

# An Efficient and Numerically Stable Method for Computing Interval Availability Distribution Bounds\*

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## Abstract

The paper develops a method, called *bounding regenerative transformation*, for the computation with numerical stability and well-controlled error of bounds for the interval availability distribution of systems modeled by finite (homogeneous) continuous-time Markov chain models with a particular structure. The method requires the selection of a regenerative state and is targeted at a class of models, class  $C'_1$ , with a “natural” selection for the regenerative state. For class  $C'_1$  models, bounds tightness can be traded-off with computational cost through a control parameter  $D_C$ , with the option  $D_C = 1$  yielding the smallest computational cost. For large class  $C'_1$  models and the selection  $D_C = 1$ , the method will often have a small computational cost relative to the model size and, with additional conditions, seems to yield tight bounds for any time interval or not small time intervals, depending on the initial probability distribution of the model. Class  $C'_1$  models with those additional conditions include both exact and bounding failure/repair models of coherent fault-tolerant systems with exponential failure and repair time distributions and repair in every state with failed components with failure rates much smaller than repair rates.

**Keywords:** Engineering, Probability, Markov processes, Reliability: availability.

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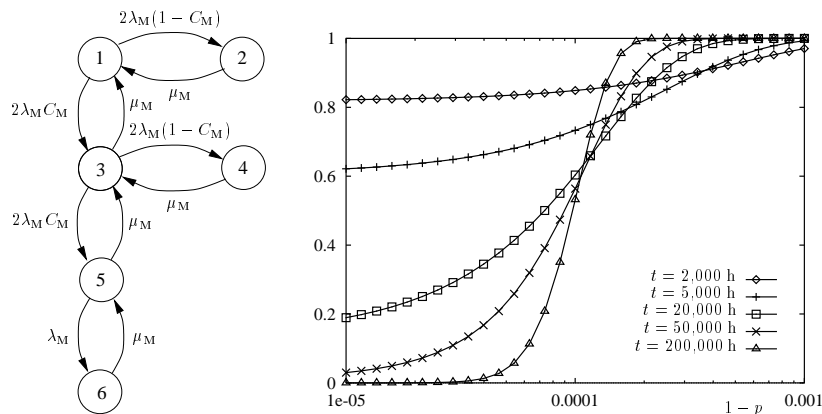


Figure 1: State diagram of continuous time Markov chain modeling a repairable fault-tolerant system using the pair-and-spare technique (left) and behavior of  $IAVCD(t, p)$  (right).

## 1 Introduction

The distribution of the interval availability, i.e. the distribution of the fraction of time in a time interval in which a system is up, is a dependability measure of practical interest. This is because the measure quantifies the probability with which a given availability level over a time interval can be guaranteed by the provider of a system to the system's user. To illustrate a typical behavior of the measure, Figure 1 plots the interval availability complementary distribution  $IAVCD(t, p)$  (probability that the fraction of time in the time interval  $[0, t]$  in which the system is up is  $> p$ ) of a fault-tolerant system using the pair and spare technique [8] in which active modules fail with rate  $\lambda_M = 10^{-3} \text{ h}^{-1}$ , the spare module does not fail, the failure of an active module is covered with probability  $C_M = 0.95$ , failed modules are repaired by a single repairman with rate  $\mu_M = 1 \text{ h}^{-1}$ , and modules do not fail when the system is down, for several values of  $t$  and values of  $p$  around the steady-state availability  $SSA = 0.9999$ , assuming that initially all modules are unfailed. The figure also gives the state diagram of the (homogeneous) continuous-time Markov chain (CTMC) modeling the system. The up states are the states 1, 3, and 5. As predicted by renewal reward process and regenerative process theories (see, for instance, [10]), for  $t \rightarrow \infty$ ,  $IAVCD(t, p)$  has an asymptotic shape with  $IAVCD(t, p) = 1$  for  $p < SSA$  and  $IAVCD(t, p) = 0$  for  $p > SSA$ , but the convergence to that asymptotic shape is very slow, making meaningful the computation of the measure for very large values of  $t$ .

Computing the interval availability distribution of a fault-tolerant system modeled by a CTMC is a challenging problem [3, 7, 10, 11, 12, 13, 14, 15, 17]. The first effort is reported in [17], where a closed form integral expression was obtained for a two-state model. In [10], randomization was used to obtain the distribution of the up time in a time interval of the same two-state model. The first method able to deal with arbitrary finite CTMC models was developed by de Souza e Silva and Gail [15] using randomization. Goyal and Tantawi [7] developed a numerical approximate method without error bounds. Sericola [14] obtained a closed form solution in terms of growing size matrices. Rubino and Sericola [11] developed an efficient numerical method for the particular case

in which up and down periods are independent one by one and of each other. Rubino and Sericola [12] developed two algorithms reducing the computational requirements of the randomization-based method developed in [15]. The first of such algorithms reduces the time requirements; the second one reduces the storage requirements. This second algorithm was reviewed in [13] as Algorithm A, where it was taken as starting point to develop another algorithm (Algorithm B), which is competitive when the number of up states of the model is small and, furthermore, can deal with some class of CTMC models with denumerable infinite state spaces. Finally, a method, which will be called in this paper *regenerative transformation*, has been developed by Carrasco [3] which covers finite CTMC models with a particular structure. In the method, a truncated transformed CTMC model is built which, with an appropriate subset of up states, has the same interval availability distribution as the original model with an arbitrarily small error and that truncated transformed CTMC model is solved using Algorithm A of [13]. The method requires the selection of a regenerative state and, as Algorithm A of [13] and all other randomization-based methods, is numerically stable and computes  $IAVCD(t, p)$  with well-controlled error. For a class of models, class  $C_1$ , including both exact and bounding failure/repair models of coherent fault-tolerant systems with exponential failure and repair time distributions and repair in every state with failed components with failure rates much smaller than repair rates, and a given “natural” selection for the regenerative state, theoretical results are available assessing the performance of the method in terms of visible model characteristics, and, for large models of that class, the method can be significantly less costly than previously available methods capable of dealing with arbitrary finite CTMC models.

All currently available methods for computing  $IAVCD(t, p)$  tend to be expensive for large  $t$ . With that motivation, in this paper, we take the regenerative transformation method as starting point to develop a potentially less costly method, called *bounding regenerative transformation*, for computing bounds for the measure. The method requires the selection of a regenerative state and is targeted at a class of models, class  $C'_1$ , which is a subclass of model class  $C_1$ , with a given “natural” selection for the regenerative state. In the method, the original CTMC model is transformed into lower bounding and upper bounding models by scaling the transition rates from up states different from the regenerative state according to a parameter  $D_C$  controlling the tightness of the bounds, and those models are solved by regenerative transformation. A more efficient implementation exists for the case  $D_C = 1$  when both bounds have to be computed and an additional condition is satisfied. For class  $C'_1$  models, bounds tightness can be traded-off with computational cost through the control parameter  $D_C$ , with the option  $D_C = 1$  yielding the smallest computational cost. For large class  $C'_1$  models and the selection  $D_C = 1$ , the method will often have a small computational cost relative to the model size and, with additional conditions, seems to yield tight bounds for any time interval or not small time intervals, depending on the initial probability distribution of the model. Class  $C'_1$  models with those additional conditions include both exact and bounding failure/repair models of coherent fault-tolerant systems with exponential failure and repair time distributions and repair in every state with failed components with failure rates much smaller than repair rates.

The rest of the paper is organized as follows. Section 2 discusses the computational cost of Algorithm A of [13], which broadly speaking can be considered the current randomization-based state-of-the-art method for computing the interval availability distribution for arbitrary finite CTMC

models, and reviews the regenerative transformation method. Section 3 describes the bounding regenerative transformation method, specifying the CTMC models covered by the method, describing the model class  $C'_1$  at which the method is targeted, motivating the method, showing that it indeed obtains bounds, and arguing that it can be relatively inexpensive. Within that section, Section 3.2 justifies and describes the more efficient particular implementation of the method. Section 4 analyzes the performance of the method using a representative large class  $C'_1$  model and compares using that example the computational cost of the method with those of Algorithm A of [13] and regenerative transformation. We also illustrate in that section that, often, bounding regenerative transformation with  $D_C = 1$  will provide bounds for large  $C'_1$  models at a small computational cost relative to the model size and that, under additional conditions, the bounds seem to be tight for any time interval or not small time intervals, depending on the initial probability distribution of the model. The Appendix includes three lengthy proofs.

## 2 Preliminaries

Let  $X = \{X(t); t \geq 0\}$  be a CTMC with state space  $\Omega$  partitioned into the set of up states  $U$  and the set of down states  $D$ . The interval availability at time  $t$ ,  $\text{IAV}(t)$ , is defined as the fraction of time in the time interval  $[0, t]$  in which the system is up, i.e.

$$\text{IAV}(t) = \frac{1}{t} \int_0^t \mathbf{1}_{X(\tau) \in U} d\tau,$$

where  $\mathbf{1}_c$  denotes the indicator function returning the value 1 when condition  $c$  is satisfied and the value 0 otherwise. In this paper, we target the computation of bounds for the interval availability complementary distribution

$$\text{IAVCD}(t, p) = P[\text{IAV}(t) > p],$$

where  $t > 0$  and  $0 < p < 1$ .

Algorithm A of [13] (Algorithm A in the sequel) can, broadly speaking, be considered the current randomization-based state-of-the-art method for computing the measure for arbitrary finite CTMC models with infinitesimal generator. Assume  $\max_{i \in \Omega} \lambda_i > 0$ , where  $\lambda_i$  is the output rate from state  $i$  of  $X$ . That method, as most of currently available methods for computing the measure, is based on the randomization construct. In that construct (see, for instance, [9]), the given CTMC  $X$  is interpreted in terms of a discrete-time Markov chain subordinated to a Poisson process with arrival rate  $\Lambda \geq \max_{i \in \Omega} \lambda_i$ . For not too small  $X$ , the method has an approximate flop count  $NC'(2T + 2|\Omega|)$ , where  $N$  and  $C'$  are truncation parameters and  $T$  is the number of transitions of  $X$ . An important feature of the method is that it is numerically stable, the only important error source being the truncation error. The truncation parameters  $N$  and  $C'$  are given by

$$N = \min \left\{ n \geq 0 : \sum_{k=n+1}^{\infty} \frac{(\Lambda t)^k}{k!} e^{-\Lambda t} \leq \frac{\varepsilon}{2} \right\},$$

$$C'' = \begin{cases} \max \left\{ c : 0 \leq c \leq N \wedge \sum_{k=0}^c \frac{((1-p)\Lambda t)^k}{k!} e^{-(1-p)\Lambda t} \leq \frac{\varepsilon}{4} \right\} & \text{if } e^{-(1-p)\Lambda t} \leq \frac{\varepsilon}{4} \\ -1 & \text{if } e^{-(1-p)\Lambda t} > \frac{\varepsilon}{4} \end{cases},$$

$$C' = \begin{cases} \min \left\{ N, \min \left\{ c \geq 0 : \sum_{k=c+1}^{\infty} \frac{((1-p)\Lambda t)^k}{k!} e^{-(1-p)\Lambda t} \leq \frac{\varepsilon}{4} \right\} \right\} & \text{if } C'' \neq -1 \\ \min \left\{ c \geq 0 : \sum_{k=c+1}^{\infty} \frac{((1-p)\Lambda t)^k}{k!} e^{-(1-p)\Lambda t} \leq \frac{\varepsilon}{2} \right\} & \text{if } C'' = -1 \end{cases},$$

where  $\varepsilon$  is the required truncation error. The truncation parameters  $N$  and  $C'$  increase with  $\Lambda$ , making  $\Lambda = \max_{i \in \Omega} \lambda_i$  the best selection for  $\Lambda$ . Using the well-known result (see, for instance, [10, Theorem 3.3.5]) that the number of arrivals in the time interval  $[0, t]$  of a Poisson process with arrival rate  $\Lambda$  has for  $\Lambda t \rightarrow \infty$  an asymptotic normal distribution with mean and variance  $\Lambda t$ , for large  $\Lambda t$  and  $\varepsilon \ll 1$ , the required  $N$  will be  $\approx \Lambda t$  and, then, the method will be very costly if the model is large. As an example, for the model considered in Section 4, which has 646,646 states, 15,578,290 transitions, and  $\Lambda \approx 2.25 \text{ h}^{-1}$ , we can estimate a flop count of  $8.25 \times 10^{13}$  when the method is run with a single target  $(t, p)$  pair with  $t = 20,000 \text{ h}$  and  $p = 0.9995$  and a truncation error requirement  $\varepsilon = 10^{-8}$ , which yields  $N = 46,241$  and  $C' = 55$ .

The regenerative transformation method developed in [3] was an effort to reduce the high relative computational cost in terms of CPU time of Algorithm A. The method requires the selection of a regenerative state  $r$  and is targeted at a particular class of models, class  $C_1$ , including both exact and bounding failure/repair models of coherent fault-tolerant systems with exponential failure and repair time distributions and repair in every state with failed components with failure rates much smaller than repair rates, with a “natural” selection for the regenerative state. Since the method developed in this paper for computing bounds for  $\text{IAVCD}(t, p)$  is based on regenerative transformation, in the remaining of this section we will review the regenerative transformation method. Let  $\alpha_i = P[X(0) = i]$  and let  $\lambda_{i,j}$  denote the transition rate of  $X$  from state  $i$  to state  $j$ . Given  $B \subset \Omega$ , let  $\alpha_B = \sum_{i \in B} \alpha_i$  denote the initial probability of  $X$  in subset  $B$  and, given  $i \in \Omega$  and  $B \subset \Omega - \{i\}$ , let  $\lambda_{i,B} = \sum_{j \in B} \lambda_{i,j}$  denote the transition rate of  $X$  from state  $i$  to subset  $B$ . Letting  $S' = S - \{r\}$ ,  $U_S = U \cap S$ ,  $D_S = D \cap S$ ,  $U'_S = U_S - \{r\}$ , and  $D'_S = D_S - \{r\}$ , the method will cover CTMCs  $X$  with infinitesimal generator and selections for  $r$  satisfying the conditions

- C1.  $\Omega$  is finite.
- C2. Either  $\Omega = S$  or  $\Omega = S \cup \{f\}$ ,  $f$  being an absorbing state.
- C3.  $|S| \geq 2$ .
- C4. Either all states in  $S$  are transient or  $X$  has a single recurrent class of states  $F \subset S$ .
- C5. All states are reachable (from some state with nonnull initial probability).
- C6.  $U \neq \emptyset$  and  $D \neq \emptyset$ .
- C7.  $\max_{i \in U} \lambda_i > 0$  and  $\max_{i \in D} \lambda_i > 0$ .

- C8.  $r \in S$  and, if  $X$  has a single recurrent class of states  $F \subset S$ ,  $r \in F$ .
- C9. If  $U'_S \neq \emptyset$ ,  $\lambda_{r,U'_S} > 0$ .
- C10. If  $U'_S \neq \emptyset$ ,  $\alpha_{D'_S} > 0$  and  $\alpha_{U'_S} = 0$ ,  $\lambda_{i,U'_S} > 0$  for some  $i \in D'_S$  with  $\alpha_i > 0$ .

Note that conditions C2, C4 and C8 and the required specification of the regenerative state  $r$  “force” the subset  $S$  and, if existent, the state  $f$ . More specifically, if  $X$  does not have any absorbing state,  $S$  must be  $\Omega$  and  $f$  does not exist; if  $X$  has a single absorbing state  $a$  and  $r \neq a$ , then  $S$  must be  $\Omega - \{a\}$  and  $f$  must be  $a$ ; if  $X$  has a single absorbing state  $a$  and  $r = a$ , then  $S$  must be  $\Omega$  and  $f$  does not exist; if  $X$  has two absorbing states  $a, b$  and  $r$  is one of them, say  $a$ , then  $S$  must be  $\Omega - \{b\}$  and  $f$  must be  $b$ ; finally, if  $X$  has more than two absorbing states or has two absorbing states but none of them is  $r$ , then no selections for  $S$  and  $f$  exist satisfying the conditions. Conditions C3, C6 and C7 are mild, in the sense that when they are not satisfied computation of  $\text{IAVCD}(t, p)$  either is trivial or can be reduced to a simpler problem. Thus, assuming  $U \neq \emptyset$  and  $\max_{i \in U} \lambda_i = 0$ , all up states would be absorbing and  $\text{IAVCD}(t, p)$  would be equal to  $P[X((1-p)t) \in U]$ . Similarly, assuming  $D \neq \emptyset$  and  $\max_{i \in D} \lambda_i = 0$ , all down states would be absorbing and  $\text{IAVCD}(t, p)$  would be equal to  $P[X(pt) \in U]$ . Condition C5 can be trivialized by deleting unreachable states. Finally, conditions C9 and C10 can be circumvented by adding to  $X$  a tiny transition rate  $\lambda \leq 10^{-10} \varepsilon / (2t_{\max})$  between an appropriate pair of states, where  $\varepsilon$  is the allowed error and  $t_{\max}$  is the largest time  $t$  at which  $\text{IAVCD}(t, p)$  has to be computed, with a negligible impact on  $\text{IAVCD}(t, p)$  no greater than  $10^{-10} \varepsilon$  (see [3]). The possibility that  $X$  has an absorbing state  $f$  is allowed to cover bounding models [16], which are useful for systems for which an exact model would have a state space of unmanageable size. A bounding model would have a state space  $\Omega = S \cup \{f\}$ , where  $S$  is a subset of the state space of the exact model and  $f$  is an absorbing state in which the bounding model is whenever the exact model has visited some state outside  $S$ . The initial probability distribution in  $S$  would be as in the exact model and the initial probability of  $f$  would be the probability that initially the exact model is outside  $S$ . Considering  $f$  to be a down/up state results in an  $\text{IAVCD}(t, p)$  measure for the bounding model which bounds from below/above the  $\text{IAVCD}(t, p)$  measure of the exact model.

The model class  $C_1$  at which the regenerative transformation method is targeted includes all CTMCs  $X$  with infinitesimal generator satisfying conditions C1–C7 and the condition

- C11. A partition  $U_0 \cup U_1 \cup \dots \cup U_{N_C}$  for  $U_S$  exists satisfying the properties
- P1.  $U_0 = \{o\}$  (i.e.  $|U_0| = 1$ ).
- P2. If  $X$  has a single recurrent class of states  $F \subset S$ ,  $o \in F$ .
- P3. If  $|U_S| \geq 2$ ,  $\lambda_{o,U_1 \cup \dots \cup U_{N_C}} > 0$ .
- P4. If  $|U_S| \geq 2$ ,  $\alpha_{D_S} > 0$  and  $\alpha_{U_1 \cup \dots \cup U_{N_C}} = 0$ ,  $\lambda_{i,U_1 \cup \dots \cup U_{N_C}} > 0$  for some  $i \in D_S$  with  $\alpha_i > 0$ .
- P5. If  $N_C > 0$ ,  $\max_{0 \leq k \leq N_C} \max_{i \in U_k} \lambda_{i,U_k - \{i\} \cup U_{k+1} \cup \dots \cup U_{N_C} \cup D_S}$  is significantly smaller than  $\min_{0 < k \leq N_C} \min_{i \in U_k} \lambda_{i,U_0 \cup \dots \cup U_{k-1}} > 0$  if  $\Omega = S$   
or  $\min_{0 < k \leq N_C} \min_{i \in U_k} \lambda_{i,U_0 \cup \dots \cup U_{k-1} \cup \{f\}} > 0$  if  $\Omega = S \cup \{f\}$ .

