

REGULARITY OF THE BRUNOVSKY-KRONECKER STRATIFICATION

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Abstract. We study the partition of the set of pairs of matrices according to the Brunovsky-Kronecker type. We show that it is a constructible stratification, and that it is Whitney regular when the second matrix is a column matrix. We give an application to the obtention of bifurcation diagrams for few-parameter generic families of linear systems.

Key words. Linear system, smooth family, versal deformation, transversality, stratification, bifurcation.

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1. Introduction. We shall be dealing with the geometric study of the perturbations of linear (time-invariant) systems $\dot{x} = Ax + Bu$. Or, more generally, with the study of the local structure of the space of parameters of a generic family of linear systems.

A system $\dot{x} = Ax + Bu$ is identified in a natural way with the pair of matrices (A, B) , so that in this paper we consider the space of pairs of matrices of this kind (and the block-similarity equivalence between them), and we study their perturbations. Then, we follow the method introduced by Arnold [1] to study perturbations of square matrices, mainly based on the consideration of versal deformations (see also ([13], (8.1.1)) and ([15], V)). A versal deformation for a pair of matrices have been explicitly obtained in [5].

A key role is played by the canonical reduced form of a pair of matrices, usually called Brunovsky form. However, in this paper we will refer to it as the Brunovsky-Kronecker form of a pair (A, B) , because it can be easily derived (see (6.5.2) of [10]) from the Kronecker form of the matrix pencil $(\lambda I + A, B)$. In fact, from this point of view of matrix pencils, equivalent results to those in [5] about versal deformations of linear system have been shown by [2], [3] and [4].

The central goal is to construct a “stratification” of the space of pairs of matrices, that is to say, a partition of it in a locally finite (see remark 6.3 (iii)) family of manifolds, called “strata”. As a starting point, the space of pairs of matrices is partitioned into block-similarity classes; that is to say, into the orbits by the action of the feedback group. Thus, the perturbations of a given system are obtained from the local description of this partition, which in turn can be derived from the above mentioned versal deformations. For example, metrical information about these perturbations is obtained in [4].

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However, as for square matrices, this partition is not locally finite. Following Arnold's method, we consider "strata" formed by the uncountable union of the orbits which have the same discrete invariants, but different eigenvalues. Therefore, the local description of this new partition informs about the perturbation of the controllability indices, etc., of a system. The key point lies in showing that each stratum is a (regular) manifold, so that the new partition is in fact a stratification, which we call BK- stratification. Again, the proof follows from the local description given by the versal deformation.

Finally, given a parametrized family of linear systems, the BK- stratification induces a partition in the space of parameters, known as the bifurcation diagram of the family. In general, if M is a manifold with a stratification $M = \cup_i X_i$, and $f(w) \in M$, $w \in W$, is a W -parametrized differentiable family on it (that is to say, W is another manifold and $f : W \rightarrow M$ a differentiable map), then the partition $W = \cup_i f^{-1}(X_i)$ is called the "bifurcation diagram" of this family. The bifurcation diagram of a family of linear systems gives precise information about the qualitative properties of the systems arising in the family, and about the effects of local perturbations of the parameters. For example, assume that the bifurcation diagram of an 1-parametrized family (say $W = \mathbb{C}$) consist of a simple point (say $w = 0$). It means that only two types of discrete invariants appear in this family, having very different "stability properties": the type corresponding to $w \neq 0$ is "stable" under little perturbations of w , whereas the other one disappears under any sligth perturbations of the value $w = 0$ of the parameter. In particular, in the numerical computation of the discrete invariants of the element corresponding to $w = 0$ it is probable that one obtains in fact the type corresponding to $w \neq 0$, but no other type can be expected.

Nevertheless some additional conditions are necessary in order to ensure that the bifurcation diagram is in fact a stratification, and that its local structure can be derived from the above description of the BK-stratification. Transversality provides the required conditions. In fact, because of the generalized implicit function theorem, if the parametrized family is transversal to all the strata, then the bifurcation diagram is also a stratification which besides has the following important property: the codimension of each one of its strata in the parameter space is the same that the codimension of the corresponding stratum in the BK-stratification. Then, for a transversal family we have a precise limitation about the possibility of changes of the local structure of the family.

The question is then: are there "many" families transversal to the BK-stratification? At this point is where the Whitney regularity conditions play a key role. In fact, if the stratification is Whitney regular, a theorem of Thom states that the set of families which are transversal to the stratification is open and dense; this gives sense to qualifying as generic such families: this is to say, "almost" all families are transversal to the stratification.

For example, we will see (example 8.2.i) that any generic 1-parametrized family of single-controlled systems has the bifurcation diagram referred above (a single point); hence, it can never arise three or more different types of discrete invariants in a 1-parametrized family of single-controlled systems. Also, we will see (example 8.2.ii) that the only singularity in a generic 2-parametrized bifurcation diagram is the normal crossing. Notice, for example, that in the bifurcation diagram of a generic

2-parametrized family of square matrices (see [1]) appears also cusp singularities.

More in general, one can list all the possible generic bifurcation diagrams for $1, 2, \dots$ parameters (see example 8.2). This is clearly interesting from an experimental point of view, because any other bifurcation diagram is “experimentally impossible”. Or, following an Arnold’s remark, if a parametrized family contains systems of more complicated BK-form, or the bifurcation diagram has more complicated singularities, or ..., then they can be removed by an arbitrarily small perturbation of the family.

There are some theoretical tools to verify the Whitney conditions. For example, according to a theorem by Whitney, a (locally finite) algebraic stratification is Whitney regular if the strata are orbits under the action of an algebraic group. Such is the case of the stratification which arises in the classification of linear systems under the action of the full linear group in the space of state variables ([14]; see also [7]). This is also the case of the stratification arising in the classification of reachable linear systems acted on by the feedback group ([16], [17]).

But the situation is more complicated if, as in our case, the partition in orbits is not locally finite, and one must consider strata formed by an infinite union of them. This is the case of the original study of Arnold: the square matrices stratified according to their Jordan type. See [8] for a detailed proof that this stratification is Whitney regular.

In our case, following these techniques, we prove that the BK- stratification of single-controlled linear systems is Whitney regular. Then, as we said, we can obtain precise descriptions of the bifurcation diagrams of generic parametrized families of this kind of systems.

The organization of this paper is as follows. Section 2 is preliminary, and it contains the general definitions and notations needed in the sequel. In section 1 we define the Brunovsky-Kronecker partition of the set $\mathcal{M}_{n,m}$. Previously, we recall the Brunovsky-Kronecker form as well as the way of calculating its invariants. In section 4 we remark that the block-similarity classes can be considered to be the orbits with regard to the action of a Lie group. Some properties can be derived from this representation: constructibility, regularity, homogeneity,... Section 5 is devoted to showing a fundamental property of local triviality along the orbits, and to giving explicitly the minitransversal variety to the orbit mentioned above. This is the key point in order to prove, in section 6, that the strata are submanifolds of $\mathcal{M}_{n,m}$ (th. 6.2). Moreover, we show that they are constructible and connected. In section 7, we tackle the main theorem (th. 7.5): the Brunovsky-Kronecker stratification is Whitney regular when $m = 1$. Also, we remark (7.6) that it verifies the frontier condition for any m . Finally, in section 8 we enumerate the singularities of bifurcation diagrams of few-parameter generic families of linear systems.

2. Preliminaires. 2.1 For every integer p , we will denote by M_p the space of p -square complex matrices, and by $Gl(p)$ the *linear group* formed by the invertible matrices of M_p . Also, we will denote $M_{p \times q}$ the space of rectangular complex matrices, having p rows and q columns. We will deal mainly with pairs of matrices (A, B) such

that $A \in M_n$ and $B \in M_{n \times m}$. In all the paper, $\mathcal{M}_{n,m}$ will denote the space of such pairs of matrices; $\mathcal{M}_{n,m} = M_n \times M_{n \times m}$.

