NON-COMPLETE INTERSECTION PRIME IDEALS IN DIMENSION 3

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ABSTRACT. We describe prime ideals of height 2 minimally generated by 3 elements in a Gorenstein, Nagata local ring of Krull dimension 3 and multiplicity at most 3. This subject is related to a conjecture of Y. Shimoda and to a long-standing problem of J. Sally.

1. INTRODUCTION

It is not known whether a Noetherian local ring, such that all its prime ideals different from the maximal ideal are complete intersections, has Krull dimension at most 2. This problem was posed by Y. Shimoda and still remains unanswered in its full generality. In fact, it is a partial version of a more general question of J. Sally's, namely, that the existence of a uniform bound on the minimal number of generators of all its prime ideals is equivalent to the dimension of the ring being at most 2.

Note that, in the Shimoda problem, one may assume without loss of generality that the local ring is Cohen-Macaulay and has dimension at most 3 (see [5] for more details, and particularly, [5, Remarks 2.2 and 2.4]). Similarly, one may ask whether one can display a prime ideal of height 2 and minimally generated by at least 3 elements in a Cohen-Macaulay local ring of dimension 3. By a result due to M. Miller [9, Theorem 2.1], under reasonably general hypotheses, a local domain of dimension at least 4 containing a field possesses an abundance of prime ideals of height 2 that are not complete intersections.

The purpose of the paper is threefold: to generalise the results obtained in the first part of [5], to give simpler proofs, and finally, to display a wide collection of examples to illustrate the range of behaviour that occurs.

Let $(R,\mathfrak{m},k)$ be a Cohen-Macaulay local ring, with k infinite, $\dim R=3$ and multiplicity $e(R)$. Let $(x,y,z)R$ be a minimal reduction of $\mathfrak{m}$. We ask for k to be infinite just to ensure that $\mathfrak{m}$ has a minimal reduction minimally generated by three elements (see [3, Remark 4.5.9]). If R is regular local, we do not need such an hypothesis, as $\mathfrak{m}$ is then its own minimal reduction.

Take $a=(a_1,a_2,a_3)\in\mathbb{N}_+^3$ and $b=(b_1,b_2,b_3)\in\mathbb{N}_+^3$, where $\mathbb{N}_+$ denotes the set of positive integers; set $\mathbb{N}=\{0\}\cup\mathbb{N}_+$. Let $c=a+b$, $c=(c_1,c_2,c_3)$. Let $\mathcal{M}$ be the matrix

$$\mathcal{M}=\begin{pmatrix} x^{a_1} & y^{a_2} & z^{a_3} \\ y^{b_2} & z^{b_3} & x^{b_1}\end{pmatrix},$$

and $v_1=x^{c_1}-y^{b_2}z^{a_3}$, $v_2=y^{c_2}-x^{a_1}z^{b_3}$ and $v_3=z^{c_3}-x^{b_1}y^{a_2}$, the 2×2 minors of $\mathcal{M}$ up to a change of sign. Consider $I=(v_1,v_2,v_3)R$, the determinantal ideal generated by the 2×2 minors of $\mathcal{M}$. Then I is a non-Gorenstein height-unmixed ideal of height two, minimally generated by three elements (see [11], where these ideals were called Herzog-Northcott ideals, or HN ideals for short).

Throughout the paper we fix this notation, and $(R,\mathfrak{m},k)$ and I will be defined as above. Under additional assumptions on R, we will study the minimal primary decomposition of I and prove that either I itself is prime or else I has a minimal prime which is not a complete intersection, thus leading to the existence of prime ideals of height 2 and minimally generated by at least 3 elements.

Set $m_1=c_2c_3-a_2b_3$, $m_2=c_1c_3-a_3b_1$, $m_3=c_1c_2-a_1b_2$ and $m=(m_1,m_2,m_3)\in\mathbb{N}_+^3$. Note that each $m_i\geq 3$. We will always suppose that $m_1\leq m_2\leq m_3$ and that $\gcd(m_1,m_2,m_3)=1$

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(changing a to (b_2, b_1, b_3) and b to (a_2, a_1, a_3) changes m to (m_2, m_1, m_3); similarly, changing a to (b_1, b_3, b_2) and b to (a_1, a_3, a_2) changes m to (m_1, m_3, m_2)). Let $\mathcal{S}(I) = \langle m_1, m_2, m_3\rangle$ denote the numerical semigroup generated by m_1, m_2, m_3 (see, e.g., [12]).

Recall that a numerical semigroup $\mathcal{S}$ is a subset of $\mathbb{N}$, closed under addition, with $0 \in \mathcal{S}$, and such that $G(\mathcal{S}) := \mathbb{N} \setminus \mathcal{S}$, the set of *gaps* of $\mathcal{S}$, is finite. The cardinality of $G(\mathcal{S})$ is denoted by $g(\mathcal{S})$ and is called the *genus* of $\mathcal{S}$. The *Frobenius number* $F(\mathcal{S})$ of $\mathcal{S}$ is the greatest integer in $G(\mathcal{S})$. One can prove that $g(\mathcal{S}) \geq (F(\mathcal{S}) + 1)/2$. Moreover, $\mathcal{S}$ is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it, and $\mathcal{S}$ is *symmetric* if it is irreducible and $F(\mathcal{S})$ is odd. Alternatively, $\mathcal{S}$ is symmetric if and only if $g(\mathcal{S}) = (F(\mathcal{S})+1)/2$ (cf. [12, Lemma 2.14 and Corollary 4.5]). Let $\{m_1 < m_2 < \ldots < m_r\}$ be the (necessarily unique) minimal system of generators of a numerical semigroup $\mathcal{S}$. The *multiplicity* of $\mathcal{S}$ is defined by the expression $\mathrm{mult}(\mathcal{S}) = m_1$ and the *embedding dimension* of $\mathcal{S}$ is defined by the expression $\mathrm{embed}(\mathcal{S}) = r$ (see [12, Theorem 2.7 and Proposition 2.10]). Every numerical semigroup of emdedding dimension two is symmetric ([12, Corollary 4.7]).

For any other unexplained notation, we refer to [3] or [6]. Our main result is as follows. Note that a minimal prime over I is necessarily of height two.

Theorem. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein, Nagata local ring, with k infinite, and $\dim R = 3$. Let $(x, y, z)R$ be a minimal reduction of $\mathfrak{m}$. Let $I = (x^{c_1} - y^{b_2}z^{a_3}, y^{c_2} - x^{a_1}z^{b_3}, z^{c_3} - x^{b_1}y^{a_2})R$. Suppose that $\mathcal{S}(I) = \langle m_1, m_2, m_3\rangle$ is not contained in any symmetric semigroup $\mathcal{S}$ with $\mathrm{mult}(\mathcal{S}) = m_1$. If $e(R) \leq 3$, then either I is prime, or else there exists a minimal prime $\mathfrak{p}$ over I such that $\mathfrak{p}$ is not a complete intersection.*

This result generalises [5, Proposition 2.8], since on the one hand, a complete Noetherian local ring R is Nagata (see [7, Chapter 12, § 31, Corollary 2]) and, on the other hand, we do not need the ring to be a domain or contain the residue field. As a consequence, it generalises the main result in [5], since the hypotheses of [5, Theorem 2.3] imply that R is Gorenstein and Nagata. In other words, we obtain the following result. Recall that a Noetherian local ring is *Shimoda* if every prime ideal in the punctured spectrum is of the principal class.

