

Degree in Mathematics

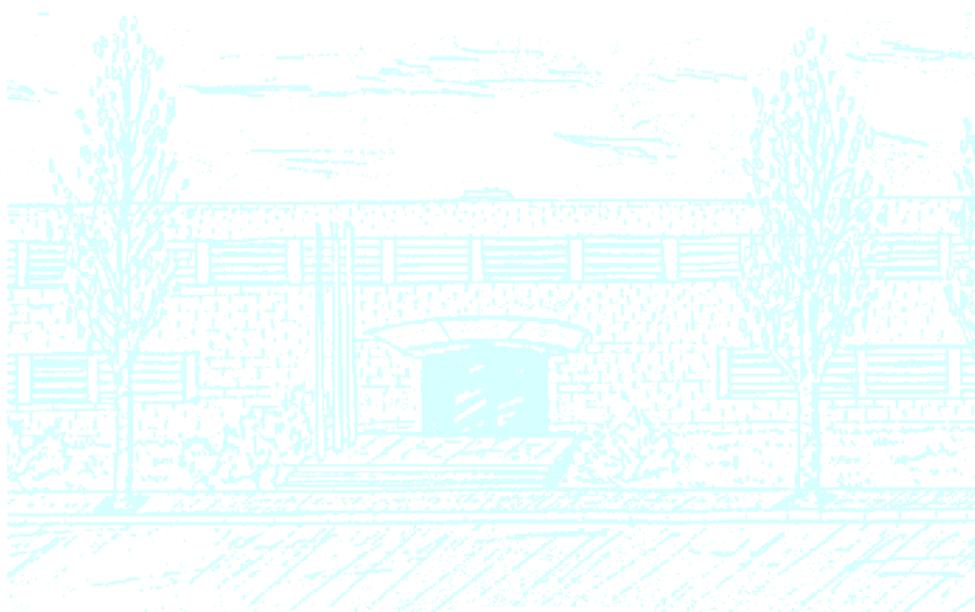
Title: The fundamental group and some applications in robotics.

Author: Homs Moreno, Josep.

Advisor: Pascual Gainza, Pere.

Department: Department I of Applied Mathematics.

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Bachelor degree thesis

**The fundamental group and some
applications in robotics.**

Josep Homs Moreno

Advisor: Pere Pascual Gainza

Abstract

The aim of this project is to study important techniques to determine if two topological spaces are homeomorphic (or homotopy equivalent) or not. Such techniques are the ones related with the fundamental group of a topological space. This project is aimed to cover widely the study of this area of knowledge, always in a very intuitive approach. In chapters one and two there is a motivation of the fundamental group concept. Chapter three shows some important computations throughout some examples. Chapter four introduces the Seifert-Van Kampen theorem and shows its power throughout more examples. This theorem will give some characterisations of really important topological spaces such as topological graphs, connected compact surfaces and torus knots among others. The proof of the Seifert-Van Kampen theorem is given in chapter five. Finally, chapter six contains a brief analysis of a research paper. In this paper there appear lots of the concepts covered in the project in an applied framework, different from the strictly theoretic one. This project uses concepts seen in introductory courses of topology and group theory.

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Chapter 1

Review of homotopy of maps

In this chapter we introduce all the concepts that will appear throughout the project. It is extremely important to fully understand them since they are the cornerstone of the fundamental group theory. Therefore, it is wise to study several examples in order to fully understand what motivates all definitions and what limitations they have.

1. Definitions

DEFINITION 1.1 (Curve). Consider $I = [0, 1] \subset \mathbb{R}$ and X a topological space. A curve or a path is a continuous function $\sigma : I \rightarrow X$. If $\sigma(0) = \sigma(1)$, then the curve is called a closed curve or a loop.

DEFINITION 1.2 (Homotopic maps). Consider two topological spaces X and Y and two continuous maps $f, g : X \rightarrow Y$. The maps f and g are said to be homotopic if there exists a continuous map $F : X \times I \rightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. In this case we will use the notation $f \simeq g$. The function F is called an homotopy between f and g .

The intuitive idea is that the function f is continuously deformed to the function g , as can be seen in figure (1).

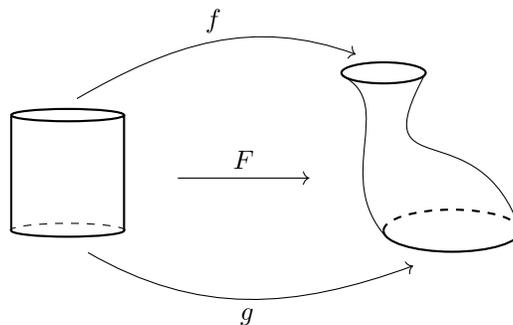


FIG. 1.

From this definition, the first result arises. Any path $\sigma : I \rightarrow X$ is homotopic to the constant curve $\varepsilon_{\sigma(0)}(t) = \sigma(0), \forall t \in I$, via the homotopy $F(s, t) = \sigma(st)$. Considering a fixed time t_0 , $F(s, t_0)$ is the path $\sigma(s)$, where $s \in [0, t_0]$. Therefore, this homotopy shortens gradually (and continuously) the path into a point. Hence, all curves are homotopic to a constant curve. The next definition avoids this situation.

DEFINITION 1.3 (Relative homotopy). *Consider $A \subseteq X$ a subspace of the space X . Two continuous maps $f, g : X \rightarrow Y$ are said to be homotopic relatively to A if there exists an homotopy $F : X \times I \rightarrow Y$ from f to g such that $F(a, t) = f(a) \forall t \in I$ and $\forall a \in A$. Note that in this case $f(a) = g(a)$. The notation used is $f \simeq_{rel(A)} g$ or simply $f \simeq_A g$.*

The concept of relative homotopy is apparently more restrictive than the concept of homotopy. Returning to the previous example, we have seen that $\sigma \simeq \varepsilon_{\sigma(0)}$ via the homotopy F , but this homotopy is not relative to $\{0, 1\}$ because for $t \neq 1$, $F(1, t) = \sigma(t) \neq \sigma(1)$. In the case of curves, the relative homotopy imposes the edges of the curve to be fixed. In figure (2) there are two paths homotopic relatively to $\{0, 1\}$ and it gives a visual example of how an homotopy deforms one curve to another.

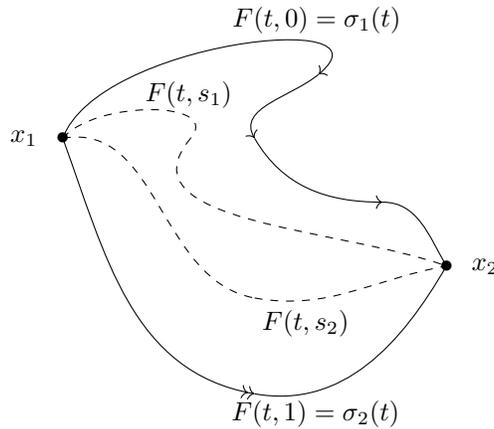


FIG. 2.

Remark that $\sigma_1, \sigma_2, F(-, s_1)$ and $F(-, s_2)$ are all homotopic relatively to $\{0, 1\}$ to each other. Remark also that if $A = \emptyset$, then the homotopy relative to A is just an homotopy.

On the other hand, figure (3) shows an example of two paths that are not homotopic relatively to $\{0, 1\}$. Both curves are in the space X , which is the rectangle without the circle. The reason why they are not homotopic relatively to $\{0, 1\}$ is that there is a *hole* that do not allow one curve to transform to the other continuously, that is, without *ripping* it.

Thanks to this definition we can generalise the concept of homeomorphism.

DEFINITION 1.4 (Homotopy equivalence). *Two given spaces X and Y are said to be homotopy equivalent or of the same homotopy type if there exist two maps*

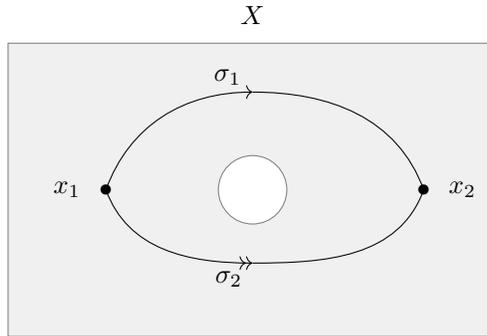


FIG. 3.

$f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_Y$. Two such functions f and g are called homotopy equivalences. In this case, we will use the evocative notation $X \simeq Y$.

Two homeomorphic spaces are homotopy equivalent, but the reciprocal is not always true. Therefore, the concept of homotopy equivalence is laxer than the concept of homeomorphism. However, we will see that this notion plays a highly important role in the fundamental group. The next two examples are very relevant particular cases of homotopy equivalence.

For the first example, consider the disc $D_3 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\} \subset \mathbb{R}^3$ and a point $\{y\} \subset D_3$. These two spaces are not homeomorphic because any map from D_3 to $\{y\}$ is not bijective, but they are of the same homotopy type. Consider the inclusion map $i : \{y\} \rightarrow D_3$ defined by $i(y) = y$ and the constant map $g : D_3 \rightarrow \{y\}$. On the one hand, $g \circ i = id_{\{y\}}$, so $g \circ i \simeq id_{\{y\}}$. On the other hand, $F : D_n \times I \rightarrow \{y\}$ defined as $F(x, t) = tx + (1-t)y$ is an homotopy from $i \circ g$ to id_{D_n} because F is continuous, $F(x, 0) = y = i(g(y))$ and $F(x, 1) = x = id_{D_n}$ ¹. Here it has been proved that D_3 is of the same homotopy type of a point. This example motivates the following definition.

DEFINITION 1.5 (Contractible space). *A space X is said to be contractible if it is homotopy equivalent to a point.*

Intuitively, a space is contractible if it can be deformed to a point within itself. For example, every convex subspace of \mathbb{R}^n is contractible via the straight line homotopy. In particular, \mathbb{R}^n and any convex subspace are contractible. The proof is exactly the same as in the previous example changing D_3 for any convex space. Also as a corollary, the image of any curve (closed or non-closed) $\sigma(I)$ in a contractible space is homotopy equivalent to a point. Finally, as we saw at the beginning of the section, any curve σ is homotopic to a constant curve $\varepsilon_{\sigma(0)}$. Hence, the set $\sigma(I) \subset X$ is contractible to the set $\{\sigma(0)\}$ (even if X is not convex).

¹Note that an homotopy of this type, i.e., $F(x, t) = tg(x) + (1-t)f(x)$ is called the *linear homotopy* or the *straight line homotopy*.

For the second example, consider the cylinder and the circle

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, -1 \leq z \leq 1\}$$

$$S^1 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1, z = 0\}$$

Consider also the inclusion map $i : S^1 \rightarrow C$ and the map $r : C \rightarrow S^1$ defined by $r(x, y, z) = (x, y, 0)$. On the one hand, $r \circ i = id_{S^1}$, so $r \circ i \simeq id_{S^1}$. On the other hand, $F((x, y, z), t) = (x, y, tz)$ is an homotopy between $i \circ r$ and id_C .

This is an example of what is called a *deformation retract*. A deformation retract is a particular case of homotopy equivalence and it will be highly important as we will see in further examples. Thus, let us introduce two more definitions.