2.2 Throughout this paper the term *manifold* is used as an abbreviation for complex differentiable manifold. Let X, Y be manifolds, $f : X \rightarrow Y$ a differentiable map and Z a submanifold of Y . We recall (see, for example, [12]) that the map f is said to be *transversal* to the submanifold Z if at each point x in $f^{-1}(Z)$ the following relation holds

$$df_x(T_x X) + T_y Z = T_y Y$$

where $y = f(x)$ and $T_x X, T_y Z, T_y Y$ are the corresponding tangent spaces.

In particular, if f is the inclusion map the above condition merely states that

$$T_x X + T_x Z = T_x Y$$

for every $x \in X \cap Z$. We say then that the submanifolds X and Z are *transversal* (or *minitransversal* if the above sum is direct).

We will use extensively throughout the paper the following result: If the differentiable map $f : X \rightarrow Y$ is transversal to a submanifold $Z \subset Y$ then $f^{-1}(Z)$ is a submanifold of X . Moreover, the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y .

In particular, if X and Z are submanifolds of Y which are transversal, then $X \cap Z$ is a submanifold of Y and $\text{codim}(X \cap Z) = \text{codim } X + \text{codim } Z$.

2.3 A *stratification* of a subset V of a manifold M is a partition $\cup_i X_i$ of V into submanifolds X_i of M (called the *strata*) which satisfies the *local finiteness condition*: every point in V has a neighbourhood in M which meets only finitely many strata.

Let W, M be manifolds, $f : W \rightarrow M$ a differentiable map, and $V = \cup_i X_i$ a stratification of M . We say that f is *transversal* to V if it is transversal to each stratum X_i of V .

According to [1] and [8], the space M_p can be partitioned into a finite number of *Segre strata*, each one formed by the matrices having the same *Segre symbol* (or the same *Jordan type*). Thus, each stratum is the uncountable union of similarity classes, differing only in the values of the distinct eigenvalues. If $C \in M_p$, we denote $\sigma_S(C)$, $\mathcal{O}_S(C)$, $E_S(C)$ the Segre symbol, the Segre orbit and the Segre stratum of C respectively. If $J \in M_p$ is a Jordan matrix, we denote $\Gamma_S(J)$ the versal deformation of J described in [1]. In fact, it is a linear subvariety of M_p , minitransversal to $\mathcal{O}_S(J)$ at J .

2.4 Let M be a finite dimensional vector space, X, Y submanifolds of M and $x \in X \cap \overline{Y}$. We say that Y is *Whitney regular* over X at x when the following condition holds: let $(x_i), (y_i)$ be sequences in X, Y respectively, both converging to x , with $x_i \neq y_i$ for all i . Take L_i to be the line spanned by $x_i - y_i$ and T_i to be the tangent space $T_{y_i} Y$. If (L_i) converges to L (in the Grassmannian of 1-dimensional subspaces of M) and T_i converges to T (in the Grassmannian of q -dimensional subspaces of M , $q = \dim Y$), then $L \subseteq T$.

It is not difficult to see that this condition is invariant under diffeomorphisms. Hence, we can define in an obvious way the Whitney regular condition when M is a manifold.

We say that Y is Whitney regular over X when it is so at every point of $X \cap \overline{Y}$.

Finally, let $\cup_i X_i$ be a stratification of a subset V of a manifold M . We say that this stratification is Whitney regular when every stratum X_i is Whitney regular over X_j ($i \neq j$).

2.5 A closed algebraic subset of \mathbb{C}^n is the set of zeroes of a finite set of polynomials. An open algebraic set of \mathbb{C}^n is the complementary of a closed algebraic subset of \mathbb{C}^n .

A subset S of \mathbb{C}^n is *constructible* if S is a disjoint union $T_1 \cup \dots \cup T_k$, where T_i is locally closed, i.e., it is the intersection of a closed algebraic set with an open algebraic set.

We call *rational mapping* a mapping of a subset of \mathbb{C}^p into \mathbb{C}^q whose components are rational functions, with denominators nowhere zero on the domain. We will need the following theorem of Chevalley: *the image of a constructible set under a rational map is constructible*.

We refer to [8] for some complementary definitions and the results which are needed in what follows.

3. The Brunovsky-Kronecker Stratification. We shall partition $\mathcal{M}_{n,m}$ into a finite number of subsets, each one formed by all the pairs of matrices having the same controllability indices and the same Segre symbol (of the Jordan block in the Brunovsky-Kronecker reduced form). Hence, each one of these subsets is an uncountable union of block-similarity classes, differing only in the values of the eigenvalues. In section 6 we shall prove that this partition is in fact a constructible regular stratification. So, we refer to it as the Brunovsky-Kronecker stratification of $\mathcal{M}_{n,m}$. Often, we will abbreviate Brunovsky-Kronecker as BK: BK-stratification, BK-reduced form,...

We recall that two pairs of matrices (A, B) , (A', B') of $\mathcal{M}_{n,m}$ are called *block-similar* and we write $(A, B) \sim (A', B')$ if there exist $P \in Gl(n)$, $Q \in Gl(m)$ and $R \in M_{n \times m}$ such that

$$(A', B') = P^{-1}(A, B) \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$$

It is an equivalence relation. It is well known that $(A, B) \sim (A', B')$ if and only if $(\lambda I + A, B)$ is strictament equivalent to $(\lambda I + A', B')$. However, notice that the block-similarity equivalence only involves the parameters defining the system.

Each pair $(A, B) \in \mathcal{M}_{n,m}$ can be reduced to its *BK-canonical form*

$$\left(\left(\begin{array}{cc} N & 0 \\ 0 & J \end{array} \right), \left(\begin{array}{cc} E & 0 \\ 0 & 0 \end{array} \right) \right)$$

where

a) $N = \text{diag}(N_1, \dots, N_r)$, N_i the nilpotent k_i -matrix
$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}$$

b) $E = \begin{pmatrix} E_1 & & & \\ & E_2 & & \\ & & \ddots & \\ & & & E_r \end{pmatrix}$, $E_i = (0 \dots 01)^t \in M_{k_i \times 1}$

c) J is a Jordan matrix; we shall write

$$J = \text{diag}(J_1, \dots, J_\nu), J_i = \text{diag}(J_{\sigma_1(i)}, J_{\sigma_2(i)}, \dots), \quad 1 \leq i \leq \nu,$$

being $J_{\sigma_j(i)}$ the $\sigma_j(i)$ -matrix

$$J_{\sigma_j(i)} = \begin{pmatrix} \lambda_i & 1 & 0 & \dots & 0 \\ 0 & \lambda_i & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 0 & 0 & 0 & \dots & \lambda_i \end{pmatrix}, \quad 1 \leq i \leq \nu, j = 1, 2, \dots$$

The integers k_i are called the *controllability indices* of (A, B) . We assume $k_1 \geq \dots \geq k_r$. We shall write $k = (k_1, \dots, k_r)$, $s = k_1 + \dots + k_r$. We call $\lambda_1, \dots, \lambda_\nu$ the *eigenvalues* of (A, B) . And we write $\sigma(i) = (\sigma_1(i), \sigma_2(i), \dots)$ for the Segre characteristic of λ_i and $\delta_i = \sigma_1(i) + \sigma_2(i) + \dots$. We assume $\sigma_1(i) \geq \sigma_2(i) \geq \dots$, for every $1 \leq i \leq \nu$. We recall (see [6]):

PROPOSITION 3.1.

