Highly eccentric Hip–Hop solutions of the $2N$–Body problem

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Abstract

We show the existence of families of hip–hop solutions in the equal–mass $2N$–body problem which are close to highly eccentric planar elliptic homographic motions of $2N$ bodies plus small perpendicular non–harmonic oscillations. By introducing a parameter ϵ, the homographic motion and the small amplitude oscillations can be uncoupled into a purely Keplerian homographic motion of fixed period and a vertical oscillation described by a Hill type equation. Small changes in the eccentricity induce large variations in the period of the perpendicular oscillation and give rise, via a Bolzano argument, to resonant periodic solutions of the uncoupled system in a rotating frame. For small $\epsilon \neq 0$, the topological transversality persists and Brouwer's fixed point theorem shows the existence of this kind of solutions in the full system.

1 Introduction

Hip–hop solutions of the equal–mass $2N$–body problem are periodic solutions in which all the bodies move in such a way that their positions in configuration space are at the vertices of a regular antiprism for all time.

A regular antiprism is a polyhedron formed by two congruent regular N–gons perpendicular to the line joining their centers and such that their orthogonal projections along this line form a regular $2N$–gon, i.e. one of the N–gons has been rotated an angle π/N on its own plane. The polyhedron is completed by connecting both N–gons, which we call the *bases*, by an alternating band of isosceles triangles. The symmetries of the equations of motion when all the masses are equal ensure that if at a given time t_0 the $2N$ bodies are on the vertices of a regular antiprism and the velocities satisfy the appropriate conditions of symmetry, then they will stay forever on the vertices of an antiprism.

If we take the line joining the centers of the bases to be the z-axis and the center of mass at the origin, then the picture of a hip–hop solution is similar to having two equal planar homographic elliptic solutions on parallel planes, each one rotated through half a central angle with respect to the other, together with an oscillatory motion of the planes along their common perpendicular. The planes will coincide at a given time with opposite velocities, separate in opposite directions, reach a maximum distance and fall again to coincide. The orthogonal projection of both N–gons on the $z = 0$ plane will always be a regular rotating $2N$–gon of variable size. A hip–hop solution is a periodic solution of this type, where periodic has the usual meaning of periodic in an ad–hoc rotating reference frame (see [2], [6]).

A number of results on hip–hop solutions have been obtained by means of variational methods. With these methods it is possible to find solutions that do not depend on a small parameter (see [3], [6] and the references therein for more details). In [2], the authors show that Poincare's argument of analytic continuation can be used to add vertical oscillations to the circular motion of $2N$ bodies of equal mass occupying the vertices of a regular $2N$–gon, and prove the existence of families of hip–hop solutions with eccentricity close to zero. An infinite number of these orbits are 3D choreographies, i.e. all the bodies move on the same non–planar curve at equally spaced time intervals.

In this work we prove the existence of hip–hop solutions for values of the eccentricity close to 1. Roughly speaking, these solutions are obtained by introducing a small parameter ϵ in order to rescale the z variable and uncouple the equations of motion into a homographic motion and a vertical oscillation. The homographic motion is not affected by the vertical motion, but the period of the vertical oscillation is greatly affected by small changes in the eccentricity of the homographic motion when the eccentricity is close to unity. As the vertical oscillation period depends continuously on the eccentricity, a standard Bolzano type argument shows the existence of periodic solutions in the uncoupled problem.

When $\epsilon \neq 0$ is small, the homographic motion is perturbed by the vertical motion, but the topological transversality can be shown to persist using Brouwer's fixed point theorem (see [4] for a general reference). These solutions are probably among the very spectacular orbits computed numerically by Terracini and shown at the Celmec IV meeting in Viterbo (2004).

2 Equations of motion

Consider $2N$ bodies of equal mass m moving under Newton's law and let $(\mathbf{r}_i, \dot{\mathbf{r}}_i)$, $i = 1, \cdots, 2N$, be their positions and velocities in $\mathbb{R}^3$. The equations of motion of the $2N$–body problem, in a normalized system of units, are

$$\ddot{\mathbf{r}}_i = \sum_{k=1, k\neq i}^{2N} \frac{\mathbf{r}_k - \mathbf{r}_i}{r_{ki}^3}, \quad i = 1, \ldots, 2N, \tag{1}$$

where $r_{ki} = |\mathbf{r}_k - \mathbf{r}_i|$.

