Title: Generalisation of Sylvester's problem
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I became convinced that if only one tried hard enough—or were clever enough!—every mathematical mystery could be resolved through geometric reasoning.

Tristan Needham
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Abstract

**Keywords:** Discrete Geometry, Incidence and Arrangement Problems, Sylvester-Gallai-Type Problems, Computational Geometry, Combinatorial Geometry

**MSC2010:** 52C35, 68U05, 05C99

Let \( P \) be a set of \( n \) points in the projective space of dimension \( d \) with the property that not all the points are contained in an hyperplane and any \( d \) points span an hyperplane. Let an ordinary hyperplane of \( P \) be an hyperplane passing through exactly \( d \) points of \( P \). We show that if \( d \) is 3, and \( n \) is large and even, then there are precisely \( n \left\lfloor \frac{n-1}{4} \right\rfloor \) ordinary planes. Indeed, we describe the exact extremisers for this problem. We also find the number of ordinary hyperplanes for small \( n \) and \( d \), and lower and upper bounds of this number.
Contents

1 Introduction 3

2 General statements 7
   2.1 Basic results ................................................. 7
   2.2 The number of ordinary hyperplanes for small \( n \) ............. 9
   2.3 Structure theorems ............................................. 13
   2.4 Counting the number of \( k \)-hyperplanes with the computer .... 14

3 Planar case 15
   3.1 Enhanced upper bound for \( e_2(n) \) .............................. 15
   3.2 Melchior’s proof of Moser’s lower bound ......................... 18
   3.3 The number of ordinary lines for \( n \) small ...................... 21
   3.4 The number of ordinary lines for \( n \) large ...................... 23

4 3-space case 25
   4.1 Enhanced upper bound in 3-space ................................ 25
   4.2 The number of ordinary planes for small \( n \) .................... 29
   4.3 The number of ordinary planes for large \( n \) .................... 40
Chapter 1

Introduction

The Sylvester’s Problem has been posed by James Joseph Sylvester (Figure 1.1) in 1893 in Educational Times [Syl93]:

Let \( n \) given points have the property that the line joining any two of them passes through a third point of the set. Must the \( n \) points all lie on one line?

The affirmative answer to this question was given by Gallai forty years later.

**Theorem 1.1** (Sylvester-Gallai Theorem). *Suppose that \( P \) is a finite set of points in the plane, not all on one line. Then there is an ordinary line spanned by \( P \), that is to say a line in \( P \) containing exactly two points.*

To prove it here we can deduce it from Theorem 3.4 (Moser), which is a stronger result.

The Sylvester-Gallai Theorem is not true in all geometries, for example the Fano plane (Figure 1.2(a)), that is a finite geometry with 7 points and 7 lines with every
line containing 3 points, hasn’t any ordinary line. Thus, the Fano plane can’t be
embedded in the real projective plane with all it’s lines drawn as straight lines. In
the complex projective plane there exist a configuration with nine points and nine
lines, Hesse’s configuration \[\text{1.2(b)}\] (the inflection points of a cubic) that has all the
lines containing 3 points, which implies that every line is non-ordinary.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12.png}
\caption{Figure (1.2)}
\end{figure}

Once we have answered the question of Sylvester, the following step is to minimise
the number of ordinary lines in a given set of \(n\) points, not all on a line. Some of
the results in this direction are in articles [CS93], [CM68], [GT13].

Next, we can try to find an analogue of the Sylvester problem in higher dimen-
sions, but we must add some restrictions, because there exist examples that have no
ordinary hyperplanes. In 3-space for example we have the following. Let \(P\) be the
configuration of points distributed in two skew lines, having at least 3 points on each
line. This configuration has no ordinary planes, that is to say a plane containing
exactly three points. This configuration was discovered by Motzkin [Mot51], and
all the planes of \(P\) have at least 4 points. Thus, one must add some rule to en-
sure at least one ordinary plane. The survey article by Borwein and Moser [BM90],
and subsequently the book of problems by Brass, Moser and Pach [BMP05], have a
section on generalisation of Sylvester’s problem to higher dimensional spaces. They
define an ordinary hyperplane as a hyperplane in which all but exactly one point are
contained in a \((d - 2)\)-dimensional affine subspace.

With this definition the following facts have been found. Motzkin [Mot51] and
Hansen [Han65], proved that any noncoplanar finite set of points in \(d\)-space has an
ordinary hyperplane. Let \(oh_d(n)\) be the minimum number of ordinary hyperplanes
spanned by a set of \(n\) points in \(d\)-space, Hansen [Han80] found the lower bound,
\(oh_3(n) \geq 2n/5\). There is no reasonable conjecture for the exact value of \(oh_d(n)\) for
\(d \geq 3\).
Simeon Ball [Bal13] proposed another generalisation to higher dimensional spaces. Let \( P \) be a set of points in \( d \)-space with the property that not all the points of \( P \) are contained in a hyperplane and any \( d \) points of \( P \) span a hyperplane. We say that a hyperplane \( H \) is an ordinary hyperplane if \( H \) contains precisely \( d \) points of \( P \), for any set \( P \) of points. Let \( e_d(n) \) denote the minimal number of ordinary hyperplanes of \( P \), minimising over all such sets of points \( P \) with \( |P| = n \). With this definition the case for \( d = 2 \) coincides with the definition of the Sylvester problem. This is the generalisation that we are going to study in this thesis, and is heavily inspired by the article of Simeon Ball.

The generalisation of Sylvester Problem that we consider can be studied with basic tools, some of them are from projective geometry and some from combinatorial inequalities. We use inequalities to find the exact value of \( e_d(n) \) for small \( n \), first we give lower and upper bounds to this value, then when this lower and upper bounds coincide the exact value follows. Projective geometry simplifies the configurations allowing points at infinity, the proofs are more concise, and we can consider the dual space. We shall see the dual space when we apply Euler’s Formula to proof Theorem 3.4. But we can formulate all the results in the \( d \)-dimensional affine space \( \mathbb{R}^d \), instead of the \( d \)-dimensional projective space \( \mathbb{P}^d_\mathbb{R} \). Applying a generic projective transformation we can move all the points at infinity of the configuration to the affine space, without affecting the number of ordinary hyperplanes.
Chapter 2

General statements

2.1 Basic results

Let $\mathbb{P}^d_\mathbb{R}$ denote the $d$-dimensional projective space over $\mathbb{R}$. Let $P$ be a subset of $n$ points of $\mathbb{P}^d_\mathbb{R}$ with the property that every $d$ points of $P$ span a hyperplane and $P$ is not contained in an hyperplane. We will always assume that $P$ has this property without mentioning it. From the following Lemma, it can be deduced that there is at least one ordinary hyperplane for every $n$ and $d \geq 3$, because we have $e_2(n) \geq 1$ by Theorem 1.1. Thus we have answered the Sylvester question for $d \geq 3$.

Figure (2.1)
Lemma 2.1. For \( d \geq 3 \),
\[
e_d(n) \geq \frac{n}{d} e_{d-1}(n-1).
\]

Proof. Let \( x \in P \). Project \( P \) from \( x \) to a set \( P' \) of \( n-1 \) points of \( \mathbb{P}^{d-1}_\mathbb{R} \). Then it’s true that, every \( d-1 \) points of \( P' \) spans a hyperplane of \( \mathbb{P}^{d-1}_\mathbb{R} \), and not all the points are contained in a hyperplane of \( \mathbb{P}^{d-1}_\mathbb{R} \). So, we observe that in \( P' \) there are at least \( e_{d-1}(n-1) \) ordinary hyperplanes of \( \mathbb{P}^{d-1}_\mathbb{R} \). Thus, \( x \) is contained in at least \( e_{d-1}(n-1) \) ordinary hyperplanes of \( \mathbb{P}^{d}_\mathbb{R} \). Figure 2.1 makes this easy to see.

If we do this reasoning for every point of \( P \), having in mind that every ordinary hyperplane contains exactly \( d \) points of \( P \), and \( P \) has \( n \) points, we get the bound. \( \square \)

Since \( e_d(n) \) is an integer, if we apply repeatedly the previous lemma we have the following

Lemma 2.2. For \( d \geq 3 \),
\[
e_d(n) \geq \left\lfloor \frac{n}{d} \left\lfloor \frac{n-1}{d-1} \left\lfloor \frac{n-2}{d-2} \cdots \left\lfloor \frac{n-d+3}{3} e_2(n-d+2) \right\rfloor \cdots \right\rfloor \right\rfloor.
\]

Theorem 2.1. For \( n \neq d+5 \),
\[
e_d(n) \geq \left\lfloor \frac{n}{d} \left\lfloor \frac{n-1}{d-1} \left\lfloor \frac{n-2}{d-2} \cdots \left\lfloor \frac{n-d+3}{3} \left\lfloor \frac{6(n-d+2)}{13} \right\rfloor \right\rfloor \cdots \right\rfloor \right\rfloor.
\]

Proof. This follows by the lower bound of Csima and Sawyer [CS93]: if \( n \neq 7 \) then \( e_2(n) \geq 6n/13 \). \( \square \)

We can construct a trivial set of points that is generalizable to a space of any dimension, and calculating the ordinary hyperplanes of it, we find an upper bound of \( e_d(n) \).