Corollary. *Let $(R, \mathfrak{m}, k)$ be a Shimoda ring of dimension $d \geq 2$. Then $d = 2$ provided that either R is regular, or else R is Gorenstein, Nagata, k is infinite and $e(R) \leq 3$.*

We finish the paper with examples that show that each one of the particular cases arising in the main theorem can occur.

2. PRELIMINARY RESULTS

We start by substantiating some remarks on the multiplicity of R and R/I.

Remark 2.1. We first observe that R/I is a one-dimensional Cohen-Macaulay ring. Next we remark that xR/I is a minimal reduction of $\mathfrak{m}R/I$. Indeed, and with an obvious abuse of notation, in R/I one has the following equalities: $x^{c_1} = y^{b_2}z^{a_3}$, $y^{c_2} = x^{a_1}z^{b_3}$ and $z^{c_3} = x^{b_1}y^{a_2}$. Then it is easy to check that $y^{c_2c_3} = x^{m_2}y^{a_2b_3}$. Since y is not a zero divisor in R/I, $y^{m_1} = x^{m_2}$ and x^{m_2} belongs to $(xR/I)^{m_1}$ since $m_1 \leq m_2$. Therefore $y \in \overline{xR/I}$, the integral closure of the ideal xR/I. Analogously, one can check that $z^{m_1} = x^{m_3} \in (xR/I)^{m_1}$, so $z \in \overline{xR/I}$. Hence xR/I is a reduction of $(x, y, z)R/I$. Since $(x, y, z)R/I$ is a reduction of $\mathfrak{m}R/I$, then xR/I is a reduction of $\mathfrak{m}R/I$. Since $\dim R/I = 1$, xR/I is a minimal reduction of $\mathfrak{m}R/I$. Observe also that $x + I$ forms a regular sequence in R/I. In particular,

$$\begin{aligned} e(R/I) &= e_{R/I}(\mathfrak{m}R/I; R/I) = e_{R/I}(xR/I; R/I) = \\ &= e_{R/I}(x + I; R/I) = \mathrm{length}_{R/I}((R/I)/(x + I)R/I) = \mathrm{length}_R(R/(xR + I)). \end{aligned}$$

Analogously, if $\mathfrak{p}$ is a minimal prime over I, then $xR/\mathfrak{p}$ is a minimal reduction of $\mathfrak{m}R/\mathfrak{p}$ and $e(R/\mathfrak{p}) = e_{R/\mathfrak{p}}(xR/\mathfrak{p}; R/\mathfrak{p}) = \mathrm{length}_R(R/(xR + \mathfrak{p}))$.

Lemma 2.2. *$e(R/I) = m_1 e(R)$.*

Proof. By Remark 2.1, $e(R/I) = \text{length}_R(R/(xR + I))$, where $xR + I = (x, y^{c_2}, y^{b_2}z^{a_3}, z^{c_3})R$. With $S = R/xR$, note that $R/(xR + I) \cong S/(y^{c_2}, y^{b_2}z^{a_3}, z^{c_3})S$. In the two-dimensional Cohen-Macaulay local ring S, and with an obvious abuse of notation, y, z is a regular sequence and a system of parameters. By [5, Lemma 2.9], $\text{length}_R(R/(xR+I)) = \text{length}_S(S/(y^{c_2}, y^{b_2}z^{a_3}, z^{c_3})S) = m_1 \text{length}_S(S/(y, z)S)$. Since $S/(y, z)S \cong R/(x, y, z)R$ and $(x, y, z)R$ is a minimal reduction of $\mathfrak{m}$, then $\text{length}_S(S/(y, z)S) = \text{length}_R(R/(x, y, z)R) = e_R(x, y, z; R) = e_R(\mathfrak{m}; R) = e(R)$. □

We now fix some more notations.

Setting 2.3. For a minimal prime $\mathfrak{p}$ over I, let $D = R/\mathfrak{p}$, which is a one-dimensional Noetherian local domain with maximal ideal $\mathfrak{m}_D$, say. Let $V = \overline{D}$ be the integral closure of D in its quotient field; then V is a Dedekind domain by the Krull-Akizuki Theorem. If Q is a maximal ideal of V, then V_Q is a DVR. Let $\mathfrak{m}_{V_Q} = QV_Q$ denote its maximal ideal, k_{V_Q} its residue field and ν_Q its valuation. If V is local, let $\mathfrak{m}_V$ denote its maximal ideal, k_V its residue field and ν its valuation. If V is local and $k = k_V$ under the natural identification, one says that k is *residually rational.* If R is a Nagata ring, then V is a finitely generated D-module.

Proposition 2.4. *Let $\mathfrak{p}$ be a minimal prime over I. Then the following hold.*

- (a) *For any Q, there exists $\eta = \eta(Q) \in \mathbb{N}_+$ such that $(\nu_Q(x), \nu_Q(y), \nu_Q(z)) = \eta(m_1, m_2, m_3)$.*
- (b) *$e(D) > 1$.*

Suppose that, in addition, R is Nagata. Then the following hold.

- (c) *$e(D) = m_1\sigma_{\mathfrak{p}}$, where $\sigma_{\mathfrak{p}} = \sum_Q \eta(Q)[k_{V_Q} : k]$.*
- (d) *$e(D) = m_1$ if and only if V is a DVR, $\eta = 1$ and k is residually rational.*
- (e) *Moreover, if $e(D) = m_1$, then D is analytically irreducible.*

Proof. Any maximal ideal Q of V contracts to $\mathfrak{m}_D$ through the natural inclusion $D \subseteq V$, so $\mathfrak{m}_D V \subseteq Q$. Therefore, in V_Q, on applying ν_Q to the equalities $x^{c_1} = y^{b_2}z^{a_3}$, $y^{c_2} = x^{a_1}z^{b_3}$ and $z^{c_3} = x^{b_1}y^{a_2}$, one gets $(\nu_Q(x), \nu_Q(y), \nu_Q(z)) = \eta(m_1, m_2, m_3)$, for some non-zero rational number $\eta = \eta(Q)$ depending on Q and $\mathfrak{p}$ (see [11, Remark 4.4]). Write $\eta = u/v$, with $u, v \in \mathbb{N}_+$. Then $v(\nu_Q(x), \nu_Q(y), \nu_Q(z)) = u(m_1, m_2, m_3)$ and, on taking the greatest common divisor, one has $v \gcd(\nu_Q(x), \nu_Q(y), \nu_Q(z)) = u \gcd(m_1, m_2, m_3) = u$. So $\gcd(\nu_Q(x), \nu_Q(y), \nu_Q(z)) = u/v$ and $\eta = u/v \in \mathbb{N}_+$.