DEFINITION 1.6 (Retraction). *A subspace A of X is called a retract of X if there exists a continuous map $r : X \rightarrow A$ such that $r|_A = r \circ i = id_A$ (where i is the inclusion map $i : A \rightarrow X$). The function r is called a retraction.*

DEFINITION 1.7 (Deformation retract). *Let $A \subseteq X$ be a subspace. A is said to be a deformation retract of X if there exists a retraction $r : X \rightarrow A$ such that $i \circ r \simeq id_X$. That is to say, if there exists an homotopy $F : X \times I \rightarrow X$ such that $F(x, 0) = id_X(x) = x$ and $F(x, 1) \in A$, $\forall x \in X$.*

For example, consider $X = \mathbb{R}^n \setminus \{(0, 0)\}$ and $A = S^{n-1} \subset \mathbb{R}^n$. Then the map $N : X \rightarrow A$ defined by

$$N(z) = \frac{z}{\|z\|}$$

is a retraction. Moreover, it is also a deformation retract. The linear homotopy

$$F : \mathbb{R}^n \setminus \{(0, 0)\} \times I \rightarrow \mathbb{R}^n \setminus \{(0, 0)\},$$

$$F(x, t) = t \frac{x}{\|x\|} + (1 - t)x$$

proves that $i \circ r \simeq id_{\mathbb{R}^n \setminus \{(0, 0)\}}$.

As we saw in earlier examples, if A is a deformation retract of X , then A and X are homotopy equivalent.

First of all notice that a deformation retraction is a special case of a retraction. A retract is a way of deforming X into A . A deformation retract forces the points x of X to travel from x to $r(x) \in A$ continuously along a curve. This last example illustrates the explanation.

Consider the example in figure (4). X is a rectangle without two little circles and $A = \{x_0\}$. Sets $Y_1 = \sigma_1(I)$ and $Y_2 = \sigma_2(I)$ are the image of two closed curves based at x_0 .

It is important the fact that both curves are closed and based at x_0 . That is $\sigma_i(0) = \sigma_i(1) = x_0$ for $i = 1, 2$. Here, A is a retract of both Y_1 and Y_2 via the constant retraction $r(x, y) = x_0$. Moreover, A is a deformation retract of Y_2 by using the straight line homotopy. However, the linear homotopy can not be used in Y_1 because there are straight lines that would intersect the circle, which do not belong to our space X . Notice that the circle on top plays no role in our discussion because curves σ_i do not enclose that circle. We state that A is not a deformation retract of Y_1 . The proof of this statement uses what is called the *index* of a curve,

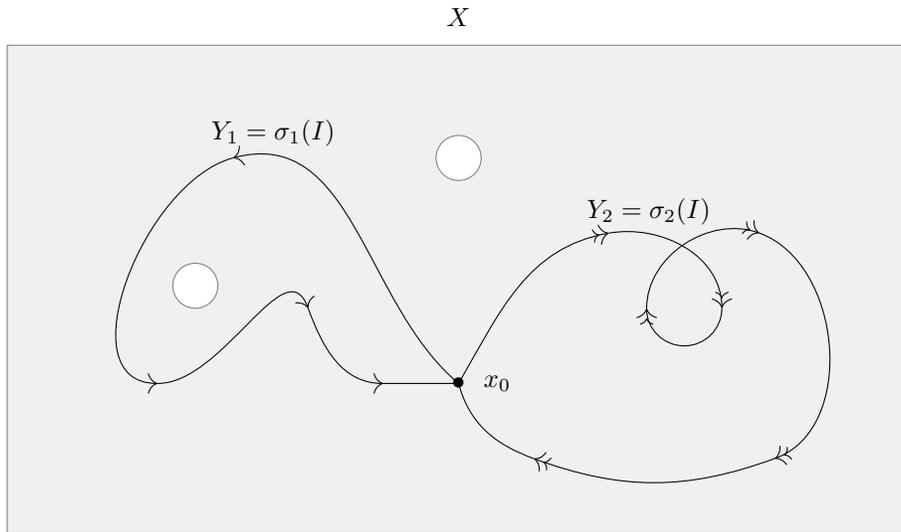


FIG. 4.

which is usually introduced in introductory topology course. However, it can be understood with an intuitive explanation. If A was a deformation retract, then all points (x, y) of Y_1 should describe a path from (x, y) to x_0 within X . This movement should be continuous in the variables $(x, y, t) \in X \times I$, i.e, this movement should not *rip* the curve. As the figure suggests, there is no way to perform this movement without ripping the curve. So, in all (and always intuitively), a retract can *rip* the space whereas the deformation retract can not. This example suggest that Y_1 and Y_2 can not be homotopy equivalent within X . The fact that Y_1 encloses a hole do not allow Y_1 to be deformed continuously to Y_2 .

2. Basic homotopy properties

There are some useful properties to take into account.

PROPOSITION 1.8. *The relation \simeq_A defined in the set of all continuous maps from X to Y is an equivalence relation. Notice that as a corollary, by considering $A = \emptyset$, we obtain that the homotopy of functions is an equivalence relation.*

PROOF. Consider two spaces X, Y and a $A \subseteq X$ a subspace.

- (1) *Reflexivity:* A function $f : X \rightarrow Y$ is homotopic to itself relatively to A via the homotopy $F(x, t) := f(x)$.
- (2) *Symmetry:* If F is the homotopy relative to A from f to g , then the map $G(x, t) = F(x, 1 - t)$ is an homotopy relative to A from g to f .

- (3) *Transitivity*: If F is the homotopy of $f \simeq_A g$ and G is the homotopy of $g \simeq_A h$, then the homotopy of $f \simeq_A h$ is

$$H(x, t) = \begin{cases} F(x, 2t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ G(x, 2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The application H is continuous except, maybe, at $t = \frac{1}{2}$. However,

$$H\left(x, \frac{1}{2}\right) = \begin{cases} F(x, 1) = g(x) & \forall x \in X, \\ G(x, 0) = g(x) & \forall x \in X. \end{cases}$$

Both F and G coincide in $t = \frac{1}{2}$ for all $x \in X$, so by the pasting lemma, H is continuous. Moreover,

$$\begin{aligned} H(x, 0) &= F(x, 0) = f(x) & \forall x \in X, \\ H(x, 1) &= G(x, 1) = h(x) & \forall x \in X, \end{aligned}$$

which means that H is an homotopy from f to h . Finally, we must check that this homotopy is relative to A . Consider $a \in A$. Then, using the hypothesis that F and G are homotopies relative to A

$$H(a, t) = \begin{cases} F(a, t) = f(a) & \forall t \in I, \\ G(a, t) = h(a) & \forall t \in I. \end{cases}$$

□

The next proposition states that the homotopy has a *proper behaviour* with the composition of functions.

PROPOSITION 1.9. *Let $f, g : X \rightarrow Y$ be functions such that $f \simeq g$. If $\psi : W \rightarrow X$ and $\varphi : Y \rightarrow Z$ are continuous maps, then $\varphi \circ f \simeq \varphi \circ g$ and $f \circ \psi \simeq g \circ \psi$.*

PROOF. Let $F : X \times I$ be the homotopy from f to g . The function $\varphi \circ F : X \times I \rightarrow Z$ is continuous (because it is a composition of continuous functions) and it is an homotopy from $\varphi \circ f$ to $\varphi \circ g$ because

$$\begin{aligned} (\varphi \circ F)(x, 0) &= \varphi(F(x, 0)) = \varphi(f(x)) = (\varphi \circ f)(x) & \text{and} \\ (\varphi \circ F)(x, 1) &= \varphi(F(x, 1)) = \varphi(g(x)) = (\varphi \circ g)(x). \end{aligned}$$

On the other hand, $F \circ (\psi \times id) : W \times I \rightarrow Y$ is an homotopy from $f \circ \psi$ to $g \circ \psi$ because

$$\begin{aligned} (F \circ (\psi \times id))(\omega, 0) &= F(\psi(\omega), 0) = f(\psi(\omega)) = (f \circ \psi)(\omega) & \text{and} \\ (F \circ (\psi \times id))(\omega, 1) &= F(\psi(\omega), 1) = g(\psi(\omega)) = (g \circ \psi)(\omega). \end{aligned} \quad \square$$

COROLLARY 1.10. *Consider $f_1, f_2 : X \rightarrow Y$ and $g_1, g_2 : Y \rightarrow Z$ with $f_1 \simeq f_2$ and $g_1 \simeq g_2$. Then,*

$$g_1 \circ f_1 \simeq g_1 \circ f_2.$$

The proof of (1.10) is immediate from proposition (1.9)

PROPOSITION 1.11. *The relation \simeq (homotopy equivalence) defined in the set of topological spaces is an equivalence relation.*

PROOF. Reflexivity. A space X is clearly homotopy equivalent to itself by considering the functions $f : X \rightarrow X$ and $g : X \rightarrow X$ of the definition (1.3) to be the identity functions.

Symmetry. If $X \simeq Y$, then there exist functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. But this is exactly the definition of $Y \simeq X$.

Transitivity. Consider $X \simeq Y \simeq Z$ and consider the functions $f_1, f_2, g_1, g_2, F_1, F_2, G_1, G_2$ defined by with

$$X \begin{array}{c} \xrightarrow{f_1} \\ \xleftarrow{f_2} \end{array} Y \begin{array}{c} \xrightarrow{g_1} \\ \xleftarrow{g_2} \end{array} Z$$

$$\begin{aligned} F_1 : f_2 \circ f_1 &\simeq id_X, & F_2 : f_1 \circ f_2 &\simeq id_Y, \\ G_1 : g_2 \circ g_1 &\simeq id_Y, & G_2 : g_1 \circ g_2 &\simeq id_Z. \end{aligned}$$

We want to find two functions $h_1 : X \rightarrow Z$ and $h_2 : Z \rightarrow X$ such that $h_2 \circ h_1 \simeq id_X$ and $h_1 \circ h_2 \simeq id_Z$. Let's find these functions using proposition (1.9).

$$\begin{aligned} g_2 \circ g_1 &\simeq id_Y \Rightarrow f_2 \circ (g_2 \circ g_1) \simeq f_2 \circ id_Y = f_2 \Rightarrow f_2 \circ (g_2 \circ g_1) \circ f_1 \simeq f_2 \circ f_1 \simeq id_X, \\ f_1 \circ f_2 &\simeq id_X \Rightarrow g_1 \circ (f_1 \circ f_2) \simeq g_1 \circ id_X = g_1 \Rightarrow g_1 \circ (f_1 \circ f_2) \circ g_2 \simeq g_1 \circ g_2 \simeq id_Z. \end{aligned}$$

Therefore

$$\begin{aligned} (f_2 \circ g_2) \circ (g_1 \circ f_1) &= f_2 \circ (g_2 \circ g_1) \circ f_1 \simeq id_X, \\ (g_1 \circ f_1) \circ (f_2 \circ g_2) &= g_1 \circ (f_1 \circ f_2) \circ g_2 \simeq id_Z. \end{aligned}$$

So the functions that we are looking for are

$$\begin{aligned} h_1 &= g_1 \circ f_1, \\ h_2 &= f_2 \circ g_2. \end{aligned}$$

□

Chapter 2

The fundamental group

Given two spaces X and Y , we want to determine whether they are homeomorphic or not. We already know some techniques to prove that two spaces are not homeomorphic using that compactness, connectivity, arcwise connectivity or separation among other properties remain invariant for homeomorphisms. The fundamental group theory pursues the same goal by analysing *holes* in the space. The way to *detect* these holes is by using closed curves. As we saw in the first chapter, paths together with relative homotopies are able to detect some holes. Our goal is to assign a group to every topological space that depends on the holes in such a way that non-isomorphic groups will lead to non-homeomorphic spaces.

1. Definition and properties

DEFINITION 2.1 (Product of paths). *Let ω be a path in X from x_0 to x_1 and σ another path in X from x_1 to x_2 (i.e. $\omega(1) = \sigma(0)$). Then we define a path $\omega * \sigma$ as*

$$(\omega * \sigma)(s) = \begin{cases} \omega(2s) & \text{if } s \in [0, \frac{1}{2}], \\ \sigma(2s - 1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

This path is called the product or composition of paths.