The natural selection for the regenerative state for class  $C_1$  models is  $r = o$ . With that natural selection, properties P2, P3 and P4 imply the fulfillment of conditions C8, C9 and C10. Model class  $C_1$  includes both exact and bounding failure/repair models of coherent fault-tolerant systems with exponential failure and repair time distributions and repair in every state with failed components with failure rates much smaller than repair rates. A partition for  $U_S$  showing that those models are in class  $C_1$  would be the partition in which  $U_k$  includes the up states with a given number of failed components, with the subsets  $U_k$  sorted following increasing numbers of failed components. Models of non-coherent fault-tolerant systems may not belong to model class  $C_1$  due to the possibility that there may be a fast repair transition going from some state in  $U_S$  to  $D_S$  and, then, property P5 may not be satisfied. Properties P2, P3 and P4 were not mentioned in [3], but they are implicitly enforced for the natural selection  $r = o$  by conditions C8, C9 and C10. The condition  $|U_S| \geq 2$  was enforced for class  $C_1$  models in [3], but the taken-out particular case  $|U_S| = 1$  was discussed there, and we have decided to include it here.

The regenerative transformation method includes two phases. In the first one, a truncated transformed CTMC model,  $V_T$ , is built which, with an appropriate subset of up states, has the same interval availability complementary distribution as  $X$  with absolute error  $\leq \varepsilon/2$ . In the second one,  $V_T$  is solved with absolute error  $\leq \varepsilon/2$  using Algorithm A. Informally,  $V_T$  is obtained by characterizing the behavior of  $X$  from  $S' = S - \{r\}$  until either hit of state  $r$  or, if existing, hit of the absorbing state  $f$ , and from  $r$  until either next hit of state  $r$  or, if existing, hit of the absorbing state  $f$ , while keeping track of the amount of time spent in  $U_S$ .

We now start describing  $V_T$  as a blackbox and how can it be built from  $X$  at the detail required by the developments to follow in Section 3. Let  $\widehat{X} = \{\widehat{X}_n, n = 0, 1, 2, \dots\}$  be the randomized discrete-time Markov chain (DTMC) of  $X$  with randomization rate  $\Lambda_U = (1 + \theta) \max_{i \in U} \lambda_i > 0$  in the states in  $U$  and randomization rate  $\Lambda_D = (1 + \theta) \max_{i \in D} \lambda_i > 0$  in the states in  $D$ , where  $\theta$  is a small quantity  $> 0$ , say  $\theta = 10^{-4}$ . The DTMC  $\widehat{X}$  has same state space and initial probability distribution as  $X$  and transition matrix  $\mathbf{P} = (P_{i,j})_{i,j \in \Omega}$ , where  $P_{i,j} = \lambda_{i,j}/\Lambda_U$ ,  $i \in U$ ,  $j \neq i$ ,  $P_{i,i} = 1 - \lambda_i/\Lambda_U$ ,  $i \in U$ ,  $P_{i,j} = \lambda_{i,j}/\Lambda_D$ ,  $i \in D$ ,  $j \neq i$ , and  $P_{i,i} = 1 - \lambda_i/\Lambda_D$ ,  $i \in D$ . Let  $\widehat{X}'$  denote a version of  $\widehat{X}$  with initial state  $r$ , and, given a DTMC  $Y$ , let  $Y_{m_1:m_2}c$ ,  $m_1, m_2 \geq 0$ , denote the predicate which is true when  $Y_n$  satisfies condition  $c$  for all  $n$ ,  $m_1 \leq n \leq m_2$  (by convention  $Y_{m_1:m_2}c$  is true for  $m_2 < m_1$ ) and let  $\#(Y_{m_1:m_2}c)$  denote the number of indices  $n$ ,  $m_1 \leq n \leq m_2$  for which  $Y_n$  satisfies condition  $c$ . Let the row vectors  $\boldsymbol{\pi}(n, k) = (\pi_i(n, k))_{i \in S}$ ,  $n \geq 0$ ,  $0 \leq k \leq n + 1$ , where  $\pi_i(n, k) = P[\widehat{X}'_n = i \wedge \widehat{X}'_{1:n} \neq r \wedge \#(\widehat{X}'_{0:n} \in U) = k]$ , and let the row vectors  $\boldsymbol{\pi}'(n, k) = (\pi'_i(n, k))_{i \in S'}$ ,  $n \geq 0$ ,  $0 \leq k \leq n + 1$ , where  $\pi'_i(n, k) = P[\widehat{X}_n = i \wedge \widehat{X}_{0:n} \neq r \wedge \#(\widehat{X}_{0:n} \in U) = k]$ . In words,  $\pi_i(n, k)$  is the probability that in the first  $n$  steps  $\widehat{X}'$  will not have entered state  $r$  and has visited  $k$  up states, and at step  $n$  is in state  $i$ ,  $i \in S$ ; and  $\pi'_i(n, k)$  is the probability that in the first  $n$  steps  $\widehat{X}$  has not visited state  $r$  and has visited  $k$  up states, and at step  $n$  is in state  $i$ ,  $i \in S'$ . Let  $a(n, k) = \sum_{i \in S} \pi_i(n, k)$ ,  $a_m(k) = \sum_{n=k-1}^{k+m-1} a(n, k)$ ,  $a'(n, k) = \sum_{i \in S'} \pi'_i(n, k)$ , and  $a'_m(k) = \sum_{n=k-1}^{k+m-1} a'(n, k)$ . The truncated transformed CTMC  $V_T$  is defined by up to three truncation parameters,  $K$ ,  $L$ , and  $C$ . The truncation parameter  $C$  is given

by

$$C = \min \left\{ c \geq 1 : \sum_{m=c+1}^{\infty} \frac{(\Lambda t q_{\max})^m}{m!} e^{-\Lambda t q_{\max}} \leq \varepsilon_1 \right\}, \quad (1)$$

where  $\Lambda = \max\{\Lambda_U, \Lambda_D\}$ ,  $tq_{\max}$  is the largest value of  $tq = t(1-p)$  at which IAVCD( $t, p$ ) has to be computed, and  $\varepsilon_1 = \varepsilon/4$  if  $U'_S \neq \emptyset$  and  $\varepsilon_1 = \varepsilon/2$  if  $U'_S = \emptyset$ . For the case  $U'_S \neq \emptyset$  and  $\alpha_{S'} > 0$ , the truncation parameters  $K$  and  $L$  are given by

$$K = \min \left\{ k \geq 2 : \alpha_{S'} a_C(k) \sum_{m=k}^{\infty} (m-k+2) \frac{(\Lambda_U t_{\max})^m}{m!} e^{-\Lambda_U t_{\max}} \leq \frac{\varepsilon}{8} \right\}, \quad (2)$$

$$L = \min \left\{ k \geq 2 : a'_C(k) \sum_{m=k}^{\infty} \frac{(\Lambda_U t_{\max})^m}{m!} e^{-\Lambda_U t_{\max}} \leq \frac{\varepsilon}{8} \right\}, \quad (3)$$

and, for the case  $U'_S \neq \emptyset$  and  $\alpha_{S'} = 0$ , the truncation parameter  $K$  is given by

$$K = \min \left\{ k \geq 2 : \alpha_{S'} a_C(k) \sum_{m=k}^{\infty} (m-k+2) \frac{(\Lambda_U t_{\max})^m}{m!} e^{-\Lambda_U t_{\max}} \leq \frac{\varepsilon}{4} \right\}, \quad (4)$$

where  $t_{\max}$  is the largest value of  $t$  at which IAVCD( $t, p$ ) has to be computed. Then,  $\mathbf{x}^B$  denoting the restriction of the row vector  $\mathbf{x}$  to the subset of indices  $B$  and  $\mathbf{0}$  denoting a row vector of appropriate dimension with all components null,  $f$ ,  $a$  and  $b$  being absorbing states, and  $\cup_c$  denoting an union to be performed when condition  $c$  is satisfied,  $V_T$  has, for the case  $\alpha_{S'} > 0$ , state space (note that conditions C2, C6 and C7 imply  $U_S \neq \emptyset$  and  $D_S \neq \emptyset$ )

$$\begin{aligned} \Omega_T = & \left\{ s_{n,k}^u : (n,k) \in D_T \wedge \boldsymbol{\pi}(n,k)^{U_S} \neq \mathbf{0} \right\} \cup \left\{ s_{n,k}^d : (n,k) \in D_T \wedge \boldsymbol{\pi}(n,k)^{D_S} \neq \mathbf{0} \right\} \\ & \cup_{U'_S \neq \emptyset} \left\{ s_{n,k}^{u'} : (n,k) \in D'_T \wedge \boldsymbol{\pi}'(n,k)^{U'_S} \neq \mathbf{0} \right\} \\ & \cup_{D'_S \neq \emptyset} \left\{ s_{n,k}^{d'} : (n,k) \in D'_T \wedge \boldsymbol{\pi}'(n,k)^{D'_S} \neq \mathbf{0} \right\} \\ & \cup_{\Omega=S \cup \{f\}} \{f\} \cup \{a\} \cup_{U'_S \neq \emptyset} \{b\}, \end{aligned}$$

and, for the case  $\alpha_{S'} = 0$ , state space

$$\begin{aligned} \Omega_T = & \left\{ s_{n,k}^u : (n,k) \in D_T \wedge \boldsymbol{\pi}(n,k)^{U_S} \neq \mathbf{0} \right\} \cup \left\{ s_{n,k}^d : (n,k) \in D_T \wedge \boldsymbol{\pi}(n,k)^{D_S} \neq \mathbf{0} \right\} \\ & \cup_{\Omega=S \cup \{f\}} \{f\} \cup \{a\} \cup_{U'_S \neq \emptyset} \{b\}, \end{aligned}$$

where, for  $U'_S \neq \emptyset$ ,

$$D_T = \{(n,k) : 0 \leq k \leq K \wedge \max\{0, k-1\} \leq n \leq k+C-1\}$$

and

$$D'_T = \{(n,k) : 0 \leq k \leq L \wedge \max\{0, k-1\} \leq n \leq k+C-1\};$$

for  $U'_S = \emptyset$  and  $r \in U_S$ ,

$$D_T = \{(n,1) : 0 \leq n \leq C\};$$

for  $U'_S = \emptyset$  and  $r \in D_S$ ,

$$D_T = \{(n,0) : 0 \leq n \leq C-1\};$$



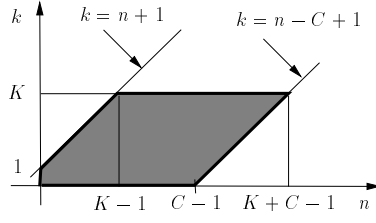


Figure 2: Domain  $D_T$  for the case  $U'_S \neq \emptyset$  (the domain includes the points in the frontier).

and, for  $U'_S = \emptyset$ ,

$$D'_T = \{(n, 0) : 0 \leq n \leq C - 1\}.$$

Figure 2 depicts the domain  $D_T$  for the case  $U'_S \neq \emptyset$ . The domain  $D'_T$  for the case  $U'_S = \emptyset$  is identical with  $K$  replaced by  $L$ . The initial probability distribution of  $V_T$  is  $P[V_T = s_{0,\cdot}^u] = \alpha_r$ ,  $P[V_T = s_{0,1}^u] = \alpha_{U'_S}$ ,  $P[V_T = s_{0,0}^d] = \alpha_{D'_S}$ ,  $P[V_T = f] = \alpha_f$ , and  $P[V_T = i] = 0$ ,  $i \notin \{s_{0,\cdot}^u, s_{0,1}^u, s_{0,0}^d, f\}$ , where  $s_{0,\cdot}^u$  denotes state  $s_{0,1}^u$  if  $r \in U_S$  and state  $s_{0,0}^d$  if  $r \in D_S$ . Note that, according to the definition of  $\pi(n, k)$ , for  $r \in U_S$ , the only state  $s_{0,k}^u$  or  $s_{0,k}^d$  present in  $\Omega_T$  is state  $s_{0,1}^u$ , and, for  $r \in D_S$ , the only state  $s_{0,k}^u$  or  $s_{0,k}^d$  present in  $\Omega_T$  is state  $s_{0,0}^d$ . It is that single state which is denoted by  $s_{0,\cdot}$ .

With  $P_{i,B}$ ,  $B \subset \Omega$  denoting  $\sum_{j \in B} P_{i,j}$ , the transition rates in  $V_T$  are as follows. Let

$$\begin{aligned} w_{n,k}^{uu} &= \sum_{i \in U_S} \pi_i(n, k) P_{i,U'_S} / \sum_{i \in U_S} \pi_i(n, k), & w_{n,k}^{ud} &= \sum_{i \in U_S} \pi_i(n, k) P_{i,D'_S} / \sum_{i \in U_S} \pi_i(n, k), \\ w_{n,k}^{du} &= \sum_{i \in D_S} \pi_i(n, k) P_{i,U'_S} / \sum_{i \in D_S} \pi_i(n, k), & w_{n,k}^{dd} &= \sum_{i \in D_S} \pi_i(n, k) P_{i,D'_S} / \sum_{i \in D_S} \pi_i(n, k), \\ q_{n,k}^u &= \sum_{i \in U_S} \pi_i(n, k) P_{i,r} / \sum_{i \in U_S} \pi_i(n, k), & q_{n,k}^d &= \sum_{i \in D_S} \pi_i(n, k) P_{i,r} / \sum_{i \in D_S} \pi_i(n, k), \\ v_{n,k}^u &= \sum_{i \in U_S} \pi_i(n, k) P_{i,f} / \sum_{i \in U_S} \pi_i(n, k), & v_{n,k}^d &= \sum_{i \in D_S} \pi_i(n, k) P_{i,f} / \sum_{i \in D_S} \pi_i(n, k), \\ w_{n,k}^{uuu} &= \sum_{i \in U'_S} \pi'_i(n, k) P_{i,U'_S} / \sum_{i \in U'_S} \pi'_i(n, k), & w_{n,k}^{udd} &= \sum_{i \in U'_S} \pi'_i(n, k) P_{i,D'_S} / \sum_{i \in U'_S} \pi'_i(n, k), \\ w_{n,k}^{duu} &= \sum_{i \in D'_S} \pi'_i(n, k) P_{i,U'_S} / \sum_{i \in D'_S} \pi'_i(n, k), & w_{n,k}^{ddd} &= \sum_{i \in D'_S} \pi'_i(n, k) P_{i,D'_S} / \sum_{i \in D'_S} \pi'_i(n, k), \\ q_{n,k}^{uu} &= \sum_{i \in U'_S} \pi'_i(n, k) P_{i,r} / \sum_{i \in U'_S} \pi'_i(n, k), & q_{n,k}^{dd} &= \sum_{i \in D'_S} \pi'_i(n, k) P_{i,r} / \sum_{i \in D'_S} \pi'_i(n, k), \\ v_{n,k}^{uu} &= \sum_{i \in U'_S} \pi'_i(n, k) P_{i,f} / \sum_{i \in U'_S} \pi'_i(n, k), & v_{n,k}^{dd} &= \sum_{i \in D'_S} \pi'_i(n, k) P_{i,f} / \sum_{i \in D'_S} \pi'_i(n, k). \end{aligned}$$

Then,

- if  $U'_S \neq \emptyset$ , each state  $s_{n,k}^u$ ,  $0 \leq k < K$ , has a transition rate  $w_{n,k}^{uu} \Lambda_U$  to state  $s_{n+1,k+1}^u$ , a transition rate  $w_{n,k}^{ud} \Lambda_U$  to state  $s_{n+1,k}^d$  if  $n \leq k + C - 2$  and to state  $a$  otherwise, a transition rate  $q_{n,k}^u \Lambda_U$  to state  $s_{0,\cdot}$  if  $s_{n,k}^u \neq s_{0,\cdot}$ , and, if  $\Omega = S \cup \{f\}$ , a transition rate  $v_{n,k}^u \Lambda_U$  to state  $f$ .
- If  $U'_S \neq \emptyset$ , each state  $s_{n,K}^u$  has a transition rate  $\Lambda_U$  to state  $b$ .
- If  $U'_S = \emptyset$ , each state  $s_{n,k}^u$  has a transition rate  $w_{n,k}^{ud} \Lambda_U$  to state  $s_{n+1,k}^d$  if  $n \leq k + C - 2$  and to state  $a$  otherwise, a transition rate  $q_{n,k}^u \Lambda_U$  to state  $s_{0,\cdot}$  if  $s_{n,k}^u \neq s_{0,\cdot}$ , and, if  $\Omega = S \cup \{f\}$ , a transition rate  $v_{n,k}^u \Lambda_U$  to state  $f$ .
- If  $U'_S \neq \emptyset$ , each state  $s_{n,k}^d$ ,  $0 \leq k < K$ , has a transition rate  $w_{n,k}^{du} \Lambda_D$  to state  $s_{n+1,k+1}^u$ , a transition rate  $w_{n,k}^{dd} \Lambda_D$  to state  $s_{n+1,k}^d$  if  $n \leq k + C - 2$  and to state  $a$  otherwise, a transition rate  $q_{n,k}^d \Lambda_D$  to state  $s_{0,\cdot}$  if  $s_{n,k}^d \neq s_{0,\cdot}$ , and, if  $\Omega = S \cup \{f\}$ , a transition rate  $v_{n,k}^d \Lambda_D$  to state  $f$ .
- If  $U'_S \neq \emptyset$ , each state  $s_{n,K}^d$  has a transition rate  $\Lambda_D$  to state  $b$ .

- If  $U'_S = \emptyset$ , each state  $s_{n,k}^d$  has a transition rate  $w_{n,k}^{dd} \Lambda_D$  to state  $s_{n+1,k}^d$  if  $n \leq k + C - 2$  and to state  $a$  otherwise, a transition rate  $q_{n,k}^d \Lambda_D$  to state  $s_{0,\cdot}$ , if  $s_{n,k}^d \neq s_{0,\cdot}$ , and, if  $\Omega = S \cup \{f\}$ , a transition rate  $v_{n,k}^d \Lambda_D$  to state  $f$ .
- If  $U'_S \neq \emptyset$ , each state  $s_{n,k}^u$ ,  $0 \leq k < L$  has a transition rate  $w_{n,k}^{uu} \Lambda_U$  to state  $s_{n+1,k+1}^u$ , a transition rate  $w_{n,k}^{ud} \Lambda_U$  to state  $s_{n+1,k}^d$  if  $n \leq k + C - 2$  and to state  $a$  otherwise, a transition rate  $q_{n,k}^u \Lambda_U$  to state  $s_{0,\cdot}$ , and, if  $\Omega = S \cup \{f\}$ , a transition rate  $v_{n,k}^u \Lambda_U$  to state  $f$ .
- If  $U'_S \neq \emptyset$ , each state  $s_{n,L}^u$  has a transition rate  $\Lambda_U$  to state  $b$ .
- If  $U'_S \neq \emptyset$ , each state  $s_{n,k}^d$ ,  $0 \leq k < L$  has a transition rate  $w_{n,k}^{du} \Lambda_D$  to state  $s_{n+1,k+1}^u$ , a transition rate  $w_{n,k}^{dd} \Lambda_D$  to state  $s_{n+1,k}^d$  if  $n \leq k + C - 2$  and to state  $a$  otherwise, a transition rate  $q_{n,k}^d \Lambda_D$  to state  $s_{0,\cdot}$ , and, if  $\Omega = S \cup \{f\}$ , a transition rate  $v_{n,k}^d \Lambda_D$  to state  $f$ .
- If  $U'_S \neq \emptyset$ , each state  $s_{n,L}^d$  has a transition rate  $\Lambda_D$  to state  $b$ .
- If  $U'_S = \emptyset$ , each state  $s_{n,k}^d$  has a transition rate  $w_{n,k}^{dd} \Lambda_D$  to state  $s_{n+1,k}^d$  if  $n \leq k + C - 2$  and to state  $a$  otherwise, a transition rate  $q_{n,k}^d \Lambda_D$  to state  $s_{0,\cdot}$ , and, if  $\Omega = S \cup \{f\}$ , a transition rate  $v_{n,k}^d \Lambda_D$  to state  $f$ .

The states which have to be considered up in  $V_T$  are the states  $s_{n,k}^u$ , the states  $s_{n,k}^u$  and state  $f$  if  $\Omega = S \cup \{f\}$  and  $f$  is an up state in  $X$ .

