A Fibonacci sequence for linear structures with two types of components*

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Abstract

We investigate binary voting systems with two types of voters and a hierarchy among the members in each type, so that members in one class have more influence or importance than members in the other class. The purpose of this paper is to count, up to isomorphism, the number of these voting systems for an arbitrary number of voters. We obtain a closed formula for the number of these systems, this formula follows a Fibonacci sequence with a smooth polynomial variation on the number of voters.

Keywords: Binary voting systems; Simple games; Two types of voters; Fibonacci sequence.

1 Introduction

We consider voting systems in which each player casts a "yes" or "no" vote, and the outcome is a collective "yes" or "no". These voting systems, known in the literature as simple games, can be very complicated. Their specialization to *symmetric* simple games, however, are simple indeed; each such game corresponds to the qualified majority rule, in which an issue is passed if and only if the number of voters in favour meets or exceeds some *threshold* or *quota*. We refer, as in [5], to this result as *May's Theorem for Simple Games* "with bias"[1]. Thus, May's Theorem with bias may be stated as follows: for each positive integer n, there is, up to isomorphism, a unique simple game that is anonymous or symmetric (i.e., voters play an equivalent rôle in the game). At least two facts are relevant of this result:

1. all symmetric games are weighted,[2]
2. the function S on the number of voters n that counts all these games is, up to isomorphism, the identity, i.e., $S(n) = n$.

There seems to be only one natural direction in which to extend symmetric games[3] within simple games. To accommodate this new class of voting systems, we make two changes to

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[1]Because May's original result (see [8] or [9]) considers only anonymous voting systems (for two alternatives) that are *neutral*: there is no built-in bias towards "yes" or "no" outcomes, so that they are treated symmetrically.

[2]Loosely speaking, a game is weighted if it can be assigned a quota and a weight for each voter, so that winning coalitions are those with the sum of the weights of their members greater than or equal to the quota.

[3]Symmetric games are also called q–out–of–n games, understanding that n is the number of voters, a weight of 1 is assigned to each voter, and the quota q is a fixed integer between 1 and n.

the symmetric simple games: we allow *two* classes of symmetric voters instead of one, and we consider that voters in one class are more influential or important than voters in the other class. The goal of this paper is to analyze the joint effect of these two changes. What voting systems are possible? How many of them are these, up to isomorphism, for a fixed number of voters?

As we shall see in this paper, the obtained results are paradoxically opposed to those described above for symmetric games, which demonstrates that the complexity of these close voting systems is considerably higher than that of symmetric games. Indeed, we will prove that the number of them is $F(n+6)-(n^2+4n+8)$, where $F(n)$ are the Fibonacci numbers[4], being n the number of voters. Hence, the number of these games is asymptotically exponential, which contrasts with the linear behavior of symmetric games.

The paper is organized as follows. Basic definitions and preliminary results are included in Section 2. Section 3 contains the main result of the paper, which is devoted to count the number of voting systems in terms of the number of voters, resulting a closed formula with asymptotic exponential behavior.

2 Preliminaries

Simple games can be viewed as models of voting systems in which a single alternative, such as a bill or an amendment, is pitted against the status quo.

Definition 2.1 *A* simple game *is a pair (N, W) in which $N = \{1, 2, \ldots, n\}$, and W is a collection of subsets of N that satisfies $N \in W$, $\emptyset \notin W$ and the monotonicity property: if $S \in W$ and $S \subseteq T \subseteq N$, then $T \in W$.*

Any set of voters is called a *coalition*, and the set N is called the *grand coalition*. Members of N are called *players* or *voters*, and the subsets of N that are in W are called *winning coalitions*. The subfamily of *minimal* winning coalitions $W^m = \{S \in W : T \subset S \Rightarrow T \notin W\}$ determines the game. The subsets of N that are not in W are called *losing coalitions*. The subfamily of *maximal* losing coalitions is $L^M = \{S \in L : S \subset T \Rightarrow T \in W\}$. A voter i is *null* in (N, W) if $i \notin S$ for all $S \in W^m$. Thus, for a non–null voter i there is at least a coalition S such that $S \in W$, $i \in S$ and $S \setminus \{i\} \in L$. Real–world examples of simple games are given by Taylor [9, 10].