By Remark 2.1, $e(D) = \text{length}_R(R/(xR + \mathfrak{p}))$. If $\text{length}_R(R/(xR + \mathfrak{p})) = 1$, then $\mathfrak{m} = xR + \mathfrak{p}$ and $R/\mathfrak{p}$ is a DVR with valuation ν, say, and uniformizing parameter x (by abuse of notation), so $\nu(x) = 1$. Applying (a), this forces $m_1 = 1$, which is in contradiction to $m_1 \geq 3$. This proves (b).

Suppose that R is Nagata. Applying Remark 2.1 and [6, Theorem 11.2.7],

$$e(D) = e_D(xD; D) = \sum_Q e_{V_Q}(xV_Q; V_Q)[k_{V_Q} : k],$$

where Q runs over the maximal ideals of V. Applying (a), $(\nu_Q(x), \nu_Q(y), \nu_Q(z)) = \eta(m_1, m_2, m_3)$, for some $\eta = \eta(Q) \in \mathbb{N}_+$. In particular, $e_{V_Q}(xV_Q; V_Q) = \text{length}(V_Q/xV_Q) = \eta(Q)m_1$. Therefore

$$e(D) = e_D(xD; D) = \sum_Q e_{V_Q}(xV_Q; V_Q)[k_{V_Q} : k] = m_1 \sum_Q \eta(Q)[k_{V_Q} : k] = m_1\sigma_{\mathfrak{p}},$$

where $\sigma_{\mathfrak{p}} = \sum_Q \eta(Q)[k_{V_Q} : k]$. Hence $e(D) = m_1\sigma_{\mathfrak{p}} \geq m_1$, and $e(D) = m_1$ is equivalent to $\sigma_{\mathfrak{p}} = 1$. Moreover, $\sigma_{\mathfrak{p}} = 1$ is equivalent to V being local and so a DVR with valuation ν say, $\eta = 1$ (i.e., $\nu(x) = m_1$, $\nu(y) = m_2$ and $\nu(z) = m_3$) and $[k_V : k] = 1$. Furthermore, in this case, D is analytically irreducible since the $\mathfrak{m}_D$-adic completion of D can be seen as a subring in the $\mathfrak{m}_V$-adic completion of V, which is a DVR, whence a domain. (For the converse statement, see [8, p. 486, Section 1].) □

Given a numerical semigroup $\mathcal{S}$ with Frobenius number $F(\mathcal{S})$, set $N(\mathcal{S}) = \{s \in \mathcal{S} \mid s < F(\mathcal{S})\}$ and $n(\mathcal{S}) = |N(\mathcal{S})|$ its cardinality. Note that $g(\mathcal{S}) + n(\mathcal{S}) = F(\mathcal{S}) + 1$. Since $g(\mathcal{S}) \geq (F(\mathcal{S})+1)/2$, it follows that $(F(\mathcal{S})+1) \geq 2n(\mathcal{S})$ (see [12, just before Proposition 2.26]).

Proposition 2.5. *Suppose that R is Nagata and that $\mathcal{S}(I)$ is not contained in any symmetric semigroup $\mathcal{S}$ with* $\operatorname{mult}(\mathcal{S}) = m_1$. *Let $\mathfrak{p}$ be a minimal prime over I such that $e(D) = m_1$. Then D is not Gorenstein.*

Proof. Observe that D cannot be a DVR since $m_1 \geq 3$. Hence the conductor $(D :_D V) \subseteq \mathfrak{m}_D$, where $V = \overline{D}$. By Remark 2.1, xD is a minimal reduction of $\mathfrak{m}_D$, so $\overline{xD} = \mathfrak{m}_D$ (see [6, Corollary 1.2.5]). By [6, Theorem 6.8.1], $\mathfrak{m}_D \subseteq \mathfrak{m}_D V = \overline{(xD)}V = xV$. By Proposition 2.4, (d), V is a DVR with uniformizing parameter t and valuation ν, say, and $\nu(x) = m_1$, $\nu(y) = m_2$ and $\nu(z) = m_3$. In particular, the numerical semigroup $\langle m_1, m_2, m_3\rangle$ is contained in the numerical semigroup $\nu(D)$. Moreover, $xV = t^{m_1}V$ and $(D :_D V) \subseteq \mathfrak{m}_D \subseteq \mathfrak{m}_D V = xV = t^{m_1}V$. Therefore, $\mathfrak{m}_D \subseteq t^{m_1}V$ and

$$\mathcal{S}(I) = \langle m_1, m_2, m_3\rangle \subseteq \nu(D) \subseteq \{0\} \cup \{n \in \mathbb{Z} \mid n \geq m_1\}. \tag{1}$$

Thus, $\nu(D)$ is a numerical semigroup containing $\mathcal{S}(I)$ and of multiplicity $\operatorname{mult}(\nu(D)) = m_1$. By hypothesis, $g(\nu(D)) > (F(\nu(D))+1)/2$ or, equivalently, $(F(\nu(D))+1) > 2n(\nu(D))$.

By Proposition 2.4, (d), k is residually rational. Applying [1, Remark, Page 40] (see also [8, Proposition 1]), we obtain $\operatorname{length}_V(V/(D :_D V)) = F(\nu(D)) + 1$ and $\operatorname{length}_D(D/(D :_D V)) = n(\nu(D))$. In particular, $\operatorname{length}_V(V/(D :_D V)) > 2\operatorname{length}_D(D/(D :_D V))$ and, by [6, Theorem 12.2.2], D cannot be Gorenstein. ☐

Now let $\mathfrak{p}$ run through $\operatorname{Min}(R/I)$, the set of minimal primes over I. Let n_I be the cardinality of $\operatorname{Min}(R/I)$. For each minimal prime $\mathfrak{p}$ over I, set $l_\mathfrak{p} = \operatorname{length}_{R_\mathfrak{p}}(R_\mathfrak{p}/I_\mathfrak{p})$. Recall from Proposition 2.4, (c), that $e(R/\mathfrak{p}) = m_1\sigma_\mathfrak{p}$.