Notice that $\omega * \sigma$ is a path indeed. It is continuous except, maybe, at $s = \frac{1}{2}$. However, $(\omega * \sigma)(\frac{1}{2}) = \omega(1) = \sigma(0)$ and therefore, by the pasting lemma, the product is continuous for all s .

PROPOSITION 2.2. *Let ω and σ be paths such that $\omega(1) = \sigma(0)$ and consider ω' and σ' paths such that $\omega \simeq_{\{0,1\}} \omega'$ and $\sigma \simeq_{\{0,1\}} \sigma'$. Then, $\omega * \sigma \simeq_{\{0,1\}} \omega' * \sigma'$.*

PROOF. By hypothesis, there exists an homotopy F from ω to ω' and an homotopy G from σ to σ' . Consider now the map $H : I \times I \rightarrow X$ defined by

$$H(s, t) = \begin{cases} F(2s, t) & \text{if } s \in [0, \frac{1}{2}] \\ G(2s - 1, t) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

This is the homotopy from $\omega * \sigma$ to $\omega' * \sigma'$ that we are looking for. First of all, H is continuous except, maybe, for $s = \frac{1}{2}$. However,

$$\begin{cases} H(\frac{1}{2}, t) = F(2\frac{1}{2}, t) = x_1, \\ H(\frac{1}{2}, t) = G(2\frac{1}{2} - 1, t) = x_1. \end{cases}$$

Both F and G coincide in $s = \frac{1}{2}$ for all $t \in I$, so by the pasting lemma H is continuous. Moreover, at time $t = 0$,

$$H(s, 0) = \begin{cases} F(2s, 0) = \omega(2s) & 0 \leq s \leq \frac{1}{2}, \\ G(2s - 1, 0) = \sigma(2s - 1) & \frac{1}{2} \leq s \leq 1, \end{cases}$$

which is exactly the definition of $(\omega * \sigma)(s)$. In the same way, at time $t = 1$

$$H(s, 1) = \begin{cases} F(2s, 1) = \omega'(2s) & 0 \leq s \leq \frac{1}{2}, \\ G(2s - 1, 1) = \sigma'(2s - 1) & \frac{1}{2} \leq s \leq 1, \end{cases}$$

which is exactly the definition of $(\omega' * \sigma')(s)$. Finally, we must check that the homotopy is relative to $\{0, 1\}$.

$$\begin{cases} H(0, t) = F(0, t) = x_0 \quad \forall t \in I, \\ H(1, t) = G(1, t) = x_2 \quad \forall t \in I. \end{cases} \quad \square$$

Notice that we proved that the proposition is true with any homotopic paths, not only with homotopic paths relative to $\{0, 1\}$.

PROPOSITION 2.3. *Consider the following relation. Two paths are said to be related if they are homotopic relatively to $\{0, 1\}$. This relation $\simeq_{\{0,1\}}$ is an equivalence relation.*

PROOF. This proof is an corollary of proposition (1.8). Just consider $A = \{0, 1\}$. \square

Proposition (2.3) allows us to consider the set of classes of homotopic paths

$$[I, X]_{\{0,1\}} := \{\text{Paths } \sigma : I \rightarrow X\} / \simeq_{\{0,1\}}.$$

Moreover, in this set we can define an internal operation

$$[\omega] * [\sigma] = [\omega * \sigma]$$

for all ω and σ such that $\omega(1) = \sigma(0)$. The internal operation is well defined by proposition (2.2), which means that the result of the operation is the same for any candidates ω of $[\omega]$ and σ of $[\sigma]$ that we choose. This operation satisfies almost the group axioms. However there are some axioms that are not fulfilled.

PROPOSITION 2.4. *The product $*$ satisfies:*

- (1) *Associativity.* If the product $[\omega] * ([\sigma] * [\rho])$ is defined, then the product $([\omega] * [\sigma]) * [\rho]$ is also defined and they are both equal.
- (2) *Existence of neutral element.* Given a point $x \in X$, consider the constant path $\varepsilon_x(s) = x$ for all $s \in I$. If ω is a path from x_0 to x_1 , then

$$[\varepsilon_{x_0}] * [\omega] = [\omega] \quad \text{and} \quad [\omega] * [\varepsilon_{x_1}] = [\omega].$$

(3) *Existence of oposite element.* If ω is a path from x_0 to x_1 , we define the path $\bar{\omega}(s) = \omega(1-s)$. Then

$$[\omega] * [\bar{\omega}] = [\varepsilon_{x_0}] \quad \text{and} \quad [\bar{\omega}] * [\omega] = [\varepsilon_{x_1}].$$

PROOF. (1) The product $[\omega] * ([\sigma] * [\rho])$ is defined if and only if the product $\omega * (\sigma * \rho)$ is defined. In this case, $\omega(1) = \sigma(0)$ and $\sigma(1) = \rho(0)$. Therefore, $(\omega * \sigma) * \rho$ is defined and hence $([\omega] * [\sigma]) * [\rho]$ is also defined. Now let's prove that

$$(\omega * \sigma) * \rho \simeq_{\{0,1\}} \omega * (\sigma * \rho).$$

First let's calculate these products

$$\begin{aligned} ((\omega * \sigma) * \rho)(s) &= \begin{cases} (\omega * \sigma)(2s) & \text{if } s \in [0, \frac{1}{2}], \\ \rho(2s-1) & \text{if } s \in [\frac{1}{2}, 1], \end{cases} \\ &= \begin{cases} \omega(4s) & \text{if } s \in [0, \frac{1}{4}], \\ \sigma(4s-1) & \text{if } s \in [\frac{1}{4}, \frac{1}{2}], \\ \rho(2s-1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases} \\ (\omega * (\sigma * \rho))(s) &= \begin{cases} \omega(2s) & \text{if } s \in [0, \frac{1}{2}], \\ (\sigma * \rho)(2s-1) & \text{if } s \in [\frac{1}{2}, 1], \end{cases} \\ &= \begin{cases} \omega(2s) & \text{if } s \in [0, \frac{1}{2}], \\ \sigma(4s-2) & \text{if } s \in [\frac{1}{2}, \frac{3}{4}], \\ \rho(4s-3) & \text{if } s \in [\frac{3}{4}, 1]. \end{cases} \end{aligned}$$

As we can see, the difference between the two paths is the velocity in which we move along ω , ρ and σ . Therefore, the homotopy just adjusts these *time lapses*

$$F(s, t) = \begin{cases} \omega\left(\frac{4s}{1+t}\right) & \text{if } s \in [0, \frac{t}{4} + \frac{1}{4}], \\ \sigma(4s-t-1) & \text{if } s \in [\frac{t}{4} + \frac{1}{4}, \frac{t}{4} + \frac{1}{2}], \\ \rho\left(\frac{ts-t-2}{2-t}\right) & \text{if } s \in [\frac{t}{4} + \frac{1}{2}, 1]. \end{cases}$$

Let's check that it is an homotopy indeed. As it is usual in this kind of proofs, it is *only* necessary to apply the pasting lemma to prove that the homotopy is continuous.

$$\begin{aligned} \text{For } s = \frac{t+1}{4} : & \begin{cases} \omega\left(\frac{4}{1+t} \frac{t+1}{4}\right) = \omega(1) = x_1, \\ \sigma(4\frac{t+1}{4} - t - 1) = \sigma(0) = x_1, \end{cases} \\ \text{For } s = \frac{t+2}{4} : & \begin{cases} \sigma\left(4\frac{t+2}{4} - t - 1\right) = \sigma(1) = x_2, \\ \rho\left(\frac{4\frac{t+2}{4} - t - 2}{2-t}\right) = \rho(0) = x_2. \end{cases} \end{aligned}$$

On the other hand, $F(s, 0) = ((\omega * \sigma) * \rho)(s)$ because

$$F(s, 0) = \begin{cases} \omega(4s) & \text{if } s \in [0, \frac{1}{4}], \\ \sigma(4s-1) & \text{if } s \in [\frac{1}{4}, \frac{1}{2}], \\ \rho(2s-1) & \text{if } s \in [\frac{1}{2}, 1], \end{cases}$$

which is the exact definition of $(\omega * \sigma) * \rho$. With the same reasoning, it can be checked that $F(s, 1) = \omega * (\sigma * \rho)$. Finally, it must be checked that the homotopy is relative to $\{0, 1\}$, that is, that the origin of ω and the ending of ρ are fixed. This can be immediately verificated by substituting

$$F(0, t) = \omega(0), \quad F(1, t) = \rho(1) \quad \text{for all } t \in I.$$

(2) We want to prove that

$$\omega * \varepsilon_{x_1} \simeq_{\{0,1\}} \omega \quad \text{and} \quad \varepsilon_{x_0} * \omega \simeq_{\{0,1\}} \omega.$$

The homotopy is $G : I \times I \rightarrow X$ defined by

$$G(s, t) = \begin{cases} \omega\left(\frac{2s}{2-t}\right) & \text{if } s \in [0, \frac{2-t}{2}], \\ x_1 & \text{if } s \in [\frac{2-t}{2}, 1]. \end{cases}$$

Let's quickly check that it is an homotopy relative to $\{0, 1\}$. The function G is continuous for all s , except, maybe, for $s = \frac{2-t}{2}$ with $t \in [0, 1]$. But

$$G\left(\frac{2-t}{2}, t\right) = \omega\left(\frac{2s}{2-t}\right)_{|s=\frac{2-t}{2}} = x_1|_{s=\frac{2-t}{2}} = x_1,$$

so it is continuous by the pasting lemma. On the other hand, $G(0, t) = \omega(t)$ and $G(1, t) = \varepsilon_{x_1}$ for all $t \in [0, 1]$. Moreover, the homotopy is relative to $\{0, 1\}$ because $G(s, 0) = \omega(0) = x_0$ and $G(s, 1) = x_1$ for all $s \in [0, 1]$.

(3) Remark that $\bar{\omega}$ is the reversed path of ω . In other words, if ω starts at x_0 and ends at x_1 , the path $\bar{\omega}(t) = \omega(1-t)$ is the reversed path that starts at x_1 and ends at x_0 . Therefore, the product of paths $\omega * \bar{\omega}$ consists on going from x_0 to x_1 along ω and then return from x_1 to x_0 along $\bar{\omega}$ as well. The homotopy is

$$H(s, t) = \begin{cases} \omega(2ts) & \text{if } s \in [0, \frac{1}{2}], \\ \omega(2t(1-s)) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

To check that H is an homotopy relative to $\{0, 1\}$ we proceed exactly in the same way as in the previous example. For this reason we omit the verification. \square

The operation $*$ is not defined for every pair of paths in $[I, X]_{\{0,1\}}$. However, this operation is defined for every pair of closed curves that start and end at some fixed point $x_0 \in X$. Finally, we reach the definition of the fundamental group.

DEFINITION 2.5 (Fundamental group). *Let X be a space and $x_0 \in X$. A path in X that starts and ends at x_0 is called a closed curve based on x_0 . The set*

$$\pi_1(X, x_0) = \{[\omega] \mid \omega : I \rightarrow X, \omega(0) = \omega(1) = x_0\}$$

is called the fundamental group of X relative to x_0 ¹.

Notice that proposition (2.4) gives $\pi_1(X, x_0)$ a structure of group, so the name of this set is justified.

PROPOSITION 2.6. *The set $\pi_1(X, x_0)$ with the product $*$ is a group.*

PROOF. First of all notice that the fundamental group is well-defined thanks to proposition (2.3). What's more, the operation $*$ is defined for every pair of elements of the set. The fact that it is a group is an immediate consequence of proposition (2.4). \square

¹When talking about the fundamental group we will usually use the more convenient notation \simeq instead of $\simeq_{\{0,1\}}$. The reader must bear in mind that despite the notation, we are always referring to the homotopy relative to $\{0, 1\}$.