The "desirability" relation defined on the set of voters represents a way to make precise the idea that a particular voting system may give one voter more influence than another. Isbell already used it in [7].

Definition 2.2 *Let (N, W) be a simple game.*

(i) Player i is more desirable *than j ($i \succsim j$, in short) in (N, W) iff*

$$S \cup \{j\} \in W \;\Rightarrow\; S \cup \{i\} \in W, \qquad \text{for all } S \subseteq N \setminus \{i, j\}.$$

(ii) Players i and j are equally desirable *($i \approx j$, in short) in (N, W) iff*

$$S \cup \{i\} \in W \Leftrightarrow S \cup \{j\} \in W, \qquad \text{for all } S \subseteq N \setminus \{i, j\}.$$

(iii) Player i is strictly more desirable *than player j ($i \succ j$, in short) in (N, W) iff i is more desirable than j, but i and j are not equally desirable.*

Definition 2.3 *A simple game (N, W) is* complete *or* linear *if the desirability relation is a complete preordering.*

[4]The Fibonacci numbers are defined by the following recurrence relation: $F(0) = 0$, $F(1) = 1$, and $F(n) = F(n-1) + F(n-2)$ for all $n > 1$.

In the field of Boolean algebra, complete games correspond to 2-monotonic positive Boolean functions, which were already considered in [6]. The problem of identifying this type of functions by using polynomial-time recognition have been treated in [1, 2]. In a complete simple game we may decompose N in a collection of subsets, called classes, $N_1 > N_2 > \cdots > N_t$, forming a partition of N, and understanding that if $i \in N_p$ and $j \in N_q$ then: $p = q$ if and only if $i \approx j$; and $p < q$ if and only if $i \succ j$.

Now we are going to define the δ-ordering, introduced in [4] for an arbitrary number of types of voters.

Definition 2.4 *Let n be the number of players of a complete simple game (N, W) with two types of players $N_1 > N_2$. Let $n_1 = |N_1|$ and let $n_2 = |N_2|$, where $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$ with $n_1 + n_2 = n$. Then the* rectangle of $(n_1 + 1) \times (n_2 + 1)$ profiles *for (N, W) is:*

$$I_{n_1} \times I_{n_2} = \{(m_1, m_2) \in (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}) \,:\, m_1 \leq n_1, m_2 \leq n_2\}.$$

In $I_{n_1} \times I_{n_2}$, the δ-ordering *given by the comparison of partial sums is:*

$$(p_1, p_2)\, \delta\, (m_1, m_2) \quad \text{if and only if} \quad p_1 \geq m_1 \;\; \textit{and} \;\; p_1 + p_2 \geq m_1 + m_2.$$

It is not difficult to check that the pair $(I_{n_1} \times I_{n_2}, \delta)$ is a distributive lattice that possesses a maximum element (n_1, n_2) and a minimum element $(0, 0)$. The profiles in $I_{n_1} \times I_{n_2}$ can be completely ordered by the *lexicographical ordering*: profile (p_1, p_2) is lexicographically greater than (m_1, m_2) if either $p_1 > m_1$, or $p_1 = m_1$ with $p_2 > m_2$.

Definition 2.5 *Two simple games (N, W) and (N', W') are said to be* isomorphic *if there is a bijective map $f : N \to N'$ such that $S \in W$ if and only if $f(S) \in W'$; f is called an isomorphism of simple games.*

The following known result has three parts. The first part shows how to associate a vector (n_1, n_2) and a matrix $\mathcal{M}$ to a complete simple game (N, W), and describes the restrictions that these parameters need to fulfill. The second part establishes that every pair of isomorphic complete simple games (N, W) and (N', W') corresponds to the same associated vector (n_1, n_2) and matrix $\mathcal{M}$ (uniqueness). The third part shows that a vector (n_1, n_2) and a matrix $\mathcal{M}$ fulfilling the conditions in Part A always correspond to a complete simple game (N, W) (existence).