(i) *The controllability indices of (A, B) , (k_1, \dots, k_r) are the conjugate partition of (r_1, \dots, r_k) where $r_i = \rho_i - \rho_{i-1}$, $i \geq 1$*

$$\rho_i = \text{rank} \begin{pmatrix} B & AB & \dots & A^{i-1}B \end{pmatrix}, \quad i \geq 1, \quad \rho_0 = 0$$

and k is the smallest integer such that $\rho_{k+1} = \rho_k$.

(ii) *The eigenvalues of (A, B) are the complex numbers λ for which there exist $v \in \mathbb{C}^n$, $v \neq 0$ such that $A^t v = \lambda v$, $B^t v = 0$. Or equivalently, such that $\text{rank}(A - \lambda I_n, B) < n$. For each eigenvalue λ , its Segre characteristic is the conjugate partition of $[n - \nu_1, \nu_1 - \nu_2, \dots]$ where*

$$\nu_j = \text{rank} \begin{pmatrix} B & AB & \dots & A^{k_1-1}B & (A - \lambda I_n)^j \end{pmatrix}$$

and k_1 is the smallest integer such that

$$\text{rank} \begin{pmatrix} B & AB & \dots & A^{k_1}B \end{pmatrix} = \text{rank} \begin{pmatrix} B & AB & \dots & A^{k_1-1}B \end{pmatrix}.$$

DEFINITION 3.2. *With the above notation, the BK-symbol (k, σ) of the pair (A, B) is formed by the controllability indices and the Segre symbol of J :*

$$(k, \sigma) = (k_1, \dots, k_r; \sigma(1), \dots, \sigma(\nu))$$

A BK-stratum, $E(k, \sigma)$, in $\mathcal{M}_{n,m}$ consists of all pairs of matrices having a given BK-symbol (k, σ) . We denote $E(A, B)$ the BK-stratum of the pair (A, B) . We denote Σ the partition $\cup_{k,\sigma} E(k, \sigma)$ of $\mathcal{M}_{n,m}$, which will be called the BK-stratification.

When we write (k, σ) we do not exclude the possibility that $(k, \sigma) = (k)$, that is to say, there is no Jordan matrix J or that $(k, \sigma) = (\sigma)$, which means that $B = 0$.

Notice that there are only finitely many BK-strata. Each one is an uncountable union of block-similarity equivalence classes, differing only in the values of the eigenvalues $\lambda_1, \dots, \lambda_\nu$.

4. The Orbits. If one considers the Lie subgroup of $Gl(n+m)$

$$\mathcal{G} = \left\{ \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} \mid P \in Gl(n), Q \in Gl(m), R \in M_{m \times n} \right\}$$

called the *state feedback group*, and its action in $\mathcal{M}_{n,m}$ defined by

$$\alpha \left(\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, (A, B) \right) = P^{-1}(A, B) \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$$

the equivalence class of a (A, B) with regard to the block-similarity is just its *orbit* under this action:

$$\mathcal{O}(A, B) = \{\alpha(g, (A, B)), g \in \mathcal{G}\}$$

Following the reasoning in [8], one can prove (see [5]):

PROPOSITION 4.1. *Orbits $\mathcal{O}(A, B)$ are constructible complex submanifolds of $\mathcal{M}_{n,m}$*

A local parametrization of $\mathcal{O}(A, B)$ at (A, B) can be obtained as follows: let $\mathcal{E}st(A, B)$ be the stabilizer of (A, B) under the action of α

$$\mathcal{E}st(A, B) = \{g \in \mathcal{G} \mid \alpha(g, (A, B)) = (A, B)\}$$

and let $V \subset \mathcal{G}$ a submanifold of \mathcal{G} minitransversal to $\mathcal{E}st(A, B)$ at the identity I . Then, it suffices to define $\alpha_1 : V \rightarrow \mathcal{O}(A, B)$ by $\alpha_1(g) = \alpha(g, (A, B))$.

PROPOSITION 4.2. ([5], (II.1.9)). *With the notations in section 1, the codimension of the orbit $\mathcal{O}(A, B)$ is:*

$$\begin{aligned} \text{codim } \mathcal{O}(A, B) &= r(n-s) + (m-r)(s-r) + (m-r)(n-s) + \\ &+ \sum_{1 \leq i, j \leq r} \max\{0, k_j - k_i - 1\} + \\ &+ \sum_{1 \leq i \leq \nu} (\sigma_1(i) + 3\sigma_2(i) + 5\sigma_3(i) + \dots) \end{aligned}$$

If $(A', B') \in \mathcal{O}(A, B)$, and we take $g_0 \in \mathcal{G}$ such that $\alpha(g_0, (A, B)) = (A', B')$, then the map $\alpha_2((X, Y)) = \alpha(g_0, (X, Y))$ gives the following homogeneity property which will be used to reduce our study to pairs of matrices in the BK-reduced form

PROPOSITION 4.3. *Given any two block-similar pairs of matrices (A, B) and (A', B') , there exists a diffeomorphism $\alpha_2 : \mathcal{M}_{n,m} \rightarrow \mathcal{M}_{n,m}$ which preserves the orbits and such that $\alpha_2(A, B) = (A', B')$.*

5. Local triviality along the orbits. A central point in the method of [8] is to reduce the study of the stratification to that of its intersection with a submanifold Γ transversal to the orbits. The key point is the selection of submanifold Γ , in order to have an appropriate description of its intersection with the BK-stratification (see (5.3)). In our case, we select Γ as the versal deformation constructed in [4].

The following lemma provides the desired local trivialisation along the orbits. It can be proved by means of the inverse function theorem (see, for example, (I.2.5) in [5]).

LEMMA 5.1. Let $(A, B) \in \mathcal{M}_{n,m}$. $\mathcal{O}(A, B)$ its orbit and $\Gamma = (A, B) + T$ a linear variety minitransversal to $\mathcal{O}(A, B)$:

$$\mathcal{M}_{n,m} = T \oplus T_{(A,B)}\mathcal{O}(A, B)$$

Let $\alpha_1 : V \rightarrow \mathcal{O}(A, B)$ be a local parametrization of $\mathcal{O}(A, B)$ at (A, B) , and denote by $\alpha_1(V)$ simply as $\mathcal{O}(A, B)$. Then the mapping

$$\begin{aligned} \beta : \Gamma \times \mathcal{O}(A, B) &\rightarrow \mathcal{M}_{n,m} \\ \beta((A, B) + (X, Y), (A', B')) &= \alpha(\alpha_1^{-1}(A', B'), (A, B) + (X, Y)) \end{aligned}$$

is a local diffeomorphism at $((A, B), J)$ which preserves the orbits.

In this paper, we will apply this lemma taking Γ to be the linear variety obtained in [5] (II.2.2). With the notations in section 1 we have:

THEOREM 5.2. Let (A, B) be a BK-matrix

$$(A, B) = \left(\left(\begin{array}{cc} N & 0 \\ 0 & J \end{array} \right), \left(\begin{array}{cc} E & 0 \\ 0 & 0 \end{array} \right) \right)$$

and let us consider the set T of pairs of matrices (X, Y) of the form:

$$X = \left(\begin{array}{cc} 0 & 0 \\ X_1^2 & X_2^2 \end{array} \right), \quad Y = \left(\begin{array}{cc} Y_1^1 & Y_2^1 \\ 0 & Y_2^2 \end{array} \right)$$

where the block-decomposition corresponds to that of (A, B) , and:

i) All the entries in X_1^2 are zero except

$$x_i^{s+1}, \dots, x_i^n, \quad i = 1, k_1 + 1, \dots, k_1 + \dots + k_{r-1} + 1$$

ii) The matrices X_2^2 are such that the variety $\Gamma_S(J) = J + \{X_2^2\}$ of M_{n-s} is the miniversal deformation of J given in ([1] (4.4)).

iii) All the entries in Y_1^1 are zero except

$$y_i^j, \quad 2 \leq i \leq r, \quad k_1 + \dots + k_{i-2} + k_i + 1 \leq j \leq k_1 + \dots + k_{i-2} + k_{i-1} - 1$$

(provided that $k_i \leq k_{i-1} - 2$).

iv) Y_2^1 is such that

$$y_{r+1}^i = \dots = y_m^i = 0, \quad i = k_1, k_1 + k_2, \dots, k_1 + \dots + k_r$$

v) All the entries in Y_2^2 are arbitrary.