Theorem 2.2. For \( n \geq d+2 \),
\[
e_d(n) \leq \binom{n-1}{d-1}.
\]

Proof. To prove this bound, we construct a set of points \( P = P' \cup x \), where \( P' \) are points contained in the hyperplane at infinity and \( x \) the origin, and then counting the ordinary hyperplanes of the set the bound follows.

Let \( P' = \{ (0,1,t,t^2,\ldots,t^{d-1}) \mid t \in T \} \), where \( T \) is a subset of \( \mathbb{R} \) of size \( n-1 \), be the set of \( n-1 \) points on the hyperplane at infinity \( H_\infty \). We observe that every \( d-1 \) points of \( P' \) span a hyperplane of \( H_\infty \), because the determinant of any \( d-1 \) points is a Vandermonde determinant different from zero.

Let \( x \) be the origin, \( x = (1,0,\ldots,0) \), we can verify that \( x \notin P' \). It’s easy to compute the number of ordinary hyperplanes of \( P = P' \cup x \), that is \( \binom{n-1}{d-1} \). \( \square \)
2.2. THE NUMBER OF ORDINARY HYPERPLANES FOR SMALL N

For $d = 3$ the set $P'$ consists of points of a parabola at the plane at infinity. We shall improve on this theorem when $d = 3$ in Chapter 4, finding general sets that have less ordinary planes than $\binom{n-1}{d-1}$.

A hyperplane passing through exactly $i$ points of $P$ is called a $i$-hyperplane. We denote $t_i$ the number of hyperplanes containing $i$ points of $P$. The following counting lemma is an easy relation satisfied by the $t_i$ numbers. It’s useful for computing the number of ordinary hyperplanes for small $n$ and $d$.

Lemma 2.3.

$$\sum_{i=d}^{n-1} \binom{i}{d} t_i = \binom{n}{d}.$$  

Proof. By counting the number of $d$-subsets in two ways. $\square$

2.2 The number of ordinary hyperplanes for small $n$

Lemma 2.4. For every $(d + 2)$-subset of $P$, there exist at most one hyperplane $H$, such that $|H \cap P| = d + 1$.

Proof. Let $Q$ be a $(d + 2)$-subset of $P$ and $p_1, p_2 \in P$ two different points. Suppose there are two different hyperplanes $H$ and $H'$, with the property that $Q_1 = Q \setminus \{p_1\}$ spans $H$ and $Q_2 = Q \setminus \{p_2\}$ spans $H'$. Then $Q_1 \cap Q_2$ is a subset of size $d$ that spans a unique hyperplane, so $H$ and $H'$, in fact, are equal. $\square$

This is an inequality that follows from counting $(d + 2)$-subsets.

Lemma 2.5.

$$\sum_{i=1}^{n-d-1} (n - d - i) \binom{d + i}{d + 1} t_{d+i} \leq \binom{n}{d + 2}.$$  

Proof. Suppose $H$ is a $(d + i)$-hyperplane, for some $i \geq 1$. We can reduce the $(d + i)$-subset of $H$ to a $(d + 1)$-subset in $\binom{d+i}{d+1}$ ways. Adding a point of $S \setminus H$ from one of these $(d + 1)$-subsets of $H$, we obtain a distinct $(d + 2)$-subset by the previous lemma (Lemma 2.4). So the number of $(d + 2)$-subsets we can construct in this way is at least $(n - d - i) \binom{d+i}{d+1}$. Therefore, if we do this counting for every $(d + i)$-hyperplane, the bound follows. $\square$

With the previous inequality we prove the following lower bound.

Theorem 2.3.

$$e_d(n) \geq \left\lceil \binom{n}{d} - \frac{d + 1}{d + 2} \binom{n}{d + 1} \right\rceil.$$
Proof. By counting $d$-subsets (Lemma 2.3),
\[
\sum_{i=0}^{n-d-1} \binom{d+i}{i} t_{d+i} = \binom{n}{d}.
\]
Therefore,
\[
t_d + \sum_{i=1}^{n-d-1} \frac{d+1}{i} \binom{d+i}{i-1} t_{d+i} = \binom{n}{d}.
\]
Since $(n-d-i)/(n-d-1) \geq 1/i$,
\[
t_d + \frac{d+1}{n-d-1} \sum_{i=1}^{n-d-1} (n-d-i) \binom{d+i}{i-1} t_{d+i} \geq \binom{n}{d}.
\]
Lemma 2.5 implies,
\[
t_d + \frac{d+1}{n-d-1} \binom{n}{d+2} \geq \binom{n}{d}
\]
and the result follows. \qed

The previous lower bound is useful only for $n \leq 2d$, because for values of $n > 2d$ it doesn’t improve our lower bound of Lemma 2.2.

**Theorem 2.4.**
\[
e_d(d+2) = \binom{d+1}{2}.
\]

**Proof.** By using the general upper bound (Theorem 2.2) and Theorem 2.3. \qed

**Theorem 2.5.** If $d$ is even then
\[
e_d(d+3) = \binom{d+2}{3}.
\]

**Proof.** The trivial set of Theorem 2.2 gives
\[
e_d(d+3) \leq \binom{d+3-1}{d-1} = \binom{d+2}{3},
\]
so we have to see that
\[
e_d(d+3) \geq \binom{d+2}{3}.
\]
By Lemma 2.5,
\[
2t_{d+1} + (d+2)t_{d+2} \leq d + 3
\]
If \( t_{d+2} = 1 \) then \( t_{d+1} = 0 \) and \( P \) has the same number of ordinary hyperplanes as the trivial set, 
\[
 t_d = \binom{d+2}{3} 
\]
If \( t_{d+2} = 0 \) then 
\[
 t_{d+1} \leq \frac{(d+2)}{2}, \tag{2.1} 
\]
because \( d \) is even. Applying Lemma 2.3 we have the equality 
\[
 t_d + (d+1)t_{d+1} = \binom{d+3}{3}. \tag{2.2} 
\]
Therefore substituting (2.1) in (2.2) we obtain, 
\[
 t_d \geq \binom{d+2}{3}. 
\]

**Theorem 2.6.** If \( d \) is odd then 
\[
 e_d(d+3) = \frac{1}{6}(d+3)(d+1)(d-1). 
\]

**Proof.** By the improved lower bound for small \( n \) of Theorem 2.3 \( e_d(d+3) \geq \frac{1}{6}(d+3)(d+1)(d-1) \).

So we only have to construct a set of \( d+3 \) points with \( \frac{1}{6}(d+3)(d+1)(d-1) \) ordinary hyperplanes. Let \( u_1, \ldots, u_{d+1} \) be \( d+1 \) points of \( \mathbb{P}^d_{\mathbb{R}} \) which span \( \mathbb{P}^d_{\mathbb{R}} \). Let 
\[
 P = \{u_1, \ldots, u_{d+1}, u, v\}, 
\]
where 
\[
 u = u_1 + \cdots + u_d, 
\]
\[
 v = \alpha_1(u_1 + u_2) + \cdots + \alpha_{(d-1)/2}(u_{d-2} + u_{d-1}) + u_{d+1} 
\]
and \( \alpha_1, \ldots, \alpha_{(d-1)/2} \) are distinct elements of \( \mathbb{R} \).

So for the set \( P \) there are two \((d+1)\)-hyperplanes \( \langle u, u_1, \ldots, u_d \rangle \) and \( \langle v, u_1, \ldots, u_{d-1}, u_{d+1} \rangle \).

Moreover, 
\[
 v - \alpha_1 u = (\alpha_2 - \alpha_1)(u_3 + u_4) + \cdots + (\alpha_{(d-1)/2} - \alpha_1)(u_{d-2} + u_{d-1}) - \alpha_1 u_d + u_{d+1}, 
\]
so \( \langle u, v, u_3, \ldots, u_{d+1} \rangle \) is also a \((d+1)\)-hyperplane of \( P \). This argument can be generalised, if we consider \( v - \alpha_i u \) for \( i = 2, \ldots, (d-1)/2 \), we find that at least the number of \((d+1)\)-hyperplanes is 
\[
 \tau_{d+1} \geq (d+3)/2. 
\]
Lemma 2.3 gives 
\[ \tau_d \leq \frac{1}{6}(d + 3)(d + 1)(d - 1). \]

Therefore, \( \tau_d = \frac{1}{6}(d + 3)(d + 1)(d - 1) \).