Corollary 2.6. *Suppose that R is Nagata. Then $e(R) = \sum_\mathfrak{p} \sigma_\mathfrak{p} l_\mathfrak{p}$. In particular, $n_I \leq e(R)$. Moreover, for small values of $e(R)$, we have the following possibilities.*

- (a) *If $e(R) = 1$, then $n_I = 1$,* $\operatorname{Min}(R/I) = \{\mathfrak{p}\}$, $(\sigma_\mathfrak{p}, l_\mathfrak{p}) = (1,1)$ *and $I = \mathfrak{p}$ is prime with $e(R/\mathfrak{p}) = m_1$.*
- (b) *Suppose that $e(R) = 2$. Then either*
  - $(b.1)$ $n_I = 1$, $\operatorname{Min}(R/I) = \{\mathfrak{p}\}$, $(\sigma_\mathfrak{p}, l_\mathfrak{p}) = (2,1)$ *and $I = \mathfrak{p}$ is prime with $e(R/\mathfrak{p}) = 2m_1$, or*
  - $(b.2)$ $n_I = 1$, $\operatorname{Min}(R/I) = \{\mathfrak{p}\}$, $(\sigma_\mathfrak{p}, l_\mathfrak{p}) = (1,2)$ *and I is $\mathfrak{p}$-primary with $e(R/\mathfrak{p}) = m_1$, or*
  - $(b.3)$ $n_I = 2$, $\operatorname{Min}(R/I) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $(\sigma_{\mathfrak{p}_i}, l_{\mathfrak{p}_i}) = (1,1)$ *for $i = 1,2$, and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$ with each $e(R/\mathfrak{p}_i) = m_1$.*
- (c) *Suppose that $e(R) = 3$. Then either*
  - $(c.1)$ $n_I = 1$, $\operatorname{Min}(R/I) = \{\mathfrak{p}\}$, $(\sigma_\mathfrak{p}, l_\mathfrak{p}) = (3,1)$ *and $I = \mathfrak{p}$ is prime with $e(R/\mathfrak{p}) = 3m_1$, or*
  - $(c.2)$ $n_I = 1$, $\operatorname{Min}(R/I) = \{\mathfrak{p}\}$, $(\sigma_\mathfrak{p}, l_\mathfrak{p}) = (1,3)$ *and I is $\mathfrak{p}$-primary with $e(R/\mathfrak{p}) = m_1$, or*
  - $(c.3)$ $n_I = 2$, $\operatorname{Min}(R/I) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $(\sigma_{\mathfrak{p}_1}, l_{\mathfrak{p}_1}) = (1,2)$, $(\sigma_{\mathfrak{p}_2}, l_{\mathfrak{p}_2}) = (1,1)$ *and $I = \mathfrak{q}_1 \cap \mathfrak{p}_2$ with $\mathfrak{q}_1$ a $\mathfrak{p}_1$-primary ideal and each $e(R/\mathfrak{p}_i) = m_1$, or*
  - $(c.4)$ $n_I = 2$, $\operatorname{Min}(R/I) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $(\sigma_{\mathfrak{p}_1}, l_{\mathfrak{p}_1}) = (2,1)$, $(\sigma_{\mathfrak{p}_2}, l_{\mathfrak{p}_2}) = (1,1)$ *and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$ with $e(R/\mathfrak{p}_1) = 2m_1$ and $e(R/\mathfrak{p}_2) = m_1$, or*
  - $(c.5)$ $n_I = 3$, $\operatorname{Min}(R/I) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$, $(\sigma_{\mathfrak{p}_i}, l_{\mathfrak{p}_i}) = (1,1)$ *for $i = 1,2,3$, and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$ with each $e(R/\mathfrak{p}_i) = m_1$.*

In particular, if $e(R) \leq 3$, then either I is prime, or else there exists a minimal prime $\mathfrak{p}$ over I such that $e(D) = m_1$, with D not Gorenstein, provided that $\mathcal{S}(I)$ is not contained in any symmetric semigroup $\mathcal{S}$ with $\operatorname{mult}(\mathcal{S}) = m_1$.

Proof. By Lemma 2.2, the Associativity Law of Multiplicities and Proposition 2.4,

$$m_1 e(R) = e(R/I) = e_{R/I}(xR/I; R/I) = \sum_\mathfrak{p} e_{R/\mathfrak{p}}(xR/\mathfrak{p}; R/\mathfrak{p})\operatorname{length}_{R_\mathfrak{p}}(R_\mathfrak{p}/I_\mathfrak{p}) = m_1 \sum_\mathfrak{p} \sigma_\mathfrak{p} l_\mathfrak{p}.$$

Thus $e(R) = \sum_{\mathfrak{p}} \sigma_{\mathfrak{p}} l_{\mathfrak{p}}$. In particular, $n_I \le e(R)$. If $e(R) = 1$, one deduces that I has a unique minimal prime $\mathfrak{p}$ and that, for such $\mathfrak{p}$, $\text{length}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/I_{\mathfrak{p}}) = 1$, so $I = \mathfrak{p}$. (See [5, Proposition 2.6]; recall that, for a Cohen-Macaulay local ring R, $e(R) = 1$ is equivalent to R being a regular local ring: cf. [10, Theorem 40.6 and Corollary 25.3] or [6, Exercise 11.8].) The rest of the assertions follow analogously. One finishes by applying Propositions 2.4, (c), and 2.5. ☐

Example 2.7. Let $(R, \mathfrak{m}, k)$ be a Cohen-Macaulay, Nagata local ring, with k infinite, and $\dim R = 3$. Let $(x, y, z)R$ be a minimal reduction of $\mathfrak{m}$. Let $I = (x^3 - yz, y^2 - xz, z^2 - x^2y)R$. If $e(R) \le 3$, then either I is prime, or else there exists a minimal prime $\mathfrak{p}$ over I such that D is not Gorenstein with $e(D) = 3$, these two cases overlapping precisely when $e(R) = 1$. (See Section 4 to note that each of the two possibilities can occur.) Moreover, in the latter case, D is an almost Gorenstein ring and the canonical ideal ω_D of D is minimally generated by two elements.

Proof. Note that $\mathcal{S}(I) = \langle 3, 4, 5\rangle$ is not contained in any symmetric semigroup $\mathcal{S}$ with $\text{mult}(\mathcal{S}) = 3$. By Corollary 2.6, either I is prime, or else

(2) there exists a minimal prime $\mathfrak{p}$ over I such that $e(D) = 3$ and D is not Gorenstein.

In the latter case (2), by Proposition 2.4, k is residually rational and such D is analytically irreducible.

Suppose that (2) holds. Then the chain of inclusions (1) in Proposition 2.5 must be a chain of equalities, so $\nu(D) = \langle 3, 4, 5\rangle$. Note that $F(\nu(D)) = 2$. So $\text{length}_V(V/(D :_D V)) = F(\nu(D)) + 1 = 3$. Since V is a DVR, it follows that $(D :_D V) = t^3V$, so $(D :_D V) = \mathfrak{m}_D = xV = t^3V$. In particular, $\mathfrak{m}_D V \subseteq D$ and D is an almost Gorenstein ring (see [4, Corollary 3.12]; see also [2]).

Since $(D :_D V) = \mathfrak{m}_D$, then $\text{length}_D(D/(D :_D V)) = 1$. Furthermore, D analytically irreducible implies that D admits a canonical ideal ω_D (see, e.g., [4, Proposition 2.7]). By [6, Theorem 12.2.3], and since k is residually rational,

$$\begin{aligned} 3 = \text{length}_V(V/(D :_D V)) = \text{length}_D(V/(D :_D V)) \ge \\ 2\text{length}_D(D/(D :_D V)) + \mu(\omega_D) - 1 = 1 + \mu(\omega_D), \end{aligned}$$

where μ stands for "minimal number of generators". Therefore, $\mu(\omega_D) \le 2$. Since D is not Gorenstein, this forces $\mu(\omega_D) = 2$ (alternatively, this follows also from the definition of "almost Gorenstein" in [2, page 418]). ☐

3. MAIN THEOREM

Now, we can state and prove the main result of the paper. We keep the same notations.