A question that naturally pops up is whether the fundamental group changes when the base point x_0 changes. The next proposition gives an answer.

DEFINITION 2.7. Consider $x_0, x_1 \in X$ and the respective fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$. Suppose that there is a path $\alpha : I \rightarrow X$ from x_0 to x_1 (i.e., $\sigma(0) = x_0$ and $\sigma(1) = x_1$). We define the function $\hat{\alpha} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ by

$$\hat{\alpha}([\omega]) = [\bar{\alpha}] * [\omega] * [\alpha].$$

Notice that if ω is a closed curve based on x_0 , then $\bar{\alpha}([\omega])$ is a closed curve based on x_1 . The trajectory of the curve $\hat{\alpha}([\omega])$ is to move from x_1 to x_0 , then move all along ω and finally return from x_0 to x_1 .

PROPOSITION 2.8. The function $\hat{\alpha} : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ is a group isomorphism.

PROOF. First we must check that $\bar{\alpha}$ is a group morphism.

$$\begin{aligned} \hat{\alpha}([\omega] * [\sigma]) &= [\bar{\alpha}] * [\omega * \sigma] * [\alpha] = [\bar{\alpha}] * [\omega * \alpha * \bar{\alpha} * \sigma] * [\alpha] \\ &= ([\bar{\alpha}] * [\omega] * [\alpha]) * ([\bar{\alpha}] * [\sigma] * [\alpha]) = \hat{\alpha}([\omega]) * \hat{\alpha}([\sigma]). \end{aligned}$$

Finally, the morphism $\hat{\beta}$ induced by $\beta = \bar{\alpha}$ is the inverse morphism of $\hat{\alpha}$.

$$\begin{aligned} (\hat{\beta} \circ \hat{\alpha})([\omega]) &= \hat{\beta}([\bar{\alpha} * \omega * \alpha]) = [\bar{\beta}] * [\bar{\alpha} * \omega * \alpha] * [\beta] \\ &= [\alpha] * [\bar{\alpha} * \omega * \alpha] * [\bar{\alpha}] = ([\alpha] * [\bar{\alpha}]) * [\omega] * ([\alpha] * [\bar{\alpha}]) = [\omega]. \end{aligned}$$

The same reasoning yields that $(\hat{\alpha} \circ \hat{\beta})([\omega]) = [\omega]$ and hence, $\hat{\alpha}$ is an isomorphism. \square

As an immediate consequence, two points $x_0, x_1 \in X$ connected by a path have the same fundamental group, where the same means isomorphic. Thanks to proposition (2.8), the definition (2.9) is well-defined.

DEFINITION 2.9 (Simply connected space). A space X is said to be simply connected if it is path connected and its fundamental group $\pi_1(X, x_0) = 1$ for some point $x_0 \in X$.

As an example, contractible spaces are simply connected. Sure enough, a contractible space is homotopy equivalent to a point. Therefore, any curve in this space is homotopy to constant curve relative to $\{0, 1\}$. As a particular case, $\pi_1(\mathbb{R}^n, x_0) = 1$ for every $x_0 \in \mathbb{R}^n$.

2. Homeomorphism invariance

The next step is to find a relation between a function $f : X \rightarrow Y$ and the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(Y, f(x_0))$. Consider $f : X \rightarrow Y$ a function and $x_0 \in X$, $y_0 = f(x_0)$. A compact way to express these hypothesis is by writing $f : (X, x_0) \rightarrow (Y, y_0)$. Notice that if ω is a closed curve based on x_0 , then $f \circ \omega$ is a closed curve based on y_0 .

DEFINITION 2.10. Let $f : (X, x_0) \longrightarrow (Y, y_0)$ be a continuous map. We can define a function $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ by $f_*([\omega]) = [f \circ \omega]$. The function f_* is called the morphism induced by f relative to x_0 .

The next proposition justifies the name of f_* .

PROPOSITION 2.11. The function $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ is well-defined and is a group morphism.

PROOF. The function f_* is well-defined thanks to proposition (1.9). This means that if $[\omega] = [\omega']$, then $[f \circ \omega] = [f \circ \omega']$. To see that f_* is a morphism, it must be checked that $f_*([\omega] * [\sigma]) = f_*([\omega]) * f_*([\sigma])$. Notice that

$$(\omega * \sigma)(s) = \begin{cases} \omega(2s) & \text{if } s \in [0, \frac{1}{2}], \\ \sigma(2s - 1) & \text{if } s \in [\frac{1}{2}, 1]. \end{cases}$$

and

$$(f \circ (\omega * \sigma))(s) = \begin{cases} (f \circ \omega)(2s) & \text{if } s \in [0, \frac{1}{2}], \\ (f \circ \sigma)(2s - 1) & \text{if } s \in [\frac{1}{2}, 1], \end{cases}$$

which is exactly the definition of $(f \circ \omega) * (f \circ \sigma)$. Therefore,

$$\begin{aligned} f_*([\omega] * [\sigma]) &= f_*([f \circ (\omega * \sigma)]) = [f \circ (\omega * \sigma)] = \\ &= [(f \circ \omega) * (f \circ \sigma)] = [f \circ \omega] * [f \circ \sigma] = f_*([\omega]) * f_*([\sigma]). \end{aligned} \quad \square$$

Finally, the so mentioned relation is shown in theorem (2.12)

THEOREM 2.12 (Homeomorphism invariance). If f is an homeomorphism, then f_* is a group isomorphism, which means that $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

PROOF. The proof is not difficult at all. It only requires finding the inverse function of f_* . Intuitively, the inverse function of f_* should be the one induced by the inverse function of the homeomorphism f .

- (1) If $f : (X, x_0) \longrightarrow (Y, y_0)$ and $g : (Y, y_0) \longrightarrow (Z, z_0)$, then

$$(g \circ f)_* : \pi_1(X, x_0) \longrightarrow \pi_1(Z, z_0)$$

satisfies that $(g \circ f)_* = g_* \circ f_*$. This is true because

$$(g \circ f)_*([\omega]) = [g \circ f \circ \omega] = g_*([f \circ \omega]) = (g_* \circ f_*)([\omega]).$$

- (2) On the other hand, the identity function $id : (X, x_0) \longrightarrow (X, x_0)$ induces the identity function in $\pi_1(X, x_0)$:

$$(id)_* = id_{\pi_1(X, x_0)}.$$

This is true because

$$id_*([\omega]) = [id \circ \omega] = [\omega] = id([\omega]).$$

- (3) Finally, if $f : (X, x_0) \longrightarrow (Y, y_0)$ is an homeomorphism and $g : (Y, y_0) \longrightarrow (X, x_0)$ is its inverse function, then

$$\begin{aligned} g_* \circ f_* &= (g \circ f)_* = (id_X)_* = id_{\pi_1(X, x_0)} \quad \text{and} \\ f_* \circ g_* &= (f \circ g)_* = (id_Y)_* = id_{\pi_1(Y, y_0)}. \end{aligned}$$

Therefore, both groups are isomorphic because f_* and g_* are each other's inverse morphisms.

□

3. Homotopy invariance

Theorem (2.12) has a more general formulation involving homotopies. The theorem asserts that it is not necessary to ask for a homeomorphism to get an isomorphism between fundamental groups, it is only necessary to ask for homotopy equivalence. The reader will check in further sections that thanks to this laxer view, the calculus of a fundamental group becomes easier. Let's formalise this theorem.

PROPOSITION 2.13. *Let $f, g : X \rightarrow Y$ be continuous maps, $F : f \simeq g$ an homotopy from f to g and $\alpha : I \rightarrow Y$ the path $\alpha(t) = F(x_0, t)$ from $f(x_0)$ to $g(x_0)$. Then, $g_* = \hat{\alpha} \circ f_*$, i.e. the diagram in figure (1) is commutative:*

$$\begin{array}{ccc} & & \pi_1(Y, f(x_0)) \\ & \nearrow^{f_*} & \downarrow \hat{\alpha} \\ \pi_1(X, x_0) & & \pi_1(Y, g(x_0)) \\ & \searrow_{g_*} & \end{array}$$

FIG. 1.

PROOF. Checking that $g_*[\omega] = (\hat{\alpha} \circ f_*)([\omega])$ is equivalent to check that $g \circ \omega \simeq \bar{\alpha} * (f \circ \omega) * \alpha$ because

$$\begin{aligned} g_*[\omega] &= [g \circ \omega] \text{ and} \\ (\hat{\alpha} \circ f_*)([\omega]) &= \hat{\alpha}([f \circ \omega]) = [\bar{\alpha}] * [f \circ \omega] * [\alpha] = [\bar{\alpha} * (f \circ \omega) * \alpha]. \end{aligned}$$

First, the path $\bar{\alpha} * (f \circ \omega) * \alpha$ is

$$\begin{aligned} ((\bar{\alpha} * (f \circ \omega)) * \alpha)(s) &= \begin{cases} \bar{\alpha}(4s) & s \in [0, \frac{1}{4}], \\ (f \circ \omega)(4s - 1) & s \in [\frac{1}{4}, \frac{1}{2}], \\ \alpha(2s - 1) & s \in [\frac{1}{2}, 1], \end{cases} \\ &= \begin{cases} \alpha(1 - 4s) & s \in [0, \frac{1}{4}], \\ (f \circ \omega)(4s - 1) & s \in [\frac{1}{4}, \frac{1}{2}], \\ \alpha(2s - 1) & s \in [\frac{1}{2}, 1]. \end{cases} \end{aligned}$$

By hypothesis, $\alpha(t) = F(x_0, t)$, so the last equality can be rewritten as

$$((\bar{\alpha} * (f \circ \omega)) * \alpha)(s) = \begin{cases} F(x_0, 1 - 4s) & s \in [0, \frac{1}{4}], \\ F(\omega(4s - 1), 0) & s \in [\frac{1}{4}, \frac{1}{2}], \\ F(x_0, 2s - 1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

Let's note $H : I \times I \longrightarrow I$ the homotopy that we want to find from $g \circ \omega$ to $\bar{\alpha} * (f \circ \omega) * \alpha$. Last equality can be slightly modified to get the homotopy H .

$$H(s, t) = \begin{cases} F(x_0, (1 - 4s)(1 - t)) & s \in [0, \frac{1}{4}], \\ F(\omega(4s - 1), t) & s \in [\frac{1}{4}, \frac{1}{2}], \\ F(x_0, h(s, t)) & s \in [\frac{1}{2}, 1], \end{cases}$$

where $h : I \times I \longrightarrow I$ is a function that must verify:

$$\begin{aligned} h(s, 0) &= 2s - 1 \text{ (because we want } F(x_0, 2s - 1) \text{ for } t = 0), \\ h(s, 1) &= 1 \text{ (because we want } F(x_0, 1) = g(x_0) \text{ for } t = 1), \\ h\left(\frac{1}{2}, t\right) &= t \text{ (because we want } H \text{ to be continuous).} \end{aligned}$$

Such a function is not so difficult to find. For example, imposing these conditions in a function like $h(s, t) = as + bt + cst + d$, it results that $a = 2$, $d = -1$, $c = -2$ and $b = 2$. So, in all,

$$H(s, t) = \begin{cases} F(x_0, (1 - 4s)(1 - t)) & s \in [0, \frac{1}{4}], \\ F(\omega(4s - 1), t) & s \in [\frac{1}{4}, \frac{1}{2}], \\ F(x_0, 2(s(1 - t) + t) - 1) & s \in [\frac{1}{2}, 1]. \end{cases}$$

It remains to verify that it is the homotopy that we want.