**Theorem 2.7.**

\( e_4(8) \geq 25 \)

**Proof.** By Lemma 2.3

\[ 3\tau_5 + 12\tau_6 + 21\tau_7 \leq 28 \]

and Lemma 2.3 implies

\[ \tau_4 + 5\tau_5 + 15\tau_6 + 35\tau_7 = 70. \]

Therefore,

\[ 3\tau_4 \geq 70 + 15\tau_6. \]

and so \( \tau_4 \geq 24 \). If \( \tau_4 = 24 \) then \( \tau_6 = 0 \) and then by Lemma 2.3 we have \( \tau_5 \notin \mathbb{Z} \). Hence \( \tau_4 \geq 25 \).

In a similar way, can be proved the following theorem.

**Theorem 2.8.**

\( e_5(9) \geq 54 \)

In Table 2.1 we list the values of \( e_d(n) \), for small \( n \) and \( d \). It uses some results that are proven in the next sections. The values that are not known exactly they have the lower and upper bounds of the number. The first row that corresponds to the planar case comes from Table 3.1. The other entries follow from the projection lower bound (Lemma 2.1), the trivial set upper bound (Theorem 2.2), the enhanced upper bound for \( d = 3 \) (Theorem’s 4.1–4.4), the exact values of \( e_d(n) \) for small \( n \) (Theorem’s 2.4–2.6 and Theorem 4.5), and improved lower bounds (Theorem 4.6 and Theorem’s 2.7–2.8).

<table>
<thead>
<tr>
<th>( n )</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_2(n) )</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>( e_3(n) )</td>
<td>.</td>
<td>4</td>
<td>6</td>
<td>8</td>
<td>11</td>
<td>8</td>
<td>16..22</td>
<td>20</td>
<td>19..31</td>
<td>24</td>
<td>26..51</td>
</tr>
<tr>
<td>( e_4(n) )</td>
<td>.</td>
<td>.</td>
<td>5</td>
<td>10</td>
<td>20</td>
<td>25..35</td>
<td>18..56</td>
<td>30..84</td>
<td>55..120</td>
<td>57..165</td>
<td>78..220</td>
</tr>
<tr>
<td>( e_5(n) )</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>6</td>
<td>15</td>
<td>32</td>
<td>54..70</td>
<td>36..126</td>
<td>66..210</td>
<td>132..330</td>
<td>149..495</td>
</tr>
<tr>
<td>( e_6(n) )</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>7</td>
<td>21</td>
<td>56</td>
<td>90..126</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>( e_7(n) )</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>8</td>
<td>28</td>
<td>80</td>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

Table (2.1)
2.3  Structure theorems

Green and Tao in [GT13, Proposition 6.13] prove the following theorem, that states that the minimal configurations are contained in either a line, the union of an irreducible conic and a line, or an irreducible cubic curve. By [GT13, Proposition 5.3], we can take the constants $\beta = 2^{87}$ and $\gamma = 6$.

**Theorem 2.9.** Suppose that $P$ is a set of $n$ points in $\mathbb{P}^2_{\mathbb{R}}$ having less than $kn$ ordinary lines, where $1 \leq k \leq \alpha (\log \log n)^{\alpha}$ for some constant $\alpha$. If $n \geq 100$, then there are constants $\beta$ and $\gamma$ such that $P$ differs in at most $\beta k \gamma$ points from a subset of either a line, the union of an irreducible conic and a line, or an irreducible cubic curve.

By projection, we can prove a generalised version of Theorem 2.9 in the following theorem, that says that the minimal configurations are contained in an algebraic variety of degree at most three. We denote by $V(f_1, \ldots, f_r)$ the set of common zeros of $P_{\mathbb{R}}$ of the homogeneous polynomials $f_1, \ldots, f_r \in \mathbb{R}[X_0, \ldots, X_d]$.

**Theorem 2.10.** Suppose that $P$ is a set of $n$ points in $\mathbb{P}^d_{\mathbb{R}}$ having less than $k(n-1)^{d-1}/d!$ ordinary hyperplanes, where $1 \leq k \leq \alpha (\log \log n)^{\alpha}$ for some constant $\alpha$. If $n \geq 100$, then there are constants $\beta$ and $\gamma$ such that $P$ differs in at most $2^{d-3} \beta k \gamma$ points from a subset of an algebraic curve $V(f_1, \ldots, f_{d-1})$, where $f_i(X_0, X_1, X_{i+1})$ is a non-zero homogeneous polynomial of degree at most three.

**Proof.** We do the proof by induction on $d$. The case $d = 2$ is true by Theorem 2.9.

Every ordinary hyperplane is incident with $d$ points of $P$, and since there are $n$ points in $P$, there is a point $p_d \in P$ on at most

$$\frac{d}{n} \cdot \frac{k(n-1)^{d-1}}{d!} < \frac{k(n-1)^{d-2}}{(d-1)!}$$

ordinary hyperplanes. Again by the same counting, excluding the point $p_d$, there is another point $p_{d-1} \in P$ on at most $k(n-1)^{d-2}/(d-1)!$ ordinary hyperplanes. Then we extend $\{p_d, p_{d-1}\}$ to a basis $B = \{p_0, \ldots, p_d\}$ of the whole space with homogeneous coordinates $(X_0, \ldots, X_d)$.

Suppose that the theorem is true for dimension $d-1$, considering the projection of $P$ from $p_d$, we have a configuration with $n - 1$ points in $\mathbb{P}^d_{\mathbb{R}}$ with homogeneous coordinates $(X_0, \ldots, X_{d-1})$. Applying the case for dimension $d-1$, there are $d-2$ non-zero homogeneous polynomials of degree at most three

$$f_1(X_0, X_1, X_2), \ldots, f_{d-2}(X_0, X_1, X_{d-1}),$$

for which all but $2^{d-3} \beta k \gamma$ points of $P$ are zeros.

The last polynomial $f_{d-1}$ is obtained in the following way, consider the projection of $P$ from $p_{d-1}$, we have a configuration with $n-1$ points in $\mathbb{P}^d_{\mathbb{R}}$ with homogeneous
coordinates \((X_0, \ldots, X_{d-2}, X_d)\). Applying the case for dimension \(d - 1\), there are \(d - 2\) non-zero homogeneous polynomials of degree at most three

\[ g_1(X_0, X_1, X_2), \ldots, g_{d-3}(X_0, X_1, X_{d-2}), g_{d-2}(X_0, X_1, X_d), \]

for which all but \(2^{d-3} \beta k^\gamma\) points of \(P\) are zeros. Naming \(f_{d-1} = g_{d-2}\), all but \(2 \cdot 2^{d-3} \beta k^\gamma\) points of \(P\) are contained in \(V(f_1, \ldots, f_{d-1})\).

\[ \Box \]

### 2.4 Counting the number of \(k\)-hyperplanes with the computer

If we want to count the number of ordinary hyperplanes of a set \(P\), when \(d \geq 4\) and \(n\) is not so large, it can be useful to check the computed value with the computer. Here is a simple algorithm that does the task. The calculations should be done in a computational software that supports symbolic calculations, like Maxima or Mathematica, because some configurations of points have coordinates that contain irrational numbers.

**Algorithm 2.1** Calculate the \(t_k\) numbers

**Input:** a set of points \(P = \{p_1, p_2, \ldots, p_n\} \subset \mathbb{P}^d\)

**Output:** a list of the numbers \(t_d, t_{d+1}, \ldots, t_{n-1}\).

1. \(S = \{d\text{-subsets of } P\}\).
2. if \(P\) doesn’t span \(\mathbb{P}^d\) or \(\exists S_i\) such that doesn’t span an hyperplane then **abort**
3. \(\triangleright \) Initialise the list of \(t_i\) numbers at 0.
4. \(t = 0\)
5. for \(i = 1\) to \(\binom{n}{d}\) do
   6. if \(S_i\) is not **computed** then
      7. for \(j = 1\) to \(n\) do
         8. \(\triangleright \) Get all the points in \(P\) that are in the hyperplane \(H\) spanned by \(S_i\)
            9. if \(p_j\) is in \(H\) then append \(p_j\) to \(S_i\)
      10. \(k = |S_i|\)
      11. \(t_k = t_k + 1\)
Chapter 3

Planar case

Let $P$ be a subset of $n$ points of $\mathbb{P}_R^2$, not all on one line. In this chapter, we will always assume that $P$ has this property without mentioning it. Some figures of this chapter contain lines that some are dotted and the other aren’t, the lines that are dotted are ordinary lines. The lines that contain points at infinity are ended with an arrow pointing at the point at infinity.