As we mentioned in the introduction, we are looking for hip–hop solutions of the $2N$–body problem, i.e. solutions such that the bodies are on the vertices of an antiprism for all time. Due to the symmetries involved, it is enough to know the position of one of the bodies. Thus, in a suitable reference system, if $\mathbf{r} = \mathbf{r}_1(t) = (x, y, z), \dot{\mathbf{r}} = \dot{\mathbf{r}}_1(t) = (\dot{x}, \dot{y}, \dot{z})$ are the position and velocity of one of the bodies, then the position and velocity of the i–th body, for $i = 2, \ldots, 2N$, is

$$\mathbf{r}_i = R^{i-1}\mathbf{r}_1, \qquad \dot{\mathbf{r}}_i = R^{i-1}\dot{\mathbf{r}}_1,$$

where R is a rotation plus a reflection given by the matrix

$$R = \begin{pmatrix} \cos(\frac{\pi}{N}) & -\sin(\frac{\pi}{N}) & 0 \\ \sin(\frac{\pi}{N}) & \cos(\frac{\pi}{N}) & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{2}$$

These equations are written in a frame such that the center of mass is at the origin and the bases (the two sets of N bodies) of the antiprism are parallel to the $z = 0$ plane. There is clearly no loss of generality in doing so and it gives a lot of insight into the physics of the problem.

The study of hip–hop solutions is reduced to the study of the three–degree of freedom system given in Proposition 1 (see [2] for details).

Proposition 1 *The vector* $(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_{2N}) = (\mathbf{r}, R\mathbf{r}, \ldots, R^{2N-1}\mathbf{r})$*, where* R *is the matrix given in (2), is a solution of the* $2N$*–body problem (1) if and only if* $\mathbf{r}(t)$ *satisfies the equation*

$$\ddot{\mathbf{r}} = \sum_{k=1}^{2N-1} \frac{(R^k - I)\mathbf{r}}{|(R^k - I)\mathbf{r}|^3}, \tag{3}$$

where I *is the identity matrix.*

The equations (3) can be written as the differential system associated to the Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) - \frac{1}{2}\sum_{k=1}^{2N-1} \frac{1}{\sqrt{4(x^2 + y^2)\sin^2\left(\frac{k\pi}{2N}\right) + ((-1)^k - 1)^2 z^2}}, \tag{4}$$

where p_x, p_y and p_z are the momenta conjugated to the x, y, z coordinates. We introduce cylindrical coordinates (r, ϕ, d) by means of the change

$$x = r\cos\phi, \quad y = r\sin\phi, \quad z = d,$$

and the Hamiltonian (4) becomes

$$\mathcal{H} = \frac{1}{2}(p_r^2 + \frac{p_\phi^2}{r^2} + p_d^2) - \frac{1}{2}\sum_{k=1}^{2N-1} \frac{1}{\sqrt{4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2}}, \tag{5}$$

where p_r, p_ϕ and p_d are the momenta associated to the cylindrical coordinates.

The function $\mathcal{H}$ does not depend on ϕ, which means that $\dot{p}_\phi = 0$. As the angular momentum $p_\phi = \Phi$ is constant it can be computed from the initial conditions and then the variable ϕ can be obtained by means of a quadrature from the equation

$$\dot{\phi} = \frac{\Phi}{r^2}. \tag{6}$$

Let us call *the reduced problem* that consisting only of the equations for the r and d variables given by

$$\begin{aligned} \ddot{r} &= \frac{\Phi^2}{r^3} - 2r \sum_{k=1}^{2N-1} \frac{\sin^2\left(\frac{k\pi}{2N}\right)}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2\right)^{3/2}}, \\ \ddot{d} &= -\frac{d}{2} \sum_{k=1}^{2N-1} \frac{\left((-1)^k - 1\right)^2}{\left(4r^2\sin^2\left(\frac{k\pi}{2N}\right) + \left((-1)^k - 1\right)^2 d^2\right)^{3/2}}, \end{aligned} \tag{7}$$

and *the complete problem* the whole set of equations (6) and (7). Our first aim is to find periodic solutions of the reduced problem. These solutions will be periodic or quasi–periodic solutions of the complete problem depending on the commensurability of their period T with $\phi(T)$.

As the equations of motion of the reduced problem (7) are invariant by the symmetry

$$\mathcal{S} : (t, r, d, \dot{r}, \dot{d}) \;\longrightarrow\; (-t, r, -d, -\dot{r}, \dot{d}),$$

we have the following well–known proposition, which provides sufficient conditions for symmetric periodic solutions to exist.

Proposition 2 *Let* $\mathbf{q}(t) = (r(t), d(t), \dot{r}(t), \dot{d}(t))$ *be a solution of the equations (7). If* $\mathbf{q}(t)$ *satisfies* $d(0) = \dot{r}(0) = 0$ *and* $d(\tau) = \dot{r}(\tau) = 0$ *at some instant of time* $t = \tau$*, then* $\mathbf{q}(t)$ *is a symmetric periodic solution of period* 2τ*.*

Other symmetries are possible (see for instance [2]). We do not intend to explore, in the present paper, all possible symmetries but, indeed, to show that for high eccentricities hip–hop solutions can be shown to exist by means of topological arguments.