Then $\Gamma = (A, B) + T$ verifies the hypothesis in lemma (3.1): $\mathcal{M}_{n,m} = T \oplus T_{(A,B)} \mathcal{O}(A, B)$.

For this particular variety Γ , it is easy to discuss how the BK-symbol of $(A, B) + (X, Y)$ varies according to the values of the entries of (X, Y) . For instance (see [4] for more details):

- i) If $X_2^2 \neq 0$, $X_1^2 = Y = 0$, then the controllability indices k_1, \dots, k_r are constant, but the eigenvalues of J or its Segre symbol can vary as in [1].
- ii) If $X_1^2 \neq 0$, $Y = 0$, then r is constant, but s increases
- iii) If $Y_2^1 \neq 0$ or $Y_2^2 \neq 0$, then r increases
- iv) If $Y_1^1 \neq 0$, $Y_2^1 = Y_2^2 = 0$, then r and s are constant, but k_1, \dots, k_r vary (in fact, the differences $k_i - k_{i+1}$ tend to decrease)

In particular:

PROPOSITION 5.3. *With the notation in (5.2):*

- a) If $(X, Y) \neq (0, 0)$, then $(A, B) + (X, Y)$ is not in $\mathcal{O}(A, B)$.
- b) $(A, B) + (X, Y)$ is in $E(A, B)$ if and only if $X_1^2 = Y = 0$, and $J + X_2^2$ has the same Segre symbol as J .

6. The Strata. Next, we are going to tackle the regularity of the BK-strata (6.2), and their dimension (6.3). As we have said above the lemma (5.1) plays an essential role. In fact, thanks to it we can reduce the problem to the intersection with the variety Γ in (5.2). Then, the result follows from the description of this intersection in (5.3), and the regularity of the Segre strata. Other properties of the BK-strata are presented in (6.4) and (6.5). From them, it is possible to give an autonomous proof of the regularity of the BK-strata (without using those of the Segre strata). Moreover, they are of interest in their own right.

LEMMA 6.1. Let be $(A, B) \in \mathcal{M}_{n,m}$, $\mathcal{O}(A, B)$ its orbit, $E(A, B)$ its stratum, and Γ as in (5.1). Then, in a neighbourhood of (A, B) , $E(A, B)$ is a submanifold of $\mathcal{M}_{n,m}$ if and only if $E(A, B) \cap \Gamma$ is.

Proof. Assume that $E(A, B)$ is regular at (A, B) . Being Γ transversal to $\mathcal{O}(A, B)$, it is also transversal to $E(A, B)$. Hence, $E(A, B) \cap \Gamma$ is regular at (A, B) .

Conversely, assume that $E(A, B) \cap \Gamma$ is regular at (A, B) . According to (5.1), we have

$$E(A, B) = \beta((E(A, B) \cap \Gamma) \times \mathcal{O}(A, B))$$

locally at (A, B) . Therefore, $E(A, B)$ is regular at (A, B) . \square

THEOREM 6.2. *Any BK-stratum is a submanifold of $\mathcal{M}_{n,m}$.*

Proof. Let be $(A, B) \in \mathcal{M}_{n,m}$, $\mathcal{O}(A, B)$ its orbit and $E(A, B)$ its BK-stratum. We must prove that $E(A, B)$ is regular at (A, B) .

Because of (4.3), we can assume that (A, B) is a BK-matrix:

$$(A, B) = \left(\left(\begin{array}{cc} N & 0 \\ 0 & J \end{array} \right), \left(\begin{array}{cc} E & 0 \\ 0 & 0 \end{array} \right) \right).$$

By (6.1) it is sufficient to prove that $E(A, B) \cap \Gamma$ is regular at (A, B) , where Γ is the particular one in (5.2). From (5.3) it follows that $E(A, B) \cap \Gamma$ is formed by pairs of the form

$$\left(\left(\begin{array}{cc} N & 0 \\ 0 & J + X_2^2 \end{array} \right), \left(\begin{array}{cc} E & 0 \\ 0 & 0 \end{array} \right) \right)$$

such that $J + X_2^2$ has the Segre symbol of J , or equivalently, such that $J + X_2^2$ belongs to the Segre stratum $E_S(J)$ of J .

Therefore, the mapping $\phi : M_{n-s} \rightarrow \mathcal{M}_{n,m}$ defined by

$$\phi(C) = \left(\left(\begin{array}{cc} N & 0 \\ 0 & C \end{array} \right), \left(\begin{array}{cc} E & 0 \\ 0 & 0 \end{array} \right) \right)$$

is a diffeomorphism such that

$$\phi(E_S(J) \cap \Gamma_S(J)) = E(A, B) \cap \Gamma$$

[8] proves that the Segre strata are regular. Hence, $E_S(J) \cap \Gamma_S(J)$ is regular at J , (we recall that $\Gamma_S(J)$ is a linear variety transversal to the Segre orbit of J , and hence also transversal to $E_S(J)$ at J), and the proof is completed. \square

PROPOSITION 6.3. *Let be $(A, B) \in \mathcal{M}_{n,m}$, $\mathcal{O}(A, B)$ its orbit and $E(A, B)$ its stratum. Then:*

$$\dim E(A, B) = \nu + \dim \mathcal{O}(A, B)$$

where ν is the number of distinct eigenvalues of (A, B) .

Proof. It is sufficient to bear in mind that, in the above proof, $\dim(E_S(J) \cap \Gamma_S(J)) = \nu$ (see [1]). \square

The next result is needed for the proof of lemma (7.4) which it is a crucial step in the proof of the main theorem of this paper.

PROPOSITION 6.4. *The BK-strata are constructible connected sets.*

Proof. Let $E(k, \sigma)$ be the BK-stratum corresponding to the BK-symbol $(k, \sigma) = (k, \sigma_1, \dots, \sigma_\nu)$. Let us consider the set $\mathbb{C}^{(\nu)} = \{(\lambda_1, \dots, \lambda_\nu) \mid \lambda_i \neq \lambda_j\} \subset \mathbb{C}^\nu$. For each $(\lambda_1, \dots, \lambda_\nu) \in \mathbb{C}^{(\nu)}$, let be $BK(k, \sigma; (\lambda_1, \dots, \lambda_\nu))$ the BK-matrix of $E(k, \sigma)$ having eigenvalues $\lambda_1, \dots, \lambda_\nu$. Finally, let us consider the mapping $\psi : \mathcal{G} \times \mathbb{C}^{(\nu)} \rightarrow \mathcal{M}_{n,m}$ defined by

$$\psi(g, (\lambda_1, \dots, \lambda_\nu)) = \alpha(g, BK(k, \sigma; (\lambda_1, \dots, \lambda_\nu)))$$

Obviously, $\mathcal{G} \times \mathbb{C}^{(\nu)}$ is a constructible set, ψ is a rational map, and $\psi(\mathcal{G} \times \mathbb{C}^{(\nu)}) = E(k, \sigma)$, so that, according to the Chevalley theorem, $E(k, \sigma)$ is a constructible set. Moreover, it is connected because ψ is continuous and $\mathcal{G} \times \mathbb{C}^{(\nu)}$ is connected. \square

PROPOSITION 6.5. *The action of \mathbf{C} in $\mathcal{M}_{n,m}$ defined by $(\lambda, (A, B)) \longrightarrow (A + \lambda I, B)$ preserves the BK-strata. That is to say, if $E(k, \sigma)$ is a BK-stratum, $(A, B) \in E(k, \sigma)$ and $\lambda \in \mathbb{C}$, then:*

- 1) μ is an eigenvalue of (A, B) if and only if $\mu + \lambda$ is an eigenvalue of $(A + \lambda I, B)$.
- 2) $(A + \lambda I, B) \in E(k, \sigma)$.

