \rho \frac{\dot{a}}{a})$$,

and using (2):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P),$$

obtaining (2.34).

Considering $M_p = \frac{1}{\sqrt{8\pi G}}$, $H = \frac{\dot{a}}{a}$ and $H^2 + \dot{H} = \frac{\ddot{a}}{a}$ we can write (2.33) and (2.34) as:

$$H^2 + \frac{k}{a^2} = \frac{\rho}{3M_p^2}$$  \hspace{1cm} (2.39)

$$H^2 + \dot{H} = -\frac{1}{6M_p^2} (\rho + 3P).$$  \hspace{1cm} (2.40)

If we substract (2.40) - (2.39) we obtain and with some algebra we obtain the Raychaudhuri Equation:

$$\dot{H} = \frac{k}{a^2} - \frac{1}{2} \frac{1}{M_p^2} (\rho + P).$$  \hspace{1cm} (2.41)

If take the derivative of the Friedman Equation (2.33)with respect the time, we obtain:

$$2H\dot{H} - \frac{k}{a^2} H = \frac{1}{3M_p^2} \ddot{\rho}.$$  \hspace{1cm} (2.42)
with this last and with the Raychauduri Equation (2.42) we get:

\[ \frac{1}{3M_p^2} \dot{\rho} = -\frac{1}{M_p^2} (\rho + P)H. \] (2.43)

Finally, we can rewrite that arriving to the Conservation Equation:

\[ \dot{\rho} = -3H(\rho + P). \] (2.44)

Summing up, on the flat case with \( M_p = 1 \), the constituents equations of this model are (with \( k = 0 \)), are given on Table 2.1.

| \( H^2 = \frac{\rho}{3} \) | Friedman Equation |
| \( \dot{H} = -\frac{1}{2} (\rho + P) \) | Raychaudhuri Equation |
| \( \dot{\rho} = -3H(\rho + P) \) | Conservation Equation |
| \( P = P(\rho) \) | EoS |

Tab. 2.1: Constituent Equations
We have studied in the previous section the Einstein equation. At the beginning Einstein put by hand a constant \( \Lambda \) in such a way that the Universe was static (although unstable). The value of \( \Lambda \) that makes the Universe static is just:

\[
\Lambda_E = 4\pi G \rho
\]  
(3.1)

The expansion of the Universe was discovered in the late 1920s, and after that, the static solutions of the Universe were no reasonable. However, after that, \( \Lambda \neq 0 \) has continue being an object of study, since one cannot exclude all the models with \( \Lambda \neq 0 \) on observational grounds.

One of the first models that agree with the expansion of the Universe was the Lemaître model (1927) were the Universe has positive spatial curvature (\( k = 1 \)). In this case, one can show, that the expansion parameter is always increasing. Although data has shown that this is not an explanation for the redshift evolution of quasars, it is of historical interest, so, it was one of the first models that exhibit a Big Bang singularity. We must think that when Lemaître’s work was published the results of Hubble were just becoming known, and because of this most people use to credit Lemaître as the father of the Big Bang Cosmology. [6]. In 1922, A. Friedmann derived the properties of (2.33) and (2.34), but his work was not known at this time, because his model was not static and the expansion of the Universe was not known at that moment.

The Universe named Friedmann-Lemaître-Robertson-Walker, i.e., FLRW Universe is a Universe that uses the Robertson-Walker Metric, since it is homogeneous and isotropic. Furthermore, this model considers an adiabatically expansion of the Universe with null variation of Entropy.

As we have explain in the previous Chapter, the Robertson-Walker metric (2.5), has a constant \( k \) related with the curvature of the Universe, that can be scaled to \( k = -1, 0, 1 \), being \( k = 0 \) the corresponding \( k \) for a flat Universe. We can argue that the recent observational results of WMAP shows that the Universe is close to flat-Euclidean \( (k = -0.0027^{+0.0039}_{-0.0038}) [8] \), so, the main important issue to study is the behaviour for a flat Universe, in our case, a flat FLRW Universe. We have finished the previous Chapter with the table (2.1) that shows the Constituent Equations of our model. We have four equations that are not independent between them.
The pressure is related with the energy density, by the Equation of State. And the Friedmann Equation relates the Hubble parameter with the energy density. Then, we realize that we have only one free variable.

The FLRW Universe uses as hypothesis the adiabatically expansion with null variation of entropy, this in fact means, that the process can be reversible. "As far as mathematics is concerned, one can just as well specify final conditions, at some remote future time and evolve backwards in time. Mathematically, final conditions are just as good as initial ones for determining the evolution of the system"[7]. Thanks to the observational Cosmology, we can have a value for the Hubble parameter for the Present Universe, that is named $H_0$. Because of this, we are going to study the behaviour using the variable $H$ as the free variable, in the sense, that is the variable for which we have some accurate knowledge, and we can use that knowledge of the value $H_0$ as a condition.

The results of the WMAP observations shows that the $H_0$ has a value of $H_0 = (67.8 \pm 0.9) \text{km} \text{s}^{-1} \text{Mpc}^{-1}$. [9].

In the following Chapters, we will find the behaviour of the Hubble parameter as a function of time $H(t)$, for a linear equation of state, in Chapter 4, and for a non-linear equation of state, in Chapter 5. As a result, we will find that there are some cases, with special times for which the Hubble parameter has a singularity time. These means, that there is a time where the Hubble parameter and/or its derivatives diverge to $\infty$. That time is what we name a time singularity in Cosmology. A time singularity can be future or past, depending on the shape of the Equation of State.

We will take a break at this point in order to understand the meaning of this kind of singularities. The time singularity that we will find for $H$, shows where $H$ is defined, as it shows a infinity discontinuity for $H(t_s)$. Following this argument, the cases where $H$ present a future singularity, will be defined from $(-\infty, t_s)$, and the cases where $H$ present a past singularity, will be defined from $(t_s, \infty)$. Furthermore, we have seen that $H = \frac{\dot{a}}{a}$, and the spatial part of the RW metric is multiplied by the scalar factor $a(t)$. In some sense, we can think that this singularities shows us where we can use the model of FLRW Universe (the metric will be nonwell-defined for this time singularities).

We have developed a general classification of the singularities for a linear and for a non-linear EoS (in our case a polynomial EoS). In Cosmology, people use to work with a Equation for a perfect fluid of the kind $P = \omega \rho$, that we have used it in Chapter 4. Later, in Chapter 5, we develop the same analysis but using as equation of state a polynomial relation between pressure and energy density of the kind $P = -\rho + A \rho^\alpha$. This Equation of State holds for different barotropic fluids, i. e. a fluid whose pressure only depends on the energy density, and for special values of $\alpha$ and $A$ it will behave as a perfect fluid again.
Looking in the literature we can found a classification for the future singularities [4] where $t < t_s$:

1. **Type I (Big Rip):** for $t \to t_s$, $a \to \infty$, $\rho \to \infty$ and $|P| \to \infty$

2. **Type II (Sudden):** $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|P| \to \infty$

3. **Type III (Big Freeze):** $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|P| \to \infty$

4. **Type IV (Generalized Sudden):** $t \to t_s$, $a \to a_s$, $\rho \to 0$ and $|P| \to 0$

\text{derivatives of } H \text{ diverge.}

In Chapter 5, we will find some analogous singularities in the past, where $t > t_s$ that in order to be consistent, we are going to define in an analogous way:

1. **Type I (Big Bang):** for $t \to t_s$, $a \to 0$, $\rho \to \infty$ and $|P| \to \infty$

2. **Type II (Past Sudden):** $t \to t_s$, $a \to a_s$, $\rho \to \rho_s$ and $|P| \to \infty$

3. **Type III (Big Hottest):** $t \to t_s$, $a \to a_s$, $\rho \to \infty$ and $|P| \to \infty$

4. **Type IV (Generalized Past Sudden):** $t \to t_s$, $a \to a_s$, $\rho \to 0$ and $|P| \to 0$

\text{derivatives of } H \text{ diverge.}
4. SINGULARITIES IN THE FLAT FLRW FOR A LINEAR EOS

If we impose a linear EoS of a perfect fluid for this first model, i.e. \( P = \omega \rho \), we can rewrite the continuity equation as:

\[ \dot{\rho} = -3H(\omega + 1)\rho, \quad (4.1) \]

and the Raychaudhuri Equation as:

\[ \dot{H} = -\frac{1}{2}(\omega + 1)\rho. \quad (4.2) \]

We can see that when \( \omega = -1 \) we have a critical (or fix) point, so this point doesn’t depend on time. We can distinguish three cases: 1) \( \omega = -1 \), 2) \( \omega < -1 \), 3) \( \omega > -1 \):

1. \( \omega = -1 \).

\[ H^2 = \frac{\rho}{3}, \quad \dot{\rho} = 0, \quad \dot{H} = 0 \]

In this case:

The Hubble parameter is a constant, meaning that the Universe is expanding at a constant rate.