In equation (7) the initial conditions $d(0) = \dot{d}(0) = 0$ will result in a planar (in the $z = 0$ plane) homographic motion of the $2N$ bodies. We intend to add a small vertical motion in the z direction and show the existence of periodic 3–dimensional hip–hop solutions. When the amplitude of the vertical motion tends to zero, the planar motion can be uncoupled from the system; the uncoupling can be accomplished through a rescaling of the variable d as follows. If we substitute εd for d in the equations (7) we obtain (see [2] for more details)

$$\begin{aligned} \ddot{r} &= \frac{\Phi^2}{r^3} - \frac{K_N^2}{r^2} + O(\epsilon^2), \\ \ddot{d} &= -\frac{S_N^2}{r^3}\, d + O(\epsilon^2), \end{aligned} \tag{8}$$

where

$$K_N^2 = \frac{1}{4} \sum_{k=1}^{2N-1} \frac{1}{\sin\left(\frac{k\pi}{2N}\right)}, \qquad S_N^2 = \frac{1}{64} \sum_{k=1}^{2N-1} \frac{((-1)^k - 1)^4}{\sin^3\left(\frac{k\pi}{2N}\right)}. \tag{9}$$

For ϵ small we intend to treat the above system as a perturbation of the case $\epsilon = 0$. It must be borne in mind, however, that the expansions in ϵ are valid only if r is bounded away from zero because coefficients involving terms of the form $1/r^n$ are likely to appear in the expansions. As the equations are analytic, we can say that given $r^* > 0$, the right–hand sides of equations (8) hold good uniformly in ϵ for $r > r^*$.

The next Lemma provides a useful bound on the constants S_N^2 and K_N^2 which will be useful later on.

Lemma 1 *Let S_N and K_N the sums defined in (9). Then, for all integers $N > 1$,*

$$\frac{S_N^2}{K_N^2} \geq \frac{2}{(2(N-1)+\sin(\frac{\pi}{2N}))\sin^2(\frac{\pi}{2N})} \geq \frac{8}{4+\sqrt{2}}.$$

Proof. Let us define $\delta_k = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}$. Each sum can be written as follows

$$4S_N^2 = \delta_N + 2\sum_{k=1}^{N-1} \frac{\delta_k}{\sin^3(\frac{k\pi}{2N})}, \qquad 4K_N^2 = 1 + 2\sum_{k=1}^{N-1} \frac{1}{\sin(\frac{k\pi}{2N})},$$

and then

$$\frac{S_N^2}{K_N^2} \geq \frac{\frac{2}{\sin^3(\frac{\pi}{2N})}}{1+\frac{2(N-1)}{\sin(\frac{\pi}{2N})}} = \frac{2}{(2N-2+\sin(\frac{\pi}{2N}))\sin^2(\frac{\pi}{2N})}.$$

Notice that the function $f(x) = (\frac{\pi}{x} - 2 + \sin x)\sin^2 x$ is monotone increasing in $[0, \pi/4]$ and thus

$$\frac{S_N^2}{K_N^2} \geq \frac{2}{f(\pi/4)} = \frac{8}{4+\sqrt{2}}.$$

■

3 Periodic solutions of the uncoupled system

Recall that $2N$ is the number of bodies in the problem. As we always think of a fixed N, we will suppress all references to N in what follows. So, the constants S_N and K_N will be written simply as S and K from now on.

If we set $\epsilon = 0$ in equations (8) we get

$$\ddot{r} = \frac{\Phi^2}{r^3} - \frac{K^2}{r^2}, \tag{10}$$

$$\ddot{d} = -\frac{S^2}{r^3}d, \tag{11}$$

where the first equation is just the radial motion of a Kepler problem (only the radial part is written here, the angular part can always be obtained through the angular momentum integral) and the second is a classical linear Hill's equation which represents (due to the rescaling introduced in the previous Section) the small vertical oscillation of the bodies. Ideally we could solve equation (10) for $r(t)$, substitute the result in equation (11) and solve the latter for $d(t)$.

Let us denote by $r_p(t, a, e)$ the solution of equation (10) with semiaxis a and eccentricity e starting at the pericenter of the orbit at $t = 0$, i.e. $r_p(0, a, e) = a(1-e)$ and $\dot{r}_p(0, a, e) = 0$. The function $r_p(t, a, e)$ is given by

$$r_p(t, a, e) = a(1 - e\cos E(t)), \tag{12}$$

where $E(t)$ is defined implicitly by Kepler's equation $E(t) - e\sin E(t) = M(t)$, with $M(t) = Ka^{-3/2}t$. The magnitudes E and M are called traditionally *eccentric anomaly* and *mean anomaly*. A general reference for all issues concerning Kepler's equation is [1].