To illustrate the “structure” of  $V_T$ , Figure 3 gives an sketch of the state diagram of  $V_T$  for the case  $\Omega = S \cup \{f\}$ ,  $r \in U_S$ ,  $U'_S \neq \emptyset$ ,  $D'_S \neq \emptyset$ ,  $\alpha_{U'_S} > 0$ , and  $\alpha_{D'_S} > 0$ , with truncation parameters  $K = 3$ ,  $L = 3$ , and  $C = 3$ . In that case, since  $r \in U_S$ ,  $s_{0,\cdot} = s_{0,1}^u$  and state  $s_{0,0}^d$  is not present. We include in the state space all possible candidate states  $s_{n,k}^u$ ,  $s_{n,k}^d$ ,  $s_{n,k}^u$ ,  $s_{n,k}^d$  subject to the considered particular case, taking into account the formal definition of  $\Omega_T$ . States  $s_{n,k}^u$ ,  $(n, k) \in D_T$  and states  $s_{n,k}^d$ ,  $(n, k) \in D_T$  which are always (for the considered particular case) outside  $\Omega_T$  are indicated by dotted circles. Similarly, states  $s_{n,k}^u$ ,  $(n, k) \in D'_T$  and states  $s_{n,k}^d$ ,  $(n, k) \in D'_T$  which are always outside  $\Omega_T$  are indicated with dotted circles. The initial probability distribution of  $V_T$  is  $P[V_T(0) = s_{0,1}^u] = \alpha_r$ ,  $P[V_T(0) = s_{0,1}^u] = \alpha_{U'_S}$ ,  $P[V_T(0) = s_{0,0}^d] = \alpha_{D'_S}$ ,  $P[V_T(0) = f] = \alpha_f$ ,  $P[V_T(0) = i] = 0$ ,  $i \notin \{s_{0,1}^u, s_{0,1}^u, s_{0,0}^d, f\}$ . For the sake of readability, we do not plot the arrows corresponding to the transition rates to states  $f$  and  $s_{0,1}^u$ . There is a transition rate with value  $q_{n,k}^u \Lambda_U$  from every state  $s_{n,k}^u$ ,  $n > 0$ ,  $k < K = 3$  to state  $s_{0,1}^u$ , a transition rate with value  $q_{n,k}^d \Lambda_D$  from every state  $s_{n,k}^d$ ,  $k < K = 3$  to state  $s_{0,1}^u$ , a transition rate with value  $q_{n,k}^u \Lambda_U$  from every state  $s_{n,k}^u$ ,  $k < L = 3$  to state  $s_{0,1}^u$ , and a transition rate with value  $q_{n,k}^d \Lambda_D$  from every state  $s_{n,k}^d$ ,  $k < L = 3$  to state  $s_{0,1}^u$ . Finally, there is a transition rate with value  $v_{n,k}^u \Lambda_U$  from every state  $s_{n,k}^u$ ,  $k < K = 3$  to state  $f$ , a transition rate with value  $v_{n,k}^d \Lambda_D$  from every state  $s_{n,k}^d$ ,  $k < K = 3$  to state  $f$ , a transition rate with value  $v_{n,k}^u \Lambda_U$  from every state  $s_{n,k}^u$ ,  $k < L = 3$  to state  $f$ , and a transition rate with value  $v_{n,k}^d \Lambda_D$  from every state  $s_{n,k}^d$ ,  $k < L = 3$  to state  $f$ .

The construction of  $V_T$  requires the computation of  $\pi(n, k)$ ,  $(n, k) \in D_T$  and, if  $\alpha_{S'} > 0$ ,  $\pi'(n, k)$ ,  $(n, k) \in D'_T$ .  $\mathbf{P}_{B,C}$  denoting the subblock of  $\mathbf{P}$  collecting the elements with index pairs

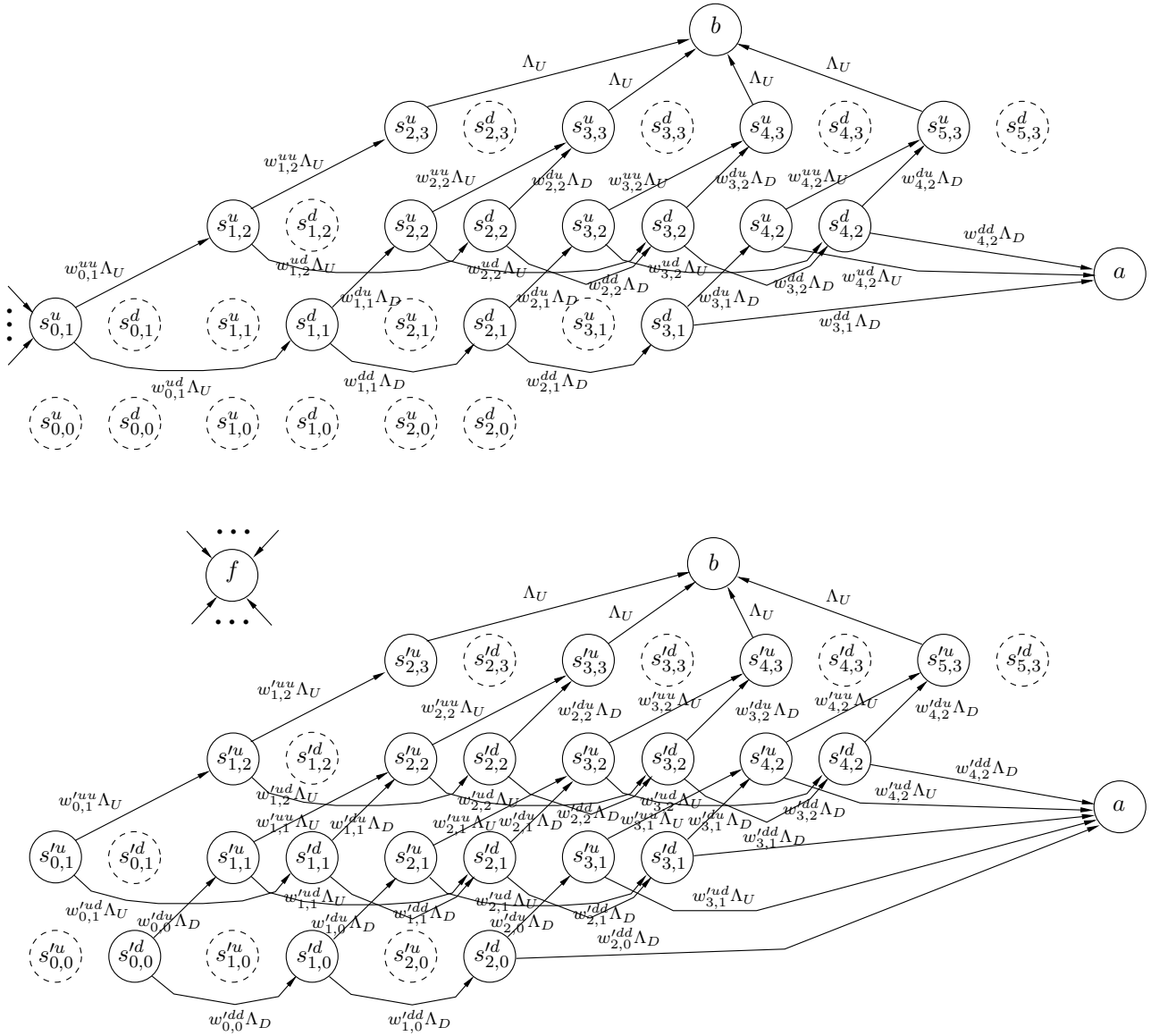


Figure 3: Sketch of the state diagram of  $V_T$  for the case  $\Omega = S \cup \{f\}$ ,  $r \in U_S$ ,  $U'_S \neq \emptyset$ ,  $D'_S \neq \emptyset$ ,  $\alpha_{U'_S} > 0$ , and  $\alpha_{D'_S} > 0$ , with truncation parameters  $K = 3$ ,  $L = 3$ , and  $C = 3$ .

in  $B \times C$ , the required row vectors  $\boldsymbol{\pi}(n, k)$  can be obtained, for increasing  $k$  and for each  $k$  for increasing  $n$ , using the recurrences

$$\boldsymbol{\pi}(n, k)^{U'_S} = \boldsymbol{\pi}(n-1, k-1) \mathbf{P}_{S, U'_S}, \quad n \geq 1, 1 \leq k \leq n+1, \quad (5)$$

$$\boldsymbol{\pi}(n, k)^{D'_S} = \boldsymbol{\pi}(n-1, k) \mathbf{P}_{S, D'_S}, \quad n \geq 1, 0 \leq k \leq n \quad (6)$$

and

$$\pi_r(n, k) = 0, \quad n \geq 1, 0 \leq k \leq n+1, \quad (7)$$

$$\pi_r(0, 0) = \mathbf{1}_{r \in D_S}, \quad (8)$$

$$\pi_r(0, 1) = \mathbf{1}_{r \in U_S}, \quad (9)$$

$$\boldsymbol{\pi}(0, k)^{U'_S} = \mathbf{0}, \quad 0 \leq k \leq 1, \quad (10)$$

$$\boldsymbol{\pi}(n, 0)^{U'_S} = \mathbf{0}, \quad n \geq 1, \quad (11)$$

$$\boldsymbol{\pi}(0, k)^{D'_S} = \mathbf{0}, \quad 0 \leq k \leq 1. \quad (12)$$

$$\boldsymbol{\pi}(n, n+1)^{D'_S} = \mathbf{0}, \quad n \geq 1. \quad (13)$$

Similarly,  $\boldsymbol{\alpha}$  denoting the row vector  $(\alpha_i)_{i \in \Omega}$ , the required row vectors  $\boldsymbol{\pi}'(n, k)$  can be obtained, for increasing  $k$  and for each  $k$  for increasing  $n$ , using the recurrences

$$\boldsymbol{\pi}'(n, k)^{U'_S} = \boldsymbol{\pi}'(n-1, k-1) \mathbf{P}_{S', U'_S}, \quad n \geq 1, 1 \leq k \leq n+1, \quad (14)$$

$$\boldsymbol{\pi}'(n, k)^{D'_S} = \boldsymbol{\pi}'(n-1, k) \mathbf{P}_{S', D'_S}, \quad n \geq 1, 0 \leq k \leq n \quad (15)$$

and

$$\boldsymbol{\pi}'(0, 0)^{U'_S} = \mathbf{0}, \quad (16)$$

$$\boldsymbol{\pi}'(0, 1)^{U'_S} = \boldsymbol{\alpha}^{U'_S}, \quad (17)$$

$$\boldsymbol{\pi}'(n, 0)^{U'_S} = \mathbf{0}, \quad n \geq 1, \quad (18)$$

$$\boldsymbol{\pi}'(0, 0)^{D'_S} = \boldsymbol{\alpha}^{D'_S}, \quad (19)$$

$$\boldsymbol{\pi}'(0, 1)^{D'_S} = \mathbf{0}, \quad (20)$$

$$\boldsymbol{\pi}'(n, n+1)^{D'_S} = \mathbf{0}, \quad n \geq 1. \quad (21)$$

For the case  $U'_S \neq \emptyset$  and not too small models, the model transformation phase has an approximate flop count  $CK(2T + M|\Omega|) + \mathbf{1}_{\alpha_{S'} > 0} CL(2T + M|\Omega|)$ , where  $T$  is the number of transitions of  $X$ ,  $M = 11$  if  $\Omega = S \cup \{f\}$  and  $M = 9$  if  $\Omega = S$ . A salient feature is that, in that case, the truncation parameters  $K$  and  $L$  are smooth functions of  $t$ . More specifically,  $K$  is  $O(\log(\Lambda_U t / \varepsilon))$  and, for  $\alpha_{S'} > 0$ ,  $L$  is  $O(\log(\Lambda_U t / \varepsilon))$ . For class  $C_1$  models with  $|U_S| \geq 2$  and the selection  $r = o$  we have the following additional result [3], where  $c(n) \sim d(n)$  for  $n \rightarrow \infty$  denotes  $\lim_{n \rightarrow \infty} c(n)/d(n) = 1$ .

**Theorem 1.** *For class  $C_1$  models with  $|U_S| \geq 2$  and the selection  $r = o$ ,  $a_C(n) \leq (C+1)h(n)$  and  $a'_C(n) \leq \alpha_{S'}(C+1)h'(n)$ , where, for  $n \rightarrow \infty$ ,  $h(n) \sim B \binom{n-1}{p-1} \rho^n$  and  $h'(n) \sim B' \binom{n-1}{p'-1} \rho'^n$ , with  $B > 0$ ,  $B' > 0$ ,  $p, p'$  integers  $\geq 1$ ,  $\rho, \rho' \approx 1 - 1/R'$ , and  $R' = \max_{i \in U_S} \lambda_i / \min_{i \in U_S - \{o\}} \lambda_i$ .*

According to Theorem 1, for class  $C_1$  models with  $|U_S| \geq 2$  and the selection  $r = o$ , the values of the truncation parameters  $K$  and  $L$  should be mainly determined by the parameter  $R'$ : the closer  $R'$  to 1, the smaller the truncation parameters  $K$  and  $L$  should be. In fact [3], as a rule of thumb, for  $R' \gg 1$ ,  $K$  and  $L$  can be roughly bounded from above by  $30R'$ . Often,  $\max_{i \in \Omega} \lambda_i t_{q_{\max}}$  will be moderate and the truncation parameter  $C$  will be moderate. In that case, both the computational cost in terms of CPU time of the model transformation phase relative to the size of  $X$  and the size of  $V_T$  will be moderate if  $R'$  has a moderate value, and, if  $X$  is large, the method will have a moderate computational cost in terms of CPU time relative to the size of  $X$ , much smaller than the computational cost of Algorithm A when  $\max_{i \in \Omega} \lambda_i t$  is large. For the case  $|U_S| = 1$ , the selection  $r = o$  yields  $U'_S = \emptyset$ , and both the computational cost in terms of CPU time of the model transformation phase relative to the size of  $X$  and the size of  $V_T$  will be always small if  $\max_{i \in \Omega} \lambda_i t_{q_{\max}}$  is moderate, and, for large  $X$ , the method will also have a small computational cost in terms of CPU time relative to size of  $X$ , much smaller than the computational cost of Algorithm A.

### 3 The Bounding Regenerative Transformation Method

We will start by identifying the CTMC models covered by bounding regenerative transformation and the model class  $C'_1$  at which the method is targeted. Then, we will motivate and justify the method and will describe it in the general case. A separate subsection will be dedicated to justify and describe a more efficient implementation of the method which is available for the case  $D_C = 1$  when both bounds have to be computed and an additional condition is satisfied.

#### 3.1 Motivation and general case

The bounding regenerative transformation method covers the same class of CTMC models and selections for the regenerative state  $r$  as the regenerative transformation method (conditions C1–C10) with the additional condition:

$$\text{C12. } U'_S \neq \emptyset.$$

The additional condition is imposed because for  $U'_S = \emptyset$  there is no up state whose transition rates have to be scaled and the bounding regenerative transformation method would be reduced to the regenerative transformation method.

The method is targeted at a model class  $C'_1$  with a “natural” selection for the regenerative state  $r$ . Model class  $C'_1$  is a subclass of model class  $C_1$  defined by conditions C1–C7 and the conditions

$$\text{C13. } |U_S| \geq 2.$$

C14. A partition  $U_0 \cup U_1 \cup \dots \cup U_{N_C}$  for  $U_S$  exists satisfying the properties:

- P1.  $U_0 = \{o\}$  (i.e.  $|U_0| = 1$ ).
- P2. If  $X$  has a single recurrent class of states  $F \subset S$ ,  $o \in F$ .
- P3.  $\lambda_{o, U_1 \cup \dots \cup U_{N_C}} > 0$ .
- P4. If  $\alpha_{D_S} > 0$  and  $\alpha_{U_1 \cup \dots \cup U_{N_C}} = 0$ ,  $\lambda_{i, U_1 \cup \dots \cup U_{N_C}} > 0$  for some  $i \in D_S$  with  $\alpha_i > 0$ .
- P5.  $\max_{0 \leq k \leq N_C} \max_{i \in U_k} \lambda_{i, U_k - \{i\} \cup U_{k+1} \cup \dots \cup U_{N_C} \cup D_S}$  is significantly smaller than  $\min_{0 < k \leq N_C} \min_{i \in U_k} \lambda_{i, U_0 \cup \dots \cup U_{k-1}} > 0$  if  $\Omega = S$   
or  $\min_{0 < k \leq N_C} \min_{i \in U_k} \lambda_{i, U_0 \cup \dots \cup U_{k-1} \cup \{f\}} > 0$  if  $\Omega = S \cup \{f\}$ .
- P6.  $\lambda_o \leq \min_{i \in U_1 \cup \dots \cup U_{N_C}} \lambda_i$ .

The natural selection for the regenerative state for class  $C'_1$  models is  $r = o$ . Since class  $C'_1$  is a subclass of class  $C_1$  and, for any model in class  $C'_1$ ,  $|U_S| \geq 2$  and  $\max_{i \in U_S} \lambda_i = \max_{i \in U_S - \{o\}} \lambda_i$  because of property P6 of the partition for  $U_S$ , we have, from Theorem 1:

**Theorem 2.** For class  $C'_1$  models and the selection  $r = o$ ,  $a_C(n) \leq (C + 1)h(n)$  and  $a'_C(n) \leq \alpha_{S'}(C + 1)h'(n)$ , where, for  $n \rightarrow \infty$ ,  $h(n) \sim B \binom{n-1}{p-1} \rho^n$  and  $h'(n) \sim B' \binom{n-1}{p'-1} \rho'^n$ , with  $B > 0$ ,  $B' > 0$ ,  $p, p'$  integers  $\geq 1$ ,  $\rho, \rho' \approx 1 - 1/R''$ , and  $R'' = \max_{i \in U_S - \{o\}} \lambda_i / \min_{i \in U_S - \{o\}} \lambda_i$ .

The bounding regenerative transformation method is motivated by Theorem 2 and is based on the following result. See, for instance, [9] for the definitions of conservative and uniformizable CTMCs with denumerable state space. They are CTMCs with denumerable state space in which the output rate from any state  $i$  is equal to the sum of the transition rates from  $i$  and in which the output rates are uniformly bounded from above. Any finite CTMC with infinitesimal generator is both conservative and uniformizable. Although we will only use the result for finite CTMCs with infinitesimal generator, that restriction does not lead to a simpler proof.

**Theorem 3.** Let  $W$  be a conservative, uniformizable CTMC with denumerable state space  $\Omega$ , subset of “up” states  $U$  and transition rates  $\lambda_{i,j}$ ,  $i, j \in \Omega$ ,  $j \neq i$  and let  $W'$  be another conservative, uniformizable CTMC with same state space, same initial probability distribution, same subset of “up” states, same transition rates from non-“up” states as  $X$ , and transition rates from “up” states  $\lambda'_{i,j} = \beta_i \lambda_{i,j}$ ,  $i \in U$ ,  $j \in \Omega$ ,  $j \neq i$ ,  $0 < \beta_i \leq 1$ . Let IAVCD( $t, p$ ) be the complementary interval availability distribution of  $W$ , i.e.  $\text{IAVCD}(t, p) = P[(\int_0^t \mathbf{1}_{W(\tau) \in U} d\tau) / t > p]$ ,  $t > 0$ ,  $0 < p < 1$ . Let IAVCD'( $t, p$ ) be the complementary interval availability distribution of  $W'$ , i.e.  $\text{IAVCD}'(t, p) = P[(\int_0^t \mathbf{1}_{W'(\tau) \in U} d\tau) / t > p]$ ,  $t > 0$ ,  $0 < p < 1$ . Then,  $\text{IAVCD}'(t, p) \geq \text{IAVCD}(t, p)$ .