3.1 Enhanced upper bound for $e_2(n)$

There are sets of points in the plane that have few ordinary lines, if $n \geq 6$ this configurations give the following upper bound

$$e_2(n) \leq \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod{2} \\ 3\lfloor n/4 \rfloor & \text{if } n \equiv 1 \pmod{2} \end{cases}$$

that in the following theorems we’ll prove.
Theorem 3.1. If $n \equiv 0 \pmod{2}$ and $n \geq 6$, then
\[
e_2(n) \leq \frac{n}{2}
\]

Proof. Let $P$ be the vertices of a regular $n/2$-gon, and the points at the line at infinity corresponding to the $n/2$ directions determined by pairs of vertices of the regular polygon. In Figure 3.1 we can see this set of points for $n = 12$. These $n$ points form a set that we’ll call $X_n$, and was discovered by Böröczky (as cited in [CM68]). Through each vertex of the polygon there is an ordinary line, so there are $n/2$ ordinary lines.

\[
\Box
\]
Theorem 3.2. If $n \equiv 1 \pmod{4}$ and $n \geq 9$, then
\[ e_2(n) \leq 3 \frac{n - 1}{4} \]

Proof. Let $P$ be the set $X_{n-1}$ plus the point $C$ at the center of the regular polygon, as in Figure 3.2(a). Consider the lines that span the points of $P$. There is an ordinary line through each of the vertices of the $\frac{n-1}{2}$-regular polygon, and $\frac{n-1}{4}$ ordinary lines through $C$. Therefore, in total there are $3 \frac{n-1}{4}$ ordinary lines.

Theorem 3.3. If $n \equiv 3 \pmod{4}$ and $n \geq 7$, then
\[ e_2(n) \leq 3 \frac{n - 3}{4} \]

Proof. Let $P$ be the set $X_{n+1}$ minus a point $D$ on the line at infinity, that corresponds to a direction not determined by an edge of the $\frac{n+1}{2}$-gon, as in Figure 3.2(b). Consider the lines that span the points of $P$. There is an ordinary line through all but two vertices of the $\frac{n+1}{2}$-gon and $\frac{n-3}{4}$ ordinary lines in the direction of $D$. Therefore, in total there are $3 \frac{n-3}{4}$ ordinary lines.

Remark 3.1. There is another Böröczky example if $n \equiv 1 \pmod{4}$ and $n \geq 9$, it is formed by the set $X_{n+1}$ minus any of the $\frac{n+1}{2}$ on the line at infinity, there are $3 \frac{n-1}{4}$ ordinary lines in the set, as the number of ordinary lines of the example of Theorem 3.2. The case for $n = 9$ is illustrated in Figure 3.3.
3.2 Melchior’s proof of the Sylvester-Gallai Theorem and Moser’s lower bound

Like many other problems in plane geometry, we consider the dual of our set of points in \( \mathbb{P}_R^2 \), that is a set of lines in \( \mathbb{P}_R^2 \). This will help us to prove the lower bound \( e_2(n) \geq (n + 11)/6 \) if \( n \) is even, in Theorem 3.4 that was discovered by Melchior and Moser [BM90].

Consider \( \mathbb{P}_R^2 \), where we have homogeneous coordinates \((x_0, x_1, x_2)\), and the line at infinity is \( x_0 = 0 \). Let \( l \) be a line in \( \mathbb{P}_R^2 \) given by the equation

\[
a_0x_0 + a_1x_1 + a_2x_2 = 0
\]

where \((a_0, a_1, a_2)\) is determined up to a scalar factor. If we regard \((a_0, a_1, a_2)\) as a point of \( \mathbb{P}_R^2 \), then \( l \mapsto (a_0, a_1, a_2) \) map a line in \( \mathbb{P}_R^2 \) to a point in the projective plane dual to \( \mathbb{P}_R^2 \). And in the other way we can map a point to its dual line.

To illustrate this we are going to calculate the dual of one set of points. Consider the set of points \( P = \{A, B, C, D\} \), where

\[
A = (1, 0, 0) \\
B = (1, 0, 1) \\
C = (1, 0, -1) \\
D = (1, 1, 0)
\]

that are represented in Figure 3.4(a) and the set of lines \( L = \{a, b, c, d\} \), where

\[
a: \{x_0 = 0\} \\
b: \{x_0 + x_2 = 0\} \\
c: \{x_0 - x_2 = 0\} \\
d: \{x_0 + x_1 = 0\}.
\]

Then the lines \( a, b, c, d \) are the dual of the points \( A, B, C, D \) respectively. In Figure 3.4(b) we can see these lines in perspective view of the dual plane, where the line \( a \) is the line at infinity.
3.2. MELCHIOR’S PROOF OF MOSER’S LOWER BOUND

In \( \mathbb{P}^2_R \), a line passing through precisely \( i \) points of \( P \) is called a \( i \)-line. In the dual of \( \mathbb{P}^2_R \), a point that is contained in exactly \( i \) lines of \( L \) is called an \( i \)-point. The number of \( i \)-lines is \( t_i \) and the number of \( i \)-points is \( v_i \).

By changing our point of view to the dual, it can be seen in Figure 3.4(b) that our ordinary lines go to \( 2 \)-points, because the dual of the line \( BD \) is the point obtained from the intersection of lines \( b \) and \( d \). This can be generalised, the dual of an \( i \)-line is an \( i \)-point. So counting ordinary lines is the same as counting \( 2 \)-points in the dual. We call \( L \) the set dual to \( P \), that is, a finite set of lines not all through one point.

We call a connecting line, a line that connects two points of \( P \). In general, a set of points \( P \) in \( \mathbb{P}^2_R \) may have two connecting lines that have a intersection not in the set \( P \), thus it’s difficult to apply Euler’s formula. But if we consider the dual set of lines \( L \) of \( P \), then every intersection of two lines is a point dual of a connecting line, and here Euler’s formula is useful because we can count the connecting lines with the numbers \( t_i \).

In Figure 3.4(b), we can see that there are 6 faces, 9 edges and 5 vertices, so the Euler’s formula in \( \mathbb{P}^2_R \)

\[
V - E + F = 6 - 9 + 4 = 1
\]

is true. The Euler characteristic is 1, rather than 2, because we are working on the projective plane rather than the affine plane or the sphere. The following lower bound has a beautiful proof that is combinatorial and does not use metrical properties of Euclidean geometry.

**Theorem 3.4.** If \( n \) is even, then

\[
e_2(n) > \frac{n + 11}{6}
\]
CHAPTER 3. PLANAR CASE

Proof. Consider the set $L$, its lines partition $\mathbb{P}^2_\mathbb{R}$ into faces. Let $V$, $E$ and $F$ be the number of vertices, edges and faces, respectively, in this dissection of the plane. By the Euler’s formula we have

$$V - E + F = 1 \quad (3.1)$$

Let $f_i$ denote the number of faces having exactly $i$ sides, then

$$V = \sum_{i \geq 2} v_i, \quad (3.2)$$

$$F = \sum_{i \geq 3} f_i, \quad (3.3)$$

$$2E = \sum_{i \geq 3} if_i = 2 \sum_{i \geq 2} iv_i, \quad (3.4)$$

and substituting in (3.1) we easily obtain

$$3 = 3V - E + 3F - 2E \quad (3.5)$$

$$= 3 \sum_{i \geq 2} v_i - \sum_{i \geq 2} iv_i + 3 \sum_{i \geq 3} f_i - \sum_{i \geq 3} if_i \quad (3.6)$$

$$= \sum_{i \geq 2} (3 - i)v_i + \sum_{i \geq 3} (3 - i)f_i, \quad (3.7)$$

and

$$v_2 = 3 + \sum_{i \geq 4} (i - 3)v_i + \sum_{i \geq 4} (i - 3)f_i$$

and thus

$$v_2 \geq 3 + \sum_{i \geq 4} (i - 3)v_i. \quad (3.8)$$

The dual of this inequality is

$$t_2 \geq 3 + \sum_{i \geq 4} (i - 3)t_i \quad (3.9)$$

This inequality is an intermediate result found by Melchior in 1940, that gives the following fact: in the set $P$ of points there are at least three ordinary lines, and proves the Sylvester-Gallai Theorem. Later in 1957, Moser used this inequality to prove a lower bound.

Expanding the sum in (3.9) gives,

$$t_2 \geq 3 + \sum_{i \geq 4} (i - 3)t_i = 3 + t_4 + 2t_5 + 3t_6 + \ldots \quad (3.10)$$
The number of points in \( P \) incident with a \( k \)-line for at least one \( k \neq 3 \) is at most

\[
2t_2 + 4t_4 + 5t_5 + \ldots \\
\leq 2t_2 + 4(t_4 + 2t_5 + 3t_6 + \ldots)
\]

and using (3.10)

\[
\leq 2t_2 + 4(t_2 - 3) = 6t_2 - 12
\]

If \( t_2 \leq (n + 11)/6 \) then \( 6t_2 - 12 \leq n - 1 \) and therefore at least one point of \( P \) is incident with only 3-lines, but this implies that \( n \) is odd. Therefore, if \( n \) is even then \( t_2 > (n + 11)/6 \).

\[\Box\]

### 3.3 The number of ordinary lines for \( n \) small

The following lemma is proven in [KM58, Lemma 4.1] by Kelly and Moser. We shall use it to prove some of the theorems of this section.