The parameters a and e are related to the angular momentum through the equality $\Phi^2 = a(1-e^2)K^2$ and to the initial position through $r_p(0, a, e) = a(1-e)$. The function $r_p(t, a, e)$ is periodic of period $2T$ where $T = \pi a^{3/2}/K$ is the Keplerian semi–period of the orbit.

Let $d_p(t, a, e)$ be the solution of equation (11) with $r(t) = r_p(t, a, e)$ and initial conditions $d_p(0, a, e) = 0$, $\dot{d}_p(0, a, e) = 1$. Notice that in view of the linearity of the equation (11) the solution with initial velocity equal to λ is $\lambda d_p(t, a, e)$.

We have in mind to show that for many values of the eccentricity e a zero of $\dot{r}_p(t, a, e)$ coincides with a zero of $d_p(t, a, e)$. For any positive integer m we know that $\dot{r}_p(mT, a, e) = 0$ for any value of e. By carefully adjusting the value of e, we want to get $d_p(mT, a, e) = 0$.

Given a positive integer m and $e < 1$, let $Z(m, a, e)$ be the number of zeros of $d_p(t, a, e)$ in the interval $[0, mT]$. If $e < 1$ the equations (10) and (11) are analytic and we can apply straightforward Sturm theory. The coefficient on the right hand side of the equation (11) is periodic and strictly negative, so the zeros of $d_p(t, a, e)$ are all simple, separate those of $\dot{d}_p(t, a, e)$ for $t > 0$ and there are only finitely many of them. So, $Z(m, a, e)$ is well defined (it could well happen that $Z(m, a, e) = 0$, but by no means $Z(m, a, e) = \infty$). We will see, however, that if e tends to 1, the number of zeros increases unboundedly. This means that zeros of $d_p(t, a, e)$ must be "entering" the interval $[0, mT]$ from the right and we must have $d_p(mT, a, e) = 0$ for certain values of e.

The limiting case $e = 1$ of these equations plays an important role in the analysis of the problem. For $e = 1$, the solution $E(M)$ of Kepler's equation $E - \sin E = M$ can be given in the form $E = (6M)^{1/3}\alpha(M)$, which is valid only for small values of M and where $\alpha(M)$ is a continuous function with $\alpha(0) = 1$ (see [1]). Then

$$r_p(t, a, 1) = a(1 - \cos E) = 2a\sin^2(E/2) = \frac{6^{2/3}}{2}\, a\, M^{2/3}\beta(M),$$

where $\beta(M)$ is a continuous function with $\beta(0) = 1$. Let β_0 be such that $1 < \beta_0^3 < \frac{64}{9(4+\sqrt{2})}$, and $\zeta \in (0, \pi)$ be small enough so that $\beta(M) < \beta_0$ for $M \in [0, \zeta]$. If $\gamma = \frac{S^2}{K^2}\frac{2}{9\beta_0^3}$ the inequality

$$\frac{S^2}{r_p^3(t, a, 1)} > \frac{\gamma}{t^2},$$

holds for $t \in (0, \zeta\frac{a^{3/2}}{K}]$.

Consider now the Euler's equation

$$\ddot{y} = -\frac{\gamma}{t^2}y,$$

which will be helpful in deriving properties of the function $d_p(t,a,e)$. From Lemma 1 we have $\gamma > 1/4$ and then, the solutions of the Euler's equation are of oscillatory type and given by

$$y(t) = \sqrt{t}\left(C_1 \sin(\sqrt{\gamma - 1/4}\ \log t) + C_2 \cos(\sqrt{\gamma - 1/4}\ \log t)\right), \tag{13}$$

where log is the natural logarithm and C_1, C_2 arbitrary constants.

The function $y(t)$ has infinitely many zeros with $t = 0$ as unique accumulation point, so that for any positive integer k there exists $\delta > 0$ such that $y(t)$ has at least k zeros in the interval $[\delta, \zeta a^{3/2}/K]$. See for instance [5].

In the compact defined by $t \in [\delta, \zeta a^{3/2}/K]$ and $e \in [0,1]$ we have $\lim_{e\to 1} r_p(t,a,e) = r_p(t,a,1)$ because we are away from the singularity $t = 0$. It is possible then to find $e^* < 1$ such that for $e \in [e^*, 1]$ and $t \in [\delta, \zeta a^{3/2}/K]$ the inequality

$$\frac{S^2}{r_p^3(t,a,e)} > \frac{\gamma}{t^2},$$

holds good.

As $[\delta, \zeta a^{3/2}/K] \subset [0, mT]$, Sturm's comparison theorem asserts that the function $d_p(t,a,e)$ has at least k zeros in the interval $[0, mT]$. So $Z(m,a,e) \geq k$, for $e \in [e^*, 1)$, and the following proposition is true.