- (1) The map H is continuous except, maybe, at $s = \frac{1}{4}$ and $s = \frac{1}{2}$. For the prior case,

$$\begin{aligned} F\left(x_0, 1 - 4\frac{1}{4}(1 - t)\right) &= F(x_0, t) \text{ and} \\ F\left(\omega\left(4\frac{1}{4} - 1\right), t\right) &= F(\omega(0), t) = F(x_0, t). \end{aligned}$$

For the latter case,

$$\begin{aligned} F\left(\omega\left(4\frac{1}{2} - 1\right), t\right) &= F(\omega(1), t) = F(x_0, t) \text{ and} \\ F\left(x_0, 2\left(\frac{1}{2}(1 - t) + t\right) - 1\right) &= F(x_0, t). \end{aligned}$$

These equalities are true for all $t \in I$ and hence, H is continuous by the pasting lemma.

- (2) For $t = 0$, $H(s, 0) = \bar{\alpha} * (f \circ \omega) \circ \alpha$. For $t = 1$,

$$\begin{aligned} H(s, 1) &= \begin{cases} F(x_0, 1) & s \in [0, \frac{1}{4}], \\ F(\omega(4s - 1), 1) & s \in [\frac{1}{4}, \frac{1}{2}], \\ F(x_0, 1) & s \in [\frac{1}{2}, 1], \end{cases} \\ &= \begin{cases} \varepsilon_{g(x_0)}(4s) & s \in [0, \frac{1}{4}], \\ (g \circ \omega)(4s - 1) & s \in [\frac{1}{4}, \frac{1}{2}], \\ \varepsilon_{g(x_0)}(2s - 1) & s \in [\frac{1}{2}, 1], \end{cases} \\ &= ((\varepsilon_{g(x_0)} * (g \circ \omega)) * \varepsilon_{g(x_0)})(s) \simeq (g \circ \omega)(s). \end{aligned}$$

(3) Finally, H is relative to $\{0, 1\}$ because

$$H(0, t) = F(x_0, 1) = g(x_0),$$

$$H(1, t) = F(x_0, 1) = g(x_0). \quad \square$$

COROLLARY 2.14. *Let $f, g : (X, x_0) \rightarrow (Y, y_0)$ be functions such that $f \simeq_{\{x_0\}} g$. Then $f_* = g_*$.*

PROOF. The path $\alpha(t) = F(x_0, t)$ is the constant path ε_{y_0} because $F(x_0, t) = f(x_0) = y_0$ for all $t \in I$. Finally, using proposition (2.13)

$$g_*([\omega]) = (\hat{\alpha} \circ f_*)([\omega]) = [\bar{\alpha}] * [f \circ \omega] * [\alpha] = [f \circ \omega] = f_*([\omega]).$$

Thus, $f_* = g_*$. \square

COROLLARY 2.15. *Let X and Y be homotopy equivalent via the functions $f : X \rightarrow Y$ and $g : Y \rightarrow X$. Consider that $x_0 \in X$ and $f(x_0) = y_0$. Then $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$ is an isomorphism.*

Notice that this corollary implies that $g_* : \pi_1(Y, y_0) \rightarrow \pi_1(X, x_0)$ is also an isomorphism.

PROOF. Consider F and G the homotopies from $g \circ f$ to id_X and from $f \circ g$ to id_Y respectively. Consider also $\alpha(t) = F(x_0, t)$ the path from $(g \circ f)(x_0)$ to x_0 and $\beta(t) = G(f(x_0), t)$ the path from $(f \circ g)(f(x_0))$ to $f(x_0)$. Using proposition (2.13), we deduce

$$(id_X)_* = \hat{\alpha} \circ (g \circ f)_* \quad \text{and} \quad (id_Y)_* = \hat{\beta} \circ (f \circ g)_*.$$

Using results (1) and (2) seen in the proof of theorem (2.12),

$$id_{\pi_1(X, x_0)} = \hat{\alpha} \circ g_* \circ f_* \quad \text{and} \quad id_{\pi_1(Y, f(x_0))} = \hat{\beta} \circ f_* \circ g_*.$$

Due to the fact that $\hat{\alpha}$ and $\hat{\beta}$ are isomorphisms, we deduce

$$g_* \circ f_* = \hat{\alpha}^{-1} \quad \text{and} \quad f_* \circ g_* = \hat{\beta}^{-1}.$$

What latest equalities say is that the composition of f_* and g_* is an isomorphism in both directions. It can only happen if both functions are injective and surjective. Therefore, both functions are isomorphisms. \square

Finally, we get to the mentioned theorem

THEOREM 2.16 (Homotopy invariance). *Let X and Y be path connected and homotopy equivalent spaces. Then, $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$ for all $x_0 \in X$ and for all $y_0 \in Y$.*

PROOF. Let $f : X \rightarrow Y$ be an homotopy equivalence. Therefore,

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, f(x_0))$$

is an isomorphism. On the other hand, let $\alpha : I \rightarrow I$ be a path from $f(x_0)$ to y_0 . Such a path exists because Y is path connected. Then, $\hat{\alpha} : \pi_1(X, f(x_0)) \rightarrow \pi_1(Y, y_0)$ is an isomorphism. Hence, the composition

$$\hat{\alpha} \circ f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

is an isomorphism. \square

Before the chapter finishes let's see a final example of theorem (2.16). Consider the spaces of figure (3). Let A be the lemniscate (the figure similar to an eight), $X = A \times [0, 1]$ and $Y = \mathbb{T}^2 \setminus D$, with $D \subset Y$ homeomorphic to a disc of \mathbb{R}^2 . First, it is easy to check that A is a deformation retract of X with the retraction

$$r : \begin{array}{ccc} X = A \times [0, 1] & \longrightarrow & A \\ (x, t) & \longrightarrow & (x, 0). \end{array}$$

Secondly, A is also a deformation retract of Y . We give a visual proof in figure (2).

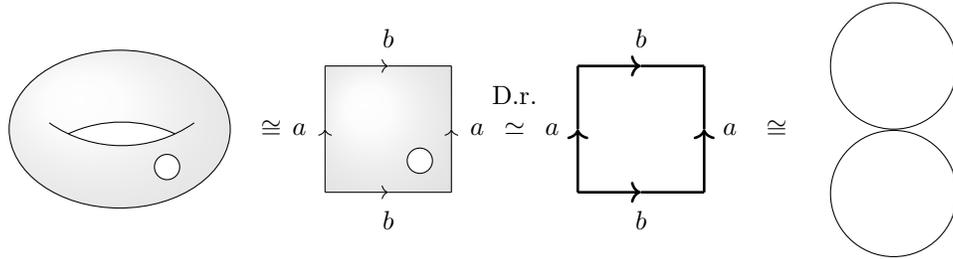


FIG. 2.

Remember that deformation retracts are particular cases of homotopy equivalence. Therefore,

$$X \simeq A \simeq Y.$$

Since \simeq induces an equivalence relation (proposition (1.11)), we get $X \simeq Y$. Notice that it is not immediate to prove directly that $X \simeq Y$. To prove it, it is easier to take an intermediate step using another set. Finally, using theorem (2.16), we conclude that

$$\pi_1(X, x_0) = \pi_1(A, a_0) = \pi_1(Y, y_0).$$

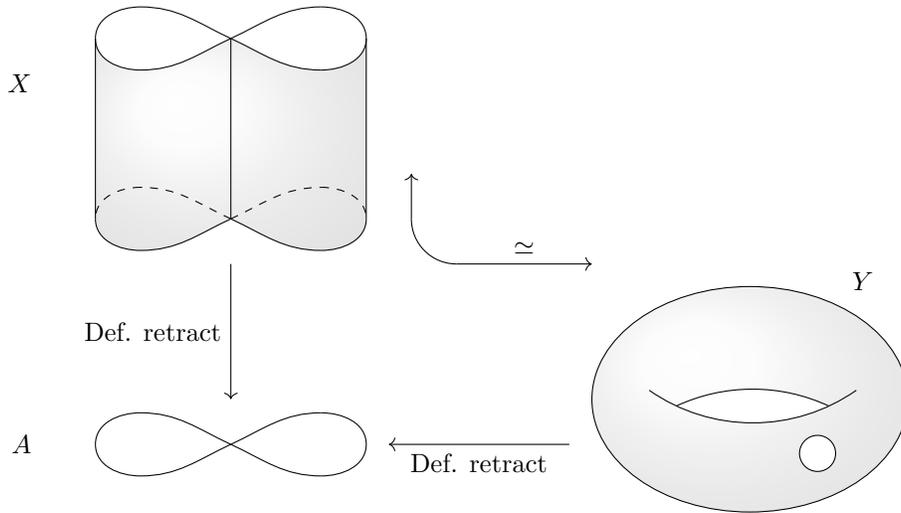


FIG. 3.

Chapter 3

Computing some fundamental groups

In this chapter there are calculations of typical fundamental groups. Computing the fundamental group is rough most times, but there are special cases in which it can be calculated. Some cases are the circle, the torus, the sphere and the projective space among others. The circle is usually introduced before the projective space because techniques used in the computation of projective space's fundamental group are generalisations of circle calculations. However, in this section we will study directly the general case so calculations of the mentioned examples will become corollaries. Finally, this section is straightforward, which means that we will only visit the necessary contents to accomplish our purpose.

1. Actions, orbit spaces, coverings and lifts

Let's first take a look at the necessary concepts about groups and actions.

DEFINITION 3.1. *Let X be a topological space and G a group (which is a topological space with discrete topology). A (left) group action of G on X is a continuous map*

$$\begin{aligned} G \times X &\longrightarrow X \\ (g, x) &\longrightarrow g \cdot x \end{aligned}$$

that satisfies

- (1) $1 \cdot x = x \quad \forall x \in X$,
- (2) $g(h \cdot x) = (gh) \cdot x \quad \forall g, h \in G$, and $\forall x \in X$.

In this case X is called a G -space.

For any fixed element $g \in G$, the application

$$\begin{aligned} g : X &\longrightarrow X \\ x &\longrightarrow g \cdot x \end{aligned}$$

is called a translation by g .

It is immediate to check that the translation by a fixed element g is bijective. The inverse of the translation by g is the translation by g^{-1} . Finally, the translation is continuous by hypothesis, so it is an homeomorphism.

DEFINITION 3.2 (Orbit space). *Let X be a G -space and $x \in X$. The orbit of x is the set*

$$Gx = \{g \cdot x \mid g \in G\},$$

i.e., is the set of all possible translations of x by elements g of G . The set of all orbits X/G with the quotient topology is called the orbit space of G on X . Equivalently, X/G is the quotient space with the equivalence relation $x \sim y$ iff there exist $g \in G$ such that $g \cdot x = y$.

From now on, we use the convenient notation gx to refer to the product $g \cdot x$.

The following examples will be revisited many times in this section:

- (1) Consider $X = \mathbb{R}$ and $G = \mathbb{Z}$. The group action is $nx = x + n$ for every $n \in \mathbb{Z}$ and for every $x \in \mathbb{R}$. The orbit of $x \in \mathbb{R}$ is $\mathbb{Z}x = \{x + n \mid n \in \mathbb{Z}\}$. Therefore, $\mathbb{R}/\mathbb{Z} = [0, 1]/\{0 \sim 1\}$, which is homeomorphic to \mathbb{S}^1 . Hence, the circle can be seen as an orbit space, $\mathbb{S}^1 \cong \mathbb{R}/\mathbb{Z}$.
- (2) Consider $X = \mathbb{S}^n$ and $G = \mathbb{Z}_2$. The group action is

$$gx = \begin{cases} x & \text{if } g = 1, \\ -x & \text{if } g = -1. \end{cases}$$

The the orbit of $x \in \mathbb{S}^n$ is $(\mathbb{Z}_2)x = \{x, -x\}$. Then, $X/G = \mathbb{S}^n/\{p \sim -p \mid p \in \mathbb{S}^n\} \cong \mathbb{P}^n$. Hence, the projective space can be seen as an orbit space.