With the following algebra:

\[ \frac{da}{a} = H dt \] and, since \( H = H_0 \),

\[ \ln \frac{a}{a_0} = H_0(t - t_0), \]

arriving to the de Sitter solution:

\[ a(t) = a_0 \exp [H_0(t - t_0)] \quad (4.3) \]

This means that the Universe is expanding or contracting, at a constant rate, although the density is constant. This is a contradiction with the intuitive idea that if something is expanding, the density must be decreasing. However, the negative pressure as \( P = -\rho \) explains this phenomenon.

2. \( \omega < -1 \).

The imposed EoS is \( P = \omega \rho \), so in this case the pressure is negative, and computing the derivatives of the Hubble Parameter and the density, we can
see that both of them increase with time. We have a phantom fluid, i.e., a fluid satisfying \( 1 + \omega < 0 \).

The equations governing the evolution of the flat FLRW Universe for a linear EoS are the following:

\[ H^2 = \frac{\rho}{3}, \quad \dot{H} = -\frac{1}{2} (\omega + 1) \rho \]

From both equations we can obtain:

\[ \dot{H} = -\frac{3}{2} (1 + \omega) H^2. \]

Solving this ordinary differential equation with some algebra

\[ \frac{\dot{H}}{H^2} = -\frac{3}{2} (1 + \omega) \rightarrow \frac{dH}{dt} = -\frac{3}{2} (1 + \omega) dt \rightarrow \frac{1}{H} - \frac{1}{H_0} = \frac{3}{2} (1 + \omega) (t - t_0), \]

we get:

\[ H = \frac{2H_0}{3(1 + \omega)(t - t_0)H_0 + 2}. \] (4.4)

This equation will be defined in all points except when the denominator is zero. The time when this occur is named singularity time, \( t_s \). We can write:

\[ t \rightarrow t_s \leftrightarrow 3 (1 + \omega) (t_s - t_0) H_0 + 2 \rightarrow 0 \]

Then:

\[ t_s - t_0 \rightarrow -\frac{2}{3(1 + \omega)H_0} \]

and taking into account that \( \omega < -1 \) we can write the singularity time as:

\[ t_s - t_0 = \frac{2}{3|1 + \omega|H_0}. \] (4.5)

So, we can see that the \( t_s > t_0 \) always, this means that the time singularity is in the future. The Hubble Parameter is then physically defined in \( (-\infty, t_s) \) (mathematically we can do an extrapolation for times greater than the singularity time, but this have not any physical meaning). In order to see the behaviour of \( a(t) \), we use the expression of the Hubble Parameter related with the scale factor \( H = \frac{\dot{a}}{a} \), and we identificate it with (4.4):

\[ \frac{da}{a} = \frac{2H_0 dt}{3(1 + \omega)(t - t_0)H_0 + 2}. \]
Developing this last, we obtain:

\[ a = a_0 \left[ \frac{3(1 + \omega)(t - t_0) + 2}{2} \right]^{\frac{2}{3(1 + \omega)}}. \]  

(4.6)

\[ \frac{2}{3(1 + \omega)} \] is negative since \( \omega < -1 \) the exponential; meaning this that in our singularity, when \( t \to t_s, a \to \infty \). Furthermore, we have notice before that \( t_s > t_0 \), so our singularity is in the future, and in this case it is named the BIG RIP singularity.

3. \( \omega > -1 \).

For \( \omega > -1 \) the pressure can be both negative and positive. In the case \(-1 < \omega < 0\) the pressure is negative. But, from the conservation equation \( \dot{\rho} = -3H(1+\omega)\rho \) we deduce that the energy density decreases for all values of \( \omega \) that satisfies \( \omega > -1 \).

Again, the equations governing the evolution of the Universe are the following ones:

\[ H^2 = \frac{\rho}{3}, \dot{H} = -\frac{1}{2} (\omega + 1) \rho. \]

From both equations we can obtain:

\[ \dot{H} = -\frac{3}{2} (1 + \omega) H^2. \]

Solving this last ordinary differential equation with some algebra,

\[ \frac{\dot{H}}{H^2} = -\frac{3}{2} (1 + \omega) \rightarrow \frac{dH}{H^2} = -\frac{3}{2} (1 + \omega) dt \rightarrow \frac{1}{H} - \frac{1}{H_0} = \frac{3}{2} (1 + \omega) (t - t_0), \]

we get:

\[ H = \frac{2H_0}{3(1 + \omega)(t - t_0)H_0 + 2}. \]  

(4.7)

In order to compute the time singularity \( t_s \) we follow the calculus:

\( t \to t_s \leftrightarrow 3 (1 + \omega) (t_s - t_0) H_0 + 2 \to 0 \),

getting a new result for the singularity time in the case \( \omega > -1 \):

\[ t_s - t_0 = -\frac{2}{3(1 + \omega)H_0}, \]

and taking into account that \( \omega > -1 \) we can write the singularity time as:

\[ t_s - t_0 \to -\frac{2}{3|1 + \omega|H_0}, \]  

(4.8)

and then, the Hubble Parameter is physically defined in \((t_s, +\infty)\).
As in the previous case, we are going to see the behaviour of $a(t)$:

$$\frac{da}{a} = \frac{2H_0 dt}{3(1 + \omega)(t - t_0)H_0 + 2}.$$ 

Developing this last we obtain with some calculus:

$$a = a_0 \left[ \frac{3(1 + \omega)(t - t_0) + 2}{2} \right]^{\frac{2}{3(1 + \omega)}} \quad (4.9)$$

In this case, $\omega > -1$, the exponential $\frac{2}{3(1 + \omega)}$ is positive; obtaining that when $t \to t_s$, $a \to 0$.

We can see that the $t_s < t_0$. This singularity is named the BIG BANG singularity and it is in the past.

Sumarizing case 2) and case 3) we can write:

$$t_s - t_0 = \pm \frac{2}{3|1 + \omega|H_0} \quad (4.10)$$

a) $t_s - t_0 > 0 \rightarrow$ BIG RIP at $t_s > t_0$. The singularity is in the future.

b) $t_s - t_0 < 0 \rightarrow$ BIG BANG at $t_s < t_0$. The singularity is in the past.

In both cases:

$$H = \frac{2H_0}{3(1 + \omega)(t - t_0)H_0 + 2} \quad (4.11)$$

$$a = a_0 \left[ \frac{3(1 + \omega)(t - t_0) + 2}{2} \right]^{\frac{2}{3(1 + \omega)}} \quad (4.12)$$

$$\dot{H} = \frac{-6H_0^2 (1 + \omega)}{[3(1 + \omega)(t - t_0)H_0 + 2]^2} = -3H \frac{1 + \omega}{3(1 + \omega)(t - t_0)H_0 + 2} \quad (4.13)$$

$$\dot{\rho} = -3H(\rho + P) = -3H\rho(\omega + 1) = -9H^3(\omega + 1) \quad (4.14)$$

$$P = \omega \rho = 3H^2 \omega \quad (4.15)$$

We are going to see that it coincides with the BIG BANG and BIG RIP concepts that we use to learn:
4. Singularities in the flat FLRW for a linear EoS

1. $t_s - t_0 > 0 \rightarrow$ BIG RIP at $t_s > t_0$.

In this case the Hubble parameter $H$ is defined in $(-\infty, t_s)$. We are recalling that, because we want to see the exact behaviour in the limit. We must take into account that when we do the limit from $t \rightarrow t_s$, we are approximating to $t_s$ from the left, following this reasoning and in order to remark it, we are going to do the limit $t \rightarrow t_s^{-}$.

If we define $t_s^-$ that $t \rightarrow t_s^- \rightarrow t_s$ in such a way that $t_s^-$ is closer to $t_s$ from the left, we can write: $H \rightarrow +\infty$.

\[
\lim_{t \rightarrow t_s^-} H = \lim_{t \rightarrow t_s^-} \frac{2H_0}{-3|1 + \omega|(t_s - t_0)H_0 + 2} = \lim_{t \rightarrow t_s^-} \frac{2H_0}{D} = +\infty, \quad (4.16)
\]

since, $t_s^- - t_0 < t_s - t_0 \rightarrow 3|1 + \omega|(t_s - t_0) < 2 \rightarrow \lim_{t \rightarrow t_s^-} D = 0^+$

Furthermore when $t \rightarrow t_s$:

$\dot{a} \rightarrow +\infty$:

\[
\lim_{t \rightarrow t_s^+} a(t) = \lim_{t \rightarrow t_s^+} \frac{a_0}{\left[-3|1 + \omega|(t - t_0)H_0 + 2\right]^{\frac{3}{2|1 + \omega|}}} = +\infty, \quad (4.17)
\]

$\dot{H} \rightarrow +\infty$.

\[
\lim_{t \rightarrow t_s^+} \dot{H} = \lim_{t \rightarrow t_s^+} \frac{-|1 + \omega|}{-3|1 + \omega|(t_s - t_0)H_0 + 2} = \lim_{t \rightarrow t_s^+} \frac{3H}{|1 + \omega|} = +\infty, \quad (4.18)
\]

since, $\lim_{t \rightarrow t_s^-} D = 0^+$.