Proposition 3 *The integer–valued function $Z(m,a,e)$ is well defined for $e < 1$ and $\lim_{e\to 1} Z(m,a,e) = +\infty$.*

The following proposition shows the existence of infinitely many periodic solutions of the uncoupled system.

Proposition 4 *Given a positive integer m, there exists a non negative integer $\overline{k}$ and an increasing sequence of eccentricities $\{e_n\}_{n\geq 1}$ converging to 1 such that the function $d_p(t,a,e_n)$ has exactly $n + \overline{k}$ zeros in the interval $t \in [0, mT]$ and $d_p(mT, a, e_n) = 0$*

Proof.

Let us consider equation (11) as a system of first order and introduce polar coordinates as follows: $d = \rho \sin\theta$, $\dot{d} = \rho\cos\theta$. Then the system becomes

$$\begin{aligned} \dot{\theta} &= \frac{S_N^2}{r_p^3(t,a,e)} \sin^2\theta + \cos^2\theta, \\ \dot{\rho} &= \left(1 - \frac{S_N^2}{r_p^3(t,a,e)}\right) \frac{\sin 2\theta}{2}\,\rho. \end{aligned} \tag{14}$$

Denote by $\theta_p(t,a,e)$ the solution of the first equation of system (14) with initial condition $\theta_p(0,a,e) = 0$. Clearly, $d_p(mT,a,e) = 0$ if and only if $\theta_p(mT,a,e) \in \pi\mathbb{Z}$. From Proposition 3 and the fact that $\theta_p(t,a,e)$ is a strictly increasing function of t and its derivative is greater than a certain positive constant, for each positive integer k, there exists e^* such that $\theta_p(mT,a,e) > k\pi$, for $e \in [e^*, 1]$ and

$$\lim_{e\to 1} \theta_p\,(mT,a,e) = \infty. \tag{15}$$

Furthermore, there exist $\overline{k}$ a non–negative integer such that

$$\overline{k} = \min\{k \in \mathbb{N}; k \geqslant 0,\ \exists e \in [0,1] \text{ such that } \theta_p\left(mT, a, e\right) = k\pi\}. \tag{16}$$

Now, for each value of a, let us consider the analytic function $e \in [0,1] \longrightarrow \theta_p(mT, a, e)$. Let us define the sequence $\{e_n\}_{n\in\mathbb{N}}$ in the following way. Firstly, we take e_0 as the minimum value of e such that is a zero of $\theta_p(mT, a, e) - \overline{k}\pi = 0$ with odd multiplicity and, secondly,

$$e_n = \min\left\{e \in [0,1]\,;\, e \geqslant e_{n-1}, \theta_p\left(mT, a, e\right) - (n+\overline{k})\pi = 0 \text{ with odd multiplicity}\right\}, \tag{17}$$

for $n \geq 1$. We observe that e_n exist due to (15). Then, the sequence $\{e_n\}_{n\in\mathbb{N}}$ is strictly increasing and converging to 1. The solution $d_p(t, a, e_n)$ has $n+\overline{k}$ zeros in the interval $[0, mT]$ and $d_p(mT, a, e_n) = 0$. ■

Clearly the eccentricities e_n and the non–negative integer $\overline{k}$ defined in Proposition 4 depend on m, but in order to keep the notation reasonable simple in what follows we will generally suppress the explicit dependence on the parameter m when not essential.

4 The reduced problem

Consider the equations for the reduced problem (8). We will show that for any of the periodic orbits of the uncoupled problem founded in Section 3 there exists $\overline{\epsilon} > 0$ such that, for any $\epsilon \in [0, \overline{\epsilon}]$, there exists a periodic solution of the reduced problem near the solution of the uncoupled system, that is, if $\epsilon \to 0$, the solution of the reduced problem tends to the solution of the uncoupled system.

Given a value of the angular momentum Φ, let $r_p^\epsilon(t, r_0, \dot{d}_0)$ and $d_p^\epsilon(t, r_0, \dot{d}_0)$ be a particular solution of equations (8) with initial conditions $r_p^\epsilon(0, r_0, \dot{d}_0) = r_0$, $\dot{r}_p^\epsilon(0, r_0, \dot{d}_0) = 0$, $d_p^\epsilon(0, r_0, \dot{d}_0) = 0$, $\dot{d}_p^\epsilon(0, r_0, \dot{d}_0) = \dot{d}_0$. Notice that due to the rescaling introduced in the variable d in Section 2, taking any other value of $\dot{d}_p^\epsilon(0, r_0, \dot{d}_0)$ would result in a different rescaled values of ϵ.