Theorem 2.6 *(Carreras and Freixas' Theorems 4.1 and 4.2 in [4] for 2 types of voters)*

Part A *Let (N, W) be a complete simple game with two nonempty classes $N_1 > N_2$, and let (n_1, n_2) be the vector defined by their cardinalities. For each coalition $S \in W$, consider the node or profile $(s_1, s_2) \in I_{n_1} \times I_{n_2}$ with components $s_k = |S \cap N_k|$ $(k = 1, 2)$. Let $\mathcal{M}$ be the matrix*

$$\mathcal{M} = \begin{pmatrix} m_{1,1} & m_{1,2} \\ \vdots & \vdots \\ m_{r,1} & m_{r,2} \end{pmatrix}$$

whose r rows are the nodes corresponding to winning coalitions which are minimal in the δ-ordering. Matrix $\mathcal{M}$ satisfies the two conditions below:

1. *$m_{i,1}, m_{i,2} \in \mathbb{N} \cup \{0\}$, $0 \leq m_{i,1} \leq n_1$ and $0 \leq m_{i,2} \leq n_2$ for all $1 \leq i \leq r$;*
2. *if $r = 1$, then $m_{1,1} > 0$ and $m_{1,2} < n_2$;*
   if $r \geq 2$, then $m_{i,1} > m_{j,1}$ and $m_{i,1} + m_{i,2} < m_{j,1} + m_{j,2}$ for all $1 \leq i < j \leq r$.[5]

[5]The lexicographic ordering chosen guarantees uniqueness under permutation of rows. This lexicographic ordering is a plausible choice which could be replaced for other alternative criteria.

Part B *(Uniqueness) Two complete simple games with two types of voters* (N, W) *and* $(N', \mathcal{W}')$ *are isomorphic if and only if* $(n_1, n_2) = (n'_1, n'_2)$ *and* $\mathcal{M} = \mathcal{M}'$.

Part C *(Existence) Given a vector* (n_1, n_2) *and a matrix* $\mathcal{M}$ *satisfying the conditions of Part A, there exists a complete simple game* (N, W) *with two types of voters associated to* (n_1, n_2) *and* $\mathcal{M}$.

For example, to illustrate how (N, W) is obtained, let $(n_1, n_2) = (2, 3)$ and $\mathcal{M} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. The set of winning profiles in $I_2 \times I_3$ is $\{(0,3),(1,2),(1,3),(2,0),(2,1),(2,3)\}$ because each of these profiles either δ-dominates $(2,0)$ or $(0,3)$; and the set of minimal winning profiles is $\{(0,3),(1,2),(2,0)\}$, because the other winning profiles can be obtained from one of these three profiles by simply adding some element in any of its components. If we take $N_1 = \{1,2\}$ and $N_2 = \{3,4,5\}$, then $N = N_1 \cup N_2$, and

$$W^m = \{\{1,2\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}\},$$

where the first coalition corresponds to profile $(2,0)$, the last coalition to profile $(0,3)$ and the remaining coalitions to profile $(1,2)$.

Theorem 2.6 is a *parametrization* theorem because it allows one to enumerate all complete games up to isomorphism by listing the possible values of certain invariants. We will see in next section that such enumeration can be done for every number of voters.

3 Counting complete games with two types of voters: Fibonacci numbers

We establish a relation between the number of non–isomorphic complete simple games with n voters of two different types and the Fibonacci numbers.

Definition 3.1 *Let* $H(n)$ *denote the number of non–isomorphic complete simple games with* n *voters of two different types.*

Note that $H(n)$ is the number of matrices

$$\begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \\ \vdots & \vdots \\ m_{r,1} & m_{r,2} \end{pmatrix}$$

such that there exists $(n_1, n_2) \in \mathbb{N} \times \mathbb{N}$, with $n = n_1 + n_2$, verifying Properties $(1) - (2)$ of Theorem 2.6-(A).

Theorem 3.2

$$H(n) = F(n+6) - (n^2 + 4n + 8),$$

where $F(n)$ *is the* n*-th Fibonacci number.*

The proof of this theorem is a consequence of some additional definitions and lemmas.

Definition 3.3 *Let* $N(a, s, b)$ *be the number of matrices with non-negative integer entries and an arbitrary number of rows* $r \geq 1$

$$\begin{pmatrix} m_{1,1} & m_{1,2} \\ m_{2,1} & m_{2,2} \\ \vdots & \vdots \\ m_{r,1} & m_{r,2} \end{pmatrix}$$

such that

1. $m_{1,1} = a$;
2. $m_{1,1} + m_{1,2} = s$;
3. $m_{r,2} = b$;
4. *if* $r \geq 2$, *then* $m_{i,1} > m_{j,1}$ *and* $m_{i,1} + m_{i,2} < m_{j,1} + m_{j,2}$ *for all* i *and* j *with* $1 \leq i < j \leq r$.