**Proof.** See the Appendix. □

Essentially, the reason why Theorem 3 holds is that scaling transition rates from up states keeping their relative values will not modify the embedded DTMC  $\Pi$  of  $W$ . Since (see, for instance, [9]) both  $W$  and  $W'$  can be interpreted in terms of  $\Pi$  by associating with the states visited by  $\Pi$  independent

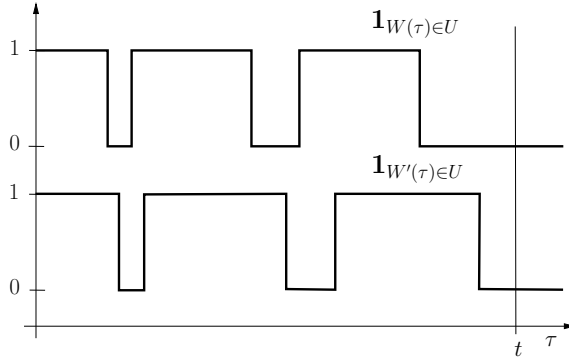


Figure 4: Comparison of corresponding realizations of  $W$  and  $W'$ .

exponential holding times with parameter equal to the output rate from the visited state, the output rates from down states are equal in  $W$  and  $W'$  and the output rates from up states are smaller in  $W'$  than in  $W$ , each realization of  $W$  will have a corresponding realization of  $W'$  differing from the former only in that the holding times in the up states will be non-smaller and, as Figure 4 illustrates, this will cause the up time in the time interval  $[0, t]$  of the realization of  $W'$  to be non-smaller than the up time in the same interval of the corresponding realization of  $W$ . Being the up time of  $W'$  in the time interval  $[0, t]$  for a realization of  $W'$  non-smaller than the up time of  $W$  in the same time interval for the corresponding realization of  $W$ , the probability that the interval availability of  $W'$  in the time interval  $[0, t]$  is greater than  $p$  will be non-smaller than the probability that the interval availability of  $W$  in the same time interval is greater than  $p$ .

According to Theorem 3, scaling up the transition rates from some up states will result in a CTMC model whose IAVCD( $t, p$ ) measure,  $\text{IAVCD}^{\text{lb}}(t, p)$ , will bound from below the IAVCD( $t, p$ ) measure of the original model. Conversely, scaling down the transition rates from some up states will result in a CTMC model whose IAVCD( $t, p$ ) measure,  $\text{IAVCD}^{\text{ub}}(t, p)$ , will bound from above the IAVCD( $t, p$ ) measure of the original model. The bounding regenerative transformation method performs such scalings in the states in  $U'_S = U_S - \{r\}$  of  $X$ , where  $r$  is the selected regenerative state, and solves the scaled models by regenerative transformation with regenerative state  $r$ . The scaling is performed so that for class  $C'_1$  models with the selection  $r = o$  the scaled models still belong to model class  $C'_1$  and have an  $R''$  parameter equal to a given control parameter  $D_C$ . Then, according to Theorem 2, the computational efficiency of the method should increase as  $D_C$  decreases. Also, in the frequent case in which  $\max_{i \in \Omega} \lambda_i t q_{\max}$  is moderate, since the scalings do not increase  $\max_{i \in \Omega} \lambda_i$ , the truncation parameter  $C$  associated with the solution of the bounding models by regenerative transformation will be moderate, and, since with  $D_C = 1$  the truncation parameters  $K$  and  $L$  associated with the solution of the bounding models should be small, for large class  $C'_1$  models and with the selections  $r = o$  and  $D_C = 1$ , the method will often obtain bounds at a small computational cost relative to the model size.

Let  $\lambda_{\min} = \min_{i \in U'_S} \lambda_i$  and  $\lambda_{\max} = \max_{i \in U'_S} \lambda_i$ . Note that, for the CTMC models and selections for the regenerative state  $r$  covered by bounding regenerative transformation,  $S$  cannot include any absorbing state: by condition C4 it can include at most one and by condition C8 that

one should be  $r$ , in contradiction with  $\lambda_{r,U'_S} > 0$  (conditions C9 and C12). Then,  $U'_S \subset S$  does not include any absorbing state and  $\lambda_{\max} \geq \lambda_{\min} > 0$ . The control parameter  $D_C$  is required to satisfy  $1 \leq D_C < \lambda_{\max}/\lambda_{\min}$ . The scaling up yielding the lower bounding model,  $X^{\text{lb}}$ , is defined by  $\lambda_{i,j}^{\text{lb}} = \lambda_{i,j}(\lambda_i^{\text{lb}}/\lambda_i)$ ,  $\lambda_i^{\text{lb}} = \max\{\lambda_i, \lambda_{\max}/D_C\}$ ,  $i \in U'_S$ ,  $j \neq i$ , where  $\lambda_i^{\text{lb}}$  and  $\lambda_{i,j}^{\text{lb}}$  are, respectively, the output rates and transition rates of  $X^{\text{lb}}$ . The scaling down yielding the upper bounding model,  $X^{\text{ub}}$ , is defined by  $\lambda_{i,j}^{\text{ub}} = \lambda_{i,j}(\lambda_i^{\text{ub}}/\lambda_i)$ ,  $\lambda_i^{\text{ub}} = \min\{\lambda_i, D_C \lambda_{\min}\}$ ,  $i \in U'_S$ ,  $j \neq i$ , where  $\lambda_i^{\text{ub}}$  and  $\lambda_{i,j}^{\text{ub}}$  are, respectively, the output rates and transition rates of  $X^{\text{ub}}$ . Note that, since larger values of  $D_C$  potentially yield smaller values for the output rates from some up states in  $X^{\text{lb}}$  and potentially yield larger values for the output rates from some up states in  $X^{\text{ub}}$ , according to Theorem 3, the larger the control parameter  $D_C$  the tighter the obtained bounds will be. Thus, for class  $C'_1$  models with the selection  $r = o$ , the control parameter  $D_C$  allows to trade off computational cost with bounds tightness. In the case  $\lambda_{\max} = \lambda_{\min}$  no selection for  $D_C$  would be possible, but, in that case, for class  $C'_1$  models and the selection  $r = o$  the parameter  $R''$  will be equal to 1 and, by Theorem 2, the truncation parameters  $K$  and  $L$  should be small. Then, when  $\max_{i \in \Omega} \lambda_i t_{q_{\max}}$  is moderate, the truncation parameter  $C$  will be moderate and the regenerative transformation method should have small computational cost in terms of CPU time relative to the size of  $X$  when  $X$  is large.

Since regenerative transformation is numerically stable and has good error control [3], bounding regenerative transformation will compute the bounds with numerical stability and well-controlled error.

### 3.2 Particular implementation

The particular case in which both the lower bound and the upper bound for  $\text{IAVCD}(t, p)$  have to be computed,  $D_C = 1$  and, if  $r \in U'_S$ ,  $\lambda_r \leq \min_{i \in U'_S} \lambda_i$  allows a computationally more efficient implementation of the bounding regenerative transformation method than the one described in the previous section. That more efficient implementation is based on the fact that the truncated transformed CTMC model corresponding to the solution of  $X^{\text{ub}}$  by regenerative transformation can be constructed without analyzing the randomized DTMC of  $X^{\text{ub}}$  if some quantities related to the construction of the truncated transformed CTMC model built during the application of regenerative transformation to the solution of  $X^{\text{lb}}$  are saved. We will denote with the superscript  $^{\text{lb}}$  ( $^{\text{ub}}$ ) the quantities corresponding to the first phase of regenerative transformation applied to  $X^{\text{lb}}$  ( $X^{\text{ub}}$ ).

The justification of the particular implementation is quite elaborated. However, for the  $C'_1$  model class at which the bounding regenerative transformation method is targeted, the case  $D_C = 1$  is the most interesting one, since it is in that case that the method will often have a relatively small computational cost. Also, with the natural selection  $r = o$ , the additional condition  $\lambda_r \leq \min_{i \in U'_S} \lambda_i$  will be satisfied because of property P6 of the partition for  $U'_S$ , and, often, both bounds will be of interest to “bracket” the exact solution of the model.

Let (for the CTMC models and selections for  $r$  covered by bounding regenerative transformation,  $U'_S \neq \emptyset$ ,  $D_S \neq \emptyset$ ,  $U'_S \neq \emptyset$ , and  $D_T = \{(n, k) : 0 \leq k \leq K \wedge \max\{0, k-1\} \leq n \leq k+C-1\}$ ,



$$D'_T = \{(n, k) : 0 \leq k \leq L \wedge \max\{0, k-1\} \leq n \leq k+C-1\}$$

$$E_{T,u} = \{(n, k) : (n, k) \in D_T \wedge k < K \wedge \boldsymbol{\pi}(n, k)^{U_S} \neq \mathbf{0}\},$$

$$E_{T,d} = \{(n, k) : (n, k) \in D_T \wedge k < K \wedge \boldsymbol{\pi}(n, k)^{D_S} \neq \mathbf{0}\},$$

and, assuming  $\alpha_{S'} > 0$ , let

$$E'_{T,u} = \{(n, k) : (n, k) \in D'_T \wedge k < L \wedge \boldsymbol{\pi}'(n, k)^{U'_S} \neq \mathbf{0}\}$$

and, if  $D'_S \neq \emptyset$ , let

$$E'_{T,d} = \{(n, k) : (n, k) \in D'_T \wedge k < L \wedge \boldsymbol{\pi}'(n, k)^{D'_S} \neq \mathbf{0}\}.$$

Note that  $E_{T,u}$  ( $E_{T,d}$ ) collects the pairs  $(n, k)$  corresponding to the states  $s_{n,k}^u$  ( $s_{n,k}^d$ ) in  $\Omega_T$  with  $k < K$  and  $E'_{T,u}$  ( $E'_{T,d}$ ) collects the pairs  $(n, k)$  corresponding to states  $s_{n,k}^{u'}$  ( $s_{n,k}^{d'}$ ) in  $\Omega_T$  with  $k < L$ . Then, the quantities corresponding to the first phase of regenerative transformation applied to  $X^{\text{lb}}$  which have to be saved are  $\Lambda_U^{\text{lb}}, \Lambda_D^{\text{lb}}, a^{\text{lb}}(n, k), (n, k) \in D_T^{\text{lb}}, k \geq 2$ ; if  $\alpha_{S'} > 0$ ,  $a^{\text{lb}}(n, k), (n, k) \in D_T^{\text{lb}}, k \geq 2; w_{n,k}^{u\text{ub}}, w_{n,k}^{d\text{ub}}, q_{n,k}^{\text{ub}}, (n, k) \in E_{T,u}^{\text{lb}}; \text{if } \Omega = S \cup \{f\}, v_{n,k}^{\text{ub}}, (n, k) \in E_{T,u}^{\text{lb}}; w_{n,k}^{d\text{ub}}, w_{n,k}^{d\text{ub}}, q_{n,k}^{\text{ub}}, (n, k) \in E_{T,d}^{\text{lb}}; \text{if } \Omega = S \cup \{f\}, v_{n,k}^{\text{ub}}, (n, k) \in E_{T,d}^{\text{lb}}; \text{if } \alpha_{S'} > 0, w_{n,k}^{u\text{ub}}, w_{n,k}^{d\text{ub}}, q_{n,k}^{\text{ub}}, (n, k) \in E_{T,u}^{\text{lb}}; \text{if } \alpha_{S'} > 0 \text{ and } \Omega = S \cup \{f\}, v_{n,k}^{\text{ub}}, (n, k) \in E_{T,u}^{\text{lb}}; \text{if } \alpha_{S'} > 0 \text{ and } D'_S \neq \emptyset, w_{n,k}^{d\text{ub}}, w_{n,k}^{d\text{ub}}, q_{n,k}^{\text{ub}}, (n, k) \in E_{T,d}^{\text{lb}}; \text{and, if } \alpha_{S'} > 0, D'_S \neq \emptyset \text{ and } \Omega = S \cup \{f\}, v_{n,k}^{\text{ub}}, (n, k) \in E_{T,d}^{\text{lb}}.$

Construction of the truncated transformed CTMC model of  $X^{\text{ub}}$  from those quantities is possible because: 1)  $C^{\text{ub}} \leq C^{\text{lb}}; K^{\text{ub}} \leq K^{\text{lb}}; \text{for } \alpha_{S'} > 0, L^{\text{ub}} \leq L^{\text{lb}}; \boldsymbol{\pi}^{\text{ub}}(n, k)^{U_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}^{\text{lb}}(n, k)^{U_S} \neq \mathbf{0}$  and  $\boldsymbol{\pi}^{\text{ub}}(n, k)^{D_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}^{\text{lb}}(n, k)^{D_S} \neq \mathbf{0}$ ; for  $\alpha_{S'} > 0$ ,  $\boldsymbol{\pi}'^{\text{ub}}(n, k)^{U'_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}'^{\text{lb}}(n, k)^{U'_S} \neq \mathbf{0}$ ; and, for  $\alpha_{S'} > 0$  and  $D'_S \neq \emptyset$ ,  $\boldsymbol{\pi}'^{\text{ub}}(n, k)^{D'_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}'^{\text{lb}}(n, k)^{D'_S} \neq \mathbf{0}$ , implying (see Figure 2 and the definition of  $\Omega_T$  in Section 2)  $D_T^{\text{ub}} \subset D_T^{\text{lb}}, \text{if } \alpha_{S'} > 0, D_T^{\text{ub}} \subset D_T^{\text{lb}}, \text{and } \Omega_T^{\text{ub}} \subset \Omega_T^{\text{lb}}; \text{and 2) there exist simple relationships between } \Lambda_U^{\text{ub}}, \Lambda_D^{\text{ub}}, a^{\text{ub}}(n, k), a^{\text{ub}}(n, k), w_{n,k}^{u\text{ub}}, w_{n,k}^{d\text{ub}}, q_{n,k}^{\text{ub}}, v_{n,k}^{\text{ub}}, w_{n,k}^{d\text{ub}}, w_{n,k}^{d\text{ub}}, q_{n,k}^{\text{ub}}, v_{n,k}^{\text{ub}}, w_{n,k}^{u\text{ub}}, w_{n,k}^{u\text{ub}}, q_{n,k}^{\text{ub}}, v_{n,k}^{\text{ub}}, w_{n,k}^{d\text{ub}}, w_{n,k}^{d\text{ub}}, q_{n,k}^{\text{ub}}, v_{n,k}^{\text{ub}}$  and the corresponding quantities for  $X^{\text{lb}}$ . Using those relationships, it is possible to determine (1)–(4)  $C^{\text{ub}}, K^{\text{ub}}$  and, if  $\alpha_{S'} > 0$ ,  $L^{\text{ub}}$ , and to build the truncated transformed model corresponding to the solution of  $X^{\text{ub}}$  by regenerative transformation. In the remaining of this section we will prove 1) and will obtain the mentioned relationships. The relationships are established in terms of the parameter  $R = \lambda_{\max}/\lambda_{\min}$ , with, we remember,  $\lambda_{\min} = \min_{i \in U'_S} \lambda_i$  and  $\lambda_{\max} = \max_{i \in U'_S} \lambda_i$ . Note that  $R > 1$ , since  $D_C < \lambda_{\max}/\lambda_{\min}$  and  $D_C = 1$  for the particular implementation to apply.

We start by relating  $\Lambda_U, \Lambda_D, \Lambda = \max\{\Lambda_U, \Lambda_D\}$  and the transition probabilities of the randomized DTMCs of  $X^{\text{lb}}$  and  $X^{\text{ub}}$ :

**Theorem 4.** *Assume  $D_C = 1$  and, if  $r \in U_S, \lambda_r \leq \lambda_{\min}$ . Then,  $\Lambda_U^{\text{ub}} = \Lambda_U^{\text{lb}}/R, \Lambda_D^{\text{ub}} = \Lambda_D^{\text{lb}}, \Lambda^{\text{ub}} \leq \Lambda^{\text{lb}}, \text{if } r \in U_S, P_{r,j}^{\text{ub}} = RP_{r,j}^{\text{lb}}, j \neq r, \text{if } r \in D_S, P_{r,j}^{\text{ub}} = P_{r,j}^{\text{lb}}, j \neq r, \text{and } P_{i,j}^{\text{ub}} = P_{i,j}^{\text{lb}}, i \in S'.$*

**Proof.** Since bounding regenerative transformation only modifies the transition rates from states in  $U'_S, \lambda_r^{\text{ub}} = \lambda_r^{\text{lb}} = \lambda_r, \lambda_{r,j}^{\text{ub}} = \lambda_{r,j}^{\text{lb}} = \lambda_{r,j}, j \neq r$  and, for  $i \in D_S, \lambda_i^{\text{ub}} = \lambda_i^{\text{lb}} = \lambda_i, \lambda_{i,j}^{\text{ub}} = \lambda_{i,j}^{\text{lb}} = \lambda_{i,j},$

$j \neq i$ . Being  $D_C = 1$ , for  $i \in U'_S$ ,  $\lambda_i^{\text{ub}} = \lambda_{\min}$  and  $\lambda_i^{\text{lb}} = \lambda_{\max}$ , implying, for  $i \in U'_S$ ,  $\lambda_i^{\text{ub}} = \lambda_i^{\text{lb}}/R$  and  $\lambda_{i,j}^{\text{ub}} = \lambda_{i,j}^{\text{lb}}/R$ ,  $j \neq i$ . Using  $\lambda_r^{\text{ub}} = \lambda_r^{\text{lb}} = \lambda_r$ ,  $\lambda_i^{\text{ub}} = \lambda_{\min}$ ,  $i \in U'_S$ ,  $\lambda_i^{\text{lb}} = \lambda_{\max}$ ,  $i \in U'_S$ , and  $\lambda_r \leq \lambda_{\min}$  if  $r \in U_S$ ,

$$\begin{aligned}\Lambda_U^{\text{ub}} &= (1 + \theta) \max_{i \in U_S} \lambda_i^{\text{ub}} = (1 + \theta) \max \left\{ \mathbf{1}_{r \in U_S} \lambda_r^{\text{ub}}, \max_{i \in U'_S} \lambda_i^{\text{ub}} \right\} \\ &= (1 + \theta) \max \{ \mathbf{1}_{r \in U_S} \lambda_r, \lambda_{\min} \} = (1 + \theta) \lambda_{\min} = (1 + \theta) \frac{\lambda_{\max}}{R} \\ &= \frac{1}{R} (1 + \theta) \max \{ \mathbf{1}_{r \in U_S} \lambda_r, \lambda_{\max} \} = \frac{1}{R} (1 + \theta) \max \left\{ \mathbf{1}_{r \in U_S} \lambda_r^{\text{lb}}, \max_{i \in U'_S} \lambda_i^{\text{lb}} \right\} \\ &= \frac{1}{R} (1 + \theta) \max_{i \in U_S} \lambda_i^{\text{lb}} = \frac{\Lambda_U^{\text{lb}}}{R}.\end{aligned}$$

Using  $\lambda_i^{\text{ub}} = \lambda_i^{\text{lb}}$ ,  $i \in D_S$ ,  $\Lambda_D^{\text{ub}} = (1 + \theta) \max_{i \in D_S} \lambda_i^{\text{ub}} = (1 + \theta) \max_{i \in D_S} \lambda_i^{\text{lb}} = \Lambda_D^{\text{lb}}$ . The result  $\Lambda^{\text{ub}} \leq \Lambda^{\text{lb}}$  follows immediately from  $\Lambda^{\text{ub}} = \max\{\Lambda_U^{\text{ub}}, \Lambda_D^{\text{ub}}\}$ ,  $\Lambda^{\text{lb}} = \max\{\Lambda_U^{\text{lb}}, \Lambda_D^{\text{lb}}\}$ ,  $\Lambda_U^{\text{ub}} < \Lambda_U^{\text{lb}}$ , and  $\Lambda_D^{\text{ub}} = \Lambda_D^{\text{lb}}$ . If  $r \in U_S$ , for  $j \neq r$ , using  $\lambda_{r,j}^{\text{ub}} = \lambda_{r,j}^{\text{lb}}$  and  $\Lambda_U^{\text{ub}} = \Lambda_U^{\text{lb}}/R$ , we get  $P_{r,j}^{\text{ub}} = \lambda_{r,j}^{\text{ub}}/\Lambda_U^{\text{ub}} = R\lambda_{r,j}^{\text{lb}}/\Lambda_U^{\text{lb}} = RP_{r,j}^{\text{lb}}$ . If  $r \in D_S$ , for  $j \neq r$ , using  $\lambda_{r,j}^{\text{ub}} = \lambda_{r,j}^{\text{lb}}$  and  $\Lambda_D^{\text{ub}} = \Lambda_D^{\text{lb}}$ , we get  $P_{r,j}^{\text{ub}} = \lambda_{r,j}^{\text{ub}}/\Lambda_D^{\text{ub}} = \lambda_{r,j}^{\text{lb}}/\Lambda_D^{\text{lb}} = P_{r,j}^{\text{lb}}$ . For  $i \in U'_S$  and  $j \neq i$ , using  $\lambda_{i,j}^{\text{ub}} = \lambda_{i,j}^{\text{lb}}/R$  and  $\Lambda_U^{\text{ub}} = \Lambda_U^{\text{lb}}/R$ , we get  $P_{i,j}^{\text{ub}} = \lambda_{i,j}^{\text{ub}}/\Lambda_U^{\text{ub}} = \lambda_{i,j}^{\text{lb}}/\Lambda_U^{\text{lb}} = P_{i,j}^{\text{lb}}$ , which implies, since  $\sum_{j \in \Omega} P_{i,j}^{\text{ub}} = \sum_{j \in \Omega} P_{i,j}^{\text{lb}} = 1$ ,  $P_{i,i}^{\text{ub}} = P_{i,i}^{\text{lb}}$ ,  $i \in U'_S$ . Finally, for  $i \in D'_S$  and  $j \neq i$ , using  $\lambda_{i,j}^{\text{ub}} = \lambda_{i,j}^{\text{lb}}$  and  $\Lambda_D^{\text{ub}} = \Lambda_D^{\text{lb}}$ , we get  $P_{i,j}^{\text{ub}} = \lambda_{i,j}^{\text{ub}}/\Lambda_D^{\text{ub}} = \lambda_{i,j}^{\text{lb}}/\Lambda_D^{\text{lb}} = P_{i,j}^{\text{lb}}$ , which, as before, implies  $P_{i,i}^{\text{ub}} = P_{i,i}^{\text{lb}}$ ,  $i \in D'_S$ .  $\square$

Using Theorem 4, it is possible to prove the following result, which relates the vectors  $\boldsymbol{\pi}(n, k)$  and  $\boldsymbol{\pi}'(n, k)$ .