**Lemma 3.1.** Let \( t = t_2 + \ldots + t_{n-1} \). If exactly \( n - r \) of the points lie on some line, and if \( n \geq 3r/2 \geq 3 \), then

\[
t \geq rn - (3r + 2)(r - 1)/2.
\]

**Theorem 3.5.**

\[
e_2(5) = 4
\]

**Proof.** The lower bound of Csima and Sawyer [CS93] gives \( e_2(5) \geq 3 \), and the trivial set of Theorem 2.2 gives \( e_2(5) \leq 4 \). Lemma 2.3 implies that \( 10 = t_2 + 3t_3 + 6t_4 \), and if \( t_2 = 3 \), this diophantine equation has no solution. Thus \( e_2(5) = 4 \).

\[\Box\]

**Theorem 3.6.**

\[
e_2(9) = 6
\]

**Proof.** The Böröczky example for \( n = 9 \) in Figure 3.5 has 6 ordinary lines.
The lower bound of Csima and Sawyer [CS93] gives $e_2(9) \geq \lceil \frac{6 \cdot 9}{13} \rceil = 5$, so we have to show that $e_2(9) \neq 5$. If $e_2(9) = 5$, then Lemma 2.3 implies that $31 = 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7 + 28t_8$. If $t_8 = 1$, then $t_3 = 1$, but the only configuration with an 8-line and nine points is the trivial set of Theorem 2.2 and it doesn’t have any 3-line. Thus, $t_8 = 0$. If $t_7 = 1$, then $t_5 = 1$, but a configuration with one 7-line and one 5-line has at least 11 points, therefore $t_7 = 0$. Similarly, $t_6 = 0$ and the equation reduces to $31 = 3t_3 + 6t_4 + 10t_5$. Only if $t_5 = 1$ the equation has a solution. If $t_4 \geq 2$, then the configuration is geometrically impossible with only nine points. If $t_4 = t_5 = 1$, then two configurations are possible, but with at least with nine ordinary lines each one, that is impossible because $t_2 = 5$. The only remaining possibility is $t_5 = 1$ and $t_3 = 7$, and then the number of lines is $t_2 + t_3 + t_5 = 13$. But Lemma 3.1 (with $n = 9$ and $r = 4$) gives that at least the number lines is 15. Therefore, $t_5 \neq 1$, and then $t_2(9) \neq 5$, because the diophantine equation has no solutions.

The following example, with 13 points, was found by Crowe and McKee [CM68]. It has few ordinary lines and it doesn’t satisfy the Dirac-Motzkin conjecture as we’ll see in Section 3.4. To construct the set of points, let $AB$ be the common edge of two congruent regular pentagons in the Euclidean plane. Let the vertices of the pentagons be $ABCDE$ and $ABC'D'E'$, respectively. Let $M$ be the midpoint of $AB$, and $I$, $J$, $K$, $L$ the points on the line at infinity in the directions $ED$, $AB$, $CD$, $MD$ respectively. Then the resulting configuration is draw it in Figure 3.6 and has only 6 ordinary lines. This example is used in the proof of the following theorem.

**Theorem 3.7.** $e_2(13) = 6$

**Proof.** The lower bound of Csima and Sawyer [CS93] gives $e_2(13) \geq 6$, and the set of points in Figure 3.6 has 6 ordinary lines.
Table 3.1 gives the number of ordinary lines for small $n$. The numbers are calculated with the lower bound of Csima and Sawyer [CS93], the enhanced upper bound for $d = 2$ (Theorem’s 3.1–3.3) and the results that give the exact value of $e_2(n)$ in Section 2.2 (Theorem’s 2.4–2.5) and in this section (Theorem’s 3.5–3.7).

<table>
<thead>
<tr>
<th>$n$</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
<th>17</th>
<th>18</th>
<th>19-21</th>
<th>22</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_2(n)$</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>3</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>?</td>
<td>8</td>
<td>?</td>
<td>9</td>
<td>?</td>
<td>--</td>
<td>11</td>
</tr>
</tbody>
</table>

Table (3.1)

### 3.4 The number of ordinary lines for $n$ large

In Table 3.1 we can observe that the exact values for the minimum of ordinary lines coincide with the enhanced upper bounds found in Section 3.1, except for the value of $n = 13$, that we have $e_2(13) = 6 < 3\lfloor 13/4 \rfloor = 9$. So for large $n$, seems plausible that the Böröczky examples are the configurations that reach the minimum of ordinary lines. This was proven by Green and Tao [GT13] in the following theorem.

**Theorem 3.8** (Sharp threshold for Dirac-Motzkin). *There is an $n_0$ such that the following is true. If $n \geq n_0$, then*

$$e_2(n) = \begin{cases} 
  n/2 & \text{if } n \equiv 0 \pmod{2} \\
  3\lfloor n/4 \rfloor & \text{if } n \equiv 1 \pmod{2} 
\end{cases}.$$
And if a set $P$ reach the minimum of ordinary lines then, up to a projective transformation, $P$ is one of the Böröczky examples described in Section 3.1.

**Remark 3.2.** We can observe that there is a unique extremal example unless $n \equiv 1 \pmod{4}$, in which case there are two, the examples of Theorem 3.2 and Remark 3.1. Note that all the examples of Theorem 3.3 are equivalent up to rotation, and the same is true for the examples of Remark 3.1.

And then is resolved for large $n$, by using the previous theorem, the Dirac-Motzkin conjecture.

**Theorem 3.9** (Dirac-Motzkin conjecture). Suppose that $n \geq n_0$ for a sufficiently large absolute constant $n_0$. Then,

$$e_2(n) \geq n/2$$

The known small values of $n$ for which the Dirac-Motzkin conjecture is not true are $n = 7$, where the Böröczky example gives 3 ordinary lines and $n = 13$, where the configuration of Crowe and McKee of Figure 3.6 gives 6 ordinary lines.
Chapter 4

3-space case

Let $P$ be a subset of $n$ points of $\mathbb{R}^3$ with the property that every 3 points of $P$ span a plane and $P$ is not contained in a plane. In this chapter, we will always assume that $P$ has this property without mentioning it.

4.1 Enhanced upper bound for $e_3(n)$

In this section we prove the following upper bound for the minimum number of ordinary planes of a set of $n$ points $P \subset \mathbb{R}^3$ and $n \geq 8$,

$$e_3(n) \leq \begin{cases} \left\lfloor n \frac{n-1}{4} \right\rfloor, & \text{if } n \equiv 0 \pmod{2} \\ \frac{3}{8}n^2 - n + \frac{5}{8}, & \text{if } n \equiv 1 \pmod{4} \\ \frac{3}{8}n^2 - \frac{3}{2}n + \frac{17}{8}, & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

This upper bound uses the vertices of regular prisms and the vertices of regular prisms without one vertex, that are sets of points with few ordinary planes, as the following theorems show. We can check that this upper bound is strictly lower than the general upper bound of Theorem 2.2 for $n \geq 8$, so it improves the general upper bound.
CHAPTER 4. 3-SPACE CASE

Figure (4.1)

Theorem 4.1. If \( n \equiv 0 \pmod{4} \) and \( n \geq 8 \), then

\[
e_3(n) \leq \frac{1}{4}n^2 - n.
\]

Proof. The idea is to construct an example of a set of \( n \) points in 3-space with \( \frac{1}{4}n^2 - n \) ordinary planes.

Let \( P \) be the vertices of a prism with regular \( n/2 \)-gons as bases, as in Figure 4.1(a). Ordinary planes have exactly three points, two in one polygon and one in the other. Consider a point \( x \), we are going to count the ordinary planes that contain \( x \) and two points in the other polygon.

Let \( l \) be the tangent of the circumscribed circle of the polygon that pass through \( x \). Consider the ordinary plane \( H \) that contains \( x \) and two points of the other polygon, \( y \) and \( z \), that are contained in a line parallel to \( l \), we have \( \frac{1}{4}n - 1 \) ordinary planes of this type. These are all the ordinary planes that contain \( x \) and two points in the other polygon. Because if we pick other \( y, z \), as in Figure 4.1(b) that are contained in a line not parallel to \( l \), then we have that a point \( w \) of the polygon of \( x \) is contained in the plane \( H \), and then it is not ordinary.

By symmetry, since there are \( n \) points in \( P \), we have that there are exactly \( n(\frac{1}{4}n - 1) \) ordinary planes. \( \square \)

The prism examples with \( n \equiv 0 \pmod{2} \) that have few ordinary planes are related by projection, to the planar Böröczky examples that have few ordinary lines, as is described in the following Remark.