Lemma 3.4 *Let* $N(a, s, b)$ *be as defined above, then*

$$N(a, s, b) = \begin{cases} 0, & \text{if } s - a > b; \\ 1, & \text{if } s - a = b; \\ \sum_{k=0}^{a-1} \binom{b+a-2-s}{k}, & \text{if } s - a < b. \end{cases}$$

Proof: We consider three cases depending on the relation between a, s and b.
Case 1: $s - a > b$.

By definition, it is clear that $N(a, s, b)$ is equal to 0 whenever $s - a > b$.

Case 2: $s - a = b$.

Note that $s - a = b$ implies $r = 1$. In fact, we just have one matrix with one row

$$(a \quad s - a) \equiv (a \quad b).$$

Thus, $N(a, s, s - a) = 1$.
Case 3: $s - a < b$.

It is clear that $s - a < b$ implies $r > 1$, and $N(a, s, b)$ is equal to the number of matrices with $r - 1$ rows

$$\begin{pmatrix} m_{2,1} & m_{2,2} \\ \vdots & \vdots \\ m_{r,1} & b \end{pmatrix}$$

such that verify (from Definition 3.3):

1. $0 \leq m_{2,1} < a$;
2. $m_{2,1} + m_{2,2} > s$;
3. if $r \geq 3$, then $m_{i,1} > m_{j,1}$ and $m_{i,1} + m_{i,2} < m_{j,1} + m_{j,2}$ for all i and j with $2 \leq i < j \leq r$.

That is,

$$N(a, s, b) = \sum_{i=0}^{a-1} \sum_{j>s} N(i, j, b).$$

On the other hand, since $N(i, j, b) = 0$ if $j > i + b$ (*Case 1*), we have that

$$N(a, s, b) = \sum_{i=0}^{a-1} \sum_{j=s+1}^{i+b} N(i, j, b).$$

Now, by *mathematical induction* we are going to prove the equality

$$\sum_{i=0}^{a-1} \sum_{j=s+1}^{i+b} N(i, j, b) = \sum_{k=0}^{a-1} \binom{b + a - 2 - s}{k}.$$

By *induction hypothesis* [*i.h.*], it is clear that

$$\begin{aligned}\sum_{i=0}^{a-1}\sum_{j=s+1}^{i+b} N(i,j,b) &= \sum_{i=0}^{a-1}\sum_{j=s+1}^{b+i-1} N(i,j,b) + f(a,s,b)\\ &\overset{[i.h.]}{=} \sum_{i=0}^{a-1}\sum_{j=s+1}^{b+i-1}\sum_{k=0}^{i-1}\tbinom{b+i-2-j}{k} + f(a,s,b)\end{aligned}$$

where we define

$$f(a,s,b) := \sum_{i=0}^{a-1} N(i,b+i,b) \quad \text{if and only if } (s+1 \le b+i).$$

Note that $s+1 \le b+i \iff s+1-b \le i$; thus, it is clear that (by definition):

$$f(a,s,b) = \begin{cases} a, & \text{if } (s<b) \iff (s+1-b \le 0)\\ \\ a+b-1-s, & \text{if } (s \ge b) \iff (s+1-b > 0)\end{cases} \tag{1}$$

From now on we will use some elementary combinatorial equalities like [3]

$$\sum_{\alpha \le \beta}\binom{\alpha}{\gamma} = \binom{\beta+1}{\gamma+1}, \qquad \text{for all } \gamma \ge 0.$$

First, we consider the following equalities

$$\begin{aligned}\sum_{j=s+1}^{b+i-1}\sum_{k=0}^{i-1}\tbinom{b+i-2-j}{k} &= \sum_{j\ge s+1}\sum_{k\le i-1}\tbinom{b+i-2-j}{k} = \sum_{k\le i-1}\sum_{j\ge s+1}\tbinom{b+i-2-j}{k}\\ &= \sum_{k\le i-1}\sum_{z\le b+i-3-s}\tbinom{z}{k} = \sum_{k=0}^{i-1}\tbinom{b+i-2-s}{k+1}\end{aligned}$$