**Proposition 1.** *Assume  $D_C = 1$  and, if  $r \in U_S$ ,  $\lambda_r \leq \lambda_{\min}$ . Then, if  $r \in U_S$ ,  $\boldsymbol{\pi}^{\text{ub}}(0, k) = \boldsymbol{\pi}^{\text{lb}}(0, k)$ ,  $0 \leq k \leq 1$  and  $\boldsymbol{\pi}^{\text{ub}}(n, k) = R\boldsymbol{\pi}^{\text{lb}}(n, k)$ ,  $n \geq 1$ ,  $0 \leq k \leq n + 1$ ; if  $r \in D_S$ ,  $\boldsymbol{\pi}^{\text{ub}}(n, k) = \boldsymbol{\pi}^{\text{lb}}(n, k)$ ,  $n \geq 0$ ,  $0 \leq k \leq n + 1$ ; and, for  $\alpha_{S'} > 0$ ,  $\boldsymbol{\pi}^{\text{ub}}(n, k) = \boldsymbol{\pi}^{\text{lb}}(n, k)$ ,  $n \geq 0$ ,  $0 \leq k \leq n + 1$ .*

**Proof.** See the Appendix.  $\square$

Note that Proposition 1 implies, as required, that  $\boldsymbol{\pi}^{\text{ub}}(n, k)^{U_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}^{\text{lb}}(n, k)^{U_S} \neq \mathbf{0}$ ,  $\boldsymbol{\pi}^{\text{ub}}(n, k)^{D_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}^{\text{lb}}(n, k)^{D_S} \neq \mathbf{0}$ , for  $\alpha_{S'} > 0$ ,  $\boldsymbol{\pi}^{\text{ub}}(n, k)^{U'_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}^{\text{lb}}(n, k)^{U'_S} \neq \mathbf{0}$ , and, for  $\alpha_{S'} > 0$  and  $D'_S \neq \emptyset$ ,  $\boldsymbol{\pi}^{\text{ub}}(n, k)^{D'_S} \neq \mathbf{0}$  if and only if  $\boldsymbol{\pi}^{\text{lb}}(n, k)^{D'_S} \neq \mathbf{0}$ . The following result, relating the quantities  $a(n, k)$  and  $a'(n, k)$  is an immediate consequence of Proposition 1, taking into account  $a(n, k) = \sum_{i \in S} \pi_i(n, k)$  and  $a'(n, k) = \sum_{i \in S'} \pi'_i(n, k)$ .

**Theorem 5.** *Assume  $D_C = 1$  and, if  $r \in U_S$ ,  $\lambda_r \leq \lambda_{\min}$ . Then, if  $r \in U_S$ ,  $a^{\text{ub}}(0, k) = a^{\text{lb}}(0, k)$ ,  $0 \leq k \leq 1$  and  $a^{\text{ub}}(n, k) = Ra^{\text{lb}}(n, k)$ ,  $n \geq 1$ ,  $0 \leq k \leq n + 1$ ; if  $r \in D_S$ ,  $a^{\text{ub}}(n, k) = a^{\text{lb}}(n, k)$ ,  $n \geq 0$ ,  $0 \leq k \leq n + 1$ ; and, for  $\alpha_{S'} > 0$ ,  $a^{\text{ub}}(n, k) = a^{\text{lb}}(n, k)$ ,  $n \geq 0$ ,  $0 \leq k \leq n + 1$ .*

The following lemma is needed to prove the sought result concerning the truncation parameters  $C$ ,  $K$  and  $L$ . A similar lemma was used in [1].

**Lemma 1.** *For  $x > 0$ ,  $k \geq 2$  and  $R > 1$ ,*

$$\frac{1}{R} \sum_{m=k}^{\infty} (m - k + 2) e^{-Rx} \frac{(Rx)^m}{m!} > \sum_{m=k}^{\infty} (m - k + 2) e^{-x} \frac{x^m}{m!}.$$

**Proof.** See the Appendix.  $\square$

Using Theorems 4 and 5 and Lemma 1, it is possible to prove the following theorem, which relates  $C^{\text{ub}}$  with  $C^{\text{lb}}$ ,  $K^{\text{ub}}$  with  $K^{\text{lb}}$  and, for  $\alpha_{S'} > 0$ ,  $L^{\text{ub}}$  with  $L^{\text{lb}}$ .

**Theorem 6.** Assume  $D_C = 1$  and, if  $r \in U_S$ ,  $\lambda_r \leq \lambda_{\min}$ . Then,  $C^{\text{ub}} \leq C^{\text{lb}}$ ,  $K^{\text{ub}} \leq K^{\text{lb}}$  and, for  $\alpha_{S'} > 0$ ,  $L^{\text{ub}} \leq L^{\text{lb}}$ .

**Proof.** That  $C^{\text{ub}} \leq C^{\text{lb}}$  follows from (1), the fact that  $\sum_{m=c+1}^{\infty} e^{-\Lambda t_{\max}} (\Lambda t_{\max})^m / m!$ ,  $c \geq 1$  is increasing with  $\Lambda$  (because it is the probability that the number of arrivals of a Poisson process with rate 1 in the time interval  $[0, \Lambda t_{\max}]$  will be greater than  $c$ ) and (Theorem 4)  $\Lambda^{\text{ub}} \leq \Lambda^{\text{lb}}$ . Assume  $r \in U_S$ . Using, then, Theorem 5, for  $k \geq 2$ ,  $a_{C^{\text{ub}}}^{\text{ub}}(k) = \sum_{n=k-1}^{k+C^{\text{ub}}-1} a^{\text{ub}}(n, k) = R \sum_{n=k-1}^{k+C^{\text{ub}}-1} a^{\text{lb}}(n, k) \leq R \sum_{n=k-1}^{k+C^{\text{lb}}-1} a^{\text{lb}}(n, k) = R a_{C^{\text{lb}}}^{\text{lb}}(k)$ , which, combined with (Theorem 4)  $\Lambda_U^{\text{ub}} = \Lambda_U^{\text{lb}}/R$  and Lemma 1 with  $x = \Lambda_U^{\text{ub}} t_{\max}$  gives

$$\begin{aligned} & \alpha_S a_{C^{\text{lb}}}^{\text{lb}}(k) \sum_{m=k}^{\infty} (m-k+2) \frac{(\Lambda_U^{\text{lb}} t_{\max})^m}{m!} e^{-\Lambda_U^{\text{lb}} t_{\max}} \\ & \geq \alpha_S \frac{a_{C^{\text{ub}}}^{\text{ub}}(k)}{R} \sum_{m=k}^{\infty} (m-k+2) \frac{(R \Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-R \Lambda_U^{\text{ub}} t_{\max}} \\ & > \alpha_S a_{C^{\text{ub}}}^{\text{ub}}(k) \sum_{m=k}^{\infty} (m-k+2) \frac{(\Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-\Lambda_U^{\text{ub}} t_{\max}}, \quad k \geq 2, \end{aligned}$$

and, then, it follows from (2) and (4) that, for  $r \in U_S$ ,  $K^{\text{ub}} \leq K^{\text{lb}}$ . Assume now  $r \in D_S$ . Using, then, Theorem 5, for  $k \geq 2$ ,  $a_{C^{\text{ub}}}^{\text{ub}}(k) = \sum_{n=k-1}^{k+C^{\text{ub}}-1} a^{\text{ub}}(n, k) = \sum_{n=k-1}^{k+C^{\text{ub}}-1} a^{\text{lb}}(n, k) \leq \sum_{n=k-1}^{k+C^{\text{lb}}-1} a^{\text{lb}}(n, k) = a_{C^{\text{lb}}}^{\text{lb}}(k)$ , which, combined with  $\Lambda_U^{\text{ub}} = \Lambda_U^{\text{lb}}/R$  and using  $R > 1$  gives

$$\begin{aligned} & \alpha_S a_{C^{\text{lb}}}^{\text{lb}}(k) \sum_{m=k}^{\infty} (m-k+2) \frac{(\Lambda_U^{\text{lb}} t_{\max})^m}{m!} e^{-\Lambda_U^{\text{lb}} t_{\max}} \\ & \geq \alpha_S a_{C^{\text{ub}}}^{\text{ub}}(k) \sum_{m=k}^{\infty} (m-k+2) \frac{(R \Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-R \Lambda_U^{\text{ub}} t_{\max}} \\ & = \alpha_S a_{C^{\text{ub}}}^{\text{ub}}(k) \sum_{n=k}^{\infty} \sum_{m=n}^{\infty} \frac{(R \Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-R \Lambda_U^{\text{ub}} t_{\max}} + \sum_{m=k}^{\infty} \frac{(R \Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-R \Lambda_U^{\text{ub}} t_{\max}} \\ & > \alpha_S a_{C^{\text{ub}}}^{\text{ub}}(k) \sum_{n=k}^{\infty} \sum_{m=n}^{\infty} \frac{(\Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-\Lambda_U^{\text{ub}} t_{\max}} + \sum_{m=k}^{\infty} \frac{(\Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-\Lambda_U^{\text{ub}} t_{\max}} \\ & = \alpha_S a_{C^{\text{ub}}}^{\text{ub}}(k) \sum_{m=k}^{\infty} (m-k+2) \frac{(\Lambda_U^{\text{ub}} t_{\max})^m}{m!} e^{-\Lambda_U^{\text{ub}} t_{\max}}, \quad k \geq 2, \end{aligned}$$

and, then, it similarly follows from (2) and (4) that, for  $r \in D_S$ ,  $K^{\text{ub}} \leq K^{\text{lb}}$ . Finally, for  $\alpha_{S'} > 0$ , using Theorem 5, for  $k \geq 2$ ,  $a_{C^{\text{ub}}}^{\text{ub}}(k) = \sum_{n=k-1}^{k+C^{\text{ub}}-1} a^{\text{ub}}(n, k) = \sum_{n=k-1}^{k+C^{\text{ub}}-1} a^{\text{lb}}(n, k) \leq \sum_{n=k-1}^{k+C^{\text{lb}}-1} a^{\text{lb}}(n, k) = a_{C^{\text{lb}}}^{\text{lb}}(k)$  and, then, it follows from (3), the fact that

$\sum_{m=k}^{\infty} e^{-\Lambda_U t_{\max}} (\Lambda_U t_{\max})^m / m!$  is increasing with  $\Lambda_U$  and  $\Lambda_U^{\text{ub}} < \Lambda_U^{\text{lb}}$  (because, by Theorem 4,  $\Lambda_U^{\text{ub}} = \Lambda_U^{\text{lb}} / R$ , and  $R > 1$ ) that  $L^{\text{ub}} \leq L^{\text{lb}}$ .  $\square$

Finally, the following theorem relates the quantities  $w_{n,k}^{uu}, w_{n,k}^{ud}, q_{n,k}^u, v_{n,k}^u, w_{n,k}^{du}, w_{n,k}^{dd}, q_{n,k}^d, v_{n,k}^d, w_{n,k}^{uu}, w_{n,k}^{ud}, q_{n,k}^u, v_{n,k}^u, w_{n,k}^{du}, w_{n,k}^{dd}, q_{n,k}^d, v_{n,k}^d$ . Its proof uses Theorem 4 and Proposition 1.

**Theorem 7.** *Assume  $D_C = 1$  and, if  $r \in U_S$ ,  $\lambda_r \leq \lambda_{\min}$ . Then, if  $r \in U_S$ ,  $w_{0,1}^{uuub} = R w_{0,1}^{uulb}$ ,  $w_{0,1}^{udub} = R w_{0,1}^{udlb}$  and, if  $\Omega = S \cup \{f\}$ ,  $v_{0,1}^{uub} = R v_{0,1}^{ulb}$ ; if  $r \in D_S$ ,  $w_{0,0}^{duub} = w_{0,0}^{dulb}$ ,  $w_{0,0}^{ddub} = w_{0,0}^{ddlb}$  and, if  $\Omega = S \cup \{f\}$ ,  $v_{0,0}^{dub} = v_{0,0}^{dlb}$ ; for  $(n, k) \in E_{T,u}^{\text{ub}} - \{(0, 1)\}$ ,  $w_{n,k}^{uuub} = w_{n,k}^{uulb}$ ,  $w_{n,k}^{udub} = w_{n,k}^{udlb}$ ,  $q_{n,k}^{uub} = q_{n,k}^{ulb}$  and, if  $\Omega = S \cup \{f\}$ ,  $v_{n,k}^{uub} = v_{n,k}^{ulb}$ ; and for  $(n, k) \in E_{T,d}^{\text{ub}} - \{(0, 0)\}$ ,  $w_{n,k}^{duub} = w_{n,k}^{dulb}$ ,  $w_{n,k}^{ddub} = w_{n,k}^{ddlb}$  and, if  $\Omega = S \cup \{f\}$ ,  $v_{n,k}^{dub} = v_{n,k}^{dlb}$ . Finally, if  $\alpha_{S'} > 0$ , for  $(n, k) \in E_{T,w}^{\text{ub}}$   $w_{n,k}^{uuub} = w_{n,k}^{uulb}$ ,  $w_{n,k}^{udub} = w_{n,k}^{udlb}$ ,  $q_{n,k}^{uub} = q_{n,k}^{ulb}$  and, if  $\Omega = S \cup \{f\}$ ,  $v_{n,k}^{uub} = v_{n,k}^{ulb}$ ; and, for  $(n, k) \in E_{T,d'}^{\text{ub}}$   $w_{n,k}^{duub} = w_{n,k}^{dulb}$ ,  $w_{n,k}^{ddub} = w_{n,k}^{ddlb}$ ,  $q_{n,k}^{dub} = q_{n,k}^{dlb}$  and, if  $\Omega = S \cup \{f\}$ ,  $v_{n,k}^{dub} = v_{n,k}^{dlb}$ .*

**Proof.** An immediate consequence of the equations for  $w_{n,k}^{uu}, w_{n,k}^{ud}, w_{n,k}^{du}, w_{n,k}^{dd}, q_{n,k}^u, q_{n,k}^d, v_{n,k}^u, v_{n,k}^d, w_{n,k}^{uu}, w_{n,k}^{ud}, w_{n,k}^{du}, w_{n,k}^{dd}, q_{n,k}^u, q_{n,k}^d, v_{n,k}^u, v_{n,k}^d$  given in Section 2, Theorem 4, and Proposition 1, noting that: 1) if  $r \in U_S$ , according to (9), (10) and (12), the only non-null component of  $\pi^{\text{ub}}(0, 1)^{U_S}$  is  $\pi_r^{\text{ub}}(0, 1)$  and the only non-null component of  $\pi^{\text{lb}}(0, 1)^{U_S}$  is  $\pi_r^{\text{lb}}(0, 1)$ ; 2) if  $r \in D_S$ , according to (8), (10) and (12), the only non-null component of  $\pi^{\text{ub}}(0, 0)^{D_S}$  is  $\pi_r^{\text{ub}}(0, 0)$  and the only non-null component of  $\pi^{\text{lb}}(0, 0)^{D_S}$  is  $\pi_r^{\text{lb}}(0, 0)$ ; 3) according to (7), for  $n \geq 1$ ,  $\pi_r^{\text{ub}}(n, k) = \pi_r^{\text{lb}}(n, k) = 0$ ; and 4) all  $(n, k)$  pairs in  $E_{T,u}^{\text{ub}} - \{(0, 1)\}$  and  $E_{T,u}^{\text{lb}} - \{(0, 1)\}$  verify  $n \geq 1$  and all  $(n, k)$  pairs in  $E_{T,d}^{\text{lb}} - \{(0, 0)\}$  and  $E_{T,d}^{\text{ub}} - \{(0, 0)\}$  verify  $n \geq 1$ .  $\square$

## 4 Numerical Analysis

In this section we show, using a representative large model in that class, that, for large class  $C'_1$  models when  $\max_{i \in \Omega} \lambda_i t_{q_{\max}}$  is moderate, bounding regenerative transformation with the selection  $D_C = 1$  will compute bounds for the IAVCD( $t, p$ ) measure at a small computational cost in terms of CPU time relative to the size of  $X$ , much smaller than the computational costs at which both regenerative transformation and Algorithm A are able to compute the measure when  $\max_{i \in \Omega} \lambda_i t$  is large. We also discuss under which conditions the obtained bounds with the selection  $D_C = 1$  are tight and illustrate the trade-off in those models between bounds tightness and computational cost controlled by the parameter  $D_C$ .