- (3) Consider $X = \mathbb{R}^2$ and $G = \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2$. The action is $(n, m)(x, y) = (x + n, y + m)$. The orbit of (x, y) is $\mathbb{Z}^2(x, y) = \{(x + n, y + m) \mid n, m \in \mathbb{Z}\}$. Thus, $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$. In general, the n -dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ can be seen an orbit space.

PROPOSITION 3.3. *Let X be a G -space. The quotient projection $\pi : X \rightarrow X/G$ is an open map.*

PROOF. Let $U \subseteq X$ be an open set. We shall prove that $\pi(U)$ is an open set in X/G with the quotient topology, which is equivalent to prove that $\pi^{-1}(\pi(U))$ is an open set of X . By definition,

$$\pi^{-1}(\pi(U)) = \bigcup_{g \in G} gU.$$

Since g is a homeomorphism, so gU is an open set and therefore, the arbitrary union of open sets is an open set. \square

DEFINITION 3.4 (Properly discontinuous action). *Let X be a G -space. The action of G on X is said to be properly discontinuous if for all $x \in X$ there exist an open set U with $x \in U$ such that*

$$g \cdot U \cap g' \cdot U = \emptyset$$

for all $g, g' \in G$ with $g \neq g'$.

It is easy to check that all actions in previous examples are properly discontinuous. Let's move on to the necessary concepts of covering spaces.

DEFINITION 3.5 (covering space). *Let \tilde{X} and X be two topological spaces and $p : \tilde{X} \rightarrow X$ a continuous map. The function p is said to be a covering if*

- (1) p is surjective.
 (2) For all $x \in X$ there is an open set U with $x \in U$ such that

$$p^{-1}(U) = \bigcup_{j \in J} U_j$$

for some collection $\{U_j \subseteq \tilde{X} \mid j \in J\}$ satisfying

- (a) U_j is open for all $j \in J$.
 (b) $U_j \cap U_k = \emptyset$.
 (c) $p|_{U_j} : U_j \rightarrow U$ is an homeomorphism.

The space \tilde{X} is called a covering space of X , X is called the base space, p is called a covering map and the whole object $p : \tilde{X} \rightarrow X$ is called a covering.

Here there are some examples of coverings to understand the concept.

- Let X be the bottom plane X and \tilde{X} be copies of X at different heights and parallel to X of figure (1). The map p is the vertical projection to the bottom plane. Then, if $U \subsetneq X$ is an open set, $p^{-1}(U)$ is the union of disjoint open sets U_i of \tilde{X} . What's more, p is an homeomorphism once it is restricted to any of the open disjoint subsets of \tilde{X} .

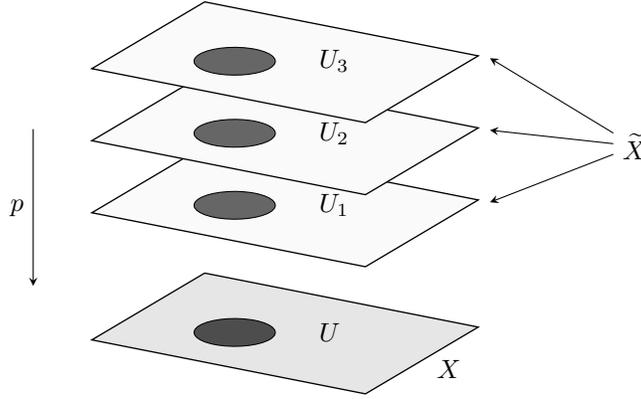


FIG. 1.

- The set \mathbb{R} is a covering space of \mathbb{S}^1 with the covering map

$$p(t) = (\cos(2\pi t), \sin(2\pi t)).$$

A covering space can be easily understood with the example of the circle. We can think of the real line \mathbb{R} as an helix spread out above the circle like in figure (2). For example, consider $(1, 0) \in \mathbb{S}^1$. Then $p^{-1}(1, 0) = 2\pi k$ for all $k \in \mathbb{Z}$. In the same way, if $U \subset \mathbb{S}^1$ is a *small* open set, $p^{-1}(U)$ is a union of disjoint open sets. The function p is surjective (but not injective), but it is an homeomorphism once it is restricted to any of the open disjoint subsets. This is a non-trivial example in which the preimage of an open set is an infinite union of sets.

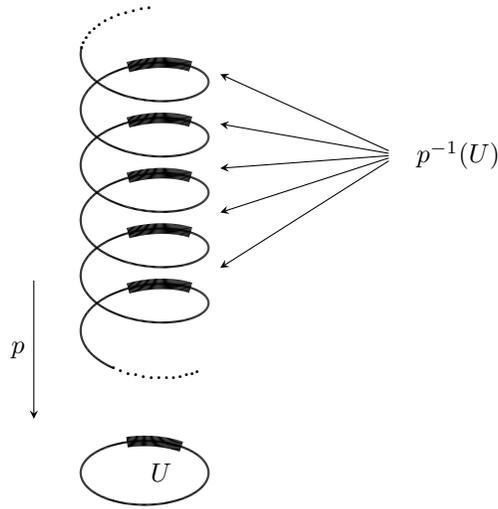


FIG. 2.

- Another non-trivial example is the one in figure (3). Consider $X = \mathbb{S}^1$ to be the circle and consider \tilde{X} to be the border of a Möbius strip, which is homeomorphic to \mathbb{S}^1 . The projection can be understood easily in the figure. The preimage of an open set corresponds to the union of two open sets of \tilde{X} . This is a non-trivial example in which the preimage of an open set is a finite union of sets.

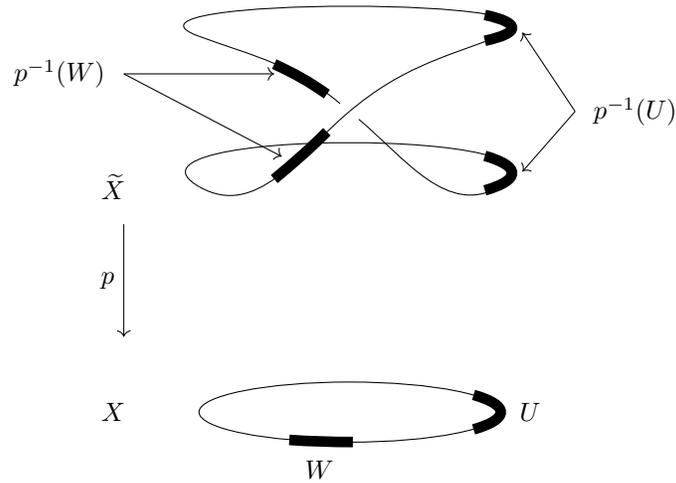


FIG. 3.

PROPOSITION 3.6. *Let X be a G -space with a properly discontinuous action. Then the projection to the quotient space $p : X \rightarrow X/G$ is a covering.*

PROOF. The quotient projection $p : X \rightarrow X/G$ is surjective and continuous. Furthermore, p is open by proposition (3.3). Let $x \in X$ and U be an open neighbourhood of x that satisfies the proper discontinuity condition. Since p is an open map, $p(U)$ is an open neighbourhood of $p(x) = Gx$. Thus,

$$p^{-1}(p(U)) = \bigcup_{g \in G} gU,$$

with $\{gU \mid g \in G\}$ being disjoint open subsets of X thanks to the proper discontinuity. In addition,

$$p|_{gU} : gU \rightarrow p(U)$$

is a continuous open and bijective map and hence an homeomorphism. \square

The following statements result immediately from proposition (3.6):

- (1) The set \mathbb{R} is a covering space of \mathbb{R}/\mathbb{Z} . The covering map is

$$\begin{aligned} p : \mathbb{R} &\longrightarrow \mathbb{R}/\mathbb{Z} \\ t &\longrightarrow [t] = \{t + k \mid k \in \mathbb{Z}\}. \end{aligned}$$

- (2) The set \mathbb{S}^n is a covering space of $\mathbb{S}^n/(\mathbb{Z}_2)$. The covering map is

$$\begin{aligned} p : \mathbb{S}^n &\longrightarrow \mathbb{S}^n/(\mathbb{Z}_2) \\ p &\longrightarrow [p] = \{p, -p\}. \end{aligned}$$

- (3) The set \mathbb{R}^n is a covering space of $\mathbb{R}^n/\mathbb{Z}^n$. The covering map is

$$\begin{aligned} p : \mathbb{R}^n &\longrightarrow \mathbb{R}^n/\mathbb{Z}^n \\ (x_1, \dots, x_n) &\longrightarrow [(x_1, \dots, x_n)] = \{(x_1 + k_1, \dots, x_n + k_n) \mid k_i \in \mathbb{Z}\}. \end{aligned}$$

Let's revisit the example of figure (2). Consider a path $f : I \rightarrow \mathbb{S}^1$. We can think of this path as the trajectory of a particle that moves along the circle. Intuitively, moving around the circle is equivalent to move along the helix and project this movement to the circle. Rigorously, we can define a function (called a *lift* of f) $\tilde{f} : I \rightarrow \mathbb{R}$ to the covering space such that $p \circ \tilde{f} = f$. This is exactly what theorem (3.7) talks about.

THEOREM 3.7 (Existence and uniqueness of lifts). *Let $p : \tilde{X} \rightarrow X$ be a covering. Consider a path $f : I \rightarrow X$ and a point $a \in \tilde{X}$ with $p(a) = f(0)$. Then there exists a unique path $\tilde{f} : I \rightarrow \tilde{X}$ such that $(p \circ \tilde{f}) = f$ and $\tilde{f}(0) = a$. Such an \tilde{f} is called a lift of f .*

PROOF. By hypothesis, for every point $x \in X$ there exist an open set $U_x \subseteq X$ (which contains x) such that

$$p^{-1}(U_x) = \bigcup_{k \in K} V_k$$

is a union of disjoint open subsets of \tilde{X} such that for every $k \in K$

$$p|_{V_k} : V_k \rightarrow U_x$$

is an homeomorphism. Due to the fact that f is continuous and U_x is an open set, $f^{-1}(U_x)$ is an open set of I , so it is a collection of open intervals (maybe an arbitrary union). Therefore, the set $\{f^{-1}(U_x) \mid x \in X\}$ can be expressed in the

form $\{(x_j, y_j) \cap I \mid j \in J\}$. This set of intervals is an open cover of the compact set I . Therefore, there exist an open finite subcover that can be expressed in the form

$$\{[0, t_1 + \varepsilon_1), (t_2 - \varepsilon_2, t_2 + \varepsilon_2), \dots, (t_n - \varepsilon_n, 0]\},$$

with the condition that $t_i + \varepsilon_i > t_{i+1} - \varepsilon_{i+1}$ for all $i = 1, 2, \dots, n - 1$. Last condition guarantees that this subcover really covers I completely. Now we choose $a_i \in (t_{i+1} - \varepsilon_{i+1}, t_i + \varepsilon_i)$ for $i = 1, 2, \dots, n - 1$. It is clear that $0 = a_0 < a_1 < \dots < a_n = 1$. Notice some important facts:

- (1) $[a_i, a_{i+1}] \subset (t_{i+1} - \varepsilon_{i+1}, t_{i+1} + \varepsilon_{i+1}) \subset f^{-1}(U_x)$ for some $x \in X$. This last inclusion is because the interval $(t_{i+1} - \varepsilon_{i+1}, t_{i+1} + \varepsilon_{i+1})$ belongs to the former cover.
- (2) We will use the following notation

$$S_i := f((t_{i+1} - \varepsilon_{i+1}, t_{i+1} + \varepsilon_{i+1})).$$

- (3) It is clear that $f([a_i, a_{i+1}]) \subset U_x$. What's more, from (1), $f([a_i, a_{i+1}]) \subset S_i \subset U_x$ for some $x \in X$. Thus, S_i verifies that $p^{-1}(S_i)$ is the union of disjoint subsets of \tilde{X} such that each subset is homeomorphic to S_i via the restriction of p .