Remark 4.1. Let \( n \equiv 0 \pmod{4} \). We can project the prism with regular \( n/2 \)-gons as bases from any point \( x \), to the Böröczky example of Theorem 3.3. For example, Figure 3.2(a) is the projection of the prism for \( n = 12 \), from a vertex to the Böröczky set \( X_{11} \).
Theorem 4.2. If \( n \equiv 3 \pmod{4} \) and \( n \geq 11 \), then
\[
e_3(n) \leq \frac{3}{8} n^2 - \frac{3}{2} n + \frac{17}{8}
\]

Proof. Let \( P \) be the vertices of a prism with regular \( m/2 \)-gons as bases, where \( m \geq 12 \) and \( m \equiv 0 \pmod{4} \). By Lemma 2.3
\[
t_3 + 4t_4 + \left( \frac{m/2}{3} \right) t_{m/2} = \left( \frac{m}{3} \right)
\]
Using Theorem 4.1 we have that \( t_3 = \frac{1}{4} m^2 - m \), certainly \( t_{m/2} = 2 \) and solving equation (4.1) for \( t_4 \) we obtain,
\[
t_4 = \frac{m^3}{32} - \frac{m^2}{8} + \frac{m}{4}
\]
The set of points that will give us the upper bound is the prism \( P \) with one vertex \( x \) removed, that has \( n = m - 1 \) points. The number of ordinary planes is
\[
\frac{1}{4} m^2 - m - A + B
\]
where \( A \) is the number of ordinary planes of the prism containing \( x \) and \( B \) is the number of 4-planes of the prism containing \( x \). By symmetry, \( A = 3t_3/m \) and \( B = 4t_4/m \), therefore the number of ordinary planes of \( P \setminus x \) is
\[
\frac{3m^2}{8} - \frac{9m}{4} + 4 = \frac{3n^2}{8} - \frac{3n}{2} + \frac{17}{8}
\]

The previous theorem cannot be applied for the case \( n = 7 \) because in the prism with \( m = 8 \) vertices the \((m/2)\)-planes are in fact the 4-planes, and in equation (4.1) we’ll have \( t_{m/2} = 0 \).

If \( n \equiv 2 \pmod{4} \) the regular prism gives the improved upper bound too.

Theorem 4.3. If \( n \equiv 2 \pmod{4} \) and \( n \geq 10 \), then
\[
e_3(n) \leq \frac{1}{4} n^2 - \frac{1}{2} n
\]
Proof. Let \( P \) be the vertices of a prism with regular \( n/2 \)-gons as bases, as in Figure 4.1(a). Consider a point \( x \), we are going to count the ordinary planes that contain \( x \) and two points in the other polygon.

Let \( l \) be the tangent of the circumscribed circle of the polygon that pass through \( x \). Consider the ordinary plane \( H \) that contain \( x \) and two points of the other polygon,
y and z, that are contained in a line parallel to l, we have \( \frac{1}{4}n - \frac{1}{2} \) ordinary planes of this type. These are all the ordinary planes that contain \( x \) and two points in the other polygon. Because if we pick other \( y, z \), as in Figure 4.1(b) that are contained in a line not parallel to \( l \), then we have that a point \( w \) of the polygon of \( x \) is contained in the plane \( H \), and then it is not ordinary.

By symmetry, since there are \( n \) points in \( P \), we have that there are exactly \( n(\frac{1}{4}n - \frac{1}{2}) \) ordinary planes. \( \square \)

**Remark 4.2.** If \( n \equiv 2 \pmod{4} \), there is another example that has the same number of ordinary planes as the prism with regular \( n/2 \)-gons as bases. It’s the right antiprism with \( n \) vertices. A right antiprism with \( n \) vertices is a polyhedron composed of two parallel copies of a regular \( n/2 \)-gon, where one base is twisted by an angle of \( 180/n \) degrees respect the other and the line connecting the base centers is perpendicular to the base planes. Figure 4.2 shows an example for \( n = 10 \). To count the number of ordinary planes which has the right antiprism, we can use the same argument of the proof of the previous theorem.

![Figure 4.2](image)

**Remark 4.3.** Let \( n \equiv 2 \pmod{4} \). We can project the prism with regular \( n/2 \)-gons as bases or the right antiprism with \( n \) vertices, from any point \( x \) of these sets, to the Böröczky example of Remark 3.1. For example, Figure 3.3 is the projection of the prism for \( n = 10 \), from a vertex to a Böröczky set \( X_9 \).

**Theorem 4.4.** If \( n \equiv 1 \pmod{4} \) and \( n \geq 9 \), then

\[
e_3(n) \leq \frac{3}{8}n^2 - n + \frac{5}{8}
\]

**Proof.** Let \( P \) be the vertices of a prism with regular \( m/2 \)-gons as bases, where \( m \geq 10 \) and \( m \equiv 0 \pmod{4} \). By Lemma 2.3

\[
t_3 + 4t_4 + \binom{m/2}{3} t_{m/2} = \binom{m}{3}
\]
Using the previous theorem we have that $t_3 = \frac{1}{4}m^2 - \frac{1}{2}m$, certainly $t_{m/2} = 2$ and solving equation (4.2) for $t_4$ we obtain,

$$\frac{m^3}{32} - \frac{m^2}{8} + \frac{m}{8}$$

The set of points that will give us the upper bound is the prism $P$ with one vertex $x$ removed, that has $n = m - 1$ points. The number of ordinary planes is

$$\frac{1}{4}m^2 - m - A + B$$

where $A$ is the number of ordinary planes of the prism containing $x$ and $B$ is the number of 4-planes of the prism containing $x$. By symmetry, $A = 3t_3/m$ and $B = 4t_4/m$, therefore the number of ordinary planes of $P \setminus x$ is

$$\frac{3m^2}{8} - \frac{7m}{4} + 2 = \frac{3n^2}{8} - n + \frac{5}{8}$$

\[\square\]

### 4.2 The number of ordinary planes for small $n$

Theorem 4.5.

$$e_3(7) = 11$$

*Proof.* The cube with a vertex deleted has 11 ordinary planes because we can count in Figure 4.3, $t_4 = 6$, $t_5 = 0$ and $t_6 = 0$, and then applying Lemma 2.3 we get $t_3 = 11$.

![Figure (4.3)](image)
Therefore, it remains to prove the lower bound, $e_3(7) \geq 11$. Let $P$ be a set of points, and consider four points of $P$ that span a plane $H$, then the other three points span a plane $H'$. This two planes are different and intersect in a line $l$. Now we distinguish three different cases for the number of the points of $l$ that are in $P$:

1. The line $l$ contains no point of $P$. This is the case of the cube with a vertex deleted that is draw it in Figure 4.3. In this case $t_5 = 0$ and $t_6 = 0$, because suppose $H''$ is a plane with 5 points, then it must contain 3 points of $H \cap P$ or 3 points of $H' \cap P$, therefore $H''$ must be equal to $H$ or $H'$, and this planes have less than 5 points.

2. The line $l$ contains one point of $P$. Let this point be $z$, the two possible configurations are in Figure 4.4. We have $t_6 = 0$, because $H$ and $H'$ have less than 6 points, and if $H''$ is a 6-plane, then it must have three points of $H \cap H'' \cap P$, but then this three points would be in a line, and this is impossible. The point $z$ cannot belong to a $k$-plane for $k \geq 4$, different from $H$ or $H'$, because such a plane would contain either two points of $H \setminus \{z\}$ or two points of $H' \setminus \{z\}$ and must therefore be either $H$ or $H'$.
4.2. THE NUMBER OF ORDINARY PLANES FOR SMALL $N$

The configuration of Figure 4.5(a) has two 4-planes if the lines 23 and 14 are parallel to 56, and $H$ is a 5-plane, so $t_4 \leq 2$ and $t_5 = 1$, thus in this case $t_3 \geq 35 - 8 - 10 = 17$, by Lemma 2.3.

And the configuration of Figure 4.5(b) has five 4-planes if the pairs of lines (12,46), (23,56) and (13,45) are parallel. Therefore, $t_4 \leq 5$. We have that $t_5 = 0$, because suppose $H''$ is a plane with 5 points, then it must contain 3 points of $H \cap P$ or 3 points of $H' \cap P$, therefore $H''$ must be equal to $H$ or $H'$, and these planes have less than 5 points. Then, Lemma 2.3 implies $t_3 \geq 35 - 20 = 15$.

<table>
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<td>H'</td>
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(a)

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<tr>
<td>H'</td>
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</table>

(b)

Figure (4.5)

3. The line $l$ contains two points of $P$. Let these points be $z$ and $z'$, the three possible configurations are in Figure 4.6. The configuration of Figure 4.6(a) is, in fact, the trivial set of Theorem 2.2, so it has \( \binom{6}{2} = 15 \) ordinary planes.

The other two configurations, are the same because either $H$ or $H'$ contains five points and the other contains four points. We can suppose that $H'$ contains five points, Figure 4.6(c). Then the points $z$ and $z'$ cannot belong to any $k$-plane for $k \geq 4$, since such a plane would contain either two points of $H \setminus \{z, z'\}$ or two points of $H' \setminus \{z, z'\}$ and must therefore be either $H$ or $H'$. This configuration has at most two 4-planes if the lines 12 and 34 are parallel, therefore $t_4 \leq 2$, $t_5 = 1$ and Lemma 2.3 implies $t_3 \geq 35 - 8 - 10 = 17$. 