Theorem 3.1. *Let $(R, \mathfrak{m}, k)$ be a Gorenstein, Nagata local ring, with k infinite, and $\dim R = 3$. Let $(x, y, z)R$ be a minimal reduction of $\mathfrak{m}$. Let $I = (x^{c_1} - y^{b_2}z^{a_3}, y^{c_2} - x^{a_1}z^{b_3}, z^{c_3} - x^{b_1}y^{a_2})R$. Suppose that $\mathcal{S}(I) = \langle m_1, m_2, m_3\rangle$ is not contained in any symmetric semigroup $\mathcal{S}$ with $\text{mult}(\mathcal{S}) = m_1$. If $e(R) \le 3$, then either I is prime, or else there exists a minimal prime $\mathfrak{p}$ over I such that $\mathfrak{p}$ is not a complete intersection.*

Proof. By Corollary 2.6, either I is prime, or else there exists a minimal prime $\mathfrak{p}$ over I such that D is not Gorenstein. In particular, since R is Gorenstein, $\mathfrak{p}$ cannot be a complete intersection ([3, Proposition 3.1.19]). ☐

The following result clarifies the hypothesis "$\mathcal{S}(I)$ not contained in any symmetric semigroup $\mathcal{S}$ with $\text{mult}(\mathcal{S}) = m_1$". Let $\mathcal{T}$ be the numerical semigroup $\mathcal{T} = \langle m_1, m_2, m_3\rangle$ with $3 \le m_1 \le m_2 \le m_3$ and $\gcd(m_1, m_2, m_3) = 1$. In particular $\text{mult}(\mathcal{T}) = m_1$ and $\text{embed}(\mathcal{T}) \le 3$. If $\text{embed}(\mathcal{T}) = 2$, then $\mathcal{T}$ is symmetric (see [12, Corollary 4.5]). Therefore, in order to fulfill the hypotheses of Proposition 2.5 and Theorem 3.1, we can suppose that $\text{embed}(\mathcal{T}) = 3$. Hence $m_1 < m_2 < m_3$.

Proposition 3.2. *Let $\mathcal{T}=\langle m_1,m_2,m_3\rangle$ be a numerical semigroup with $3\leq m_1<m_2<m_3$ and $\gcd(m_1,m_2,m_3)=1$. Suppose that $\operatorname{embed}(\mathcal{T})=3$. Let $\Delta=\{\langle 3,4,5\rangle,\langle 3,5,7\rangle,\langle 4,5,7\rangle,\langle 4,7,9\rangle\}$. Then $\mathcal{T}$ is not contained in any symmetric semigroup $\mathcal{S}$ with $\operatorname{mult}(\mathcal{S})=m_1$ if and only if $\mathcal{T}\in\Delta$.*

Proof. The "if" implication is a simple check. We now prove the "only if" implication. Take $\mathcal{T}=\langle m_1,m_2,m_3\rangle$ and suppose that $\mathcal{T}\not\in\Delta$. Let us show that $\mathcal{T}$ is contained in a symmetric semigroup $\mathcal{S}$ with $\operatorname{mult}(\mathcal{S})=m_1$.

Observe that since $\operatorname{embed}(\mathcal{T})=3$, then $m_3\not\in\langle m_1,m_2\rangle$ and $m_3>m_2$. For the sake of simplicity, set $BG(m_1,m_2)=G(\langle m_1,m_2\rangle)\cap\{m\in\mathbb{N}_+\mid m>m_2\}$, where $G(\langle m_1,m_2\rangle)$ is the set of gaps of $\langle m_1,m_2\rangle$ (BG standing for "big gaps"). Thus $m_3\in BG(m_1,m_2)$.

Suppose that $m_1=3$ and $m_2=4$. Then $m_3\in BG(m_1,m_2)=\{5\}$, in contradiction to $\mathcal{T}\not\in\Delta$. Analogously, if $m_1=3$ and $m_2=5$, then $m_3\in BG(m_1,m_2)=\{7\}$, in contradiction to $\mathcal{T}\not\in\Delta$. Therefore, if $m_1=3$, then $m_2\geq 6$ and $\mathcal{T}\subseteq\langle 3,4\rangle=\{0,3,4,6,\mapsto\}$, which is symmetric.

Suppose that $m_1=4$. Set $\mathcal{S}_1=\langle 4,5,6\rangle=\{0,4,5,6,8,\mapsto\}$, $\mathcal{S}_2=\langle 4,6,7\rangle=\{0,4,6,7,8,10,\mapsto\}$, which are symmetric. Let us prove that either $\mathcal{T}\subseteq\mathcal{S}_1$, or else $\mathcal{T}\subseteq\mathcal{S}_2$. Indeed, if $m_2=5$, then $m_3\in BG(m_1,m_2)=\{6,7,11\}$. Since $\mathcal{T}\not\in\Delta$, $m_3\in\{6,11\}$ and $\mathcal{T}\subseteq\mathcal{S}_1$. Suppose that $m_2=6$. If $m_3=9$, then $\mathcal{T}\subseteq\mathcal{S}_1$. If $m_3\neq 9$, then $\mathcal{T}\subseteq\mathcal{S}_2$. Suppose that $m_2=7$. Since $\mathcal{T}\not\in\Delta$, then $m_3\neq 9$ and $\mathcal{T}\subseteq\mathcal{S}_2$. If $m_1=4$ and $m_2\geq 8$, then $\mathcal{T}\subseteq\mathcal{S}_1$.

Suppose that $m_1\geq 5$. Take $\mathcal{S}_1=\langle m_1,m_1+1,\dots,2m_1-2\rangle$ and $\mathcal{S}_2=\langle m_1,m_1+2,\dots,2m_1-1\rangle$. One can check that $F(\mathcal{S}_1)=2m_1-1$, $F(\mathcal{S}_2)=2m_1+1$, and that $\mathcal{S}_1$ and $\mathcal{S}_2$ are symmetric.

If $m_2\geq 2m_1$, then $\mathcal{T}\subseteq\mathcal{S}_1$. Suppose that $m_2=2m_1-1$. If $m_3\neq 2m_1+1$, then $\mathcal{T}\subseteq\mathcal{S}_2$. If $m_3=2m_1+1$, then $\mathcal{T}\subseteq\langle m_1,2m_1-1,2m_1+1,\dots,3m_1-4,3m_1-2\rangle$, which is symmetric (with Frobenius number $4m_1-3$).