Proof. 1) According to (3.1), μ is an eigenvalue of (A, B) if and only if $\text{rank}(A - \mu I, B) < n$. And obviously it is equivalent to $\text{rank}((A + \lambda I) - (\mu + \lambda)I, B) < n$.

2) To see that (A, B) and $(A + \lambda I, B)$ have the same controllability indices, it is sufficient to prove that

$$\text{rank}(B, AB, A^2B, \dots, A^iB) = \text{rank}(B, (A + \lambda I)B, \dots, (A + \lambda I)^iB)$$

for any $1 \leq i \leq n$. And this follows immediately from the fact that $(A + \lambda I)^j B$ is a linear combination of $B, AB, \dots, A^j B$, for any j .

Finally, to see that (A, B) and $(A + \lambda I, B)$ have the same Segre symbol, it is sufficient to prove that

$$\begin{aligned} \text{rank}(B, AB, \dots, A^{k_1-1}B, (A - \mu I)^j) &= \\ &= \text{rank}(B, (A + \lambda I)B, \dots, (A + \lambda I)^{k_1-1}B, ((A + \lambda I) - (\mu + \lambda)I)^j) \end{aligned}$$

for any $1 \leq j \leq n$. And we argue as above to conclude the proof. \square

7. Regularity of the BK-stratification ($m = 1$). Finally, we are going to prove that the BK-stratification $\Sigma = \cup_{\sigma} E(\sigma)$ is Whitney regular (7.5) when $m = 1$. As for the regularity of the BK-strata. Again, the key point is the application of lemma (5.1) to prove (7.1) which reduces the problem to the stratification induced by the intersection with the variety Γ in (5.2). But now this induced stratification is not a Segre stratification, so that we cannot conclude simply by using the results in [8]. Previously we will study the particular case of strata called simple (7.2); since they have a particular homogeneity property (7.3), the regularity over them follows from the Whitney theorem (7.4). The general case can be viewed as a product of simple strata, because of the explicit descriptions in (5.3). In addition, we remark (7.6) that for any $m \geq 1$ Σ satisfies the frontier condition.

Given $(A, B) \in \mathcal{M}_{n,m}$, and Γ as in (5.1), let us consider the induced stratification $\cup_{k,\sigma} (E(k, \sigma) \cap \Gamma)$ of a sufficiently small neighbourhood in Γ of (A, B) , which we denote $\Sigma \cap \Gamma$. Notice that it is well defined because Γ is not only transversal to $\mathcal{O}(A, B)$, but also to every orbit sufficiently close to (A, B) (see (5.1)), and hence to every BK-stratum sufficiently close to (A, B) .

LEMMA 7.1. With the above notation, Σ is Whitney regular over $E(A, B)$ at (A, B) if and only if $\Sigma \cap \Gamma$ is Whitney regular over $E(A, B) \cap \Gamma$ at (A, B) .

Proof. The necessity follows (see [8] I (1.4)) from the transversality of Γ to every stratum sufficiently close to (A, B) , which we have remarked above. For the sufficiency, we use again (5.1):

$$E(k, \sigma) = \beta((E(k, \sigma) \cap \Gamma) \times \mathcal{O}(A, B))$$

locally in (A, B) . Obviously, if $\cup_{k, \sigma}(E(k, \sigma) \cap \Gamma)$ is Whitney regular over $E(A, B) \cap \Gamma$ at (A, B) , the product stratification $\cup_{k, \sigma}((E(k, \sigma) \cap \Gamma) \times \mathcal{O}(A, B))$ is also Whitney regular over $(E(A, B) \cap \Gamma) \times \mathcal{O}(A, B)$ at (A, B) . And, by means of the above diffeomorphism β , the proof is finished. \square

DEFINITION 7.2. A pair of matrices $(A, B) \in \mathcal{M}_{n, m}$ is called simple if it has one eigenvalue, at most. A stratum $E(k, \sigma)$ is called simple if its elements are simple.

The simple strata verify a particular homogeneity property, in some sense the converse to the one in (6.4):

PROPOSITION 7.3. Let $E(k, \sigma)$ be a simple stratum. For any $(A, B), (A', B') \in E(k, \sigma)$, there exists a diffeomorphism f of $\mathcal{M}_{n, m}$, preserving strata, and such that: $f(A, B) = (A', B')$.

Proof. If $(k, \sigma) = (k)$, it is trivial. If $(k, \sigma) = (k, \sigma(1))$ or $(k, \sigma) = (\sigma(1))$, let λ, λ' be the eigenvalues of (A, B) and (A', B') respectively. Then, because of (6.5), the pair $(A + (\lambda' - \lambda)I, B)$ is block similar to (A', B') . Hence, there exists $g_0 \in \mathcal{G}$ such that $(A', B') = \alpha(g_0, (A + (\lambda' - \lambda)I, B))$. It is straightforward that the mapping

$$f(X, Y) = \alpha(g_0, (X + (\lambda' - \lambda)I, Y))$$

verifies the desired conditions. \square

The Whitney theorem states that any stratum of a constructible locally finite stratification has a Whitney regular point. Hence, any stratum $E(k, \sigma)$ has a point (A, B) such that Σ is Whitney regular over $E(k, \sigma)$ at (A, B) . In the particular case where $E(k, \sigma)$ is simple, the above homogeneity property implies that all the points of $E(k, \sigma)$ are Whitney regular. Therefore, we have the following lemma, as a first step to tackle the main theorem:

LEMMA 7.4. Σ is Whitney regular over any simple BK-stratum.

THEOREM 7.5. If $m = 1$ the Brunovsky-Kronecker stratification is Whitney regular.

Proof. Let be $(A, B) \in \mathcal{M}_{n, 1}$, $\mathcal{O}(A, B)$ its orbit and $E(A, B)$ its stratum. We shall prove that Σ is Whitney regular over $E(A, B)$ at (A, B) . Let Γ be the linear variety defined in (5.2), and let us denote with the same symbol the neighbourhood of (A, B) where the isomorphism in (5.1) holds. Then, according to (7.1), it is sufficient to prove that $\Sigma \cap \Gamma$ is Whitney regular over $E(A, B) \cap \Gamma$ at (A, B) .

Because of (4.3), we can assume that (A, B) is a BK-matrix. Firstly, we suppose that $B \neq 0$. We will discuss later the case $B = 0$. Since $B \neq 0$ we have that

$$(A, B) = \left(\left(\begin{array}{cc} N & 0 \\ 0 & J \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right), J = \text{diag}(J_1, \dots, J_\nu),$$

as in (3.2). Remark that N has only one block and $E = (0 \ \dots \ 0 \ 1)^t \in M_{s \times 1}$.