The example is a CTMC model of a fault-tolerant storage system made up of ten 5-level RAID subsystems, each one comprising eight disks, two redundant disk controllers and two redundant power supplies (see Figure 5). The power supplies work in cold standby redundancy. The system is up if all RAID subsystems are up. A RAID subsystem is up if, ignoring coverage faults, at least one controller is unfailed, at least one power supply is unfailed, and at least seven disks have updated data (when a failed disk is repaired in an up subsystem, a reconstruction process fills the repaired disk with data consistent with the data stored in the remaining seven disks). Disks in up subsystems

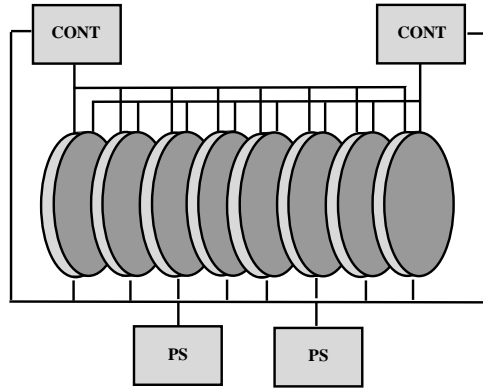


Figure 5: Architecture of the RAID subsystem.

fail with rate  $4 \times 10^{-6} \text{ h}^{-1}$  if no disk is under reconstruction and with rate  $6 \times 10^{-6} \text{ h}^{-1}$  if one disk is under reconstruction, controllers in up subsystems fail with rate  $2 \times 10^{-5} \text{ h}^{-1}$  if the subsystem has two unfailed controllers and with rate  $3 \times 10^{-5} \text{ h}^{-1}$  if the subsystem has one unfailed controller, the active power supply of an up subsystem fails with rate  $2 \times 10^{-5} \text{ h}^{-1}$ , the coverage to controller failures is 0.95, and the coverage to power supply failures is 0.98. Disks are reconstructed with rate  $0.125 \text{ h}^{-1}$ . Components of down subsystems do not fail. It is assumed the availability of an unlimited number of repairmen to repair failed components in up RAID subsystems. However, there is only a repairman to recover down RAID subsystems. The repair rate of failed components in up RAID subsystems is  $0.05 \text{ h}^{-1}$  and down subsystems are brought to a fully operational state with no component failed and all disks with consistent data with rate  $0.10 \text{ h}^{-1}$ . In case several RAID subsystems are down, the repairman selects at random the one to be brought up. Advantage is taken of the fact that all RAID subsystems have identical behavior to reduce the size of the state space of the model. A more detailed description of the model can be found in [2]. The model is quite large: 646,646 states and 15,578,290 transitions. The model has no absorbing state and, then, illustrates the case  $\Omega = S$ . A partition for the subset of up states,  $U_S$ , showing that the model is in class  $C'_1$  is  $U_S = U_0 \cup U_1 \cup \dots \cup U_{40}$ ,  $U_k = \{s \in U_S : N_C(s) + 2N_D(s) + N_P(s) + N_R(s) = k\}$ , where  $N_C(s)$  is the number of up RAID subsystems with one failed controller in state  $s$ ,  $N_D(s)$  is the number of up RAID subsystems with one failed disk in state  $s$ ,  $N_P(s)$  is the number of up RAID subsystems with one failed power supply in state  $s$ , and  $N_R(s)$  is the number of up RAID subsystems with one disk under reconstruction in state  $s$ . We will start by assuming that the system is initially in the state in which all RAID subsystems are in their fully operational state. That state is the single state  $o$  belonging to the subset  $U_0$  and is taken as regenerative state in both bounding regenerative transformation and regenerative transformation. The steady-state availability of the system is 0.99975425, making them reasonable the choices 0.9995 and 0.9999 which we will take for  $p$ . All methods are run with a single target  $(t, p)$  pair and an allowed error  $\varepsilon = 10^{-8}$ . The bounding regenerative transformation method is requested to compute both the lower bound and the upper bound. CPU times are measured/estimated in/for a workstation with a Sun-Blade 1000 processor and 4 GB of memory (significantly larger than the memory consumption for all methods). To estimate the CPU times for Algorithm A and large  $t$  we used the approximate flop count of that method given in Section 2.

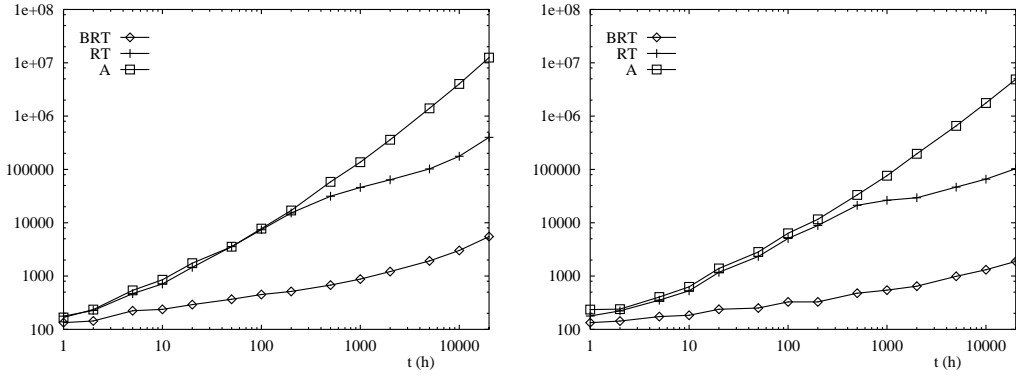


Figure 6: CPU times of bounding regenerative transformation (BRT) with  $D_C = 1$ , regenerative transformation (RT) and Algorithm A (A) for  $p = 0.9995$  (left) and  $p = 0.9999$  (right).

We start by considering the selection  $D_C = 1$  for bounding regenerative transformation. For the considered example and  $r = o$ ,  $\lambda_r$  is smaller than  $\lambda_{\min} = \min_{i \in U'_S} \lambda_i$ , and, thus, bounding regenerative transformation will use the particular, more efficient implementation discussed in Section 3.2. Table 1 gives the bounds obtained by bounding regenerative transformation (BRT), the values of the truncation parameters  $C^{\text{lb}}$  and  $K^{\text{lb}}$  defining the truncated transformed CTMC model built when, in BRT, the lower bounding model  $X^{\text{lb}}$  is solved by regenerative transformation (as discussed, the truncation parameters  $C^{\text{ub}}$  and  $K^{\text{ub}}$  defining the truncated transformed CTMC model built when the upper bounding model  $X^{\text{ub}}$  is solved are non-greater than, respectively,  $C^{\text{lb}}$  and  $K^{\text{lb}}$ ), the values of the truncation parameters  $C$  and  $K$  defining the truncated transformed CTMC model built by regenerative transformation (RT), and the values of the truncation parameters  $C'$  and  $N$  for Algorithm A (A), for  $p = 0.9995$  and  $p = 0.9999$  and increasing values of  $t$ . Figure 6 gives the CPU times consumed by the methods (for large  $t$  the CPU times of Algorithm A are enormous and they were estimated using approximate flop counts). As predicted theoretically,  $K^{\text{lb}}$  has small values. Since  $\max_{i \in \Omega} \lambda_i t(1-p)$  has moderate values (for  $p = 0.9995$  and  $t = 20,000$  h,  $\max_{i \in \Omega} \lambda_i t(1-p) \approx 22.5$ )  $C^{\text{lb}}$  has also moderate values. All this makes the CPU times consumed by BRT relatively small: for the largest  $t$  considered, 5,494 s (about 92 minutes) for  $p = 0.9995$  and 1,883 s (about 31 minutes) for  $p = 0.9999$ . Since  $C$  is identical to  $C^{\text{lb}}$  (this will always be the case), the CPU times for RT compared with those of BRT scale approximately as the truncation parameter  $K$  scales with  $K^{\text{lb}}$  and are, therefore, significantly larger for large  $t$ : for the largest  $t$  considered, 399,853 s (about 111 hours) for  $p = 0.9995$  and 103,290 s (about 29 hours) for  $p = 0.9999$ . The values of  $K$  satisfy the rough upper bound  $30R'$  mentioned at the end of Section 2, since for the example  $R' \approx 2.25/0.05 = 45$  and  $30R' \approx 1,350$ . Finally, being the model large, the truncation parameter  $N$  significantly larger than  $K^{\text{lb}}$  and  $K$  for large  $t$ , and the truncation parameter  $C'$  very similar to  $C^{\text{lb}}$  and  $C$  (this will almost always be the case) the CPU times of Algorithm A are significantly larger than the CPU times of both BRT and RT: for the largest  $t$  considered, the estimated CPU time for Algorithm A is  $1.158 \times 10^7$  s (about 134 days) for  $p = 0.9995$  and  $4.482 \times 10^6$  s (about 52 days) for  $p = 0.9999$ , and thus for large values of  $t$  and a conventional hardware platform, the example can be considered out of reach of Algorithm A.

Figure 7 gives the breakdown of the CPU times consumed by BRT into its three main com-

Table 1: Results for bounding regenerative transformation (BRT) with  $D_C = 1$ , regenerative transformation (RT) and Algorithm A (A).

$t$ (h)	$p$	BRT		RT		A			
		$\text{IAVCD}^{\text{lb}}(t, p)$	$\text{IAVCD}^{\text{ub}}(t, p)$	$C^{\text{lb}}$	$K^{\text{lb}}$	$C'$	$N$		
1	0.9995	0.99997543	0.99997600	2	8	2	13	2	15
10	0.9995	0.99975017	0.99975927	3	13	3	47	3	55
100	0.9995	0.99751052	0.99757828	6	15	6	278	5	316
1,000	0.9995	0.97644748	0.97700453	12	16	12	884	11	2,528
10,000	0.9995	0.85732856	0.86048627	36	18	36	1,009	35	23,375
20,000	0.9995	0.81889809	0.82303294	55	18	55	1,041	55	46,241
1	0.9999	0.99997542	0.99997599	2	8	2	13	2	15
10	0.9999	0.99974996	0.99975907	2	13	2	47	2	55
100	0.9999	0.99749956	0.99755676	4	15	4	278	4	316
1,000	0.9999	0.97548885	0.97606827	7	16	7	884	6	2,528
10,000	0.9999	0.79696265	0.80124391	16	18	16	1,009	15	23,375
20,000	0.9999	0.66211670	0.66861207	22	18	22	1,041	21	46,241

ponents: generation of the truncated transformed CTMC model of  $X^{\text{lb}}$ : trans (lb), solution of the truncated transformed model of  $X^{\text{lb}}$  by Algorithm A: sol (lb), and solution of the truncated transformed model of  $X^{\text{ub}}$  by Algorithm A: sol (ub). The particular implementation of BRT applies and the construction of the truncated transformed CTMC model of  $X^{\text{ub}}$  from the quantities saved during the construction of the truncated transformed CTMC model of  $X^{\text{lb}}$  consumes negligible CPU times and those CPU times are not shown. For the considered values of  $t$ , most of the CPU time consumed by the method is due to the generation of the truncated transformed CTMC model of  $X^{\text{lb}}$ , but the CPU times due to the solution by Algorithm A of the truncated transformed CTMC models increase with  $t$  faster than the CPU time consumed in the generation of the truncated transformed CTMC model of  $X^{\text{lb}}$  and, for large enough  $t$ , would dominate the computational cost of the method. As the figure clearly illustrates, the importance of those components also increases with  $1 - p$ .

The bounds obtained by BRT with  $D_C = 1$  are quite tight. Intuitively, for large  $t$ , this is because all  $X$ ,  $X^{\text{lb}}$  and  $X^{\text{ub}}$  spent most of the time in  $U_S$  in state  $o$  and the three models only differ in the holding times in the states in  $U_S - \{o\}$ . This will be the case for any class  $C'_1$  model with the selection  $r = o$  provided that the partition for  $U_S$  satisfies the additional properties

P7. For each  $i \in U_k$ ,  $0 < k \leq N_C$ ,  $\lambda_{i, U_k - \{i\} \cup U_{k+1} \cup \dots \cup U_{N_C} \cup D_S}$  if  $\Omega = S$   
or  $\lambda_{i, U_k - \{i\} \cup U_{k+1} \cup \dots \cup U_{N_C} \cup D_S \cup \{f\}}$  if  $\Omega = S \cup \{f\}$  is significantly smaller than  
 $\lambda_{i, U_0 \cup \dots \cup U_{k-1}}$ .

P8.  $\lambda_o \ll \min_{i \in U_1 \cup \dots \cup U_{N_C}} \lambda_i$ .

The reason is that P7 implies that, from any state  $i \in U_S - \{o\}$ , the embedded DTMC

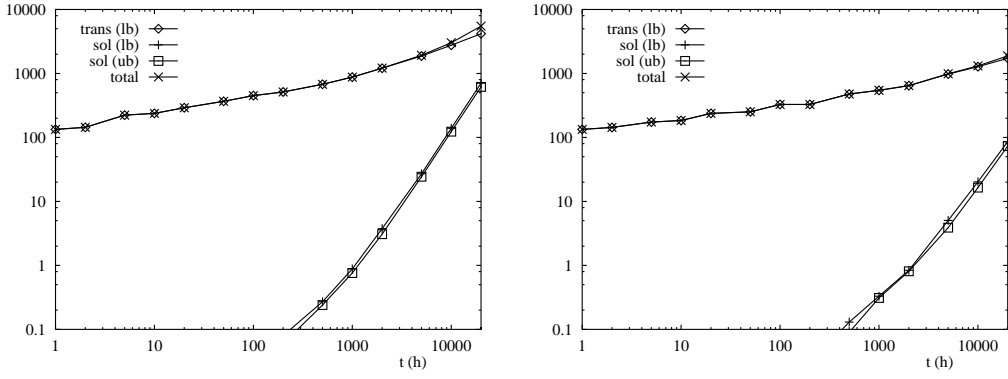


Figure 7: Breakdown of CPU times of BRT with  $D_C = 1$  for  $p = 0.9995$  (left) and  $p = 0.9999$  (right).

Table 2: Bounds obtained by BRT with  $D_C = 1$  for an initial probability distribution non concentrated in state  $o$ .

$t$ (h)	$p = 0.9995$		$p = 0.9999$	
	$\text{IAVCD}^{\text{lb}}(t, p)$	$\text{IAVCD}^{\text{ub}}(t, p)$	$\text{IAVCD}^{\text{lb}}(t, p)$	$\text{IAVCD}^{\text{ub}}(t, p)$
1	0.99905631	0.99994872	0.99905616	0.99994870
10	0.99870751	0.99954032	0.99870689	0.99953997
100	0.99647487	0.99703209	0.99645977	0.99701923
1,000	0.97547825	0.97648897	0.97448111	0.97553219
10,000	0.85677215	0.86018960	0.79620927	0.80084126
20,000	0.81853341	0.82283871	0.66154598	0.66830577

will go towards state  $o$  with almost one probability and P8 implies that each holding time in a state  $i \in U_S - \{o\}$  will be much smaller than each holding time in state  $o$ . Properties P7 and P8 are satisfied moderately by the example, since for the partition for  $U_S$  previously discussed  $\max_{0 < k \leq 40} \max_{i \in U_k} \lambda_{i, U_k - \{i\} \cup U_{k+1} \cup \dots \cup D_S} = 1.08 \times 10^{-3} \text{ h}^{-1}$ ,  $\min_{0 < k \leq 40} \min_{i \in U_k} \lambda_{i, U_0 \cup \dots \cup U_{k-1}} = 0.05 \text{ h}^{-1}$ ,  $\lambda_o = 9.2 \times 10^{-4} \text{ h}^{-1}$ , and  $\min_{0 < k \leq 40} \min_{i \in U_k} \lambda_i \approx 0.05 \text{ h}^{-1}$ . Class  $C'_1$  models with the additional properties P7 and P8 for the partition for  $U_S$  include both exact and bounding failure/repair models of coherent fault-tolerant systems with exponential failure and repair time distributions and repair in every state with failed components with failure rates much smaller than repair rates. The fact that the bounds are also tight for small  $t$  seems to have to do with the fact that all the initial probability of the CTMC model in  $U_S$  is concentrated in state  $o$ . Table 2 gives the bounds obtained by BRT with  $D_C = 1$  when the initial state of the CTMC model is the state in which one RAID subsystem has one unfailed controller, no other component failed and no disk under reconstruction and the remaining RAID subsystems are in their fully operational state. In that case, the bounds are not so tight for small values of  $t$ .

Finally, we analyze the trade-off in BRT between bounds accuracy and computational cost in terms of CPU time controlled by the parameter  $D_C$ . Table 3 gives the bounds obtained by BRT and



Table 3: Trade-off in BRT between bounds tightness and computational cost for  $t = 10,000$  h and  $p = 0.9995$  and initial state, state  $o$ .

$D_C$	$\text{IAVCD}^{\text{lb}}(t, p)$	$\text{IAVCD}^{\text{ub}}(t, p)$	$C^{\text{lb}}$	$K^{\text{lb}}$	$C^{\text{ub}}$	$K^{\text{ub}}$	CPU time (s)
1	0.85732856	0.86048627	36	18	35	18	3,002
2	0.85740339	0.86005160	36	48	35	37	13,429
10	0.85799895	0.85996905	36	277	35	221	80,027
20	0.85869229	0.85996905	36	520	35	452	158,420

the respective CPU times for  $t = 10,000$  h,  $p = 0.9995$  and increasing values of  $D_C$ , assuming that the initial state is state  $o$ . We also give  $C^{\text{lb}}$ ,  $K^{\text{lb}}$ ,  $C^{\text{ub}}$ , and  $K^{\text{ub}}$ . We can note that the bounds become moderately tighter as  $D_C$  increases but, as a result of a significant increase of  $K^{\text{lb}}$  and  $K^{\text{ub}}$ , the computational cost of the method increases sharply. Thus, the option  $D_C = 1$  seems to be the most attractive one for class  $C'_1$  models with the partition for  $U_S$  satisfying properties P7 and P8.

## 5 Conclusions

Based on a previously developed method for the computation of the interval availability distribution of systems modeled by CTMCs, the regenerative transformation method, we have developed a method called bounding regenerative transformation for the computation of bounds for that measure. The method requires the selection of a regenerative state, is numerically stable and computes the bounds with well-controlled error. For models belonging to a certain class, class  $C'_1$ , and a particular, “natural” selection for the regenerative state, the method allows to trade off bounds tightness with computational cost through a control parameter  $D_C$ . For large class  $C'_1$  models, the less expensive version will often provide bounds at a small computational cost in terms of CPU time relative to the model size. When the model satisfies additional conditions, the bounds obtained by the less expensive version of the method seem to be tight for any time interval or not small time intervals, depending on whether the initial probability distribution of the model in up states different, if existing, from the absorbing state is concentrated in the natural selection for the regenerative state or not. Class  $C'_1$  models with those additional conditions include both exact and bounding failure/repair models of coherent fault-tolerant systems with exponential failure and repair time distributions and repair in every state with failed components with failure rates much smaller than repair rates.

## Appendix

**Proof of Theorem 3.** As theoretical background for measure theory and Lebesgue integration we use [6]. The characterization of the probability space underlying a discrete time Markov chain with denumerable state space is discussed in [4]. The uniqueness of such probability space follows from Kolmogorov’s extension theorem (see, for instance, [5]). That theorem also implies the existence

and uniqueness of a denumerable product of probability spaces. Let  $\Pi = \{\Pi_n; n = 0, 1, 2, \dots\}$  be the embedded discrete-time Markov chain of  $W$  (see, for instance, [9]).  $\Pi$  has same state space and initial probability distribution as  $W$  and transition probabilities  $\psi_{i,j} = P[\Pi_{n+1} = j \mid \Pi_n = i] = \lambda_{i,j}/\lambda_i$ ,  $j \neq i$ ,  $\psi_{i,i} = P[\Pi_{n+1} = i \mid \Pi_n = i] = 0$  for the states  $i$  with  $\lambda_i = \sum_{j \in \Omega - \{i\}} \lambda_{i,j} > 0$  and  $\psi_{i,j} = P[\Pi_{n+1} = j \mid \Pi_n = i] = 0$ ,  $j \neq i$ ,  $\psi_{i,i} = P[\Pi_{n+1} = i \mid \Pi_n = i] = 1$  for the states  $i$  with  $\lambda_i = 0$ . The embedded DTMC of  $W'$  has same state space, initial probability distribution and transition probabilities as  $\Pi$  and, therefore, is probabilistically identical to  $\Pi$ . Both  $W$  and  $W'$  can be interpreted in terms of  $\Pi$ :  $\Pi$  gives the sequence of states visited by  $W$  ( $W'$ ) and each state visit has a duration given by an independent holding time variable with exponential distribution with parameter equal to the output rate from the visited state.