Second, reorganizing the previous results we have:

$$\begin{aligned}\sum_{i=0}^{a-1}\sum_{k=0}^{i-1}\tbinom{b+i-2-s}{k+1} &= \sum_{k=0}^{a-2}\sum_{i=k+1}^{a-1}\tbinom{b+i-2-s}{k+1} = \sum_{k=0}^{a-2}\sum_{y=b+k-1-s}^{b+a-3-s}\tbinom{y}{k+1}\\ &= \sum_{k=0}^{a-2}\left(\tbinom{b+a-2-s}{k+2} - \tbinom{b+k-1-s}{k+2}\right)\\ &= \sum_{k=0}^{a-2}\tbinom{b+a-2-s}{k+2} - \sum_{k=0}^{a-2}\tbinom{b+k-1-s}{k+2}\end{aligned}$$

Third, note that

$$\begin{aligned}\sum_{k=0}^{a-2}\tbinom{b+a-2-s}{k+2} &= \sum_{k=2}^{a}\tbinom{b+a-2-s}{k}\\ &= \sum_{k=0}^{a-1}\tbinom{b+a-2-s}{k} + \tbinom{b+a-2-s}{a} - 1 - (b+a-2-s)\\ &= \sum_{k=0}^{a-1}\tbinom{b+a-2-s}{k} + \tbinom{b+a-2-s}{a} - b - a + 1 + s\end{aligned} \tag{2}$$

and

$$\begin{aligned}\sum_{k=0}^{a-2}\tbinom{b+k-1-s}{k+2} &= \sum_{k=0}^{a-1}\tbinom{b+k-1-s}{b-3-s} = \sum_{p=b-1-s}^{b+a-3-s}\tbinom{p}{b-3-s}\\ &= \begin{cases}\tbinom{b+a-2-s}{b-2-s} - \tbinom{b-1-s}{b-2-s} = \tbinom{b+a-2-s}{a} + s + 1 - b,\\ \qquad\qquad \text{if } (b-1-s \ge 0) \iff (b>s)\\ \sum_{p=0}^{b+a-3-s}\tbinom{p}{b-3-s} = \tbinom{b+a-2-s}{b-2-s} = \tbinom{b+a-2-s}{a},\\ \qquad\qquad \text{if } (b-1-s<0) \iff (b \le s)\end{cases}\end{aligned} \tag{3}$$

Finally, the equality for $s-a<b$ follows from Equations (1), (2) and (3) depending on either $s<b$ or $s \geq b$.

Case 1: $s<b$:

$$\begin{aligned} N(a,s,b) = \;& a+ \\ & \left(\sum_{k=0}^{a-1}\binom{b+a-2-s}{k}+\binom{b+a-2-s}{a}-b-a+1+s\right)- \\ & \left(\binom{b+a-2-s}{a}+s+1-b\right) \\ = \;& \sum_{k=0}^{a-1}\binom{b+a-2-s}{k} \end{aligned}$$

Case 2: $s \geq b$:

$$\begin{aligned} N(a,s,b) = \;& (a+b-1-s)+ \\ & \left(\sum_{k=0}^{a-1}\binom{b+a-2-s}{k}+\binom{b+a-2-s}{a}-b-a+1+s\right)- \\ & \binom{b+a-2-s}{a} \\ = \;& \sum_{k=0}^{a-1}\binom{b+a-2-s}{k}. \end{aligned}$$

□

Definition 3.5 *Let $G(a,b)$ be the number of matrices given by $N(a,s,b)$ such that $0 \leq m_{1,1} \leq a$, $0 \leq m_{r,2} \leq b$, $a \leq m_{1,1}+m_{1,2} \leq a+b$ (equivalently, $0 \leq m_{1,2} \leq b$) and, furthemore, $m_{1,1}>0$.*