Suppose that $m_2\leq 2m_1-2$. If $m_3\neq 2m_1-1$, then $\mathcal{T}\subseteq\mathcal{S}_1$. If $m_3=2m_1-1$ and $m_2\neq m_1+1$, then $\mathcal{T}\subseteq\mathcal{S}_2$. Finally, if $m_2=m_1+1$ and $m_3=2m_1-1$, $\mathcal{T}\subseteq\langle m_1,m_1+1,m_1+4,\dots,2m_1-1\rangle$, which is symmetric (with Frobenius number $2m_1+3$). □

Remark 3.3. Recall that $a=(a_1,a_2,a_3)\in\mathbb{N}_+^3$, $b=(b_1,b_2,b_3)\in\mathbb{N}_+^3$ and $c=a+b$. Moreover $m_1=c_2c_3-a_2b_3=a_2a_3+a_3b_2+b_2b_3$, $m_2=c_1c_3-a_3b_1=a_1a_3+a_1b_3+b_1b_3$, and $m_3=c_1c_2-a_1b_2=a_1a_2+a_2b_1+b_1b_2$. It is easy to check that the following four matrices

$$\mathcal{M}_1=\begin{pmatrix} x & y & z\\ y & z & x^2\end{pmatrix},\ \mathcal{M}_2=\begin{pmatrix} x & y & z\\ y & z & x^3\end{pmatrix},\ \mathcal{M}_3=\begin{pmatrix} x^2 & y^2 & z\\ y & z & x\end{pmatrix},\ \mathcal{M}_4=\begin{pmatrix} x^3 & y^2 & z\\ y & z & x\end{pmatrix},$$

give rise to the corresponding ideals of 2×2 minors

$$\begin{aligned} I_1&=(x^3-yz,y^2-xz,z^2-x^2y)R\ ,\ I_2=(x^4-yz,y^2-xz,z^2-x^3y)R\ ,\\ I_3&=(x^3-yz,y^3-x^2z,z^2-xy^2)R\ ,\ I_4=(x^4-yz,y^3-x^3z,z^2-xy^2)R\ ,\end{aligned}$$

with $S(I_1)=\langle 3,4,5\rangle$, $S(I_2)=\langle 3,5,7\rangle$, $S(I_3)=\langle 4,5,7\rangle$ and $S(I_4)=\langle 4,7,9\rangle$, the four semigroups appearing in the set Δ.

In fact, these are the only examples with prescribed semigroup in Δ. Indeed, if $m_1=3$, then a_2, a_3, b_2 and b_3 must be equal to 1. Substituting in the expressions of m_2 and m_3 leads to a 2×2 system with solution $a_1=(1/3)(2m_2-m_3)$ and $b_1=(1/3)(2m_3-m_2)$. If $m_2=4$ and $m_3=5$, then $a_1=1$ and $b_1=2$. If $m_2=5$ and $m_2=7$, then $a_1=1$ and $b_1=3$.

If $m_1=4$, this forces either $a_2=2$ and a_3, b_2 and b_3 equal to 1, or else $b_3=2$ and a_2, a_3 and b_2 equal to 1. If $a_2=2$, substituting in the expressions of m_2 and m_3, one gets a 2×2 system with solution $a_1=(1/4)(3m_2-m_3)$ and $b_1=(1/2)(m_3-m_2)$. If $m_2=5$ and $m_3=7$, then $a_1=2$ and $b_1=1$. If $m_2=7$ and $m_3=9$, then $a_1=3$ and $b_1=1$. Finally, if $b_3=2$, substituting in the expressions of m_2 and m_3, one gets a 2×2 system with solution $a_1=(1/2)(m_2-m_3)$ and $b_1=(1/4)(3m_3-m_2)$. However $m_2<m_3$ would force $a_1<0$, which makes no sense.

4. Examples

Our next purpose is to display examples of each one of the cases in Corollary 2.6. First we fix the notations for the rest of the paper.

Setting 4.1. Let k be a field and let X, Y, Z, W, t be indeterminates over k. Set $A = k[X,Y,Z]$, $\mathfrak{m}_A = (X,Y,Z)A$ and $S = A_{\mathfrak{m}_A}$, the localization of A in $\mathfrak{m}_A$. Call $\mathfrak{m}_S$ the maximal ideal of S. Take a, b, c and $m \in \mathbb{N}^3_+$ as in Section 1 and suppose that $m_1 < m_2 < m_3$ and $\gcd(m_1,m_2,m_3) = 1$. Let $J = (X^{c_1} - Y^{b_2}Z^{a_3}, Y^{c_2} - X^{a_1}Z^{b_3}, Z^{c_3} - X^{b_1}Y^{a_2})A \subset \mathfrak{m}_A$. By [11, Theorem 7.8], J is a prime ideal of A. In fact, $J = \ker(\varphi_m : A \to k[t])$, where φ_m sends X, Y and Z to t^{m_1}, t^{m_2} and t^{m_3}, respectively. In particular, JS is a prime ideal of S.

Set $B = A[W] = K[X,Y,Z,W]$, $\mathfrak{m}_B = (X,Y,Z,W)B$ and $T = B_{\mathfrak{m}_B}$, the localization of B in $\mathfrak{m}_B$. Call $\mathfrak{m}_T$ the maximal ideal of T. By abuse of notation, we consider elements of A to be elements of B and elements of S to be elements of T. Let $n \geq 1$, $g_1,\ldots,g_n$, with $g_i \in \mathfrak{m}_A^i$ and $f = W^n + g_1W^{n-1} + \ldots + g_n \in (WB + \mathfrak{m}_AB)^n$. Note that $JB + fB \subset \mathfrak{m}_B$.

We now specify our model for the ring R and our model for the ideal I that will exemplify the results considered in the paper, particularly as regards Theorem 3.1 and Corollary 2.6. Take $R = T/fT$, the factor ring of T modulo f. Let $\mathfrak{m}_R$ denote the maximal ideal of R. Let lower-case letters x, y, z, w denote the corresponding image elements in R. Thus $\mathfrak{m}_R = (x,y,z,w)R$ and clearly $(R,\mathfrak{m}_R,k)$ is a Gorenstein, Nagata local ring of dimension $\dim R = 3$. Since w is integral over the ideal $(x,y,z)R$, then $(x,y,z)R$ is a minimal reduction of $\mathfrak{m}_R$. Now take $I = JR = (x^{c_1} - y^{b_2}z^{a_3}, y^{c_2} - x^{a_1}z^{b_3}, z^{c_3} - x^{b_1}y^{a_2})R$. Clearly $e(R) = n$, by a standard result (see [6, Example 11.2.8], say); alternatively, by calculation, since x, y, z is a regular sequence in R, then $e(R) = e_R((x,y,z)R;R) = \text{length}_R(R/(x,y,z)R)$, so, setting $T' = T/(X,Y,Z)T$,

$$e(R) = \text{length}_T(T/(X,Y,Z,f)T) = \text{length}_{T'}(T'/W^nT') = n.$$

Let us study the minimal primary decomposition of I for different particular choices of the element f. We start with the cases in Corollary 2.6 in which I is prime.