Let us consider the pairs of matrices:

$$(A_i, B_i) = \left(\left(\begin{array}{cc} N & 0 \\ 0 & J_i \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right) \in \mathcal{M}_{s+\delta_i, 1} = M_{s+\delta_i} \times M_{(s+\delta_i) \times 1}$$

for $1 \leq i \leq \nu$. And let $\Sigma_1, \dots, \Sigma_\nu$ be the BK-stratifications of the respective spaces of matrices $\mathcal{M}_{s+\delta_1, 1}, \dots, \mathcal{M}_{s+\delta_\nu, 1}$. If we denote $E_i(A_i, B_i)$ the stratum of (A_i, B_i) in Σ_i , $0 \leq i \leq \nu$, from the above lemma (7.4), it follows that Σ_i is Whitney regular over $E_i(A_i, B_i) \cap \Gamma_i$ at (A_i, B_i) , where $\Gamma_1, \dots, \Gamma_\nu$ are the respective linear varieties defined as in (5.2):

$$\Gamma_i = (A_i, B_i) + \left\{ \left(\begin{array}{cc} 0 & 0 \\ X_1^2(i) & X_2^2(i) \end{array} \right), \left(\begin{array}{c} 0 \\ 0 \end{array} \right) \right\}, \quad 1 \leq i \leq \nu$$

As above, we denote with the same symbol Γ_i the neighbourhoods of (A_i, B_i) where the corresponding isomorphisms in (5.1) hold, so that the induced stratifications $\Sigma_i \cap \Gamma_i$, $0 \leq i \leq \nu$, are well defined. In fact, according to (7.1), $\Sigma_i \cap \Gamma_i$ is Whitney regular over $E_i(A_i, B_i) \cap \Gamma_i$ at (A_i, B_i) , $0 \leq i \leq \nu$.

Now, let us consider the diffeomorphism

$$\varphi : \Gamma_1 \times \dots \times \Gamma_\nu \longrightarrow \Gamma$$

$$\begin{aligned} \varphi \left(\left(\left(\begin{array}{cc} N & 0 \\ X_1 & H_1 \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right), \dots, \left(\left(\begin{array}{cc} N & 0 \\ X_\nu & H_\nu \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right) \right) = \\ = \left(\left(\begin{array}{cccc} N & 0 & \dots & 0 \\ X_1 & H_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ X_\nu & 0 & \dots & H_\nu \end{array} \right), \left(\begin{array}{c} E \\ 0 \\ \vdots \\ 0 \end{array} \right) \right) \end{aligned}$$

where $X_i = X_1^2(i)$ and $H_i = J_i + X_2^2(i)$, $1 \leq i \leq \nu$.

We know that $\Pi = (\Sigma_1 \cap \Gamma_1) \times \dots \times (\Sigma_\nu \cap \Gamma_\nu)$ is a stratification which is Whitney regular over the stratum $(E_1(A_1, B_1) \cap \Gamma_1) \times \dots \times (E_\nu(A_\nu, B_\nu) \cap \Gamma_\nu)$ at $((A_1, B_1), \dots, (A_\nu, B_\nu))$, ([8] (1.2)). Hence, because φ is a diffeomorphism, to conclude the proof it is sufficient to show that φ preserves strata locally at (A, B) .

That is to say, given two points $z = \left(\left(\begin{array}{cc} N & 0 \\ X_i & H_i \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right)_{1 \leq i \leq \nu}$ and $z' =$

$\left(\left(\begin{array}{cc} N & 0 \\ X'_i & H'_i \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right)_{1 \leq i \leq \nu}$ belonging to the same stratum of Π , then the images $\varphi(z)$, $\varphi(z')$ belong to the same stratum of $\Sigma \cap \Gamma$, provided that they are sufficiently close to (A, B) .

To prove this, we are going to see that the controllability indices and the Segre symbol of $\varphi(z)$ and $\varphi(z')$ are the same.

(i) In order to prove the first assertion it is sufficient to show that if we write $\varphi(z) = (C, D)$

$$C = \begin{pmatrix} N & 0 & \dots & 0 \\ X_1 & H_1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ X_\nu & 0 & \dots & H_\nu \end{pmatrix}, \quad D = \begin{pmatrix} E \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and $\varphi(z') = (C', D)$, then

$$\begin{aligned} r_j &= \text{rank} (D, CD, \dots, C^{j-1}D) = \\ &= \text{rank} (D, C'D, \dots, C'^{j-1}D) = r'_j, \quad 1 \leq j \leq n. \end{aligned}$$

A simple computation shows that $r_j = r'_j = j$ for $1 \leq j \leq s$. Thus, we can assume $j > s$.

If we denote Y_i the first column of the matrix X_i , $1 \leq i \leq \nu$, it is easy to check that

$$r_j = \text{rank} \begin{pmatrix} E & NE & \dots & N^{s-1}E & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & Y_1 & H_1 Y_1 & \dots & H_1^\ell Y_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0 & Y_\nu & H_\nu Y_\nu & \dots & H_\nu^\ell Y_\nu \end{pmatrix}$$

where $\ell = j - s - 1$.

We remark that, because z and z' belong to the same stratum, one has $Y_i = 0$ if and only if $Y'_i = 0$. Hence, if $Y_i = 0$ for $1 \leq i \leq \nu$ it is obvious that $r_j = r'_j$ for $s + 1 \leq j \leq n$. Consequently, we assume now that $Y_i \neq 0$ for some i . Renumbering, if it is necessary, we can suppose that $Y_i \neq 0$ (hence $Y'_i \neq 0$) for $1 \leq i \leq t$, $t \leq \nu$.

Since $r_j \leq r_{j+1} \leq r_j + 1$, in order to see that $r_j = r'_j$ for $s < j \leq n$ it is enough to prove that if j is the first index such that $r_j = r_{j-1}$, index j has also the same property with regard to r'_j .

Therefore, let $j = s + \ell + 1$, be the first index such that

$$r_j = r_{j-1}$$

Then, there exist a_k , $0 \leq k \leq \ell - 1$, satisfying

$$\begin{pmatrix} H_1^\ell Y_1 \\ \vdots \\ H_t^\ell Y_t \end{pmatrix} + a_{\ell-1} \begin{pmatrix} H_1^{\ell-1} Y_1 \\ \vdots \\ H_t^{\ell-1} Y_t \end{pmatrix} + \dots + a_0 \begin{pmatrix} Y_1 \\ \vdots \\ Y_t \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and

$$\psi(x) = a_0 + \dots + a_{\ell-1} x^{\ell-1} + x^\ell$$

is the minimal polynomial of the vector $Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_t \end{pmatrix}$ with regard to the matrix $H = \text{diag}(H_1, \dots, H_t)$.

Now, let $\varphi_i(x)$ be the minimal polynomial of H_i , and $\psi_i(x)$ the minimal polynomial of Y_i , with regard to H_i , for $1 \leq i \leq t$. Making the transversal Γ_i still a bit smaller, if necessary, we can assume that $\text{g.c.d.}(\varphi_i(x), \varphi_k(x)) = 1$ for $i \neq k$. We are going to see that $\psi(x) = \psi_1(x) \dots \psi_t(x)$. In fact, since $\varphi_i(H_i)(Y_i) = 0$ one has that $\psi_i(x) \mid \varphi_i(x)$ for $1 \leq i \leq t$, so that $\text{g.c.d.}(\psi_i(x), \psi_k(x)) = 1$ if $i \neq k$. On the other hand, as $\psi(Y) = 0$, we have $\psi(Y_i) = 0$; hence, $\psi_i(x) \mid \psi(x)$ for $1 \leq i \leq t$. Consequently, $\psi_1(x) \dots \psi_t(x) \mid \psi(x)$. Conversely, it is easy to check that $\psi_1(H) \dots \psi_t(H)(Y) = 0$, so that $\psi(x) \mid \psi_1(x) \dots \psi_t(x)$, and the claimed equality holds.