$\dot{\rho} \rightarrow +\infty$.

\[
\lim_{t \rightarrow t_s^+} \dot{\rho} = \lim_{t \rightarrow t_s^+} -9H^3(|1 + \omega|) = \lim_{t \rightarrow t_s^+} 9H^3(|1 + \omega|) = +\infty \quad (4.19)
\]

2. $t_s - t_0 < 0 \rightarrow$ BIG BANG at $t_s < t_0$.

Analogously, we can think that when $t \rightarrow t_s$ it is going to $t_s$ from the right side since $H$ is defined from $(t_s, +\infty)$.

If we define $t_s^+$ that $t \rightarrow t_s^+ \rightarrow t_s$ in such a way that $t_s^+$ is closer to $t_s$ from the right, and taking into account that $t_s - t_0 < 0$, $|t_s^+ - t_0| < |t_s - t_0|$ obtaining in this case: $H \rightarrow +\infty$ when $t \rightarrow t_s$:

\[
\lim_{t \rightarrow t_s^+} H = \lim_{t \rightarrow t_s^+} \frac{2H_0}{3|1 + \omega|(t_s - t_0)H_0 + 2} = \lim_{t \rightarrow t_s^+} \frac{2H_0}{D} = +\infty, \quad (4.20)
\]
since, $\lim_{t \to t_s^+} D = 0^+$. 

Furthermore when $t \to t_s$:

\[ a \to 0 \]

\[
\lim_{t \to t_s^+} a(t) = \lim_{t \to t_s^+} a_0 \left[ -\frac{3|1 + \omega|(t - t_0)H_0 + 2}{2} \right]^{\frac{2}{1 + \omega}} = 0, \quad (4.21)
\]

\[ \dot{H} \to -\infty \]

\[
\lim_{t \to t_s^+} \dot{H} = \lim_{t \to t_s^+} -3H \frac{|1 + \omega|}{3|1 + \omega|(t_s - t_0)H_0 + 2} = \lim_{t \to t_s^+} -3H \frac{|1 + \omega|}{D} = -\infty, \quad (4.22)
\]

since, $\lim_{t \to t_s^+} D = 0^+$. 

\[ \dot{\rho} \to -\infty. \]

\[
\lim_{t \to t_s^+} \dot{\rho} = \lim_{t \to t_s^+} -9H^3|1 + \omega| = -\infty. \quad (4.23)
\]

This result agrees with the concepts of BIG BANG, (a singularity that starts in a point, with a scale factor $a(t)$ increasing as the energy density $\rho$ decreases), and BIG RIP (the singularity that has negative pressure and where the energy density and the scale factor of the Universe $a(t)$ tends to infinity).
5. SINGULARITIES IN THE FLAT FLRW FOR A NONLINEAR EOS

In this chapter we are going to analyze the singularities when we deal with a nonlinear EoS. In the general case we have:

$$P = -\rho + f(\rho).$$  \hspace{1cm} (5.1)

We will take a polynomial function with exponent $\alpha$:

$$P = -\rho + A\rho^\alpha,$$  \hspace{1cm} (5.2)

and then, the constituent equations are those of table 5.1:

<table>
<thead>
<tr>
<th></th>
<th>Friedmann Equation</th>
<th>Raychaudhuri Equation</th>
<th>Conservation Equation</th>
<th>EoS</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^2 = \frac{\rho}{3}$</td>
<td>$\dot{H} = -\frac{1}{2} A\rho^\alpha$</td>
<td>$\dot{\rho} = -3HA\rho^\alpha$</td>
<td>$P = -\rho + A\rho^\alpha$</td>
<td></td>
</tr>
</tbody>
</table>

Tab. 5.1: Constituent Equations with a Non-linear EoS

Using the Friedman equation with the Raychandhuri equation we obtain a ordinary differential equation that we can write as:

$$\dot{H} = -\frac{3^\alpha}{2} AH^{2\alpha} \leftrightarrow \frac{dH}{H^{2\alpha}} = -\frac{3^\alpha}{2} Adt,$$ \hspace{1cm} (5.3)

and we get:

$$\int_{H_0}^{H} \frac{dH}{H^{2\alpha}} = -\int_{t_0}^{t} \frac{3^\alpha}{2} Adt.$$ \hspace{1cm} (5.4)

In order to integrate this equation, first we are going to consider the different cases in the left side of the equation (5.4). See Table 5.2.
5. Singularities in the flat FLRW for a nonlinear EoS

1. $2\alpha \neq 1 \quad \alpha > 0 \quad \int_{H_0}^H \frac{dH}{H^{2\alpha}} = \frac{1}{2 - 2\alpha} [H^{1-2\alpha} - H_0^{1-2\alpha}]$

2. $2\alpha \neq 1 \quad \alpha < 0 \quad \int_{H_0}^H \frac{dH}{H^{-1+2\alpha}} = \frac{1}{1+2\alpha} [H_1^{1+2\alpha} - H_0^{1+2\alpha}]$

3. $2\alpha = 1 \quad \int_{H_0}^H \frac{dH}{H} = \ln \frac{H}{H_0}$

Tab. 5.2: Possible cases. Left side (5.4)

A. $A > 0 \quad -\int_{t_0}^t \frac{3\alpha}{2} A \, dt = -\frac{3\alpha}{2} |A|(t - t_0)$

B. $A < 0 \quad -\int_{t_0}^t \frac{3\alpha}{2} A \, dt = +\frac{3\alpha}{2} |A|(t - t_0)$

Tab. 5.3: Possible cases. Right side (5.4)

Now, we are going to deal with the right side of (5.4). Table 5.3.

With Table 5.2 and Table 5.3 we are able to distinguish 6 cases in Table 5.4.

1. $2\alpha \neq 1; \alpha \geq 0; A > 0 \quad \frac{1}{2 - 2\alpha} [H^{1-2\alpha} - H_0^{1-2\alpha}] = -\frac{3\alpha}{2} |A|(t - t_0)$

2. $2\alpha \neq 1; \alpha \geq 0; A < 0 \quad \frac{1}{2 - 2\alpha} [H^{1-2\alpha} - H_0^{1-2\alpha}] = +\frac{3\alpha}{2} |A|(t - t_0)$

3. $2\alpha \neq 1; \alpha < 0; A > 0 \quad \frac{1}{1+2\alpha} [H_1^{1+2\alpha} - H_0^{1+2\alpha}] = -\frac{3\alpha}{2} |A|(t - t_0)$

4. $2\alpha \neq 1; \alpha < 0; A < 0 \quad \frac{1}{1+2\alpha} [H_1^{1+2\alpha} - H_0^{1+2\alpha}] = +\frac{3\alpha}{2} |A|(t - t_0)$

5. $2\alpha = 1; A > 0 \quad \ln \frac{H}{H_0} = -\frac{3\alpha}{2} |A|(t - t_0)$

6. $2\alpha = 1; A < 0 \quad \ln \frac{H}{H_0} = +\frac{3\alpha}{2} |A|(t - t_0)$

Tab. 5.4: Possible cases for (5.4)

The Table 5.4 shows the different cases that in what follows we are going to analyze with further detail.

1. $2\alpha \neq 1; \alpha \geq 0; A > 0\quad \frac{1}{2 - 2\alpha} [H^{1-2\alpha} - H_0^{1-2\alpha}] = -\frac{3\alpha}{2} |A|(t - t_0)$
5. Singularities in the flat FLRW for a nonlinear EoS

\[ H = \left[ H_0^{1-2\alpha} - (1 - 2\alpha)\frac{3^\alpha}{2} |A|(t - t_0) \right]^{\frac{1}{1-2\alpha}} \]  

(5.5)

(a) \( \alpha > \frac{1}{2} \leftrightarrow (1 - 2\alpha) < 0 \)

We can write:

\[ H = \frac{1}{(B + C(t-t_0))^p} \]  

(5.6)

where \( p = \frac{1}{|1-2\alpha|} \), \( p \in \mathbb{R} \), \( B = \frac{1}{H_0^{\frac{1}{p}}} \), \( C = \frac{3^\alpha}{2p} |A| \). Note that \( B > 0 \) and \( C > 0 \).

i. \( p < 1 \leftrightarrow \alpha > 1 \):

If \( B + C(t_t_0) = 0 \) equivalently \( t_s - t_0 = -\frac{B}{C} \). When \( t \rightarrow t_s \),

\( H \rightarrow +\infty \). In this case we have a past singularity.