To find an improved lower bound for \( n = 9 \), first we have to prove the following lemma, that says that the Böröczky examples are the unique extremal examples. This lemma implies that if we project \( P \) from a point \( x \) that is not contained in a 5-plane, then the projection will not have a 4-line, and using Lemma 4.1, we have that the projection will not be a Böröczky example, so at least will have 5 ordinary lines, and then \( x \) is incident with at least 5 ordinary planes. This fact is useful to prove Theorem 4.6.

**Lemma 4.1.** Let \( P \subset \mathbb{P}^2_\mathbb{R} \) be a set of 8 points with 4 ordinary lines then, up to a projective transformation, \( P \) is the Böröczky example \( X_8 \) (Figure 4.7).

**Proof.** The proof is divided in two parts, in the first part we prove that \( P \) must have a 4-line, and in the second part we prove that \( P \), up to a projective transformation, is the Böröczky example \( X_8 \).
4.2. THE NUMBER OF ORDINARY PLANES FOR SMALL N

By contradiction, we’ll prove that there is a 4-line. Suppose that $t_4 = 0$, since $t_2 = 4$, Lemma 2.3 gives

$$24 = 3t_3 + 10t_5 + 15t_6 + 21t_7.$$  \hfill (4.3)

We have that $t_5 = t_6 = t_7 = 0$, because if $t_7 = 1$, then $t_3 = 1$, but a configuration with one 7-line and one 3-line has at least 9 points. If $t_6 = 1$, then $t_3 = 3$, but a configuration with one 6-line and three 3-lines has at least 10 points. If $t_5 \geq 1$, then the diophantine equation (4.3) has no solution. So, $t_5 = t_6 = t_7 = 0$, and equation (4.3) implies $t_3 = 8$.

Let $x$ be a point of $P$, then $x$ is incident with at most three 3-lines. Because if not, then we will have that $x$ is incident with 8 points or more (see Figure 4.8(a)), and $P$ would have at least nine points.

Since there are eight 3-lines and eight points, and each point is incident with at most three 3-lines, each point is incident with exactly three 3-lines. There are four 2-lines, so each point is incident with one 2-line. Therefore any point $x$ in $P$ is incident with the lines of Figure 4.8(b), this picture is called the line pencil of $x$. 
Up to isomorphism there are 4 possible configurations with 8 points having the line pencil of Figure 4.8(b). The possible combinations are drawn in Figure 4.9, that contain only the 3-lines that can be drawn as straight lines. The 2-lines are 12, 34, 56, 78. The configuration of Figure 4.9(b) is impossible, because the line pencil of 6 is 56, 246, 168, and 367, but then there would be a 4-line, the line 2367. The configuration of Figure 4.9(c) is impossible too, because the line pencil of 6 is 56, 168, 236 and 467, but then there would be a 4-line, the line 1467. And the last that is impossible is the configuration of Figure 4.9(d), because the line pencil of 6 is 56, 168, 267 and 346, but then it lacks the 2-line 34. So the only possible configuration is the one in Figure 4.9(a) and has the lines 12, 34, 56, 78, 135, 186, 174, 238, 257, 264, 367 and 458.

Now, we have to see that this configuration of Figure 4.9(a) can’t be embedded in the plane. By the fundamental theorem of projective geometry, there is a projectivity that transforms the configuration of Figure 4.9(a) to the configuration of Figure 4.10.
sending the points 1, 2, 3, 4 to the points with coordinates

\[
\begin{align*}
1 &= (0,0,1) \\
2 &= (0,1,0) \\
3 &= (1,0,0) \\
4 &= (1,1,1)
\end{align*}
\]

Thus, by the relations of incidence we deduce that the other points have coordinates

\[
\begin{align*}
5 &= (a,a,b) \\
6 &= (b-a,b,0) \\
7 &= (a,0,b) \\
8 &= (b-a,b,b-a)
\end{align*}
\]

where \(a, b \in \mathbb{R} \setminus \{0\}\). The points 4, 5 and 8 are contained in a 3-line, thus the determinant of this three points should be equal to zero, \(b^2 - ab + a^2 = 0\), but this equation doesn’t have a solution with \(a, b \in \mathbb{R} \setminus \{0\}\).

This is the second part of the proof, we are going to prove that if \(t_4 = 1\), then the configuration is, up to projective transformation, the Böröczky example \(X_8\). First, we place the 4-line in the line at infinity, \(l_\infty\), so \(P\) has 4 points in \(l_\infty\) and 4 points not in \(l_\infty\).
Consider a point in $l_\infty$, then the possible line pencils of this point are drawn in Figure 4.11, where the black point is the point in $l_\infty$ and the white points are not in $l_\infty$, and the other points at the line at infinity are not drawn in the pictures. Since, we have four 2-lines, we distinguish the following cases:

1. There is one point in $l_\infty$ of the type of Figure 4.11(c) and three points in $l_\infty$ of the type of Figure 4.11(a). This case can't be drawn in the projective plane because in Figure 4.12 we draw two black points of the type “a” and the third point of this type should be the intersection of the two dashed lines, but this point is never on the line at infinity. So, this case is impossible.

2. There are two points in $l_\infty$ of the type of Figure 4.11(a) and two points in $l_\infty$ of the type of Figure 4.11(b). In this case, $P$ is, up to projective transformation, the Böröczky example $X_8$. 

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**Figure (4.11)**

(a) ![Image](a)

(b) ![Image](b)

(c) ![Image](c)

**Figure (4.12)**

![Image](4.12)
Theorem 4.6. \( e_3(9) \geq 16 \).

Proof. First, we consider the three possibilities for the number of 5-planes separately, and we will show that \( t_3 \geq 15 \) for every case. Next, we are going to proof that \( t_3 \) can't be 15 so \( e_3(9) \geq 16 \).

1. There are two 5-planes. Here we distinguish two subcases:

(a) The intersection of the 5-planes is a line that contains one point \( x \in P \) (Figure 4.13(a)). Then, we project \( P \) from \( x \) to an 8 point configuration with two 4-lines, and 16 ordinary lines. From this, follows that \( x \) is incident with precisely 16 ordinary planes, and \( t_3 \geq 16 \).

(b) The intersection of the 5-planes is a line that contains two points \( x, y \in P \), and there is a point \( z \) not contained in the 5-planes (Figure 4.13(b)). Then, project \( P \) from \( x \) to an 8 point configuration shown in Figure 4.14. In this Figure we draw the case when the projection has the minimum of ordinary lines, that is 7. If we project from \( y \) we arrive at the same configuration, so it follows that \( x \) and \( y \) are contained in at least 7 ordinary planes each one.
Project $P$ from $z$ to a configuration with 8 points and with no 4-lines, so this configuration must have at least 5 ordinary lines. Because if it has 4 ordinary lines, then we get a contradiction from Lemma 4.1, since the extremal configuration for 8 points must have a 4-line. Therefore, $z$ is incident with at least 5 ordinary planes. Project $P$ from a point $w \in P \setminus \{x, y, z\}$ to a configuration with 8 points and at least $e_2(8) = 4$ ordinary lines, therefore $w$ is incident with at least 4 ordinary planes. Then, it follows that the number of ordinary planes of $P$ is

$$t_3 \geq \left\lceil \frac{2 \cdot 7 + 1 \cdot 5 + 6 \cdot 4}{3} \right\rceil = 15.$$

2. There is one 5-plane $H$. First we are going to prove that, at most, two points $\{x, y\}$ of the 5-plane have the property that projecting $P$ from one of this points we have the B"or"oczky set $X_8$. Consider the projection of $P$ from $x$ to a plane parallel from $H$. If this projection is the B"or"oczky set, then the projection of the points of $H$ from $x$ is formed by the points at infinity given by the directions of pairs of vertices of the square. In Figure 4.15(a) is drawn 4 lines that are incident with $x$, these lines are the directions of the vertices of a square and, then should contain the points of $H \cap P$ by the previous projection. But the same projection can be made for the point $y$ and the lines can be drawn similarly too as the figure shows. The points of $H \cap P$ must be on the intersections of these lines, and there are only two possibilities one $H \cap P$ equal to the white points (Figure 4.15(a)) and the other $H \cap P$ equal to the black points Figure 4.15(b). But this two cases are, in fact, the same configuration rotated 180 degrees.
4.2. THE NUMBER OF ORDINARY PLANES FOR SMALL N

So, if we project $P$ from a point of $(H \cap P) \setminus \{x, y\}$, we don’t project to the points at infinity given by the directions of the square. Therefore, at most we have two points in the 5-plane that project to $X_8$.