Using Proposition 2, we need to prove that there exist values of $r_0, \dot{d}_0$ and τ such that the solution of equations (8) $r_p^\epsilon(t, r_0, \dot{d}_0)$ and $d_p^\epsilon(t, r_0, \dot{d}_0)$ satisfies

$$\begin{aligned} \dot{r}_p^\epsilon\left(\tau, r_0, \dot{d}_0\right) &= 0, \\ d_p^\epsilon\left(\tau, r_0, \dot{d}_0\right) &= 0. \end{aligned}$$

In the case $\epsilon = 0$, we have the following result equivalent to Proposition 4. From now on, we denote $\overline{T} = mT$, i.e. a multiple of the Keplerian semi–period.

Proposition 5 *Given a positive integer m and for each value of the semimajor axis a, there exist sequences of initial conditions $\{r_0^n\}_{n\in\mathbb{N}}$ and momenta $\{\Phi_n\}_{n\in\mathbb{N}}$, both converging to zero, such that the solution $r_p^\epsilon(t, r_0, \dot{d}_0)$ and $d_p^\epsilon(t, r_0, \dot{d}_0)$ of equations (8) with $\epsilon = 0, \Phi = \Phi_n$ and initial conditions $r_0 = r_0^n$ and $\dot{d}_0$ any value, is a periodic solution of period $2\overline{T}$.*

Proof. Clearly, for $\epsilon = 0$, given a fixed value of the semimajor axis a and for any $r_0 \in (0, a]$, the solution $(r_p^0(t, r_0, \dot{d}_0)$ and $d_p^0(t, r_0, \dot{d}_0))$ of equations (8) coincide with the solution of the uncoupled problem introduced in Section 3. In particular, when $\dot{d}_0 = 1$, $r_p^0(t, r_0, 1) = r_p(t, a, e)$ and $d_p^0(t, r_0, 1) = d_p(t, a, e)$ for $e = 1 - (r_0/a)$ and $\Phi^2 = a(1 - e^2)K^2$. From Proposition 4 and according to (12), given any non negative integer m, there exists a sequence $\{e_n\}_{n\in\mathbb{N}}$ such that if $r_0^n = a(1 - e_n)$ then

$$\begin{aligned} \dot{r}_p^0(\overline{T}, r_0^n, 1) = \dot{r}_p(\overline{T}, a, e_n) &= 0, \\ d_p^0(\overline{T}, r_0^n, 1) = d_p(\overline{T}, a, e_n) &= 0. \end{aligned} \tag{18}$$

As it was pointed out in Section 3, due to the linearity of equation (11), the same result is also true for any value of $\dot{d}_0$. ■

In order to extend the Theorem 5 to the case $\epsilon \neq 0$ we will use topological arguments as follows. We fix a value $\dot{d}_0$ and a, and for each $n \in N$ and $m \in N$ and $\epsilon \geq 0$ we define the following vector field:

$$\mathcal{F}^\epsilon(\tau, r_0) = (\mathcal{F}_1^\epsilon(\tau, r_0), \mathcal{F}_2^\epsilon(\tau, r_0)) = ((-1)^{m+1}\dot{r}_p^\epsilon(\tau, r_0, \dot{d}_0), (-1)^{n+1+\overline{k}} d_p^\epsilon(\tau, r_0, \dot{d}_0)), \tag{19}$$

for $(\tau, r_0) \in [0, \infty) \times (0, a]$ and $\overline{k}$ defined in (16). The existence of symmetric periodic orbits of equations (8) reduces to find a singular point (τ, r_0) for the vector field $\mathcal{F}^\epsilon$.

Lemma 2 *Given a positive integer m and any value of $\dot{d}_0$ and the semimajor axis a, let $\overline{T}$, Φ_n and r_0^n, $n \geq 1$, be as in Proposition 5, $r_p^\epsilon(t, r_0, \dot{d}_0)$ and $d_p^\epsilon(t, r_0, \dot{d}_0)$ be the solution of equations (8) with $\Phi = \Phi_n$, and $\mathcal{F}^\epsilon(\tau, r_0)$ be the vector field defined in (19). Then, for any fixed value of n, there exists a box $\mathcal{D} = [\overline{T}_1, \overline{T}_2] \times [b_1, b_2] \subset (0, \infty) \times (0, a]$ and $\epsilon_n > 0$ such that $\mathcal{F}^\epsilon(\tau, r_0)$ over $\partial\mathcal{D}$ is a vector pointing towards the interior of $\mathcal{D}$ for all $\epsilon \leq \epsilon_n$, and $(\overline{T}, r_0^n) \in \mathcal{D}$ is the only zero of $\mathcal{F}^0$ in $\mathcal{D}$.*

Proof. In order to prove the first statement it suffices to show that, for $\epsilon = 0$, we can construct a box $\mathcal{D}$ such that the vector field does not vanish on $\partial\mathcal{D}$ and points inwards. As $\mathcal{F}^\epsilon(\tau, r_0)$ is continuous, a standard compacity argument shows that, there exists ϵ_n such that if $\epsilon < \epsilon_n$, then $\mathcal{F}^\epsilon$ still has both properties.