Example 4.2. Cases (a), $(b.1)$ and $(c.1)$.

(*i*) In Setting 4.1, take $f = W^n - X^{n-1}Y$. When $n \in \{1,2\}$, take $m = (m_1,m_2,m_3)$ in $\{(3,4,5),(4,5,7),(4,7,9)\}$; when $n = 3$, take m in $\{(3,4,5),(3,5,7),(4,5,7)\}$. Note that for each choice of n and m, $\gcd(nm_1,nm_2,nm_3,(n-1)m_1+m_2) = 1$. Let $P = \ker(\psi)$, where $\psi : B \to k[t]$ sends X, Y, Z and W to t^{nm_1}, t^{nm_2}, t^{nm_3} and $t^{(n-1)m_1+m_2}$, respectively. Then $P = JB + fB$. In particular, $JT + fT$ is a prime ideal of T. Thus $e(R) = n$ and I is a prime ideal of R.

(*ii*) Take $f = W^n - X^{n-1}Z$, $n = 3$, in Setting 4.1 and take m in $\{(3,4,5),(3,5,7),(4,7,9)\}$. Note that for each choice of m, $\gcd(nm_1,nm_2,nm_3,(n-1)m_1+m_3) = 1$. Let $P = \ker(\psi)$, where $\psi : B \to k[t]$ sends X, Y, Z and W to t^{nm_1}, t^{nm_2}, t^{nm_3} and $t^{(n-1)m_1+m_3}$, respectively. Then $P = JB + fB$. In particular, $JT + fT$ is a prime ideal of T. Thus $e(R) = n$ and I is a prime ideal of R.

Proof. (*i*) It suffices to adapt the proofs of [11, Remark 7.2, Lemma 7.5 and Theorem 7.8] to the ring B and the ideal $JB + fB$, with the variables X, Y, Z and W being given weights nm_1, nm_2, nm_3 and $(n-1)m_1 + m_2$, respectively. In this regard, note that $JB + fB$ is unmixed, since $J(B/fB)$ is unmixed by [11, Proposition 2.2, (b)].

(*ii*) This follows similarly, with the variables X, Y, Z and W now given weights nm_1, nm_2, nm_3 and $(n-1)m_1 + m_3$, respectively. ☐

An example covering Case $(b.1)$ when $m = (3,5,7)$ is shown in Example 4.11. Before proceeding, we need some prior observations.

Remark 4.3. Take $g \in \mathfrak{m}_A$. Then g defines a surjective evaluation map $\varphi_g : B \to A$, where φ_g fixes k, X, Y and Z and sends W to g. Note that if $p \in B \backslash \mathfrak{m}_B$, then $p(0,0,0,g(0,0,0)) = p(0,0,0,0) \neq 0$, so $\varphi_g(p) \in A \setminus \mathfrak{m}_A$, and if $q \in A \setminus \mathfrak{m}_A$, then $q \in B \setminus \mathfrak{m}_B$ and $\varphi_g(q) = q$. In particular, φ_g can be extended to a morphism, $\varphi_g : T \to S$, say, that is a retraction of the natural inclusion $S \subset T$.

Lemma 4.4. *Let* $g \in \mathfrak{m}_A$. *Then* $\ker(\varphi_g : B \to A) = (W - g)B$ *and* $\ker(\varphi_g : T \to S) = (W - g)T$. *In particular,* $JB + (W - g)B$ *is a prime ideal of height* 3 *in* B *and* $JT + (W - g)T$ *is a prime ideal of height* 3 *in* T.

Proof. That $\ker(\varphi_g : B \to A) = (W - g)B$ follows easily from the appropriate Division Algorithm. The second assertion follows since localisation is a flat functor, so kernels are preserved. In particular, since JA is a prime of height 2 in A and $\varphi_g(JB) = JA$, then (via φ_g^{-1}) $JB+(W-g)B/(W-g)B$ is a prime of height 2 in $B/(W - g)B$, so $JB + (W - g)B$ is a prime ideal of height 3 in B because $W - g$ is prime in B. Analogously, $JT + (W - g)T$ is a prime ideal of height 3 in T. □

Next we note some elementary facts about lifting a minimal primary decomposition over an ideal. We shall use these facts below without explicit mention.

Remark 4.5. Let L, K be ideals in a Noetherian ring C such that $L \supseteq K$. For $i = 1, \dots, r$, consider ideals Q_i and P_i with $P_i \supseteq Q_i \supseteq L$ such that in C/K we have the minimal primary decomposition $L/K = \cap_i Q_i/K$, where each P_i/K is a prime ideal and Q_i/K is P_i/K-primary. Then in C, $L = \cap_i Q_i$ is a minimal primary decomposition, and for $i = 1, \dots, r$, each P_i is a prime ideal and Q_i is P_i-primary. In particular, if L/K is an unmixed ideal in C/K, then L is an unmixed ideal in C.

Proof. Note that for each i, $C/P_i \simeq (C/K)/(P_i/K)$, so C/P_i is a domain. Moreover, $C/Q_i \simeq (C/K)/(Q_i/K)$, so in C/Q_i each divisor of zero is nilpotent. The remainder of the assertions follow from the basic theory of ideals in factor rings. □

Example 4.6. CASES $(b.2)$ AND $(c.2)$. Take $f = W^n$, $n \geq 1$, in Setting 4.1. Then $P = JB+WB$ is a prime ideal of B contained in $\mathfrak{m}_B$. Set $\mathfrak{p} = PR$. Then $e(R) = n$, $\mathrm{Min}(R/I) = \{\mathfrak{p}\}$, $(\sigma_{\mathfrak{p}}, l_{\mathfrak{p}}) = (1, n)$ and I is $\mathfrak{p}$-primary with $e(R/\mathfrak{p}) = m_1$.

Proof. By Lemma 4.4, P is a prime ideal of height 3. Since $I = JR$ is unmixed (see [11, Proposition 2.2]), it follows easily that PT is the unique prime minimal over $JT + fT$.

Set $U = T_{PT}$ (the localisation of T at the prime PT). Then $V = U/IU$ is a one-dimensional local domain with maximal ideal generated by the image of W in V. Hence V is a DVR. It is immediate that V/W^nV is of length n (as V-module). By definition, this length is the local length of $JT + fT$ at PT. Since $R = T/fT$, we deduce that $l_{\mathfrak{p}}$, the local length of I at its unique minimal prime $\mathfrak{p} = PR$, equals n. □

Example 4.7. CASES $(b.3)$ AND $(c.3)$. Take $f = W^{n-1}(W - X)$, $n \geq 2$, in Setting 4.1. Then $P_1 = JB + WB$ and $P_2 = JB + (W - X)B$ are prime ideals of B contained in $\mathfrak{m}_B$. Set $\mathfrak{p}_i = P_iT$, $i = 1, 2$. Then $e(R) = n$, $\mathrm{Min}(R/\mathfrak{p}) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$, $(\sigma_{\mathfrak{p}_1}, l_{\mathfrak{p}_1}) = (1, n - 1)$, $(\sigma_{\mathfrak{p}_2}, l_{\mathfrak{p}_2}) = (1, 1)$ and $I = \mathfrak{q}_1 \cap \mathfrak{p}_2$ is a minimal primary decomposition with $\mathfrak{q}_1$ a $\mathfrak{p}_1$-primary ideal and $e(R/\mathfrak{p}_i) = m_1$.