So, we have at most two points of the 5-plane that the projection of $P$ from these points is $X_8$, thus at most two points of the 5-plane are incident with 4 ordinary planes. The other points of the 5-plane are incident with at least 5 ordinary planes. And a point that is not in the 5-plane is incident with at least 5 ordinary planes too, since the projection of $P$ from this point hasn’t a 4-line. Therefore, the number of ordinary planes of $P$ is

$$t_3 \geq \left\lceil \frac{2 \cdot 4 + 3 \cdot 5 + 4 \cdot 5}{3} \right\rceil = 15.$$  

3. There isn’t any 5-plane. Then, every point is incident with at least 5 ordinary planes, since the projection from $P$ from any point hasn’t a 4-line. Therefore, the number of ordinary planes of $P$ is

$$t_3 \geq \frac{9 \cdot 5}{3} = 15.$$  

If $t_3 = 15$, then Lemma 2.3 implies that $69 = 4t_4 + 10t_5 + 20t_6 + 35t_7 + 56t_8$. This diophantine equation has no solution for nonzero $t_8$. If $t_7 = 1$, then any solution implies $t_5 \geq 1$, but a configuration with one 7-plane and at least one 5-plane has at least 10 points, therefore $t_7 = 0$ and the equation reduces to $69 = 4t_4 + 10t_5 + 20t_6$. But this equation has no solution, thus $t_3 \neq 15$ and $e_3(9) \geq 16$. \qed
4.3 The number of ordinary planes for large \( n \)

**Theorem 4.7.** There is an \( n_0 \) such that the following is true. If \( n \geq n_0 \) and even, then

\[
    e_3(n) = n \left\lfloor \frac{n-1}{4} \right\rfloor
\]

And if a set \( P \) reaches the minimum of ordinary planes then, if \( n \equiv 0 \pmod{4} \), up to a projective transformation, \( P \) is a regular prism with \( n/2 \)-gons as bases, and if \( n \equiv 2 \pmod{4} \), up to a projective transformation, \( P \) is a regular prism with \( n/2 \)-gons as bases or a right antiprism with \( n/2 \)-gons as bases (See Remark 4.2).

**Proof.** We’ll prove that \( e_3(n) = n \left\lfloor \frac{n-1}{4} \right\rfloor \) by contradiction. Suppose that there is a set \( P \) with fewer than \( n \left\lfloor \frac{n-1}{4} \right\rfloor \) ordinary planes. Let \( v_i \in P \), for \( i = 1, \ldots, n \) denote the points of the set \( P \). For \( i = 1, \ldots, n \), project the set \( P \) from a point \( v_i \), to a \( n-1 \) planar configuration, let \( k_i \) be the number of ordinary lines in this configuration. Then \( k_i \) is the number of ordinary planes that contain the vertex \( v_i \). The number of ordinary planes in \( P \) is

\[
    t_3 = \frac{k_1 + k_2 + \ldots + k_n}{3}
\]

Let \( Q \) be the prism with \( n/2 \)-gons as bases. Let \( w_i \in Q \), for \( i = 1, \ldots, n \) denote the vertices of the prism. For \( i = 1, \ldots, n \), project the set \( Q \) from a point \( w_i \), to the Böröczky set \( X_{n-1} \) that is done in Remarks 4.1 and 4.3. These Böröczky sets have an odd number of points and \( e_2(n) = 3 \left\lfloor \frac{n-1}{4} \right\rfloor \) ordinary lines, they are extremal examples by the result of Green and Tao in Theorem 3.8. The number of ordinary planes in \( Q \) is

\[
    n/3 \left[ \frac{n-1}{4} \right] = n \left\lfloor \frac{n-1}{4} \right\rfloor
\]

But if \( t_3 < n \left\lfloor \frac{n-1}{4} \right\rfloor \) then there is a \( k_i \) such that \( k_i < e_2(n) \) for some \( i \), and this is impossible.

Next, we are going to prove that, if a set \( P \) reaches the minimum of ordinary planes then, if \( n \equiv 0 \pmod{4} \), up to a projective transformation, \( P \) is a regular prism with \( n/2 \)-gons as bases, and if \( n \equiv 2 \pmod{4} \), up to a projective transformation, \( P \) is a regular prism with \( n/2 \)-gons as bases or a right antiprism with \( n/2 \)-gons as bases. Suppose that \( P \) reaches the minimum of ordinary planes, then the projections of \( P \) from any point of the set must have \( e_2(n-1) \) ordinary lines. Because all the projections must have at least \( e_2(n-1) \) ordinary lines, and if there is one projection with at least \( e_2(n-1) + 1 \) ordinary lines, then \( P \) doesn’t reach the minimum of ordinary planes, because the regular prism with \( n/2 \)-gons as bases has fewer ordinary planes.

If \( n \equiv 0 \pmod{4} \), then up to a projective transformation, any projection of \( P \) from a point of the set is the configuration of Theorem 3.3. Consider one of these
4.3. THE NUMBER OF ORDINARY PLANES FOR LARGE N

41

projections, that consists of one \((n/2 - 1)\)-line and a projective transformation of a regular \(n/2\)-gon. Thus, we have a \(n/2\)-plane \(H\) that projects to the \((n/2 - 1)\)-line. Then projecting from one point of \(P \setminus H\) to the plane \(H\), by the previous argument, since this projection has a \((n/2 - 1)\)-line too, it follows that the other \(n/2\) points of \(P \setminus H\) are contained in a plane \(H'\), and then \(t_{n/2} = 2\). Let \(P_1\) and \(P_2\) be the points of \(H \cap P\) and \(H' \cap P\) respectively.

Let \(l\) be the line that results from the intersection of \(H\) and \(H'\), and send this line to infinity by a projective transformation, from now on we will always assume that \(P\) has this transformation applied. Project \(P\) from a point of \(P_1\) to \(H'\), the projection is a regular \(n/2\)-gon formed by the points of \(P_2\) and \((n/2 - 1)\) points on the line at infinity, because the points of \(P_2\) are fixed points by this projection. In the same way, can be proved that the points of \(P_1\) form a regular \(n/2\)-gon too.

Consider the projection of \(P\) from a point of \(P_1\) to \(H'\), giving the example of Theorem 3.3. The points at the line at infinity can be obtained by projecting the points of \(P_1\) from a point \(x \in P_1\) or by projecting \(P_2\) from a point \(y \in P_2\). Thus, the directions determined by pairs of vertices of \(P_1\) or by pairs of \(P_2\) the same. We can transform \(P\) by a projective transformation so that \(P_1\) and \(P_2\) are circumscribed in a circle with radius 1, and that the line that connect the center of \(P_1\) and \(P_2\) is perpendicular to the plane \(H\).

In Figure 4.16(a) there is a hexagon and its rotation by 180/6 degrees about its center, we can see that the directions determined by pairs of vertices are the same in the two polygons, Figure 4.16(b) shows the odd case. We can determine the directions by the angle that make with the \(OX\) axis. Thus a regular polygon with \(n\) vertices determine the directions: \(\{0, \alpha, 2\alpha, \ldots, (n-1)\alpha\}\), where \(\alpha = 180/n\) degrees. If we rotate the polygon \(\beta\) degrees, where \(0 \leq \beta < 2\alpha\), then the angles of the directions of the rotated polygon are \(\{\beta, \beta + \alpha, \beta + 2\alpha, \ldots, \beta + (n-1)\alpha\}\). This implies that, if the polygon and its rotation by \(\beta\) degrees about its center determine the same directions then \(\beta = 0\) or \(\alpha\). Therefore, \(P\), is a regular prism with \(n/2\)-gons as bases or a right antiprism with \(n/2\)-gons as bases.

But it can’t be a right antiprism. Because, project \(P\) from \(x \in P_2\) to \(H\), we get the set \(X_{n+1}\) minus a point \(D\) on the line at infinity that corresponds to a direction determined by an edge of the \(n+1/2\)-gon. But this configuration is not the Böröczky example of Theorem 3.3 because the point \(D\) should be on the line at infinity that corresponds to a direction not determined by an edge of the \(n+1/2\)-gon.
If \( n \equiv 2 \) (mod 4), then up to a projective transformation, any projection of \( P \) from a point of the set could be the configuration of Theorem 3.2 or the configuration of Remark 3.1. We’ll prove that the projection, in fact, is always the one of Remark 3.1. Suppose that \( n \equiv 2 \) (mod 4), and that a projection of \( P \) from a point of the set is the one of Theorem 3.2. Then, the projection has a \( (n/2 - 1) \)-line, so \( P \) has a \( n/2 \)-plane, let this plane be \( H \). The projection of \( P \) from a point in \( P \setminus H \) to \( H \) must be the configuration of Remark 3.1, because if it was the configuration of Theorem 3.2, then we would have three colinear points in the set \( H \cap P \). But then the projection has a \( (n/2 - 1) \)-line, so \( P \) has two \( n/2 \)-planes, and therefore all the projections are of the type of Remark 3.1, because if not, then we would have three colinear points by the argument of the previous projection.

Applying the previous arguments for the case of \( n \equiv 0 \) (mod 4), we have that \( P \) is, up to a projective transformation, a regular prism with \( n/2 \)-gons as bases or a right antiprism with \( n/2 \)-gons as bases. Note, that in this case we have that \( n/2 \) is odd, so the right antiprism of 4.2 has the projections of \( P \) from points of the set equal to the Böröczky example of Remark 3.1 and then the antiprism is an extremiser.
Bibliography