Next, if we denote $\text{deg} \psi_i(x) = m_i$ it is obvious that

$$\ell = m_1 + \dots + m_t.$$

But m_i is the minimal integer such that

$$\begin{aligned} & \text{rank} \begin{pmatrix} E & NE & \dots & N^{s-1}E & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & Y_i & H_i Y_i & \dots & H_i^{m_i} Y_i \end{pmatrix} = \\ = & \text{rank} \begin{pmatrix} E & NE & \dots & N^{s-1}E & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & Y_i & H_i Y_i & \dots & H_i^{m_i-1} Y_i \end{pmatrix} = s + m_i \end{aligned}$$

Hence, taking into account that $\left(\begin{pmatrix} N & 0 \\ X_i & H_i \end{pmatrix}, \begin{pmatrix} E \\ 0 \end{pmatrix} \right)$ and $\left(\begin{pmatrix} N & 0 \\ X'_i & H'_i \end{pmatrix}, \begin{pmatrix} E \\ 0 \end{pmatrix} \right)$ belong to the same stratum, it follows that the minimal polynomial of Y'_i (first column of X'_i) has also degree m_i , $i = 1 \dots t$. Therefore, the degree of the minimal

polynomial of $Y' = \begin{pmatrix} Y'_1 \\ \vdots \\ Y'_t \end{pmatrix}$ is $m_1 + \dots + m_t = \ell$ and the first assertion above is proved.

(ii) We must now prove that the Segre symbols of $\varphi(z)$ and $\varphi(z')$ are the same.

Firstly, we remark that μ is an eigenvalue of $\varphi(z) = (C, D)$ if and only if it is an eigenvalue of $\left(\begin{pmatrix} N & 0 \\ X_i & H_i \end{pmatrix}, \begin{pmatrix} E \\ 0 \end{pmatrix} \right)$, for some $1 \leq i \leq \nu$. To see this, it is sufficient to bear in mind the characterization in (3.1): μ is an eigenvalue of $\varphi(z) = (C, D)$ if and only if $\text{rank}(C - \mu I, D) < n$; since $\text{rank} \left(\begin{pmatrix} N & 0 \end{pmatrix} (E) \right) = s$, it means that $\text{rank} \left(\begin{pmatrix} X_i & H_i - \mu I \end{pmatrix} (0) \right) < \delta_i$, for some $1 \leq i \leq \nu$; hence $\text{rank} \left(\begin{pmatrix} N - \mu I & 0 \\ X_i & H_i - \mu I \end{pmatrix} \begin{pmatrix} E \\ 0 \end{pmatrix} \right) < s + \delta_i$, so that μ is an eigenvalue of $\left(\begin{pmatrix} N & 0 \\ X_i & H_i \end{pmatrix}, \begin{pmatrix} E \\ 0 \end{pmatrix} \right)$. And the same is true conversely.

On the other hand, since $\left(\begin{pmatrix} N & 0 \\ X_i & H_i \end{pmatrix}, \begin{pmatrix} E \\ 0 \end{pmatrix} \right)$ and $\left(\begin{pmatrix} N & 0 \\ X'_i & H'_i \end{pmatrix}, \begin{pmatrix} E \\ 0 \end{pmatrix} \right)$ are in the same stratum, there is an eigenvalue μ' of the last pair with the same Segre

characteristic as that of μ . As we have just seen, μ' is also an eigenvalue of $\varphi(z')$. Then, taking into account (3.1) it can be checked that μ and μ' have the same Segre characteristic, and the theorem is completely proved if $B \neq 0$.

Now, we suppose that $B = 0$. In this case we have to make the following changes in the first part of the proof

$$\begin{aligned} & \cdot (A, B) = (J, 0) \\ & \cdot (A_i, B_i) = (J_i, 0) \\ & \cdot \Gamma_i = (J_i, 0) + \{(X_2^2(i), Y_2^2(i))\} \\ & \cdot \varphi((H_1, Z_1), \dots, (H_\nu, Z_\nu)) = \left(\left(\begin{array}{ccc} H_1 & & \\ & \ddots & \\ & & H_\nu \end{array} \right) \left(\begin{array}{c} Z_1 \\ \vdots \\ Z_\nu \end{array} \right) \right), \text{ where } Z_i = \\ & \quad Y_2^2(i). \\ & \cdot Z = ((H_1, Z_i))_{1 \leq i \leq \nu}, Z' = ((H'_1, Z'_i))_{1 \leq i \leq \nu} \end{aligned}$$

Then, the proof of (i) is quite similar to the former case with

$$C = \left(\begin{array}{ccc} H_1 & & \\ & \ddots & \\ & & H_\nu \end{array} \right), \quad D = \left(\begin{array}{c} Z_1 \\ \vdots \\ Z_\nu \end{array} \right).$$

As for (ii), the reasoning is also the same: μ is an eigenvalue of (C, D) if and only if it is an eigenvalue of (H_i, Z_i) for some i , (as in the former case we have to bear in mind that (H_i, Z_i) belongs to a neighbourhood of $(J_i, 0)$ and that J_1, \dots, J_ν have different eigenvalues). The theorem now is completely proved. \square

We recall that a stratification Σ satisfies the *frontier condition* if for any strata $E, E' \in \Sigma$ such that $\overline{E} \cap E' \neq \emptyset$ then $\overline{E} \supset E'$. That is to say, the frontier of a stratum is a union of strata. It follows from (5.5), (4.4) and [9] that the Brunovsky-Kronecker stratification satisfies the frontier condition, if $m = 1$. In fact, the theorems (4.7) and (5.6) of [11] show that this condition holds for all values of m . Therefore, we have:

PROPOSITION 7.6. *The BK-stratification verifies the frontier condition.*

REMARK 7.7. Notice that the limitation $m = 1$ in the above theorem is only used in (7.5 (i)). We conjecture that the theorem is also true for $m > 1$, but in this case our approach should be modified because Y is not a column matrix. Unfortunately, until now we have not succeeded in finding an alternative approach which works in the general case.

8. Bifurcation diagrams. The space $\mathcal{M}_{n,1}$ is equipped with a Whitney stratification, so that we can make use of the Thom transversality theorem: let W be a d -dimensional manifold, and $C^\infty(W, \mathcal{M}_{n,1})$ the space of the W -parametrized differentiable families of pairs of matrices of $\mathcal{M}_{n,1}$; that is to say, the differentiable maps $\varphi : W \rightarrow \mathcal{M}_{n,1}$; then the subset formed by the ones transversal to the BK-stratification is open and dense. These families will be called “generic”.

Let $\varphi : W \rightarrow \mathcal{M}_{n,1}$ be a generic family. For all $E(k, \sigma) \subset \mathcal{M}_{n,1}$, $\varphi^{-1}(E(k, \sigma))$

is a submanifold of W and

$$\text{codim } \varphi^{-1}(E(k, \sigma)) = \text{codim } E(k, \sigma).$$

Moreover, $\cup \varphi^{-1}(E(k, \sigma))$ is a Whitney regular stratification for W , called the *bifurcation diagram* of the (generic) family.

The local description at $w \in W$ of the bifurcation diagram can be derived from the one of a versal deformation of $(A, B) = \varphi(w)$. Because of (4.3), we can assume (A, B) to be a BK-matrix, and then consider its versal deformation Γ in (5.2).

REMARK 8.1. Notice that the linear subvariety of Γ defined by $\text{trace } X_i = 0$, for all diagonal blocks X_i of X_2^2 , is transversal to $E(A, B)$ at (A, B) (but not to $\mathcal{O}(A, B)$).