Proof. By Lemma 4.4, $P_1 = JB + WB$ and $P_2 = JB + (W - X)B$ are prime ideals of B contained in $\mathfrak{m}_B$. Since $I = JR$ is unmixed, it follows that P_i are the only minimal primes above $JT + fT$. Note that P_1 and P_2 are distinct, since $\varphi_0(P_1) \neq \varphi_0(P_2)$, as is easily seen from the fact that $X \notin J$. In particular, $W \notin P_2$ and $W - X \notin P_1$. A simple localization argument shows that $JT + fT = P_1 \cap P_2$. □

Example 4.8. CASE $(c.4)$. Take $f = (W^{n-1} - X^{n-2}Y)(W - X)$, $n = 3$, in Setting 4.1. As in Example 4.2, take $m = (m_1, m_2, m_3)$ in $\{(3, 4, 5), (4, 5, 7), (4, 7, 9)\}$. Then we claim that $P_1 = JB+(W^{n-1}-X^{n-2}Y)B$ and $P_2 = JB+(W-X)B$ are prime ideals of B contained in $\mathfrak{m}_B$. The latter holds by Lemma 4.4. To see the former, it suffices to repeat the argument of Example 4.2 (i) only now having ψ send X, Y, Z and W to $t^{(n-1)m_1}$, $t^{(n-1)m_2}$, $t^{(n-1)m_3}$ and $t^{(n-2)m_1+m_2}$, respectively. Note that in each case $\gcd((n - 1)m_1, (n - 1)m_2, (n - 1)m_3, (n - 2)m_1 + m_2) = 1$. Set $\mathfrak{p}_i = P_iT$,

$i = 1, 2$. Then $e(R) = n$, $\mathrm{Min}(R/\mathfrak{p}) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$ $(\sigma_{\mathfrak{p}_1}, l_{\mathfrak{p}_1}) = (n-1, 1)$, $(\sigma_{\mathfrak{p}_2}, l_{\mathfrak{p}_2}) = (1, 1)$ and $I = \mathfrak{p}_1 \cap \mathfrak{p}_2$ is a minimal primary decomposition with $e(R/\mathfrak{p}_1) = (n-1)m_1$ and $e(R/\mathfrak{p}_2) = m_1$. (We leave the details to the reader.)

Example 4.9. Case $(c.5)$. Take $f = W^{n-2}(W-X)(W-Y)$, $n \geq 3$, in Setting 4.1. By Lemma 4.4, $P_1 = JB + WB$, $P_2 = JB + (W-X)B$ and $P_3 = JB + (W-Y)B$ are prime ideals of B contained in $\mathfrak{m}_B$. Set $\mathfrak{p}_i = P_iT$, $i = 1, 2, 3$. Then $e(R) = n$, $\mathrm{Min}(R/\mathfrak{p}) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$, $(\sigma_{\mathfrak{p}_1}, l_{\mathfrak{p}_1}) = (1, n-2)$, $(\sigma_{\mathfrak{p}_i}, l_{\mathfrak{p}_i}) = (1, 1)$, for $i = 2, 3$, and $I = \mathfrak{q}_1 \cap \mathfrak{p}_2 \cap \mathfrak{p}_3$ is a minimal primary decomposition with $\mathfrak{q}_1$ a $\mathfrak{p}_1$-primary ideal and $e(R/\mathfrak{p}_i) = m_1$, for $i = 1, 2, 3$. (Details are left to the reader.)

Remark 4.10. We can even find examples with f a prime element in B, hence R a domain, with some restrictions on the base field k. Note that in Example 4.2, for $n = 3$, $f = W^3 - X^2Y$ is irreducible in B. Indeed, suppose that f has a factor of the form $W - g$, for some $g \in A$. Then $\varphi_g(f) = 0$, so $g^3 = X^2Y$. Since X and Y are irreducible elements in the UFD A, this yields a contradiction.

For the cases $(b.3)$ and $(c.3)$, as in Example 4.7, and with $m = (3, 4, 5)$ and $n = 2$, take $f = W^2 - XZ$, which is irreducible in B, by an analogous argument. If $\mathrm{char}(k) \neq 2$, then $I = (JR + (w-y)R) \cap (JR + (w+y)R)$ is a minimal primary decomposition.

For the case $(c.4)$, as in Example 4.8, and with $m = (4, 5, 7)$ and $n = 3$, take $f = W^3 - X^2Z$, which analogously is irreducible in B.

If k is separable and does not contain a cube root of unity different from 1, then one can show, by a rather lengthy and technical argument not given here, that $I = (JR + (w-y)R) \cap (JR + (w^2 + yw + y^2)R)$ is a minimal primary decomposition. (Hint: Extend the base field from k to $k[\lambda]$, where λ is a primitive cube root of unity. Use the properties of integral and faithfully flat extensions, together with the Cohen-Seidenberg Theorem [7, Theorem 5, pp. 33-34], particularly [7, Theorem 5, vi)].)

For the case $(c.5)$, as in Example 4.9, with $m = (4, 5, 7)$ and $n = 3$ and $f = W^3 - X^2Z$ as above, and if k contains a cube root of unity $\lambda \neq 1$ (and so three distinct cube roots of unity $1, \lambda, \lambda^2$), then $I = \cap_{j=0}^{2}(JR + (w - \lambda^j y)R)$ is a minimal primary decomposition.

Note that in these examples, for instance when $f = W^2 - XZ$, while R is a domain, it is not a UFD, since w, x and z are prime elements in R yet $w^2 = xz$. Here, $(x, w)R$ is a non-principal prime ideal of height 1 (and R is not Shimoda, see Section 1).

Example 4.11. Case $(b.1)$, $m = (3, 5, 7)$. Let k be a field of characteristic different from 2, not containing a square root of -1. Let $f = W^2 + XZ$. Then $JB + fB = JB + (W^2 + Y^2)B$. An analogue of the Hint in Remark 4.10 above shows that $JB + fB$ is a prime ideal. Thus $e(R) = 2$ and I is a prime ideal.

Remark 4.12. The examples above prove that all the cases in Corollary 2.6 and in the main theorem can occur. They also suggest that the condition $e(R) \leq 3$ is not strictly necessary. However the proof of Theorem 3.1 strongly relies on applying the Associative Law of Multiplicities for small values of $e(R)$. It seems clear then that radically different techniques will be needed in order to extend Theorem 3.1 (still in dimension 3) to the case of higher, or indeed arbitrary, multiplicities.

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