We start by constructing a common probability space  $(\mathcal{E}, \mathcal{A}, Q)$  in terms of which both  $W$  and  $W'$  can be defined. This is done by combining the probability space underlying  $\Pi$  with the probability space underlying a set of exponentially distributed independent random variables which will account (with scaling in the case of  $W'$ ) for the holding times. To simplify the proof, we will associate with absorbing states exponentially distributed holding times with non-null parameter. Let  $(\mathcal{E}_\Pi, \mathcal{A}_\Pi, Q_\Pi)$  be the probability space underlying  $\Pi$ :  $\mathcal{E}_\Pi$  is the set of infinite sequences  $\pi = (s_0^\pi, s_1^\pi, \dots)$ ,  $s_i^\pi \in \Omega$ ,  $\mathcal{A}_\Pi$  is the  $\sigma$ -algebra generated by the collection of subsets  $\mathcal{E}_\Pi^{s_0, \dots, s_n} = \{\pi = (s_0^\pi, s_1^\pi, \dots) : s_0^\pi = s_0 \wedge s_1^\pi = s_1 \wedge \dots \wedge s_n^\pi = s_n\}$ ,  $(s_0, \dots, s_n) \in \Omega^{n+1}$ ,  $n = 0, 1, \dots$  and  $Q_\Pi[\mathcal{E}_\Pi^{s_0, \dots, s_n}] = P[\Pi_0 = s_0] \psi_{s_0, s_1} \dots \psi_{s_{n-1}, s_n}$ . Let  $H_{n,s}$ ,  $n = 0, 1, \dots$ ,  $s \in \Omega$  be independent exponential random variables with parameter  $\Lambda_s$ , where  $\Lambda_s = \lambda_s$  if  $\lambda_s > 0$  and  $\Lambda_s = \Lambda^* > 0$  if  $\lambda_s = 0$ . For each random variable  $H_{n,s}$ ,  $n = 0, 1, \dots$ ,  $s \in \Omega$ , let  $([0, \infty), \mathcal{B}_{[0, \infty)}, \mu_{n,s})$  be the underlying probability space:  $\mathcal{B}_{[0, \infty)}$  is the Borel  $\sigma$ -algebra on  $[0, \infty)$  and  $\mu_{n,s}$  is the Borel probability measure defined by the distribution function of the random variable  $H_{n,s}$ . Let  $(\mathcal{E}_H, \mathcal{A}_H, \mu)$  be the product of the probability spaces  $([0, \infty), \mathcal{B}_{[0, \infty)}, \mu_{n,s})$ ,  $n = 0, 1, \dots$ ,  $s \in \Omega$ . The probability space  $(\mathcal{E}, \mathcal{A}, Q)$  is the product of the probability spaces  $(\mathcal{E}_\Pi, \mathcal{A}_\Pi, Q_\Pi)$  and  $(\mathcal{E}_H, \mathcal{A}_H, \mu)$ . With respect to  $\mathcal{E}_H$ , given a  $\omega \in \mathcal{E}_H$ ,  $h_{n,s}(\omega)$  will denote the coordinate of  $\omega$  equal to the realization of the random variable  $H_{n,s}$ . With respect to  $\mathcal{E}$ , given a  $\omega \in \mathcal{E}$ ,  $\pi(\omega) = (s_0^{\pi(\omega)}, s_1^{\pi(\omega)}, \dots)$  will denote the  $\mathcal{E}_\Pi$  coordinate of  $\omega$  and  $h_{n,s}(\omega)$  will denote the coordinate of  $\omega$  equal to the realization of the random variable  $H_{n,s}$ .

The CTMC  $W$  can be defined in terms of  $(\mathcal{E}, \mathcal{A}, Q)$  as follows. Each  $\omega \in \mathcal{E}$  gives a realization,  $W(\omega, t)$ , of  $W$

$$\begin{aligned} W(\omega, t) &= s_0^{\pi(\omega)}, & 0 \leq t < h_{0, s_0^{\pi(\omega)}}(\omega), \\ W(\omega, t) &= s_1^{\pi(\omega)}, & h_{0, s_0^{\pi(\omega)}}(\omega) \leq t < h_{0, s_0^{\pi(\omega)}}(\omega) + h_{1, s_1^{\pi(\omega)}}(\omega), \\ &\vdots \\ W(\omega, t) &= s_m^{\pi(\omega)}, & \sum_{n=0}^{m-1} h_{n, s_n^{\pi(\omega)}}(\omega) \leq t < \sum_{n=0}^m h_{n, s_n^{\pi(\omega)}}(\omega), \\ &\vdots \end{aligned}$$

Let  $L(\omega)$ ,  $\omega \in \mathcal{E}$  be the random variable defined as  $L(\omega) = \min\{l \geq 0 : \sum_{n=0}^l h_{n, s_n^{\pi(\omega)}}(\omega) > t\}$ . It is well known (see, for instance, Kijima 1997) that, being  $H_0, H_1, \dots$  independent exponential

random variables with parameters  $\lambda_0, \lambda_1, \dots$  such that  $\lambda_i > 0$  and  $\sup_{i \geq 0} \lambda_i < \infty$ ,  $\lim_{n \rightarrow \infty} H_0 + H_1 + \dots + H_n = \infty$  with probability 1, implying that  $\min\{n \geq 0 : H_0 + H_1 + \dots + H_n > t\}$  is defined with probability 1. Then,

$$\begin{aligned} Q[\{\omega \in \mathcal{E} : L(\omega) \text{ is defined}\}] &= Q\left[\left\{\omega \in \mathcal{E} : \min\left\{l \geq 0 : \sum_{n=0}^l h_{n, s_n^{\pi(\omega)}}(\omega) > t\right\} \text{ is defined}\right\}\right] \\ &= \int_{\mathcal{E}_\Pi} \mu\left[\left\{\omega' \in \mathcal{E}_H : \min\left\{l \geq 0 : \sum_{n=0}^l h_{n, s_n^{\pi(\omega')}} > t\right\} \text{ is defined}\right\}\right] dQ_\Pi(\pi) \\ &= \int_{\mathcal{E}_\Pi} dQ_\Pi(\pi) = 1. \end{aligned}$$

Let  $A$  be the subset of  $\mathcal{E}$

$$\begin{aligned} A = \left\{ \omega \in \mathcal{E} : L(\omega) \text{ is defined} \wedge \sum_{n=0}^{L(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \in U} h_{n, s_n^{\pi(\omega)}}(\omega) \right. \\ \left. + \mathbf{1}_{s_{L(\omega)}^{\pi(\omega)} \in U} \left( t - \sum_{n=0}^{L(\omega)-1} h_{n, s_n^{\pi(\omega)}}(\omega) \right) > pt \right\}. \end{aligned}$$

Since  $A$  collects, except for a subset with probability 0, all realizations of  $W$  for which the ‘‘up’’ time in the time interval  $[0, t]$  is  $> pt$ ,

$$\text{IAVCD}(t, p) = Q[A].$$

Since, given  $\beta > 0$  and being  $H$  an exponential random variable with parameter  $\lambda > 0$ ,  $H/\beta$  is an exponential random variable with parameter  $\beta\lambda$ , the CTMC  $W'$  can be defined in terms of  $(\mathcal{E}, \mathcal{A}, Q)$  as follows. Each  $\omega \in \mathcal{E}$  gives a realization,  $W'(\omega, t)$ , of  $W'$

$$\begin{aligned} W'(\omega, t) &= s_0^{\pi(\omega)}, \quad 0 \leq t < \mathbf{1}_{s_0^{\pi(\omega)} \in U} \frac{h_{0, s_0^{\pi(\omega)}}(\omega)}{\beta_{s_0^{\pi(\omega)}}} + \mathbf{1}_{s_0^{\pi(\omega)} \notin U} h_{0, s_0^{\pi(\omega)}}(\omega), \\ W'(\omega, t) &= s_1^{\pi(\omega)}, \quad \mathbf{1}_{s_0^{\pi(\omega)} \in U} \frac{h_{0, s_0^{\pi(\omega)}}(\omega)}{\beta_{s_0^{\pi(\omega)}}} + \mathbf{1}_{s_0^{\pi(\omega)} \notin U} h_{0, s_0^{\pi(\omega)}}(\omega) \\ &\leq t < \mathbf{1}_{s_0^{\pi(\omega)} \in U} \frac{h_{0, s_0^{\pi(\omega)}}(\omega)}{\beta_{s_0^{\pi(\omega)}}} + \mathbf{1}_{s_0^{\pi(\omega)} \notin U} h_{0, s_0^{\pi(\omega)}}(\omega) \\ &\quad + \mathbf{1}_{s_1^{\pi(\omega)} \in U} \frac{h_{1, s_1^{\pi(\omega)}}(\omega)}{\beta_{s_1^{\pi(\omega)}}} + \mathbf{1}_{s_1^{\pi(\omega)} \notin U} h_{1, s_1^{\pi(\omega)}}(\omega) \\ &\vdots \\ W'(\omega, t) &= s_m^{\pi(\omega)}, \quad \sum_{n=0}^{m-1} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_{n, s_n^{\pi(\omega)}}(\omega)}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_{n, s_n^{\pi(\omega)}}(\omega) \right) \\ &\leq t < \sum_{n=0}^m \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_{n, s_n^{\pi(\omega)}}(\omega)}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_{n, s_n^{\pi(\omega)}}(\omega) \right). \\ &\vdots \end{aligned}$$

Let  $L'(\omega)$ ,  $\omega \in \mathcal{E}$  be the random variable defined as

$L'(\omega) = \min\{l \geq 0 : \sum_{n=0}^l (\mathbf{1}_{s_n^{\pi(\omega)} \in U} h_{n, s_n^{\pi(\omega)}}(\omega) / \beta_{s_n^{\pi(\omega)}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_{n, s_n^{\pi(\omega)}}(\omega)) > t\}$ . It can be proved that  $L'$  is defined with probability 1 as it was proved that  $L$  was defined with probability 1.

Let  $A'$  be the subset of  $\mathcal{E}$

$$A' = \left\{ \omega \in \mathcal{E} : L'(\omega) \text{ is defined} \wedge \sum_{n=0}^{L'(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_{n, s_n^{\pi(\omega)}}(\omega)}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_{L'(\omega)}^{\pi(\omega)} \in U} \left( t - \sum_{n=0}^{L'(\omega)-1} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_{n, s_n^{\pi(\omega)}}(\omega)}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_{n, s_n^{\pi(\omega)}}(\omega) \right) \right) > pt \right\}.$$

Since  $A'$  collects, except for a subset with probability 0, all realizations of  $W'$  for which the ‘‘up’’ time in the time interval  $[0, t]$  is  $> pt$ ,

$$\text{IAVCD}'(t, p) = Q[A'].$$

To prove the theorem it suffices to show that  $A \subset A'$ . In that proof, we will use the shorthand  $h_n^{\pi(\omega)}$  for  $h_{n, s_n^{\pi(\omega)}}(\omega)$ .

First note that, being  $0 < \beta_i \leq 1$ ,  $i \in U$ ,  $\sum_{n=0}^l (\mathbf{1}_{s_n^{\pi(\omega)} \in U} h_n^{\pi(\omega)} / \beta_{s_n^{\pi(\omega)}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)}) \geq \sum_{n=0}^l h_n^{\pi(\omega)}$ , implying that  $L'(\omega)$  is defined when  $L(\omega)$  is and that, in that case,  $L'(\omega) \leq L(\omega)$ . Assuming  $L(\omega)$  and  $L'(\omega)$  defined, let

$$B(\omega) = \sum_{n=0}^{L(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \in U} h_n^{\pi(\omega)} + \mathbf{1}_{s_{L(\omega)}^{\pi(\omega)} \in U} \left( t - \sum_{n=0}^{L(\omega)-1} h_n^{\pi(\omega)} \right)$$

and let

$$B'(\omega) = \sum_{n=0}^{L'(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_n^{\pi(\omega)}}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_{L'(\omega)}^{\pi(\omega)} \in U} \left( t - \sum_{n=0}^{L'(\omega)-1} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_n^{\pi(\omega)}}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} \right) \right).$$

It suffices to show  $B'(\omega) \geq B(\omega)$ . Since

$$B(\omega) = t - C(\omega)$$

with

$$C(\omega) = \sum_{n=0}^{L(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} + \mathbf{1}_{s_{L(\omega)}^{\pi(\omega)} \notin U} \left( t - \sum_{n=0}^{L(\omega)-1} h_n^{\pi(\omega)} \right) \quad (22)$$

and

$$B'(\omega) = t - C'(\omega)$$

with

$$C'(\omega) = \sum_{n=0}^{L'(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} + \mathbf{1}_{s_{L'(\omega)}^{\pi(\omega)} \notin U} \left( t - \sum_{n=0}^{L'(\omega)-1} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_n^{\pi(\omega)}}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} \right) \right), \quad (23)$$

it suffices to show that, assuming  $L(\omega)$  and  $L'(\omega)$  defined and  $L'(\omega) \leq L(\omega)$ ,  $C'(\omega) \leq C(\omega)$ . Two cases will be considered: a)  $L'(\omega) = L(\omega)$ , and b)  $L'(\omega) < L(\omega)$ .

In case a), using (22) and (23),

$$\begin{aligned} C'(\omega) &= \sum_{n=0}^{L(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} + \mathbf{1}_{s_{L(\omega)}^{\pi(\omega)} \notin U} \left( t - \sum_{n=0}^{L(\omega)-1} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} \frac{h_n^{\pi(\omega)}}{\beta_{s_n^{\pi(\omega)}}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} \right) \right) \\ &\leq \sum_{n=0}^{L(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} + \mathbf{1}_{s_{L(\omega)}^{\pi(\omega)} \notin U} \left( t - \sum_{n=0}^{L(\omega)-1} h_n^{\pi(\omega)} \right) = C(\omega). \end{aligned}$$

In case b), assuming  $s_{L'(\omega)}^{\pi(\omega)} \notin U$ ,

$$\begin{aligned} \sum_{n=0}^{L'(\omega)} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} h_n^{\pi(\omega)} / \beta_{s_n^{\pi(\omega)}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} \right) &> t \text{ implies} \\ \sum_{n=0}^{L'(\omega)-1} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} h_n^{\pi(\omega)} / \beta_{s_n^{\pi(\omega)}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} \right) + h_{L'(\omega)}^{\pi(\omega)} &> t \text{ and} \\ \left( t - \sum_{n=0}^{L'(\omega)-1} \left( \mathbf{1}_{s_n^{\pi(\omega)} \in U} h_n^{\pi(\omega)} / \beta_{s_n^{\pi(\omega)}} + \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} \right) \right) &< h_{L'(\omega)}^{\pi(\omega)}. \text{ Using, then, (22) and (23),} \end{aligned}$$

$$\begin{aligned} C'(\omega) &\leq \sum_{n=0}^{L'(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} + \mathbf{1}_{s_{L'(\omega)}^{\pi(\omega)} \notin U} h_{L'(\omega)}^{\pi(\omega)} \leq \sum_{n=0}^{L(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} \\ &\leq \sum_{n=0}^{L(\omega)-1} \mathbf{1}_{s_n^{\pi(\omega)} \notin U} h_n^{\pi(\omega)} + \mathbf{1}_{s_{L(\omega)}^{\pi(\omega)} \notin U} \left( t - \sum_{n=0}^{L(\omega)-1} h_n^{\pi(\omega)} \right) = C(\omega). \end{aligned}$$

It remains to check that  $L_n, L'_n \in \mathcal{A}$ ,  $n = 0, 1, \dots$ , where  $L_n = \{\omega \in \mathcal{E} : L(\omega) = n\}$  and  $L'_n = \{\omega \in \mathcal{E} : L'(\omega) = n\}$  and that  $A, A' \in \mathcal{A}$ .

We start by checking that  $L_n \in \mathcal{A}$ ,  $n = 0, 1, \dots$ . Let  $F_n = \{\omega \in \mathcal{E} : \sum_{m=0}^n h_{m, s_m^{\pi(\omega)}}(\omega) > t\}$ ,  $n = 0, 1, \dots$ . Since  $L_0 = F_0$  and, for  $n \geq 1$ ,  $L_n = F_n \cap F_{n-1}^c$ , it suffices to check that  $F_n \in \mathcal{A}$ ,  $n = 0, 1, \dots$ . Let  $F^{s_0, \dots, s_n} = \{\omega \in \mathcal{E}_H : \sum_{m=0}^n h_{m, s_m}(\omega) > t\}$ . Since  $F_n = \cup_{(s_0, \dots, s_n) \in \Omega^{n+1}} \mathcal{E}_{\Pi}^{s_0, \dots, s_n} \times F^{s_0, \dots, s_n}$ ,  $\Omega^{n+1}$  is denumerable and  $\mathcal{E}_{\Pi}^{s_0, \dots, s_n} \in \mathcal{A}_{\Pi}$ , it suffices to check that  $F^{s_0, \dots, s_n} \in \mathcal{A}_H$ ,  $(s_0, \dots, s_n) \in \Omega^{n+1}$ ,  $n = 0, 1, \dots$ . This follows if  $H_n = \{(h_0, \dots, h_n) \in [0, \infty)^{n+1} : \sum_{m=0}^n h_m > t\} \in \otimes_{m=0}^n \mathcal{B}_{[0, \infty)} = \mathcal{B}_{[0, \infty)^{n+1}}$ ,  $n = 0, 1, \dots$ , which can be proved by induction on  $n$  as follows. The case  $n = 0$  is trivial since  $H_0 = (t, \infty) \in \mathcal{B}_{[0, \infty)}$ . Assume the result holds for  $n = i \geq 0$ . We have  $H_{i+1} = G_{i+1} \cup \cup_{j=0}^{i+1} J_j$ , where  $G_{i+1} = \{(h_0, \dots, h_{i+1}) \in [0, \infty)^{i+2} : h_0 > 0 \wedge \dots \wedge h_{i+1} > 0 \wedge \sum_{m=0}^{i+1} h_m > t\}$  and  $J_j = \{(h_0, \dots, h_{i+1}) \in [0, \infty)^{i+2} : h_j = 0 \wedge \sum_{m=0, m \neq j}^{i+1} h_m > t\}$ . But  $G_{i+1} \in \mathcal{B}_{[0, \infty)^{i+2}}$ , since  $G_{i+1}$  is an open subset of  $[0, \infty)^{i+2}$  and  $J_j \in \mathcal{B}_{[0, \infty)^{i+2}} = \mathcal{B}_{[0, \infty)} \otimes \mathcal{B}_{[0, \infty)^{i+1}}$ , since  $\{0\} \in \mathcal{B}_{[0, \infty)}$  and, by the induction hypothesis,  $H_i \in \mathcal{B}_{[0, \infty)^{i+1}}$ . That  $L'_n \in \mathcal{A}$ ,  $n = 0, 1, \dots$  can be checked similarly, the only difference being that  $H_n$  has to be replaced by  $H'_n = \{(h_0, \dots, h_n) \in [0, \infty)^{n+1} : \sum_{m=0}^n h_m / \alpha_m > t\}$ ,  $0 < \alpha_m \leq 1$ , which can be easily shown to belong to  $\mathcal{B}_{[0, \infty)^{n+1}}$ .