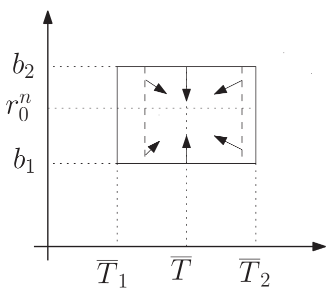

Figure 1: Vector field $\mathcal{F}^\epsilon$ defined inside the square $[\overline{T}_1, \overline{T}_2] \times [b_1, b_2]$

Clearly, from (18), we have that $\mathcal{F}^0(\overline{T}, r_0^n) = (0, 0)$, and also, for any value of $r_0 \in (0, a]$, $\mathcal{F}_1^0(\overline{T}, r_0) = 0$.

Notice that, for any value $r_0 \in (0, a]$, $\overline{T}$ is a multiple of the semi-period of the function $r_p^0(t, r_0, \dot{d}_0)$, and that $r_p^0(\overline{T}, r_0, \dot{d}_0)$ is the distance from the origin to the pericenter (for m even) or to the apocenter (for m odd). Consequently, if we consider $\tau < \overline{T}$ in a sufficiently small neighborhood of $\overline{T}$, we have $\dot{r}_p^0(\tau, r_0, \dot{d}_0) < 0$ (for m even) and $\dot{r}_p^0(\tau, r_0, \dot{d}_0) > 0$ (for

m odd). In a similar way, for $\overline{T} < \tau$, in a sufficiently small neighborhood of $\overline{T}$, we have $\dot{r}_p^0(\tau, r_0, \dot{d}_0) > 0$ (if m is even) and $\dot{r}_p^0(\tau, r_0, \dot{d}_0) < 0$ (for m odd). So we can take $\overline{T}_1$, $\overline{T}_2$, such that $\overline{T}_1 < \overline{T} < \overline{T}_2$ and for $0 < r_0 < a$ we have $\mathcal{F}_1^0(\tau, r_0) > 0$ for $\overline{T}_1 \leq \tau < \overline{T}$, and $\mathcal{F}_1^0(\tau, r_0) < 0$ for $\overline{T} < \tau \leq \overline{T}_2$ (see Figure 1).

We will show now that there exist values b_1 and b_2 such that $r_0^n \in [b_1, b_2]$ and the vector field is inward-pointing in the region $\mathcal{D}$. If it is necessary, the interval $[\overline{T}_1, \overline{T}_2]$ will be restricted. As in Proposition 4, we write

$$d_p^0(t, r_0, \dot{d}_0) = \rho_p \sin\theta_p.$$

By the definition of e_n (see (17)), for any positive integer $n \geq 1$ we have the following three consecutive multiples of π

$$\begin{aligned}\theta_p(\overline{T}, a, e_{n-1}) &= (n - 1 + \overline{k})\pi,\\ \theta_p(\overline{T}, a, e_n) &= (n + \overline{k})\pi,\\ \theta_p(\overline{T}, a, e_{n+1}) &= (n + 1 + \overline{k})\pi.\end{aligned}$$

Then, for a small enough value $h > 0$, there exist two values of the eccentricity η_1, η_2 such that $e_{n-1} < \eta_1 < e_n < \eta_2 < e_{n+1}$, $\theta_p(\overline{T}, a, \eta_1) = (\overline{k} + n)\pi - h$, and $\theta_p(\overline{T}, a, \eta_2) = (\overline{k} + n)\pi + h$. Then, if $b_1 = a(1 - \eta_1)$ and $b_2 = a(1 - \eta_2)$ the vector field verify that $\mathcal{F}_2^0(\overline{T}, b_1) < 0$ and $\mathcal{F}_2^0(\overline{T}, b_2) > 0$ (see Figure 1).

As $d_p^0(t, r_0, \dot{d}_0)$ is a continuous function, there exist $\delta_1 > 0$ such that $\mathcal{F}_2^0(\tau, b_1) < 0$ and $\mathcal{F}_2^0(\tau, b_2) > 0$ for $\left|\tau - \overline{T}\right| < \delta_1$. Now we restrict if necessary the interval $[\overline{T}_1, \overline{T}_2]$ and the proof is complete.