We shall define lifts \tilde{f}_k inductively over $[0, a_k]$ for $k = 0, 1, \dots, n$ such that $\tilde{f}_k(0) = a$. For $k = 0$ there is only one option since $[0, a_0] = \{0\}$. We define $\tilde{f}_0(0) = a$. Suppose now that we already defined a unique $\tilde{f}_k : [0, a_k] \rightarrow \tilde{X}$. Recall that

$$p^{-1}(S_i) = \bigcup_{j \in J} W_j,$$

with W_j disjoint and with $p|_{W_j} : W_j \rightarrow S_i$ an homeomorphism for every $j \in J$. Now let's see that there is a unique W among all W_j such that $\tilde{f}_k(a_k) \in W$ as figure (4) suggests.

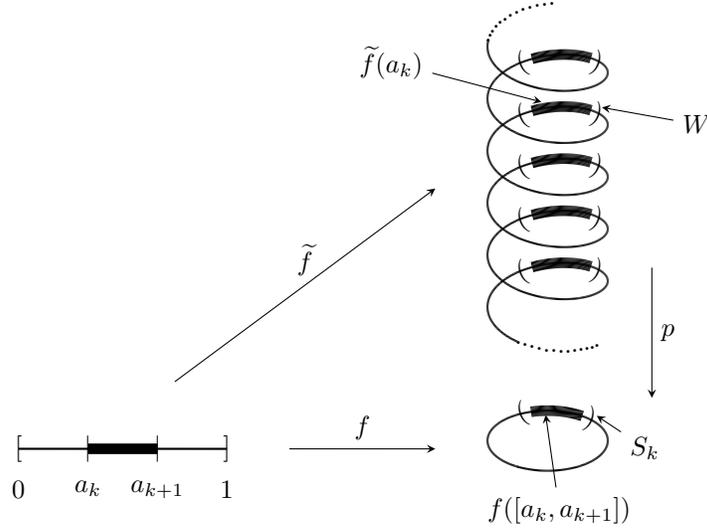


FIG. 4.

This is true because

- (1) The point $f(a_k) \in f([a_k, a_{k+1}]) \subset S_k$.
- (2) Hence, $p^{-1}(f(a_k)) \subset p^{-1}(S_k) = \bigcup_{j \in J} W_j$.
- (3) The function \tilde{f}_k satisfies $p \circ \tilde{f}_k = f$. Thus

$$\{\tilde{f}_k(a_k)\} \subseteq p^{-1}(p(\tilde{f}_k(a_k))) = p^{-1}(f(a_k)) \subseteq \bigcup_{j \in J} W_j.$$

Because the union is disjoint, there exist a unique W with $\tilde{f}_k(a_k) \in W$.

Recall that $[a_k, a_{k+1}]$ is path connected. Therefore any continuous extension \tilde{f}_{k+1} must verify that the set $\tilde{f}_{k+1}([a_k, a_{k+1}])$ is path connected. Since $\tilde{f}_k(a_k) \in W$ with W path connected, the only possibility left is that $\tilde{f}_k([a_k, a_{k+1}]) \subset W$. This open set W satisfies that $p|_W : W \rightarrow S_k$ is a homeomorphism. Thanks to this, we define the function

$$\rho : [a_k, a_{k+1}] \rightarrow W \quad \text{as} \quad \rho = (p|_W)^{-1} \circ f.$$

Finally, we define the extension

$$\tilde{f}_{k+1}(s) = \begin{cases} \tilde{f}_k(s) & 0 \leq s \leq a_k, \\ \rho(s) & a_k \leq s \leq a_{k+1}. \end{cases}$$

It only remains to check a few things:

- (1) The function \tilde{f}_{k+1} is continuous using the pasting lemma because $\rho(a_k) = ((p|_W)^{-1} \circ f)(a_k) = \tilde{f}_k(a_k)$.
- (2) The function ρ satisfies that $p|_W \circ \rho = f$. By hypothesis, \tilde{f}_k satisfies the same relation $p \circ \tilde{f}_k = f$. Thus, \tilde{f}_{k+1} also satisfies the equation. It only remains to check that the new defined lift is unique.
- (3) The function ρ is unique. If there were two functions $\rho, \tilde{\rho} : [a_k, a_{k+1}] \rightarrow \tilde{W}$, both should verify

$$\begin{cases} p \circ \rho = f|_{[a_k, a_{k+1}]}, \\ p \circ \tilde{\rho} = f|_{[a_k, a_{k+1}]}. \end{cases}$$

Because we know that $p|_W$ is an homeomorphism, then

$$\begin{cases} p|_W \circ \rho = f|_{[a_k, a_{k+1}]} \\ p|_W \circ \tilde{\rho} = f|_{[a_k, a_{k+1}]} \end{cases} \implies \begin{cases} \rho = \left((p|_W)^{-1} \circ f|_{[a_k, a_{k+1}]} \right) \\ \tilde{\rho} = \left((p|_W)^{-1} \circ f|_{[a_k, a_{k+1}]} \right) \end{cases}$$

and thus both ρ and $\tilde{\rho}$ should be equal. Because ρ and \tilde{f}_k are unique, \tilde{f}_{k+1} is unique. □

Usually, theorem (3.7) goes together with a theorem called *lifts of homotopies*. The theorem states that every homotopy has a unique lift to the covering space. We need this theorem to prove in the future that some application that we will define later on is well defined.

THEOREM 3.8 (Existence and uniqueness of lifts of homotopies of paths). *Let $p : \tilde{X} \rightarrow X$ be a covering. Consider an homotopy $F : I \times I \rightarrow X$ and a point $a \in \tilde{X}$ with $p(a) = F(0,0)$. Then there exists a unique map $\tilde{F} : I \times I \rightarrow \tilde{X}$ such that $(p \circ \tilde{F}) = F$ and $\tilde{F}(0,0) = a$.*

PROOF. The proof is analogous to the previous one. Since it is quite long, we'll give the necessary details to do it. In the previous proof, we used that I is compact to find a partition of the interval. In this case, we proceed similarly and say that I^2 is compact. Therefore, we can find

$$\begin{aligned} 0 &= a_0 < a_1 < \cdots < a_n = 1, \\ 0 &= b_0 < b_1 < \cdots < b_m = 1, \end{aligned}$$

such that $F(R_{i,j}) \subset X$, where $R_{i,j}$ is the rectangle

$$R_{i,j} = \{(t,s) \in I^2 \mid a_i \leq t \leq a_{i+1}, b_j \leq s \leq b_{j+1}\}.$$

Now, the lift \tilde{F} is also defined inductively over the rectangles

$$R_{0,0}, R_{0,1}, \dots, R_{0,m}, R_{1,0}, R_{1,1}, \dots$$

in a similar way to the previous theorem. □

2. The fundamental group of an orbit space

From now on we will consider X to be a path connected G -space with a properly discontinuous action. By proposition (3.6), $p : X \rightarrow X/G$ is a covering. The goal in this section is to find a relation between the fundamental group of X/G and the group G .

Consider $x_0 \in X$ and $y_0 = p(x_0) \in X/G$. By definition,

$$p^{-1}(y_0) = Gx_0 = \{gx_0 \mid g \in G\}.$$

Now consider f a closed curve in X/G based on y_0 , i.e., $[f] \in \pi_1(X/G, y_0)$. By theorem (3.7), there exist a unique lift \tilde{f} of f such that $\tilde{f}(0) = x_0$. The lift satisfies $p \circ \tilde{f} = f$, so

$$\{\tilde{f}(1)\} \subset p^{-1}(p(\tilde{f}(1))) = p^{-1}(f(1)) = p^{-1}(f(0)) = p^{-1}(y_0).$$

Thus, there exists a unique $g_f \in G$ such that $\tilde{f}(1) = g_f x_0$. It is unique because if $g_f x_0 = g'_f x_0 = \tilde{f}(1)$, then multiplying by the inverse of x_0 we would obtain $g_f = g'_f$. Now we must check that the function φ is well defined. In other words:

PROPOSITION 3.9. *With the same hypothesis, the function*

$$\varphi : \pi_1(X/G, y_0) \rightarrow G$$

with $\varphi([f]) = g_f$ is well-defined (i.e. it does not depend on the candidate of the class $[f]$ that we choose).

PROOF. With the same notation, consider two closed paths $f_0, f_1 : I \rightarrow X/G$ based on y_0 such that they are homotopic relatively to $\{0,1\}$. Consider their

respective unique lifts \tilde{f}_0 and \tilde{f}_1 with $\tilde{f}_0(0) = \tilde{f}_1(0) = \tilde{x}_0$. We want to check that $\tilde{f}_0(1) = \tilde{f}_1(1)$.

Consider $F : f_0 \simeq_{\{0,1\}} f_1$ the homotopy relative to $\{0,1\}$. By proposition (3.8) there exists a unique lift $\tilde{F} : I \times I \rightarrow X$ such that

$$\tilde{F}(0,0) = x_0 = \tilde{f}_0(0) = \tilde{f}_1(0).$$

From $F(t,0) = f_0(t)$, we deduce that $F(t,0)$ is a closed path on the parameter t based at $F(0,0) = f_0(0) = y_0$. Since the lift of a path is unique, we conclude that the lift $\tilde{F}(t,0) = \tilde{f}_0(t)$ with $\tilde{F}(0,0) = \tilde{f}_0(0) = x_0$. With the same reasoning, we obtain that $\tilde{F}(t,1) = \tilde{f}_1(t)$ with $\tilde{F}(0,1) = \tilde{f}_1(0) = x_0$. Recall that $F(1,s) = f_0(1) = f_1(1) = y_0$ for all $s \in I$ and hence, $\tilde{F}(1,s)$ is a path in the parameter s from $\tilde{f}_0(1)$ to $\tilde{f}_1(1)$. Notice that $p(\tilde{F}(1,s)) = F(1,s) = f_0(1) = y_0$. Therefore $\tilde{F}(1,s) \in p^{-1}(y_0) = Gy_0 \cong G$. Finally, I is connected and therefore $\tilde{F}(1,I)$ is also connected. Since G has the discrete topology, the only possibility left is that $\tilde{F}(1,I) = \{x_0\}$ is a set with only one element. Therefore, $\tilde{f}_0(1) = \tilde{f}_1(1)$. \square

PROPOSITION 3.10. *With the same hypothesis, the function*

$$\varphi : \pi_1(X/G, y_0) \rightarrow G$$

with $\varphi([f]) = g_f$ is a group homomorphism.