Lemma 3.6 *Let $G(a,b)$ be as Definition 3.5, then*

$$G(a,b)=\sum_{i=0}^{a}\sum_{k=0}^{i-1}\binom{b}{k+2}+a\,b.$$

Proof: Note that in order to fulfill $m_{1,1}>0$ we have to subtract $(a+b+1)$ matrices:

$$\begin{array}{lll} (i \;\; b) & \text{with } 0<i\leq a & \rightarrow \; a \text{ matrices,} \\ (0 \;\; j) & \text{with } 0<j\leq b & \rightarrow \; b \text{ matrices,} \\ (0 \;\; 0) & & \rightarrow \; 1 \text{ matrix.} \end{array}$$

Now, using elemetal algebraic manipulations, the equality [3] follows:

$$\begin{aligned} G(a,b) = \;& \left(\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{s=i}^{i+j}N(i,s,j)\right)-(a+b+1) \\ = \;& \left(\sum_{i=0}^{a}\sum_{j=0}^{b}\left(\left(\sum_{s=i}^{i+j-1}\sum_{k=0}^{i-1}\binom{j+i-2-s}{k}\right)+1\right)\right)-(a+b+1) \\ = \;& \sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=0}^{i-1}\sum_{s=i}^{i+j-2}\binom{j+i-2-s}{k}+\left(\sum_{i=0}^{a}\sum_{j=0}^{b}1\right)-(a+b+1) \\ = \;& \sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=0}^{i-1}\sum_{s=i}^{i+j-2}\binom{j+i-2-s}{k}+(a+1)\,(b+1)-(a+b+1) \\ = \;& \sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=0}^{i-1}\sum_{z=0}^{j-2}\binom{z}{k}+a\,b=\sum_{i=0}^{a}\sum_{j=0}^{b}\sum_{k=0}^{i-1}\binom{j-1}{k+1}+a\,b \\ = \;& \sum_{i=0}^{a}\sum_{k=0}^{i-1}\sum_{j=0}^{b-1}\binom{j}{k+1}+a\,b=\sum_{i=0}^{a}\sum_{k=0}^{i-1}\binom{b}{k+2}+a\,b. \end{aligned}$$

□

Finally, Lemma 3.7 proves Theorem 3.2.

Lemma 3.7 *Let $H(n)$ and $G(a,b)$ as defined above, then*

$$H(n) = \sum_{a=1}^{n} G(a, n-a)$$

and, furthermore,

$$H(n) = F(n+6) - (n^2 + 4n + 8).$$

Proof: From Definition 3.1 and 3.5, where $H(n)$ and $G(a,b)$ are respectively defined, it is clear that

$$H(n) = \sum_{a=1}^{n} G(a, n-a).$$

Thus, we just have to prove that

$$\sum_{a=1}^{n} G(a, n-a) = F(n+6) - (n^2 + 4n + 8).$$

In fact, taking into account that $G(0,n) = 0$, Lemma 3.6, and the known identity of Fibonacci numbers [3]

$$F(n) = \sum_{k=0}^{n-1} \binom{n-k}{k},$$

the equality results

$$\begin{aligned}
\sum_{a=1}^{n} G(a, n-a) = & \sum_{a=0}^{n} G(a, n-a) \\
= & \sum_{a=0}^{n} \left(\sum_{i=0}^{a} \sum_{k=0}^{i-1} \binom{n-a}{k+2} + a\,(n-a) \right) \\
= & \sum_{i=0}^{n} \sum_{k=0}^{i-1} \sum_{y=0}^{n-i} \binom{y}{k+2} + \binom{n+1}{3} = \sum_{i=0}^{n} \sum_{k=0}^{i-1} \binom{n-i+1}{k+3} + \binom{n+1}{3} \\
= & \sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \binom{n-i+1}{k+3} + \binom{n+1}{3} = \sum_{k=0}^{n-1} \sum_{x=1}^{n-k} \binom{x}{k+3} + \binom{n+1}{3} \\
= & \sum_{k=0}^{n-1} \binom{n-k+1}{k+4} + \binom{n+1}{3} = \sum_{k=4}^{n+3} \binom{n+5-k}{k} + \binom{n+1}{3} \\
= & \sum_{k=0}^{n+5} \binom{n+5-k}{k} - \left[\binom{n+5}{0} + \binom{n+4}{1} + \binom{n+3}{2} + \binom{n+2}{3} \right] + \binom{n+1}{3} \\
= & F(n+6) - (n^2 + 4n + 8).
\end{aligned}$$

$\square$

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