Considering \( H = \frac{a}{a} \) and integrating to obtain \( a(t) \):

\[ \ln \frac{a}{a_0} = \frac{(B+C(t-t_0))^{1-p} - B^{1-p}}{C(1-p)} \]

meaning this that we can express \( a(t) \):

\[ a = a_0 \exp \left[ \frac{(B+C(t-t_0))^{1-p} - B^{1-p}}{C(1-p)} \right] \]  

(5.7)

In this case, the scalar factor of the Universe \( a \rightarrow a_s = \text{constant} \).

We can continue the analysis to obtain the behaviour of \( \dot{H} \) at \( t \rightarrow t_s \):

\[ \dot{H} = \frac{-pC}{(B + C(t-t_0))^{p+1}}, \]  

(5.8)

and we obtain \( \dot{H} \rightarrow -\infty \) when \( t \rightarrow t_s \).

ii. \( p = 1 \leftrightarrow \alpha = 1 \) and \( A > 0 \):

Now, we can write (5.6) as,

\[ H = \frac{1}{B + C(t-t_0)} \]  

(5.9)

And we can see that the time singularity occurs for: \( t_s - t_0 = -\frac{B}{C} \), and in such time: \( H \rightarrow +\infty \). But now we have a particular solution for \( a(t) \):

\[ a(t) = a_0 \left[ \frac{B + C(t-t_0)}{B} \right]^{\frac{1}{C}} \]  

(5.10)
where the particular value of $a_s$ is:

$$a_s \to 0$$

We can see that we have obtained the same singularity in the previous chapter, where we had considered a linear EoS, the BIG BANG singularity.

iii. $p > 1 \leftrightarrow \frac{1}{2} < \alpha < 1$ In this case we consider again:

$$H = \frac{1}{(B + C(t - t_0))^p},$$

(5.11)

with $p \neq 1$, $\dot{H}$ will be the same as in the case 1.a.i:

$$\dot{H} = \frac{-pC}{(B + C(t - t_0))^{p+1}},$$

(5.12)

but, the difference with respect the case 1.a.i is $a(t)$:

$$a = a_0 \exp \left[ \frac{(B + C(t - t_0))^{1-p} - B^{1-p}}{C(1-p)} \right],$$

(5.13)

at first sight it looks closer to the one given in case 1.a.i, but taking care, we can see in this case $1 - p < 0$, and we can write:

$$a = a_0 \exp \left[ \frac{-1}{C|1-p|} \left( \frac{1}{(B + C(t - t_0))|1-p|} - \frac{1}{B|1-p|} \right) \right],$$

(5.14)

Obtaining as a result that $a \to 0 \leftrightarrow t \to t_s$, as in the previous case 1.a.ii, with the difference that in this case it tends to zero exponentially.

(b) $0 \leq \alpha < \frac{1}{2} \leftrightarrow (1 - 2\alpha) > 0$

In this case, we must distinguish between different cases in order to consider all the possible singularities.

i. $\frac{1}{1-2\alpha} = n \mid n \in \mathbb{N} \text{ and } n > 1$, what happens when: $\alpha = \frac{1}{2} \left( \frac{n-1}{n} \right)$

We can write our solution as:

$$H = (B - C(t - t_0))^n$$

(5.15)

Where in (5.15), $B = H_0^{1-2\alpha}$, $C = \frac{2(1-2\alpha)}{n}$ and both are $B > 0$ and $C > 0$.

We can see that the derivatives of $H$ respect time will not have any point in which $H$ is not defined in the real line, in other words, $H$ will only tend to $+\infty$ when the time tends to $-\infty$.

Considering $H = \frac{a}{a}$ we are able to integrate this to obtain $a(t)$:
5. Singularities in the flat FLRW for a nonlinear EoS

\[ \ln \frac{a}{a_0} = \frac{-\((B - C(t-t_0))^{n+1}\)}{C(n+1)} + \frac{B^{n+1}}{C(n+1)} \]

or:

\[ a = a_0 \exp \left[ \frac{-\((B - C(t-t_0))^{n+1} + B^{n+1}\)}{C(n+1)} \right] \] (5.16)

In this case we have that if \( H \to 0^+ \), or the same \( t_s - t_0 \to \frac{B}{C} \), then \( a \to a_s \), this happens for a future time \( t_s > t_0 \), then \( a \to a_s = \text{constant} \) such that \( a_s > a_0 \). (We have distinguished the time \( t_s \) from a singular time \( t_s \) because this is not a singular time when we consider a natural exponent, but, in the next we will find an analogous expression for the singular time when we consider rational and real numbers.

We will obtain a special value for the derivatives of \( H \) in this particular case. We can see that the first derivative is:

\[ \dot{H} = -C(B - C(t-t_0))^{n-1} \] (5.17)

The second derivative will be:

\[ \ddot{H} = C^2(B - C(t-t_0))^{n-2} \] (5.18)

And we can find a general case for an m-derivative:

\[ \frac{d^m H}{dt^m} = (-1)^m C^m (B - C(t-t_0))^{n-m} \prod_{m=1}^{m} (n - m + 1) \] (5.19)

All the derivatives of \( H \) will tend to zero at \( t_s \). The m-derivative or higher will be zero for every value of \( t \) if \( m = n + 1 \).

ii. \( \frac{1}{1-2\alpha} = x \in \mathbb{R}^+ \setminus \mathbb{N} \) and \( 0 < \alpha < \frac{1}{2} \):

\[ H = (B - C(t-t_0))^q \] (5.20)

Where we consider, as in the previous case, \( B = H_{0}^{1-2\alpha}, C = \frac{3^\alpha}{2}(1 - 2\alpha)|A| \) and remember that \( B > 0 \) and \( C > 0 \). If \( B - C(t-t_0) = 0 \to t_s - t_0 = \frac{B}{C} \), this happens for a time \( t_s > t_0 \), for a future time.

And in this case we find that:

\[ a = a_0 \exp \left[ \frac{-\((B - C(t-t_0))^{x+1} + B^{x+1}\)}{C(x + 1)} \right] \] (5.21)

Meaning that \( a \to a_s = \text{constant} \) constant when \( t \to t_s \)

Computing the first derivative of \( H \):

\[ \frac{dH}{dt} = -C(B - C(t-t_0))^{x-1}, \] (5.22)
getting in the limit for the singular time: \( \dot{H} \rightarrow 0 \). But in this particular case the exponent is not a natural number and we can find always some higher order derivative of \( H \) such that derivative goes to \( \infty \) at this point (for \( t > t_s \) the function is not defined). If we compute the general result for a \( k \)-derivative of \( H \) respect time:

\[
\frac{d^k H}{dt^k} = (-1)^k C^k (B - C(t - t_0))^{x-k} \prod_{k=1}^{k} (x - k + 1). \tag{5.23}
\]

We can notice that when \( x + 1 < k \) we will have that the \( k \)-derivative of \( H \) diverges. At this point we must take care of the sign of each factor: \( \prod_{k=1}^{k} (x - k + 1) \) will be negative for each \( k \geq x + 1 \). The factor \((-1)^k\) will be positive or negative depending if \( k \) is even or odd. Summarizing the last considerations:

- If \( k \) is odd and \( x + 1 < k \) then \( \frac{d^k H}{dt^k} \rightarrow +\infty \)
- If \( k \) is even and \( x + 1 < k \) then \( \frac{d^k H}{dt^k} \rightarrow -\infty \)

iii. The special case when \( n = 1 = \frac{1}{1-2\alpha} \leftrightarrow \alpha = 0 \)

In this case there are no singularity times:

\[
H = B - C(t - t_0) \tag{5.24}
\]

\[
a = a_0 \exp\left[B(t - t_0) - \frac{C}{2}(t - t_0)^2\right] \tag{5.25}
\]

and the derivative of \( H \):

\[
\dot{H} = -C \tag{5.26}
\]

There is an special time named \( t_* \) such that when \( t \rightarrow t_* \), \( H \rightarrow 0 \), \( a \rightarrow \text{constant} \) and \( \dot{H} = -C \)

2. \( 2\alpha \neq 1; \alpha \geq 0; A < 0 \)

\[
\frac{1}{1-2\alpha}[H^{1-2\alpha} - H_0^{1-2\alpha}] = + \left[ 2 - 3\alpha \right] |A|(t - t_0)
\]

\[
H = \left\{ H_0^{1-2\alpha} + (1 - 2\alpha) \frac{3^\alpha}{2} |A|(t - t_0) \right\}^{\frac{1}{1-2\alpha}} \tag{5.27}
\]

As the previous case, 1., we must consider different cases:
5. Singularities in the flat FLRW for a nonlinear EoS

(a) \( \alpha > \frac{1}{2} \leftrightarrow (1 - 2\alpha) < 0 \) We can write:

\[
H = \frac{1}{(B - C(t - t_0))^p}
\]  

(5.28)

where \( p = \frac{1}{1 - 2\alpha} \), \( p \in \mathbb{R} \), \( B = \frac{1}{H_0^p} \), \( C = \frac{3\alpha}{2p^2}|A| \), \( B > 0 \) and \( C > 0 \).

i. Case \( p \in \mathbb{R} \setminus \{1\} \) and \( p < 1 \leftrightarrow \alpha > 1 \):

If \( B - C(t_s - t_0) = 0 \) or \( t_s - t_0 = \frac{B}{C} \), \( H \to +\infty \). In this case we have a future singularity, \( t_s > t_0 \).