Let us check now that  $A \in \mathcal{A}$ . Let

$$A_n = \left\{ \omega \in \mathcal{E} : \sum_{m=0}^{n-1} \mathbf{1}_{s_m^{\pi(\omega)} \in U} h_{m, s_m^{\pi(\omega)}}(\omega) + \mathbf{1}_{s_n^{\pi(\omega)} \in U} \left( t - \sum_{m=0}^{n-1} h_{m, s_m^{\pi(\omega)}}(\omega) \right) > pt \right\}.$$

Since  $A = \bigcup_{n=0}^{\infty} (L_n \cap A_n)$ , it suffices to check that  $A_n \in \mathcal{A}$ ,  $n = 0, 1, \dots$ . Let

$$A^{s_0, \dots, s_n} = \left\{ \omega \in \mathcal{E}_H : \sum_{m=0}^{n-1} \mathbf{1}_{s_m \in U} h_{m, s_m}(\omega) + \mathbf{1}_{s_n \in U} \left( t - \sum_{m=0}^{n-1} h_{m, s_m}(\omega) \right) > pt \right\}.$$

Since  $A_n = \bigcup_{(s_0, \dots, s_n) \in \Omega^{n+1}} \mathcal{E}_{\Pi}^{s_0, \dots, s_n} \times A^{s_0, \dots, s_n}$ ,  $\Omega^{n+1}$  is denumerable and  $\mathcal{E}_{\Pi}^{s_0, \dots, s_n} \in \mathcal{A}_{\Pi}$ , it suffices to check that  $A^{s_0, \dots, s_n} \in \mathcal{A}_H$ ,  $(s_0, \dots, s_n) \in \Omega^{n+1}$ ,  $n = 0, 1, \dots$ . We will consider two cases: a)  $s_n \in U$  and b)  $s_n \notin U$ . In case a),  $A^{s_0, \dots, s_n} = \{\omega \in \mathcal{E}_H : \sum_{m=0}^{n-1} \mathbf{1}_{s_m \notin U} h_{m, s_m}(\omega) < (1-p)t\}$  and the result follows if  $K_n = \{(h_0, \dots, h_n) \in [0, \infty)^{n+1} : \sum_{m=0}^n h_m < t\} \in \bigotimes_{m=0}^n \mathcal{B}_{[0, \infty)} = \mathcal{B}_{[0, \infty)^{n+1}}$ ,  $n = 0, 1, \dots$ , which can be proved by induction on  $n$  as follows. The case  $n = 0$  is trivial, since  $K_0 = [0, t) \in \mathcal{B}_{[0, \infty)}$ . Assume the result holds for  $n = i \geq 0$ . We have  $K_{i+1} = M_{i+1} \cup \bigcup_{j=0}^{i+1} N_j$ , where  $M_{i+1} = \{(h_0, \dots, h_{i+1}) \in [0, \infty)^{i+2} : h_0 > 0 \wedge \dots \wedge h_{i+1} > 0 \wedge \sum_{m=0}^{i+1} h_m < t\}$  and  $N_j = \{(h_0, \dots, h_{i+1}) \in [0, \infty)^{i+2} : h_j = 0 \wedge \sum_{m=0, m \neq j}^{i+1} h_m < t\}$ . But  $M_{i+1} \in \mathcal{B}_{[0, \infty)^{i+2}}$ , since  $M_{i+1}$  is an open subset of  $[0, \infty)^{i+2}$  and  $N_j \in \mathcal{B}_{[0, \infty)^{i+2}} = \mathcal{B}_{[0, \infty)} \otimes \mathcal{B}_{[0, \infty)^{i+1}}$ , since  $\{0\} \in \mathcal{B}_{[0, \infty)}$  and, by the induction hypothesis,  $K_i \in \mathcal{B}_{[0, \infty)^{i+1}}$ . In case b),  $A^{s_0, \dots, s_n} = \{\omega \in \mathcal{E}_H : \sum_{m=0}^{n-1} \mathbf{1}_{s_m \in U} h_{m, s_m}(\omega) > pt\}$  and the result follows if  $H_n = \{(h_0, \dots, h_n) \in [0, \infty)^{n+1} : \sum_{m=0}^n h_m > t\} \in \bigotimes_{m=0}^n \mathcal{B}_{[0, \infty)}$ ,  $n = 0, 1, \dots$ , which was proved previously. That  $A' \in \mathcal{A}$  can be checked similarly, the only difference being that, in case b),  $H_n$  has to be replaced by  $H'_n = \{(h_0, \dots, h_n) \in [0, \infty)^{n+1} : \sum_{m=0}^n h_m / \alpha_m > t\}$ ,  $0 < \alpha_m \leq 1$ , which belongs to  $\mathcal{B}_{[0, \infty)^{n+1}}$ .  $\square$

**Proof of Proposition 1.** The result  $\pi^{\text{ub}}(0, k) = \pi^{\text{lb}}(0, k)$ ,  $0 \leq k \leq 1$  follows immediately from (8), (9), (10), and (12).

Let state  $r$  be numbered first in all vectors indexed by a subset including state  $r$ . The proof that, for  $r \in U_S$ ,  $\pi^{\text{ub}}(n, k) = R \pi^{\text{lb}}(n, k)$ ,  $n \geq 1$ ,  $0 \leq k \leq n+1$  is by induction on  $n$ . Consider first the base case  $n = 1$ . From (11), we have

$$\pi^{\text{ub}}(1, 0)^{U'_S} = \pi^{\text{lb}}(1, 0)^{U'_S} = \mathbf{0}. \quad (24)$$

Using (5), (8), (10), (12) and (Theorem 4)  $P_{r,j}^{\text{ub}} = R P_{r,j}^{\text{lb}}$ ,  $j \neq r$ ,

$$\begin{aligned} \pi^{\text{ub}}(1, 1)^{U'_S} &= \pi^{\text{ub}}(0, 0) \mathbf{P}_{S, U'_S}^{\text{ub}} = (\mathbf{1}_{r \in D_S} 0 \cdots 0) \mathbf{P}_{S, U'_S}^{\text{ub}} = \mathbf{1}_{r \in D_S} \mathbf{P}_{\{r\}, U'_S}^{\text{ub}} \\ &= \mathbf{1}_{r \in D_S} R \mathbf{P}_{\{r\}, U'_S}^{\text{lb}} = R (\mathbf{1}_{r \in D_S} 0 \cdots 0) \mathbf{P}_{S, U'_S}^{\text{lb}} \\ &= R \pi^{\text{lb}}(0, 0) \mathbf{P}_{S, U'_S}^{\text{lb}} = R \pi^{\text{lb}}(1, 1)^{U'_S}. \end{aligned} \quad (25)$$

Using (5), (9), (10), (12) and  $P_{r,j}^{\text{ub}} = R P_{r,j}^{\text{lb}}$ ,  $j \neq r$ ,

$$\begin{aligned} \pi^{\text{ub}}(1, 2)^{U'_S} &= \pi^{\text{ub}}(0, 1) \mathbf{P}_{S, U'_S}^{\text{ub}} = (\mathbf{1}_{r \in U_S} 0 \cdots 0) \mathbf{P}_{S, U'_S}^{\text{ub}} = \mathbf{1}_{r \in U_S} \mathbf{P}_{\{r\}, U'_S}^{\text{ub}} \\ &= \mathbf{1}_{r \in U_S} R \mathbf{P}_{\{r\}, U'_S}^{\text{lb}} = R (\mathbf{1}_{r \in U_S} 0 \cdots 0) \mathbf{P}_{S, U'_S}^{\text{lb}} \\ &= R \pi^{\text{lb}}(0, 1) \mathbf{P}_{S, U'_S}^{\text{lb}} = R \pi^{\text{lb}}(1, 2)^{U'_S}. \end{aligned} \quad (26)$$

Using (6), (8), (10), (12) and  $P_{r,j}^{\text{ub}} = RP_{r,j}^{\text{lb}}$ ,  $j \neq r$ ,

$$\begin{aligned}\pi^{\text{ub}}(1,0)^{D'_S} &= \pi^{\text{ub}}(0,0) \mathbf{P}_{S,D'_S}^{\text{ub}} = (\mathbf{1}_{r \in D_S} 0 \cdots 0) \mathbf{P}_{S,D'_S}^{\text{ub}} = \mathbf{1}_{r \in D_S} \mathbf{P}_{\{r\},D'_S}^{\text{ub}} \\ &= \mathbf{1}_{r \in D_S} R \mathbf{P}_{\{r\},D'_S}^{\text{lb}} = R (\mathbf{1}_{r \in D_S} 0 \cdots 0) \mathbf{P}_{S,D'_S}^{\text{lb}} \\ &= R \pi^{\text{lb}}(0,0) \mathbf{P}_{S,D'_S}^{\text{lb}} = R \pi^{\text{lb}}(1,0)^{D'_S}.\end{aligned}\quad (27)$$

Using (6), (9), (10), (12) and  $P_{r,j}^{\text{ub}} = RP_{r,j}^{\text{lb}}$ ,  $j \neq r$ ,

$$\begin{aligned}\pi^{\text{ub}}(1,1)^{D'_S} &= \pi^{\text{ub}}(0,1) \mathbf{P}_{S,D'_S}^{\text{ub}} = (\mathbf{1}_{r \in U_S} 0 \cdots 0) \mathbf{P}_{S,D'_S}^{\text{ub}} = \mathbf{1}_{r \in U_S} \mathbf{P}_{\{r\},D'_S}^{\text{ub}} \\ &= \mathbf{1}_{r \in U_S} R \mathbf{P}_{\{r\},D'_S}^{\text{lb}} = R (\mathbf{1}_{r \in U_S} 0 \cdots 0) \mathbf{P}_{S,D'_S}^{\text{lb}} \\ &= R \pi^{\text{lb}}(0,1) \mathbf{P}_{S,D'_S}^{\text{lb}} = R \pi^{\text{lb}}(1,1)^{D'_S}.\end{aligned}\quad (28)$$

From (13),

$$\pi^{\text{ub}}(1,2)^{D'_S} = \pi^{\text{lb}}(1,2)^{D'_S} = \mathbf{0}.\quad (29)$$

From (7),

$$\pi_r^{\text{ub}}(1,k) = \pi_r^{\text{lb}}(1,k) = 0, \quad 0 \leq k \leq 2.\quad (30)$$

Collecting (24)–(30), we have

$$\pi^{\text{ub}}(1,k) = R \pi^{\text{lb}}(1,k), \quad 0 \leq k \leq 2,$$

completing the base case. Assume  $\pi^{\text{ub}}(m,k) = R \pi^{\text{lb}}(m,k)$ ,  $m \geq 1$ ,  $0 \leq k \leq m+1$  and let us prove  $\pi^{\text{ub}}(m+1,k) = R \pi^{\text{lb}}(m+1,k)$ ,  $0 \leq k \leq m+2$ . From (7),

$$\pi_r^{\text{ub}}(m+1,k) = \pi_r^{\text{lb}}(m+1,k) = 0, \quad 0 \leq k \leq m+2.\quad (31)$$

From (11),

$$\pi^{\text{ub}}(m+1,0)^{U'_S} = \pi^{\text{lb}}(m+1,0)^{U'_S} = \mathbf{0}.\quad (32)$$

Using (5), (7), (Theorem 4)  $\mathbf{P}_{S',U'_S}^{\text{ub}} = \mathbf{P}_{S',U'_S}^{\text{lb}}$ , and the induction hypothesis,

$$\begin{aligned}\pi^{\text{ub}}(m+1,k)^{U'_S} &= \pi^{\text{ub}}(m,k-1) \mathbf{P}_{S',U'_S}^{\text{ub}} = \pi^{\text{ub}}(m,k-1)^{S'} \mathbf{P}_{S',U'_S}^{\text{ub}} \\ &= R \pi^{\text{lb}}(m,k-1)^{S'} \mathbf{P}_{S',U'_S}^{\text{lb}} = R \pi^{\text{lb}}(m,k-1) \mathbf{P}_{S',U'_S}^{\text{lb}} \\ &= R \pi^{\text{lb}}(m+1,k)^{U'_S}, \quad 1 \leq k \leq m+2.\end{aligned}\quad (33)$$

Using (6), (7), (Theorem 4)  $\mathbf{P}_{S',D'_S}^{\text{ub}} = \mathbf{P}_{S',D'_S}^{\text{lb}}$ , and the induction hypothesis,

$$\begin{aligned}\pi^{\text{ub}}(m+1,k)^{D'_S} &= \pi^{\text{ub}}(m,k) \mathbf{P}_{S',D'_S}^{\text{ub}} = \pi^{\text{ub}}(m,k)^{S'} \mathbf{P}_{S',D'_S}^{\text{ub}} \\ &= R \pi^{\text{lb}}(m,k)^{S'} \mathbf{P}_{S',D'_S}^{\text{lb}} = R \pi^{\text{lb}}(m,k) \mathbf{P}_{S',D'_S}^{\text{lb}} \\ &= R \pi^{\text{lb}}(m+1,k)^{D'_S}, \quad 0 \leq k \leq m+1.\end{aligned}\quad (34)$$

From (13),

$$\pi^{\text{ub}}(m+1,m+2)^{D'_S} = \pi^{\text{lb}}(m+1,m+2)^{D'_S} = \mathbf{0}.\quad (35)$$

Collecting (31)–(35), we have

$$\pi^{\text{ub}}(m+1,k) = R \pi^{\text{lb}}(m+1,k), \quad 0 \leq k \leq m+2,$$

completing the induction step.

The proof of the result  $\pi^{\text{ub}}(n, k) = \pi^{\text{lb}}(n, k)$ ,  $n \geq 1$ ,  $0 \leq k \leq n + 1$  for the case  $r \in D_S$  follows step by step with  $R = 1$  the proof of the result  $\pi^{\text{ub}}(n, k) = R\pi^{\text{lb}}(n, k)$ ,  $n \geq 1$ ,  $0 \leq k \leq n + 1$  for the case  $r \in U_S$  by noting that, according to Theorem 4, the only difference between the two cases is that, for  $r \in U_S$ ,  $P_{r,j}^{\text{ub}} = RP_{r,j}^{\text{lb}}$ ,  $j \neq r$  and, for  $r \in D_S$ ,  $P_{r,j}^{\text{ub}} = P_{r,j}^{\text{lb}}$ ,  $j \neq r$ .

For  $\alpha_{S'} > 0$ , the result  $\pi^{\text{ub}}(n, k) = \pi^{\text{lb}}(n, k)$ ,  $n \geq 0$ ,  $0 \leq k \leq n + 1$  follows immediately from (14)–(21), noting that  $\alpha^{\text{ub}} = \alpha^{\text{lb}}$  and, according to Theorem 4,  $\mathbf{P}_{S',U'_S}^{\text{ub}} = \mathbf{P}_{S',U'_S}^{\text{lb}}$  and  $\mathbf{P}_{S',D'_S}^{\text{ub}} = \mathbf{P}_{S',D'_S}^{\text{lb}}$ .  $\square$

**Proof of Lemma 1.** Let the function

$$f_y(z) = \frac{1}{z} \sum_{m=k}^{\infty} (m - k + 2) e^{-zy} \frac{(zy)^m}{m!},$$

with  $k \geq 2$ . We have to show  $f_x(R) > f_x(1)$  for  $x > 0$  and  $R > 1$ . Taking derivatives,

$$\begin{aligned} f'_y(z) &= -\frac{1}{z^2} \sum_{m=k}^{\infty} (m - k + 2) e^{-zy} \frac{(zy)^m}{m!} - \frac{y}{z} \sum_{m=k}^{\infty} (m - k + 2) e^{-zy} \frac{(zy)^m}{m!} \\ &\quad + \frac{y}{z} \sum_{m=k}^{\infty} (m - k + 2) e^{-zy} \frac{(zy)^{m-1}}{(m-1)!}. \end{aligned}$$

But,

$$\begin{aligned} \sum_{m=k}^{\infty} (m - k + 2) e^{-zy} \frac{(zy)^{m-1}}{(m-1)!} &= \sum_{m=k-1}^{\infty} (m - k + 3) e^{-zy} \frac{(zy)^m}{m!} \\ &= \sum_{m=k}^{\infty} (m - k + 2) e^{-zy} \frac{(zy)^m}{m!} + e^{-zy} \frac{(zy)^{k-1}}{(k-1)!} + \sum_{m=k-1}^{\infty} e^{-zy} \frac{(zy)^m}{m!}, \end{aligned}$$

yielding

$$f'_y(z) = \frac{y}{z} e^{-zy} \frac{(zy)^{k-1}}{(k-1)!} + \frac{y}{z} \sum_{m=k-1}^{\infty} e^{-zy} \frac{(zy)^m}{m!} - \frac{1}{z^2} \sum_{m=k}^{\infty} (m - k + 2) e^{-zy} \frac{(zy)^m}{m!},$$

which gives

$$f'_y(1) = ye^{-y} \frac{y^{k-1}}{(k-1)!} + y \sum_{m=k-1}^{\infty} e^{-y} \frac{y^m}{m!} - \sum_{m=k}^{\infty} (m - k + 2) e^{-y} \frac{y^m}{m!}.$$

The second and third terms can be rewritten as

$$\begin{aligned} y \sum_{m=k-1}^{\infty} e^{-y} \frac{y^m}{m!} &= \sum_{m=1}^{\infty} e^{-y} \frac{y^{k+m-1}}{(k+m-2)!}, \\ \sum_{m=k}^{\infty} (m - k + 2) e^{-y} \frac{y^m}{m!} &= \sum_{m=1}^{\infty} (m + 1) e^{-y} \frac{y^{k+m-1}}{(k+m-1)!}, \end{aligned}$$



yielding

$$\begin{aligned}
f'_y(1) &= ye^{-y} \frac{y^{k-1}}{(k-1)!} + \sum_{m=1}^{\infty} e^{-y} \frac{y^{k+m-1}}{(k+m-2)!} - \sum_{m=1}^{\infty} (m+1) e^{-y} \frac{y^{k+m-1}}{(k+m-1)!} \\
&= ye^{-y} \frac{y^{k-1}}{(k-1)!} + \sum_{m=1}^{\infty} (k+m-1) e^{-y} \frac{y^{k+m-1}}{(k+m-1)!} - \sum_{m=1}^{\infty} (m+1) e^{-y} \frac{y^{k+m-1}}{(k+m-1)!} \\
&= e^{-y} \frac{y^k}{(k-1)!} + (k-2) \sum_{m=k}^{\infty} e^{-y} \frac{y^m}{m!}.
\end{aligned}$$

This shows that  $f'_y(1)$  is  $> 0$  for  $y > 0$ . Let  $\delta = \min\{f'_y(1), x \leq y \leq xR\} > 0$  (the minimum exists because  $f'_y(1)$  is continuous in the domain). We have  $f'_y(1) \geq \delta$  for  $x \leq y \leq xR$ .

Let  $\Delta > \max_{x \leq y \leq xR, 1 \leq z \leq R} |f''_y(z)| \geq 0$  (the maximum exists because  $f''_y(z)$  and, therefore,  $|f''_y(z)|$  is continuous in the domain). Let  $q = 2\delta/\Delta > 0$ . Taylor's theorem applied to  $f_y(z)$  at  $z = 1$  gives

$$f_y(1+r) = f_y(1) + f'_y(1)r + \frac{f''_y(z)}{2}r^2, \quad r > 0, z \in [1, 1+r].$$

Then, for  $x \leq y \leq xR$ ,  $1 < 1+r \leq R$ , and  $r \leq q$ , we have

$$f_y(1+r) - f_y(1) = f'_y(1)r + \frac{f''_y(z)}{2}r^2 > \delta r - \frac{\Delta}{2}r^2 = r \left( \delta - \frac{\Delta}{2}r \right) \geq r \left( \delta - \frac{\Delta}{2}q \right) = 0,$$

implying  $f_y(1+r) > f_y(1)$  for  $x \leq y \leq xR$ ,  $1 < 1+r \leq R$  and  $r \leq q$ .

The result asserted by the lemma can be proved from the fact that  $f_y(1+r) > f_y(1)$  for  $x \leq y \leq xR$ ,  $1 < 1+r \leq R$  and  $r \leq q$  as follows. Let  $N$  be the minimum integer  $n > 0$  with  $(1+q)^n \geq R$ . We can write  $R = (1+q)^{N-1}(1+r)$  with  $r \leq q$ ,  $1 < 1+r \leq R$  and, if  $N > 1$ ,  $1+q < R$ . Then,

$$\begin{aligned}
f_x(R) &= f_x((1+q)^{N-1}(1+r)) = \frac{1}{(1+q)^{N-1}} f_{x(1+q)^{N-1}}(1+r) \\
&> \frac{1}{(1+q)^{N-1}} f_{x(1+q)^{N-1}}(1) = \frac{1}{(1+q)^{N-2}} f_{x(1+q)^{N-2}}(1+q) \\
&> \frac{1}{(1+q)^{N-2}} f_{x(1+q)^{N-2}}(1) > \cdots > f_x(1). \quad \square
\end{aligned}$$

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