EXAMPLE 8.2. As an example, we enumerate the singularities of bifurcation diagrams of few-parameter generic families with $B \neq 0$.

For the comodity of the reader we write down the codimension of $E(A, B)$ for $m = 1$ (see (4.2) and (6.3))

$$\text{codim } E(A, B) = n - k_1 + \sum_{1 \leq i \leq \nu} (\sigma_1(i) + 3\sigma_2(i) + 5\sigma_3(i) + \dots) - \nu$$

Notice that $\text{codim } E(A, B) = 0$ if and only if $n = k_1$, which corresponds to the open and dense subset of \mathcal{M}_{n-1} formed by the pairs (A, B) equivalent to the controllable pair

$$(8.1) \quad \left(\left(\begin{array}{cc} 0 & I_{n-1} \\ 0 & 0 \end{array} \right), \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \right)$$

Usually, it is also called bifurcation diagram of φ the variety $\cup \varphi^{-1}(E(k, \sigma))$ with $\text{codim } E(A, B) > 0$.

(i) *One-parameter families.*

The only stratum of codimension one in the space $\mathcal{M}_{n,1}$ is the one formed by the pairs of matrices equivalent to

$$(8.2) \quad (A, B) = \left(\left(\begin{array}{cc} N & 0 \\ 0 & \alpha \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right)$$

where N is the nilpotent $(n-1)$ -matrix, and $\alpha \in \mathbb{C}$. A transversal family is (see (8.1))

$$(8.3) \quad \left(\left(\begin{array}{cc} N & 0 \\ X & \alpha \end{array} \right), \left(\begin{array}{c} E \\ 0 \end{array} \right) \right)$$

with $X = \begin{pmatrix} x & 0 & \dots & 0 \end{pmatrix} \in M_{1 \times (n-1)}(\mathbb{C})$. One checks easily that for $x \neq 0$ the above pair (8.3) is equivalent to the pair (8.1). Therefore, the bifurcation diagram for a generic 1-parameter family is $x = \{0\}$.

Notice that x is a complex number so that $\mathbb{C} - \{0\}$ is connected.

So, for a generic 1-parameter family there are only pairs equivalent to (8.1) and for an isolated value of the parameter a pair equivalent to (8.2).

(ii) *Two-parameter families.*

The only stratum of codimension two is the one formed by the pairs of matrices equivalent to

$$(8.4) \quad \left(\left(\begin{array}{ccc} N & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{array} \right), \left(\begin{array}{c} E \\ 0 \\ 0 \end{array} \right) \right)$$

where N is the nilpotent $(n-2)$ -matrix, and $\alpha, \beta \in \mathbb{C}$. A transversal family is

$$(8.5) \quad \left(\left(\begin{array}{ccc} N & 0 & 0 \\ X & \alpha & 0 \\ Y & 0 & \beta \end{array} \right), \left(\begin{array}{c} E \\ 0 \\ 0 \end{array} \right) \right)$$

with $X = (x \ 0 \ \dots \ 0) \in M_{1 \times (n-2)}(\mathbb{C})$, $Y = (y \ 0 \ \dots \ 0) \in M_{1 \times (n-2)}(\mathbb{C})$.

Then, it is easily seen that for a pair of (8.5) there are only the following possibilities: if $x \neq 0$ and $y \neq 0$, (8.5) is equivalent to (1); if $xy = 0$, but $x \neq 0$ or $y \neq 0$, (8.5) is equivalent to (8.2). So, the bifurcation diagram for a generic 2-parameter family is the usual stratification associated to $xy = 0$, that is to say

$$\{x = y = 0\} \cup \{xy = 0, x \neq 0 \text{ or } y \neq 0\} \cup \{x \neq 0, y \neq 0\}$$

Therefore, for a generic 2-parameter family there are only three possibilities:

- The pairs corresponding to points outside the axes are equivalent to (8.1).
- To the points of the axes, except 0, correspond pairs equivalent to (8.2).
- To the origin 0 corresponds a pair equivalent to (8.4).

(iii) *Three-parameter families.*

There are two strata of codimension three:

(a) the one corresponding to

$$(8.6) \quad \left(\left(\begin{array}{cccc} N & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \gamma \end{array} \right), \left(\begin{array}{c} E \\ 0 \\ 0 \\ 0 \end{array} \right) \right)$$

where N is the nilpotent $(n-3)$ -matrix, and $\alpha, \beta, \gamma \in \mathbb{C}$.

(b) the one corresponding to

$$(8.7) \quad \left(\left(\begin{array}{ccc} N & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & \alpha \end{array} \right), \left(\begin{array}{c} E \\ 0 \\ 0 \end{array} \right) \right)$$

where N is the nilpotent $(n-2)$ -matrix, and $\alpha, \beta \in \mathbb{C}$.

Notice that the last one is the well-known umbrella singularity introduced by Whitney.

These diagrams are stratified in the following way

$$\begin{aligned}
\text{(a)} \quad & \underbrace{\{x = y = z = 0\}}_{(\alpha)} \cup \underbrace{\{x = y = 0, z \neq 0\} \cup \{x = z = 0, y \neq 0\} \cup \{y = z = 0, x \neq 0\}}_{(\beta)} \\
& \cup \underbrace{\{x = 0, y \neq 0, z \neq 0\} \cup \{y = 0, x \neq 0, x \neq 0\} \cup \{z = 0, x \neq 0, y \neq 0\}}_{(\gamma)} \\
& \cup \underbrace{\{x \neq 0, y \neq 0, z \neq 0\}}_{(\delta)}. \\
\text{(b)} \quad & \underbrace{\{x = y = z = 0\}}_{(\alpha)} \cup \underbrace{\{zx^2 - y^2 = 0, x \neq 0 \text{ or } y \neq 0\}}_{(\beta)} \cup \underbrace{\{x = y = 0, z \neq 0\}}_{(\gamma)} \\
& \cup \underbrace{\{zx^2 - y^2 \neq 0\}}_{(\delta)}.
\end{aligned}$$

to each point of the stratum α , β , γ , δ corresponds, equivalently, a pair equivalent to (8.1), (8.2), (8.4) or (8.6) for (a) or (8.1), (8.2), (8.4) or (8.7) for (b).

Therefore, a generic 3-parameter family only contains pairs equivalents to the mentioned above. Notice, moreover, that the bifurcation diagram tell us how these pairs are distributed.

REMARK 8.3.

- (i) We know that the Whitney regular condition implies that a small perturbation of a generic parametrized family will not affect the structure of the family. We think that it would be very interesting to deduce from the stratification a numerical knowledge of how far is the family of a change of structure.
- (ii) It is also worth mentioning that a generic d -parameter family cannot contain any structures of codimension $d + 1$ or higher, but it may not contain any structures of codimension d either, if the family is overparametrized. For example, the 2-parameter family (clearly overparametrized)

$$\left(\left(\begin{array}{cc} 0 & 0 \\ x + y & \alpha \end{array} \right) \left(\begin{array}{c} 1 \\ 0 \end{array} \right) \right)$$

only contains structures of codimension 0 or 1.

- (iii) There are several reasons to include the local finiteness condition in the definition of a stratification. First of all, if we remove this condition, we do not know if the fundamental theorems of Thom about stratification, such

as transversality, isotopy, ... hold. Besides, notice that without that condition the bifurcation diagram would have infinitely many components so that when we perturb a family we can get infinitely many possibilities. This is poor information. Also, if the union of strata of codimension d is a manifold of codimension $d-1$, the generic families would not be transversal to the strata. Consider, for example, a disc in \mathbb{C} which is the union of all its interior points.

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