Taking \( H = \frac{\dot{a}}{a} \) and integrating \( H(t) \) we obtain:

\[
\ln \frac{a}{a_0} = -\frac{(B - C(t - t_0)^{(1-p)})}{C(1-p)} + \frac{A^{1-p}}{C(1-p)}
\]

and we can express in this case \( a(t) \) as:

\[
a = a_0 \exp \left[ \frac{B^{1-p} - (B - C(t - t_0))^{(1-p)}}{C(1-p)} \right] 
\]  

(5.29)

When \( t \to t_s \) then \( a \to a_s = a_0 \exp \frac{B^{1-p}}{C(1-p)} \) that is a constant bigger than \( a_0 \) so \( (1 - p) > 0 \).

And, we can see that \( \dot{H} \to +\infty \) when \( t \to t_s \):

\[
\dot{H} = \frac{pC}{[B - C(t - t_0)]^{p+1}}
\]  

(5.30)

ii. Case \( p = 1 \).

If we name \( \frac{1}{1 - 2\alpha} = r \), then \( \alpha = 0 \leftrightarrow r = 1 \) and \( \alpha = 1 \leftrightarrow r = -1 \). As we are considering only the case \( \alpha > \frac{1}{2} \), then, we will have \( r = -1 \), that if we use eq.(97), it is equivalent to \( p = 1 \).

With \( p = 1 \) we are going to have again a linear equation of state: \( p = -\rho + A\rho \) and depending on the value of \( A \), in this case is \( A < 0 \), we will obtain again the value of general relativity for a linear EoS with negative pressure.

\[
H = \frac{1}{B - C(t - t_0)}
\]  

(5.31)

In this case we have the same singularity for \( H \) as before: \( t_s - t_0 = \frac{B}{C} \), \( H \to +\infty \). But in this case we have a particular solution for \( a(t) \):

\[
a(t) = a_0 \left[ \frac{B}{B - C(t - t_0)} \right]^{\frac{1}{r}}
\]  

(5.32)

In this case the particular value of \( a_s \) is:
5. Singularities in the flat FLRW for a nonlinear EoS

\( a_s \to +\infty \) when \( t \to t_s \)

And \( \dot{H} \to +\infty \) when \( t \to t_s \) since:

\[
\dot{H} = \frac{C}{B - C(t - t_0)}
\]  
(5.33)

iii. Case \( p \in \mathbb{R} \setminus \{1\} \) and \( p > 1 \leftrightarrow \alpha < 1 \) If \( B - C(t_s - t_0) = 0 \) or \( t_s - t_0 = \frac{B}{C} \), \( H \to +\infty \). In this case we have a future singularity, \( t_s > t_0 \).

Taking \( H = \frac{\dot{a}}{a} \) and integrating \( H(t) \) we obtain:

\[
\ln a_a = - \frac{B}{C} \alpha \left( \frac{1}{1 - p} \right) + \left( \frac{1}{C(1-p)} \right) C
\]

and we can express in this case \( a(t) \) as:

\[
a = a_0 \exp \left[ \left( \frac{B}{C} \alpha \right) \left( \frac{1}{1 - p} \right) + \left( \frac{1}{C(1-p)} \right) C \right]
\]  
(5.34)

When \( t \to t_s \) then \( a \to +\infty \) again, but in this case it tends to infinity exponentially with time, and in the previous potentially.

And, we can see that \( \dot{H} \to +\infty \) when \( t \to t_s \):

\[
\dot{H} = \frac{pC}{[B - C(t - t_0)]^{p+1}}
\]  
(5.35)

Note that we have considered \( B = \frac{1}{H_0^2}, C = \frac{3\alpha}{2p} |A| \) and \( B > 0, C > 0 \).

(b) \( \alpha < \frac{1}{2} \leftrightarrow (1 - 2\alpha) > 0 \). Here, we must distinguish different cases:

i. \( \frac{1}{1 - 2\alpha} = n, n \in \mathbb{N} \mid n > 1 \)

\[
H = [B + C(t - t_0)]^n,
\]  
(5.36)

where \( B = \frac{1}{H_0^2}, C = \frac{3\alpha}{2p} |A|, B > 0 \) and \( C > 0 \). In this case, the derivatives of \( H \) have no singularity. \( H \to 0 \) when \( t_s - t_0 = -\frac{B}{C} \), so it is a past point, \( t_s < t_0 \). Computing \( a(t) \):

\[
a = a_0 \exp \left[ \left( \frac{B}{C} \alpha \right) \left( \frac{1}{1 - p} \right) + \left( \frac{1}{C(1-p)} \right) C \right]
\]  
(5.37)

And computing \( \dot{H}(t) \):

\[
\dot{H} = nC [B + C(t - t_0)]^{n-1}
\]  
(5.38)

In other words, we obtain, \( H \to 0, a \to a_s = constant \) and \( H \to 0 \), when \( t \to t_s \).
ii. \( \frac{1}{1-2\alpha} = q, p \in \mathbb{R}^+ \setminus \mathbb{N} \)

\[
H = [B + C(t - t_0)]^p, \tag{5.39}
\]

where \( B = H_0^{1 \over 2} \), \( C = \frac{3^n}{2|A|} \), \( B > 0 \) and \( C > 0 \). We will have \( H \to 0 \) if \( t_s - t_0 = -\frac{B}{C} \), in the past.

But, in this case, \( p < 1 \) as \( \alpha > 0 \). So, the first derivative of \( H \) respective time will diverge: \( \dot{H} \to +\infty \) when \( t \to t_s \). And computing \( a(t) \):

\[
a = a_0 \exp \left[ \frac{(B + C(t - t_0))^{n+1} - B^{p+1}}{C(q + 1)} \right] \tag{5.40}
\]

we get: \( a \to a_s = \text{constant} \) when \( t \to t_s \). And we can show that:

\[
d^kH \over dt^k = C^k [B + C(t - t_0)]^{p-k} \prod_{k=1}^k (p - k + 1) \tag{5.41}
\]

(Note this case is different from the previous one since the derivative of \( H \) diverges)

iii. The special case \( n = 1 \). In this case we must obtain the same result as in General Relativity, so when \( n = 1 \), it means that \( \alpha = 0 \) and our EoS is linearly dependent with the density: \( P = -\rho \). In this special case, we can write:

\[
H = B + C(t - t_0) \tag{5.42}
\]

and we obtain that \( H \to 0 \), if \( t_s - t_0 = -\frac{B}{C} \), in the past.

And \( \dot{H} = C \) for every possible time.

If we compute \( a(t) \):

\[
a = a_0 \exp \left[ B(t - t_0) + \frac{C}{2}(t - t_0)^2 \right] \tag{5.43}
\]

Here we can see that there are no singularities.

3. \( 2\alpha \neq 1; \alpha < 0; A > 0 \)

In this case, our integral (62), takes the solution:

\[
\frac{1}{1+|2\alpha|} [H^{1+|2\alpha|} - H_0^{1+|2\alpha|}] = -\frac{3^n}{2} |A|(t - t_0)
\]

That we can express as:

\[
H = [B - C(t - t_0)]^p \tag{5.44}
\]

where we have considered \( p = \frac{1}{1+|2\alpha|}, B = H_0^{1 \over 2}, C = \frac{3^n}{2p} |A| \), \( B > 0 \) and \( C > 0 \).
5. Singularities in the flat FLRW for a nonlinear EoS

In cases 3rd and 4th, as $\alpha < 0$ we have that $1 + |2\alpha| > 0$. Furthermore, $p$ cannot be a natural number $n$ because $1 + |2\alpha| > 1$. We can consider a general case such that $p \in \mathbb{R}$ (we can do this if $B > C(t - t_0)$, in other words, $t \in (-\infty, t_*)$). In addition, $\alpha \neq 1$ since 1 is a natural number and $\alpha$ is strictly less than zero. (the case $\alpha = 0$ has been considered before.