Notice that the box $\mathcal{D}$ has been constructed such that $(\overline{T}, r_0^n)$ is the only zero of $\mathcal{F}^0$ in $\mathcal{D}$. ■

Theorem 1 *Let N, m be positive integers, $N \geq 2$, and let a be a real positive value and $\overline{T} = m\pi a^{3/2}/K$, where the constant $K = K_N$ is defined in (9). There exists infinite sequences $\{r_0^n\}_{n\in\mathbb{N}}$, $\{\Phi_n\}_{n\in\mathbb{N}}$ and $\{\epsilon_n\}_{n\in\mathbb{N}}$, $n \geq 1$, such that r_0^n converges to zero, $\epsilon_n > 0$ and for any fixed value of $n \geq 1$ and for any $\epsilon \in (0, \epsilon_n]$ there exist $\overline{T}^\epsilon$ and r_0^ϵ in such a way that:*

1. *when $\epsilon \to 0$ then $r_0^\epsilon \to r_0^n$ and $\overline{T}^\epsilon \to \overline{T}$,*
2. *the solution of equations (8) for $\Phi = \Phi_n$ with initial conditions $r(0) = r_0^\epsilon$, $\dot{r}(0) = 0$, $d(0) = 0$ and $\dot{d}(0) = \dot{d}_0$ is a hip–hop solution of period $2\overline{T}^\epsilon$ of the reduced problem.*

Proof. Given a value of the semimajor axis a, the existence of sequences r_0^n and Φ_n are ensured from Proposition 5. For a fixed value of $n \geq 1$, from Lemma 2 and a topological fixed point argument (see, for example, [4]), there exist a value ϵ_n such that, for all $\epsilon \leq \epsilon_n$ there exist r_0^ϵ and $\overline{T}^\epsilon$ satisfying that $\mathcal{F}^\epsilon(\overline{T}^\epsilon, r_0^\epsilon) = 0$. This proves the second part of the statement, and the first one comes from the fact that $(\overline{T}, r_0^n)$ is the only zero of $\mathcal{F}^0$ in the box $\mathcal{D}$ and an argument of continuity. ■

Theorem 1 shows the existence of periodic orbits of the reduced system (8). Clearly, each one of these periodic solutions are hip–hop solutions of the equations (7) close to the planar homographic motion, that is, the projection on the (x, y)–plane performs a precessing highly eccentric elliptic motion, and with a small vertical amplitude.

Fixed the number of the bodies $2N$, a value for m and for the semimajor axis a, it is easy to find numerically the firsts elements of the sequence $\{e_n\}_{n\in\mathbb{N}}$ (up to a certain precision). For each value e_n, a family of hip–hop solutions is born from the planar homographic motion, which can be followed numerically. In Figure 2, two characteristic curves of one of these families are shown.

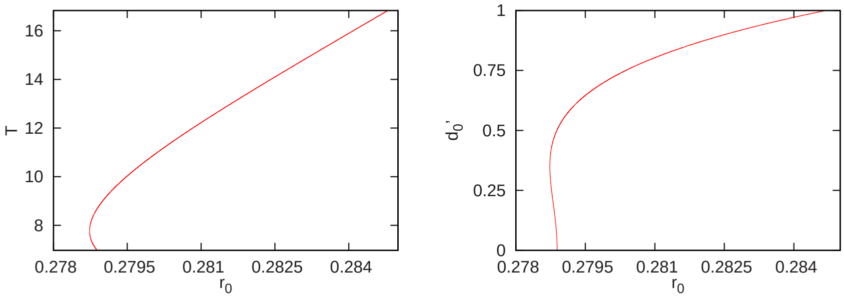

Figure 2: Characteristic curves of a family of hip–hop solutions. On the left, the $(r_0, \overline{T})$ curve; on the right, the (r_0, d_0') curve. $N = 3$, $a = 1$, $m = 3$ and the initial $e_1 = 0.721109941900484$.

In Figure 3 two hip–hop solutions are shown in (x, y, z) coordinates. They are plotted for one period of the variables (r, d) (in which the orbit is periodic). Notice that in general, these solutions are quasi–periodic in the cartesian coordinates.

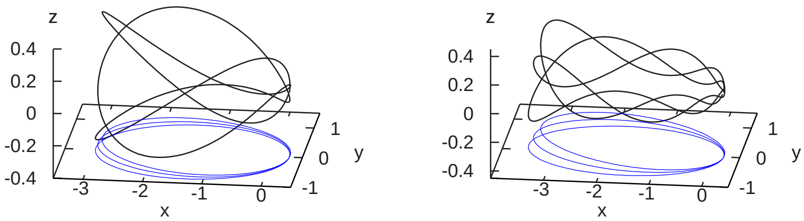

Figure 3: Hip–hop solutions for $N = 3$ and $m = 3$ (left), $N = 7$ and $m = 3$ (right) in the (x, y, z) and their projections on the (x, y)–plane.

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