With this considerations, we have only 1 case: $p \in \mathbb{R}$
This is the general case where we can write:

$$ H = [B - C(t - t_0)]^p. $$

This case doesn’t present any singularity in $H$, but the first derivative (for every value of $p$) diverges at $t_* - t_0 = \frac{B}{C}$, a future time. Since $1 + |2\alpha| > 1$, $p < 1$, and the first derivative will diverge:

$$ \dot{H} = -Cp [B - C(t - t_0)]^{p-1} $$ (5.45)

Computing $a(t)$ we obtain:

$$ a = a_0 \exp \left[ \frac{B^{p+1} - (B - C(t - t_0))^{p+1}}{C} \right] $$ (5.46)

When $t \to t_*:
- \dot{H} \to -\infty
- H \to 0
- a \to a_* = \text{constant} > a_0

4. $2\alpha \neq 1; \alpha < 0; A < 0$

In this case, our integral (62), takes the solution: $\frac{1}{1+|2\alpha|} \left[ H^{1 + |2\alpha|} - H_0^{1 + |2\alpha|} \right] = \frac{3^n}{2^p} |A| (t - t_0)$

That we can express as:

$$ H = [B + C(t - t_0)]^p $$ (5.47)

where we have considered $p = \frac{1}{1+|2\alpha|}, B = H_0^{\frac{1}{2p}}, C = \frac{3^n}{2^p} |A|$, $B > 0$ and $C > 0$.

With the considerations of 3., we have only 1 case again: $p \in \mathbb{R}$
This is the general case where we can write:

$$ H = [B + C(t - t_0)]^p. $$
5. Singularities in the flat FLRW for a nonlinear EoS

This case doesn’t present any singularity in $H$, but the first derivative derivative (for every value of $p$) that diverges at $t_s - t_0 = -\frac{B}{C}$, a past time.

$$\dot{H} = +Cp [B + C(t - t_0)]^{p-1}$$

(5.48)

Computing $a(t)$ we obtain:

$$a = a_0 \exp \left[ \frac{(B + C(t - t_0))^{p+1} - B^{p+1}}{C} \right]$$

(5.49)

- $\dot{H} \to +\infty$
- $H \to 0$
- $a \to a_s = constant < a_0$

5. $2\alpha = 1; A > 0$

The cases $2\alpha = 1$ have a particular solution for the right side of the identity (62), that is a logarithm. In this case $A$ is bigger than 0.

$$\ln \left[ \frac{H}{H_0} \right] = -\frac{3\alpha}{2} |A|(t - t_0),$$

(5.50)

So, we can express $H$ as:

$$H = H_0 \exp \left[ -\frac{3\alpha}{2} |A|(t - t_0) \right],$$

(5.51)

And, remember that $H = \frac{\dot{a}}{a}$ we can express $a(t)$ as:

$$a(t) = a_0 \exp \left\{ \frac{-2}{3\alpha |A|} H_0 \left[ \exp \left( -\frac{3\alpha}{2} |A|(t - t_0) \right) - 1 \right] \right\}$$

(5.52)

That has no singularities: in analogy to the LITTLE RIP, which we will see in (5.55), we call this model LITTLE BANG.

6. $2\alpha = 1; A < 0$

This case is analogous to the fifth case but now, we consider $A < 0$. Now, the identity (5.4) can be expressed as:

$$\ln \left[ \frac{H}{H_0} \right] = +\frac{3\alpha}{2} |A|(t - t_0),$$

(5.53)
So, we can express $H$ as:

$$H = H_0 \exp \left( \frac{3\alpha}{2} |A|(t - t_0) \right),$$  \hspace{1cm} (5.54)

And, remember that $H = \frac{3}{2}$ we can express $a(t)$ as:

$$a(t) = a_0 \exp \left\{ \frac{2}{3^\alpha |A|} H_0 \left[ \exp \left( \frac{3\alpha}{2} |A|(t - t_0) \right) - 1 \right] \right\},$$  \hspace{1cm} (5.55)

This case is called LITTLE RIP [4].

We have considered only when the equation of $H$ is Physically defined. From the Mathematical point of view we can do an extrapolation of this equation, to see how is its behaviour before or after the singularity (depending if it is a past or a future singularity).

Once we have arrived here, we have analized all the possible cases of the identity (5.4) and we are able to summarize all the possible singularities in a single table named Table(5.5):
1.a.i \( \alpha > 1 \) \( A > 0 \) \( p < 1 \) \( t \rightarrow t_s(\text{past}) \) \( H \rightarrow +\infty \) \( \dot{H} \rightarrow -\infty \) \( a \rightarrow a_s = \text{constant} \)

1.a.ii \( \alpha = 1 \) \( A > 0 \) \( p = 1 \) \( t \rightarrow t_s(\text{past}) \) \( H \rightarrow +\infty \) \( \dot{H} \rightarrow -\infty \) \( a \rightarrow 0(\text{polynomial}) \)

1.a.iii \( \frac{1}{2} < \alpha < 1 \) \( A > 0 \) \( p > 1 \) \( t \rightarrow t_s(\text{past}) \) \( H \rightarrow +\infty \) \( \dot{H} \rightarrow -\infty \) \( a \rightarrow 0(\text{exponentially}) \)

1.b.i \( 0 < \alpha < \frac{1}{2} \) \( A > 0 \) \( p \in \mathbb{N} \setminus \{1\} \) \( t \rightarrow t_s(\text{future}) \) \( H \rightarrow 0 \) \( \dot{H} \rightarrow 0 \) \( a \rightarrow a_s = \text{constant} \)

1.b.ii \( 0 < \alpha < \frac{1}{2} \) \( A > 0 \) \( p \in \mathbb{R} \setminus \mathbb{N} \) \( t \rightarrow t_s(\text{future}) \) \( H \rightarrow 0 \) \( \frac{d^k H}{dt^k} \rightarrow \pm\infty \) \( a \rightarrow a_s = \text{constant} \)

1.b.iii \( \alpha = 0 \) \( A > 0 \) \( p = 1 \) \( t \rightarrow t_s(\text{future}) \) \( H \rightarrow 0 \) \( \dot{H} \rightarrow -C \) \( a \rightarrow a_s = \text{constant} \)

2.a.i \( \alpha > 1 \) \( A < 0 \) \( p = 1 \) \( t \rightarrow t_s(\text{future}) \) \( H \rightarrow +\infty \) \( \dot{H} \rightarrow +\infty \) \( a \rightarrow a_s = \text{constant} \)

2.a.ii \( \alpha = 1 \) \( A < 0 \) \( p = 1 \) \( t \rightarrow t_s(\text{future}) \) \( H \rightarrow +\infty \) \( \dot{H} \rightarrow +\infty \) \( a \rightarrow +\infty(\text{polynomial}) \)

2.a.iii \( \frac{1}{2} < \alpha < 1 \) \( A < 0 \) \( p > 1 \) \( t \rightarrow t_s(\text{future}) \) \( H \rightarrow +\infty \) \( \dot{H} \rightarrow +\infty \) \( a \rightarrow +\infty(\text{exponentially}) \)

2.b.i \( 0 < \alpha < \frac{1}{2} \) \( A < 0 \) \( n > 1, n \in \mathbb{N} \setminus \{1\} \) \( t \rightarrow t_s(\text{past}) \) \( H \rightarrow 0 \) \( \dot{H} \rightarrow 0 \) \( a \rightarrow a_s = \text{constant} \)

2.b.ii \( 0 < \alpha < \frac{1}{2} \) \( A < 0 \) \( p > 1, p \in \mathbb{R}^+ \setminus \mathbb{N} \) \( t \rightarrow t_s(\text{past}) \) \( H \rightarrow 0 \) \( \dot{H} \rightarrow +\infty \) \( a \rightarrow a_s = \text{constant} \)

2.b.iii \( \alpha = 0 \) \( A < 0 \) \( p = 1 \) \( t \rightarrow t_s(\text{past}) \) \( H \rightarrow 0 \) \( \dot{H} \rightarrow C \) \( a \rightarrow a_s = \text{constant} \)

3. \( \alpha < 0 \) \( A > 0 \) \( - \) \( t \rightarrow t_s(\text{future}) \) \( H \rightarrow 0 \) \( \dot{H} \rightarrow -\infty \) \( a \rightarrow a_s = \text{constant} \)

4. \( \alpha < 0 \) \( A < 0 \) \( - \) \( t \rightarrow t_s(\text{past}) \) \( H \rightarrow 0 \) \( \dot{H} \rightarrow +\infty \) \( a \rightarrow a_s = \text{constant} \)

5. \( 2\alpha = 1 \) \( A > 0 \) \( - \) \( \text{PAST} \) \( - \) \( - \) \( \text{LITTLE BANG} \)

6. \( 2\alpha = 1 \) \( A < 0 \) \( - \) \( \text{FUTURE} \) \( - \) \( - \) \( \text{LITTLE RIP} \)

Tab. 5.5: Results in the cases of table 4
Summarizing our results in terms of an $A$ positive or negative, we can build the following tables 5.6 and 5.7:
### Singularities in the flat FLRW for a nonlinear EoS


<table>
<thead>
<tr>
<th>$A &lt; 0$</th>
<th>$\alpha &lt; 0$</th>
<th>$\alpha = 0$</th>
<th>$0 &lt; \alpha &lt; \frac{1}{2}$</th>
<th>$\alpha = \frac{1}{2}$</th>
<th>$\frac{1}{2} &lt; \alpha &lt; 1$</th>
<th>$\alpha = 1$</th>
<th>$\alpha &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H \to 0$</td>
<td>$H \to 0$</td>
<td>$H \to 0$</td>
<td>$H \to +\infty$ (natural exponent)</td>
<td>$H \to +\infty$</td>
<td>$H \to +\infty$</td>
<td>$H \to +\infty$</td>
<td>$H \to +\infty$</td>
</tr>
<tr>
<td>$\dot{H} \to -C$</td>
<td>$\dot{H} \to +\infty$ (no-natural exponent)</td>
<td>$\dot{H} \to +\infty$</td>
<td>$\dot{H} \to +\infty$</td>
<td>$\dot{a} \to +\infty$</td>
<td>$\dot{a} \to +\infty$</td>
<td>$\dot{a} \to +\infty$</td>
<td>$\dot{a} \to +\infty$</td>
</tr>
<tr>
<td>$a \to \text{constant}$</td>
<td>$a \to \text{constant}$</td>
<td>$a \to \text{constant}$</td>
<td>$a \to \text{constant}$</td>
<td>$a \to \text{constant}$</td>
<td>$a \to \text{constant}$</td>
<td>$a \to \text{constant}$</td>
<td>$a \to \text{constant}$</td>
</tr>
<tr>
<td>no singularity</td>
<td>no singularity</td>
<td>no singularity</td>
<td>no singularity</td>
<td>no singularity</td>
<td>no singularity</td>
<td>no singularity</td>
<td>no singularity</td>
</tr>
</tbody>
</table>

*Tab. 5.6: Singularities for $A < 0$*
<table>
<thead>
<tr>
<th>$\alpha &lt; 0$</th>
<th>$\alpha = 0$</th>
<th>$0 &lt; \alpha &lt; \frac{1}{2}$</th>
<th>$\alpha = \frac{1}{2}$</th>
<th>$\frac{1}{2} &lt; \alpha &lt; 1$</th>
<th>$\alpha = 1$</th>
<th>$\alpha &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H \to 0$</td>
<td>$\dot{H} \to C$</td>
<td>$\dot{H} \to 0$ (natural exponent)</td>
<td>$\dot{H} \to 0$ (natural exponent)</td>
<td>$\dot{H} \to \infty$</td>
<td>$\dot{H} \to \infty$</td>
<td>$\dot{H} \to \infty$</td>
</tr>
<tr>
<td>no singularity</td>
<td>$\dot{H} \to \infty$</td>
<td>$\dot{H} \to \infty$</td>
<td>$\dot{H} \to \infty$</td>
<td>$\dot{H} \to \infty$</td>
<td>$\dot{H} \to \infty$</td>
<td>$\dot{H} \to \infty$</td>
</tr>
<tr>
<td>$a \to \text{ctt}$</td>
<td>$\dot{a} \to \text{ctt}$</td>
<td>TYPE IV (non-natural exponent)</td>
<td>TYPE I (PAST)</td>
<td>TYPE III (PAST)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Tab. 5.7:** Singularities for $A > 0$
To conclude, we are going to compare our results with the ones obtained in the reference [5]. In order to do this, we are going to build a table with the results obtained by Lopez Revelles (table 6.1): We can build for example based on the results of Revelles:

\[
\begin{align*}
\alpha &= 2 & \alpha &= \frac{2}{3} & \alpha &= \frac{1}{5} & \alpha &= -2 \\
H &\propto (t - t_s)^{\frac{1}{3}} & H &\propto (t - t_s)^{-\frac{1}{3}} & H &\propto (t - t_s)^{\frac{1}{2}} & H &\propto (t - t_s)^{\frac{1}{2}} \\
\dot{H} &\propto (t - t_s)^{-\frac{1}{3}} & \dot{H} &\propto (t - t_s)^{-\frac{2}{3}} & \dot{H} &\propto (t - t_s)^{-\frac{1}{3}} & \dot{H} &\propto (t - t_s)^{-\frac{1}{3}} \\
a &\propto (t - t_s)^{\frac{1}{2}} & a &\propto (t - t_s)^{-2} & a &\propto (t - t_s)^{\frac{1}{2}} & a &\propto (t - t_s)^{\frac{1}{2}} \\
t &\to t_s & t &\to t_s & t &\to t_s & t &\to t_s \\
H &\to \infty & H &\to \infty & H &\to H_s & H &\to H_s \\
\dot{H} &\to \infty & \dot{H} &\to \infty & \dot{H} &\to H_s & \dot{H} &\to \infty \\
\rho &\to \infty & \rho &\to \infty & \rho &\to \rho_s & \rho &\to \rho_s \\
a &\to a_s & a &\to \infty & a &\to a_s & a &\to a_s \\
|P| &\to \infty & |P| &\to \infty & |P| &\to |P_s| & |P| &\to \infty \\
\text{SINGULARITY} & \text{TYPE III} & \text{TYPE I} & \text{TYPE IV} & \text{GENERALIZED SUDDEN} & \text{TYPE II} & \text{SUDDEN} \\
\text{BIG FREEZE} & \text{BIG RIP} & \text{GENERALIZED SUDDEN} & \text{SUDDEN} & \text{SUDDEN} & \text{SUDDEN} & \text{SUDDEN}
\end{align*}
\]

Tab. 6.1: Classification of Singularities

The results of A. J. Lopez Revelles in [5] indicate that there are four future singularities in the case \( P = -\rho + Ap^\alpha \) for a concrete values of \( \alpha \).

Two of them corresponds to a density that decrease with time: Type IV (Generalized Sudden) and Type II (Sudden). We can find these singularities in table 5.7 (cases of \( A > 0 \)).

The other two corresponds to a density that decreases with time: Type III (Big Freeze) and Type I (Big Rip). This corresponds to our future singularities of table 5.6 (cases of \( A < 0 \)).

We have done the development of our equations for \( H \). We recall again the Friedmann equation:

\[
H^2 = \frac{\rho}{3},
\]

in order to understand that the behaviour of \( H \) in the limit corresponds to the be-
haviour of $\rho$ in the limit with the difference that they tends to 0 or to $\infty$ with a different power, then we can consider that it is the same behaviour.

In table 5.6 for $A < 0$ we found the following singularities for $t \to t_s$:

1. PAST TYPE II. PAST SUDDEN $\rho \to 0$, $|P| \to \infty$, $a \to a_s < a_0$

2. PAST TYPE IV. PAST GENERALIZED SUDDEN. $\rho \to 0$, $|P| \to 0$ (Higher derivatives diverge), $a \to a_s < a_0$

3. FUTURE TYPE I. BIG RIP $\rho \to \infty$, $|P| \to \infty$, $a \to \infty$

4. FUTURE TYPE III. BIG FREEZE. $\rho \to \infty$, $|P| \to \infty$, $a \to a_s > a_0$

We can see that the behaviour of $\rho$ is completely antiituitive. Consider that as the time passes the scalar factor of the universe is increasing, how can we explain that $\rho$ decreases for this cases? The shape of the dynamics of the variables $\rho$ and $H$ are taken from these equations:

$$\dot{H} = -\frac{1}{2}A\rho^\alpha$$ (6.2)

$$\dot{\rho} = -3HA\rho^\alpha$$ (6.3)

These equations show that for $A < 0$, $H$ and $\rho$ increase with time, as their derivatives are both positive. Recalling that $a(t)$ in all the results increase with time: for all the cases $A < 0$, $\rho$ increases as $a(t)$ increases. This means that the past and future singularities for $A < 0$ corresponds to the behaviour of a phantom fluid.

The singularities that we have found in table 5.7 for the cases $A > 0$ are completely analogous, with the difference that they behave as a fluid, $\rho$ decreases as $a(t)$ increases. To be consistent we are going to enumerate them:

1. FUTURE TYPE II. FUTURE SUDDEN $\rho \to 0$, $|P| \to \infty$, $a \to a_s > a_0$

2. FUTURE TYPE IV. FUTURE GENERALIZED SUDDEN. $\rho \to 0$, $|P| \to 0$ (higher derivatives diverges), $a \to a_s > a_0$

3. PAST TYPE I. BIG BANG $\rho \to \infty$, $|P| \to \infty$, $a \to 0$

4. PAST TYPE III. BIG HOTTEST. $\rho \to \infty$, $|P| \to \infty$, $a \to a_s < a_0$
APPENDIX
A. COEFFICIENTS OF THE EINSTEIN TENSOR

In Chapter 2, we have done the development of the Einstein Tensor and the Friedmann and the Acceleration Equations for $k = 0$ because we are working all the time in a flat Universe. If we consider $k \neq 0$, we can do an analogous development.

Using (2.15) and (2.16):

\begin{equation}
\Gamma_{ijk} = \frac{1}{2}(g_{jk,i} + g_{ki,j} - g_{ij,k}), \quad (A.1)
\end{equation}

\begin{equation}
\Gamma_{ij}^{h} = \Gamma_{ijk}g^{kh}. \quad (A.2)
\end{equation}

Where (A.1) are named the Christoffel Symbols of first class and (A.2) are named the Christoffel Symbols of second class. We have seen that with the symbols of Christoffel we are able to compute the Ricci Tensor $R_{ab}$ and the Scalar Curvature $R$:

\begin{equation}
R_{ab} = \Gamma_{ab,c}^{c} - \Gamma_{ac,b}^{c} + \Gamma_{ab}^{e} \Gamma_{ec}^{c} - \Gamma_{ac}^{e} \Gamma_{eb}^{c}, \quad (A.3)
\end{equation}

\begin{equation}
R = g^{ab}R_{ab}. \quad (A.4)
\end{equation}

and using that we will be able to compute the Einstein tensor:

\begin{equation}
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \quad (A.5)
\end{equation}

The Einstein Equation is a covariant equation:

\begin{equation}
R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}, \quad (A.6)
\end{equation}

Although this, the results for the Einstein tensor can be different, depending the coordinate system we choose.

In this section we are going to do two developments:

1. We are going to compute the Einstein Tensor in spherical coordinates and show that although its components are different from (2.10) and (2.11), the Friedmann and the Acceleration equations are the same.
A. COEFFICIENTS OF THE EINSTEIN TENSOR

2. Once we have done the first proof, we are going to realize that the Einstein tensor is different in spherical coordinates. We are going to see that we can derive the coefficients using a conformal time, concluding that they can be written as in (2.10) and (2.11). Recall that:

\[ G_{00} = \frac{3}{a^2}(\dot{a}^2 + k), \]  
\[ G_{11} = -(2\ddot{a}a + \dot{a}^2 + k) = G_{22} = G_{33}. \]

A.1 Development in spherical coordinates

We have seen in Chapter 2 that the Robertson-Walker metric is given by (2.5) and has components in spherical coordinates:

\[
\mathbf{g}_{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & a^2 & 0 & 0 \\
0 & 0 & 1 - kr^2 & 0 \\
0 & 0 & 0 & a^2 r^2 \\
\end{pmatrix}
\]  
(A.9)

With some algebra we obtain that the Symbols of Christoffel of first class for the Robertson-Walker metric are:

\[
\Gamma_{110} = -\frac{\dot{a}a}{1 - kr^2} = -\Gamma_{101} = -\Gamma_{011} \]  
(A.10)

\[
\Gamma_{220} = -a\dot{r}^2 = -\Gamma_{202} = -\Gamma_{022} \]  
(A.11)

\[
\Gamma_{330} = -a\dot{r}^2 \sin^2 \theta = -\Gamma_{303} = -\Gamma_{033} \]  
(A.12)

\[
\Gamma_{111} = \frac{a^2 kr}{(1 - kr^2)^2} \]  
(A.13)

\[
\Gamma_{221} = -a^2 r = -\Gamma_{212} = -\Gamma_{122} \]  
(A.14)

\[
\Gamma_{331} = -a^2 r \sin^2 \theta = -\Gamma_{313} = -\Gamma_{133} \]  
(A.15)

\[
\Gamma_{332} = -a^2 r \sin \theta \cos \theta = -\Gamma_{323} = -\Gamma_{233} \]  
(A.16)

In order to compute the Christoffel symbols of second class we need:

\[
g^{\mu\nu} = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & \frac{1 - kr^2}{a^2} & 0 & 0 \\
0 & 0 & \frac{1}{a^2 r^2} & 0 \\
0 & 0 & 0 & \frac{1}{a^2 r^2 \sin \theta} \\
\end{pmatrix}
\]  
(A.17)
Obtaining the Christoffel symbols of second class:

\[ \Gamma^0_{11} = \Gamma_{110} g^{00} = \frac{\ddot{a}}{1 - kr^2} \]  
(A.18)

\[ \Gamma^0_{22} = \Gamma_{220} g^{00} = a\dot{a}r^2 \]  
(A.19)

\[ \Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{a}}{a} \]  
(A.20)

\[ \Gamma^1_{22} = -r(1 - kr^2) \]  
(A.21)

\[ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} \]  
(A.22)

\[ \Gamma^2_{33} = -\sin \theta \cos \theta \]  
(A.23)

\[ \Gamma^1_{11} = \frac{kr}{1 - kr^2} \]  
(A.24)

\[ \Gamma^0_{33} = a\dot{a}r^2 \sin^2 \theta \]  
(A.25)

\[ \Gamma^1_{33} = -r(1 - kr^2) \sin^2 \theta \]  
(A.26)

\[ \Gamma^3_{23} = \cot \theta \]  
(A.27)

Using (A.3) we can compute at this moment the Ricci Tensor for a Robertson-Walker metric in spherical coordinates:

\[ R_{00} = -\Gamma^1_{01,0} - \Gamma^2_{02,0} - \Gamma^3_{03,0} - \Gamma^1_{01} \Gamma^1_{01} - \Gamma^2_{02} \Gamma^2_{02} - \Gamma^3_{03} \Gamma^3_{03} = -3\frac{\ddot{a}}{a} \]  
(A.28)

\[ R_{11} = \frac{a\ddot{a} + 2\dot{a}^2 + 2k}{1 - kr^2} \]  
(A.29)

\[ R_{22} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \]  
(A.30)

\[ R_{33} = r^2(a\ddot{a} + 2\dot{a}^2 + 2k) \sin^2 \theta, \]  
(A.31)

and the Curvature Scalar:

\[ R = -6 \left[ \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} \right] \]  
(A.32)

We can find these results in [10]. With the Ricci Tensor and the Curvature Scalar we can find our coefficients in spherical coordinates:

\[ G_{00} = \frac{3}{a^2} (\dot{a}^2 + k), \]  
(A.33)

\[ G_{11} = -\frac{(2\ddot{a}a + \dot{a}^2 + k)}{1 - kr^2}, \]  
(A.34)
A. COEFFICIENTS OF THE EINSTEIN TENSOR

\[ G_{22} = -r^2(2\ddot{a}a + \dot{a}^2 + k), \quad (A.35) \]
\[ G_{33} = -r^2 \sin^2 \theta (2\ddot{a}a + \dot{a}^2 + k). \quad (A.36) \]

We are able to derive the Friedmann Equation and the Acceleration Equation with the Einstein Equation (2.3) that we recall here again in the following form:
\[ G_{\mu\nu} = 8\pi G T_{\mu\nu}, \quad (A.37) \]

using that in spherical coordinates que must write:
\[ T_{\mu\nu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & a^2 \rho & 0 & 0 \\ 0 & 0 & a^2 r^2 P & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin \theta P \end{bmatrix}, \quad (A.38) \]

finally obtaining the desired equations:
\[ \frac{3}{a^2}(\ddot{a}^2 + k) = 8\pi G \rho \quad (A.39) \]
\[ -\frac{2\ddot{a}a + \dot{a}^2 + k}{a^2} = 8\pi G P \quad (A.40) \]

A.2 Development with a conformal time

We have seen in chapter 2 that we can do an "easier" development in cartesian coordinates if we choose a flat Universe, i.e. \( k = 0 \). The key point is that, we have done all the development in this project for a flat Universe. But, if we want to do the same development for a general case, taking into account that the curvature can be different from zero, i.e. \( k \neq 0 \), maybe it is useful to define a conformal time.

"The conformal time can be interpreted as a clock that decelerates along with the expansion of the universe" [11].

In this section, we are going to explain that in further detail and derive the equations of the dynamics of the Universe using this metric, observing that we arrive to the same result as before.

We recall the RW metric tensor:
\[ g_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2 & 0 & 0 \\ 0 & 0 & a^2 r^2 & 0 \\ 0 & 0 & 0 & a^2 r^2 \sin^2 \theta \end{bmatrix}, \quad (A.41) \]

where we can write the metric of the spacetime:
\[ ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta + r^2 \sin^2 \theta d\psi \right) \quad (A.42) \]
At this point we define a conformal time:

\[ dt = a(\eta)d\eta \] (A.43)

with the conformal time and using a function \( \phi(r) \) we can rewrite the metric as:

\[ ds^2 = a^2(\eta) \left[ -d^2\eta + d^2\phi + f^2(\phi)d^2\Omega \right], \] (A.44)

where \( d^2\Omega = d^2\theta + \sin^2\theta d^2\psi \) The development that follows from this point to find the Coefficients of the Einstein tensor can be found in the literature [12]
BIBLIOGRAPHY

[1] José A. Díez, C. Ulises Moulinés. Fundamentos de Filosofía de la Ciencia (3a Edición actualizada) Ariel Filosofía