PROOF. Consider two closed paths $f, f' \in X/G$ based at y_0 . Consider $\widetilde{f * f'}$ the unique lift of $f * f'$ with $\widetilde{f * f'}(0) = x_0$. Consider \tilde{f} and \tilde{f}' the unique lifts of f and f' respectively, both with $\tilde{f}(0) = \tilde{f}'(0) = x_0$. Consider also $l_a(f')$ the unique lift of f' with $l_a(f')(0) = a := \tilde{f}(1) = g_f \cdot x_0$. First, we shall prove that

$$\widetilde{f * f'} = \tilde{f} * l_a(f').$$

Sure enough, $\tilde{f}(1) = g_f \cdot x_0 = l_a(f')(0)$, so the product $\tilde{f} * l_a(f')$ makes sense. What's more, the product

$$\tilde{f} * l_a(f') = \begin{cases} \tilde{f}(2s) & s \in [0, \frac{1}{2}], \\ l_a(f')(2s-1) & s \in [\frac{1}{2}, 1], \end{cases}$$

satisfies

- (1) $(\tilde{f} * l_a(f'))(0) = \tilde{f}(0) = x_0 = \widetilde{f * f'}(0)$.
- (2) Owing to the definition of a lift,

$$\begin{aligned} p \circ (\tilde{f} * l_a(f'))(s) &= \begin{cases} (p \circ \tilde{f})(2s) & s \in [0, \frac{1}{2}], \\ (p \circ l_a(f'))(2s-1) & s \in [\frac{1}{2}, 1], \end{cases} \\ &= \begin{cases} f(2s) & s \in [0, \frac{1}{2}] \\ f'(2s-1) & s \in [\frac{1}{2}, 1] \end{cases} = \widetilde{f * f'}(s). \end{aligned}$$

Then, $\tilde{f} * l_a(f')$ is a lift of $f * f'$ with the same initial point and because of the uniqueness of the lift, the equality holds. Secondly, we shall prove that $g_f \tilde{f}' = l_a(f')$ using the same reasoning.

- (1) $(g_f \tilde{f}')(0) = g_f \tilde{f}'(0) = g_f x_0 = l_a(f')(0)$.

(2) Notice that by definition, $g_f \tilde{f}'(s) \in p^{-1}(\tilde{f}'(s))$. Therefore, $p(g_f \tilde{f}'(s)) = p(\tilde{f}'(s))$ and hence

$$(p \circ (g_f \tilde{f}'))(s) = (p \circ (\tilde{f}'))(s) = f'(s).$$

Then, the equality holds. Thirdly,

$$\widetilde{f * f'}(1) = (\tilde{f} * l_a(f'))(1) = (\tilde{f} * (g_f \tilde{f}'))(1) = (g_f \tilde{f}')(1) = g_f(g_{f'}x_0) = (g_f g_{f'})x_0.$$

Finally, we deduce that $\varphi([f] * [f']) = \varphi([f * f']) = g_f g_{f'} = \varphi([f])\varphi([f'])$. \square

PROPOSITION 3.11. *The kernel of $\varphi : \pi_1(X/G, y_0) \rightarrow G$ is the subgroup*

$$p_*(\pi_1(X, x_0)).$$

PROOF. Recall that p_* is the function defined in the first section (2.10). The kernel of φ is the set of elements $[f] \in \pi_1(X/G, y_0)$ such that $\varphi([f]) = 1$. This set is exactly the set of elements $[f] \in \pi_1(X/G, y_0)$ such that $\tilde{f}(1) = x_0$. Recall that, by definition $\tilde{f}(0) = x_0$. Thus, the kernel is the set of elements $[f] \in \pi_1(X/G, y_0)$ such that \tilde{f} is a closed curve based on $x_0 \in X$. Hence, is the set of elements $[f] \in \pi_1(X/G, y_0)$ of the form $[p \circ \tilde{f}]$ for $[\tilde{f}] \in \pi_1(X, x_0)$, i.e. $p_*(\pi_1(X, x_0))$. \square

THEOREM 3.12. *The morphism $\varphi : \pi_1(X/G, y_0) \rightarrow G$ induces an isomorphism between $\pi_1(X/G, y_0)/p_*(\pi_1(X, x_0))$ and G . That is,*

$$\pi_1(X/G, y_0)/p_*(\pi_1(X, x_0)) \cong G.$$

PROOF. Thanks to the first isomorphism theorem, we only must check that the homomorphism

$$\varphi : \pi_1(X/G, y_0) \rightarrow G$$

is surjective. Given $g \in G$, consider f_g a path in X from x_0 to gx_0 (which exists because X is path connected). Therefore, $[p \circ f_g] \in \pi_1(X/G, y_0)$. By definition, $\varphi([p \circ f_g])x_0 = (\widetilde{p \circ f_g})(1)$ where $\widetilde{p \circ f_g}$ is the unique lift of $(p \circ f_g)$ with $\widetilde{p \circ f_g}(0) = x_0$. It is immediate to check that f_g is the lift of $(p \circ f_g)$. What's more, by construction $f_g(1) = gx_0$, so $\varphi([p \circ f_g]) = g$, what proves that φ is surjective. \square

COROLLARY 3.13. *If X is simply connected, then $\pi_1(X/G, y_0) \cong G$.*

PROOF. If X is simply connected, then $\pi_1(X, x_0) = \{1\}$ and $p_*(\pi_1(X, x_0)) = p_*(\{1\}) = \{1_{\pi_1(X/G, y_0)}\}$. So

$$G \cong \pi_1(X/G, y_0)/p_*(\pi_1(X, x_0)) \cong \pi_1(X/G, y_0). \quad \square$$

Finally, we can compute the fundamental group of the previous examples.

(1) The space \mathbb{S}^1/\mathbb{Z} . Moreover, \mathbb{R} is simply connected. Then,

$$\pi_1(\mathbb{S}^1, x_0) \cong \pi_1(\mathbb{R}/\mathbb{Z}, x_0) \cong \mathbb{Z}.$$

Remark that x_0 appears in both sets but they are not equal at all, as in the left hand side it belongs to the circle and in the right hand side it belongs to a quotient space. In this case, x_0 is *mute* in the meaning that it only stresses

the necessity of a point in the definition of the fundamental group. This is also applied in the following examples.

- (2) The space $\mathbb{P}^n \cong \mathbb{S}^n/\mathbb{Z}_2$. In addition, \mathbb{S}^n is simply connected¹. Moreover, there are alternative proofs which need nothing more than the definition of the fundamental group. Then,

$$\pi_1(\mathbb{P}^n, x_0) \cong \pi_1(\mathbb{S}^n/\mathbb{Z}_2, x_0) \cong \mathbb{Z}_2.$$

- (3) The space $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$. What's more, \mathbb{R}^n is simply connected. Then,

$$\pi_1(\mathbb{T}^n, x_0) \cong \pi_1(\mathbb{R}^n/\mathbb{Z}^n, x_0) \cong \mathbb{Z}^n.$$

3. The fundamental group of a torus

In this brief section we will compute the fundamental group of a torus in a different way. The theorem introduced here will be important for calculations further ahead.

THEOREM 3.14. *Let X, Y be two topological spaces. The fundamental group of the product topology $\pi_1(X \times Y, (x, y))$ is isomorphic to $\pi_1(X, x) \times \pi_1(Y, y)$.*

PROOF. Consider the product topology $X \times Y$ with its canonical projections $p_X : X \times Y \rightarrow X$, $p_Y : X \times Y \rightarrow Y$. Those projections define a morphism

$$\begin{aligned} \varphi : \pi_1(X \times Y, (x, y)) &\longrightarrow \pi_1(X, x) \times \pi_1(Y, y) \\ [\omega] &\longrightarrow (p_{X*}[\omega], p_{Y*}[\omega]) = ([p_X\omega], [p_Y\omega]). \end{aligned}$$

The morphism is well-defined by proposition (1.9). On the other hand, recall that the internal operation of the product of fundamental groups is

$$([\omega_1], [\sigma_1]) \cdot ([\omega_2], [\sigma_2]) = ([\omega_1] * [\omega_2], [\sigma_1] * [\sigma_2]).$$

Moreover, it is immediate to check that

$$[p_X \circ (\omega * \sigma)] = [(p_X \circ \omega) * (p_X \circ \sigma)]$$

by writing the definitions (and the same holds for p_Y). Using this two properties, it is immediate to confirm that φ is a group morphism indeed. To prove that it is an isomorphism let's first see that it is injective. Consider $[\omega] \in \ker \varphi$. Then, by definition, $p_X\omega$ and $p_Y\omega$ are homotopic to the constant paths ε_x and ε_y respectively. Let $F : p_X\omega \simeq \varepsilon_x$ and $G : p_Y\omega \simeq \varepsilon_y$ be such homotopies. Then, the continuous map $H : I \times I \rightarrow X \times Y$ defined by

$$H(s, t) = (F(s, t), G(s, t))$$

is an homotopy from ω to $\varepsilon_{(x,y)}$. Hence, $[\omega] = 1$. Finally, φ is surjective. Consider α and β closed curves in X and Y respectively. Then, the closed curve $\omega(t) = (\alpha(t), \beta(t)) \in X \times Y$ verifies that $\varphi([\omega]) = ([\alpha], [\beta])$. \square

COROLLARY 3.15. *The fundamental group of the torus \mathbb{T}^2 is isomorphic to \mathbb{Z}^2 .*

PROOF. The torus $\mathbb{T}^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$ and thus

$$\pi_1(\mathbb{T}^2, x_0) \cong \pi_1(\mathbb{S}^1 \times \mathbb{S}^1, (x_0, x_0)) \cong \pi_1(\mathbb{S}^1, x_0) \times \pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z} \times \mathbb{Z} = \mathbb{Z}^2. \quad \square$$

¹We did not prove yet that the n-dimensional sphere is simply connected. This proof is given in theorem (4.25)

Finally, by induction, $\pi_1(\mathbb{T}^n, x_0) \cong \mathbb{Z}^n$.

4. Comments on the results

In previous paragraphs we proved that the fundamental group of an sphere, a torus, and a projective space are not isomorphic. Hence, these topological spaces can not be homeomorphic and can not be of the same homotopy type. At this point, the reader might be asking itself what happens with the reciprocal of theorems (2.12) or (2.16). Are there non-homeomorphic (or even, non-homotopy equivalent) topological spaces with the same fundamental group? The answer is yes. For example, \mathbb{R}^n and a point $x_0 \in \mathbb{R}^n$ have the same fundamental group

$$\pi_1(\mathbb{R}^n, x_0) \cong \pi_1(\mathbb{R}^n, x_0) = 1,$$

but they are not homeomorphic since there can not exist any bijection between them. We could ask the same question for other important topological spaces such as compact spaces, connected spaces or topological manifolds among others². Let's focus our interest in topological manifolds since they are highly relevant in many mathematical areas. In all, there exist two non-homeomorphic topological manifolds with the same fundamental group? The answer is still yes. The *lens space* is a relatively easy example.

Although there are multiple definitions of the lens space, we will only present the orbit space version. Consider \mathbb{S}^3 the sphere in \mathbb{R}^4 . For simplicity with calculations, we will assume that points in \mathbb{S}^3 are of the form (z_1, z_2) , with $z_1, z_2 \in \mathbb{C}$. Let p and q be two coprime integer numbers and consider the action

$$\begin{aligned} \mathbb{Z}_p \times \mathbb{S}^3 &\longrightarrow \mathbb{S}^3 \\ (n, z_1, z_2) &\longrightarrow n \cdot (z_1, z_2) := (e^{2\pi i \frac{n}{p}} z_0, e^{2\pi i \frac{nq}{p}} z_1). \end{aligned}$$

DEFINITION 3.16 (Lens spaces). *With the previous notation, we define the lens space as*

$$L(p, q) = \mathbb{S}^3 / \mathbb{Z}_p.$$

Recall from the previous section that if $p = 2$ and $q = 1$, we obtain that $\mathbb{S}^3 / \mathbb{Z}_2$ is the projective space.

Let's check that this is an action indeed.

- (1) The first property is satisfied,

$$0 \cdot (z_0, z_1) = (e^0 z_0, e^0 z_1) = (z_0, z_1).$$

- (2) The second property is satisfied,

$$\begin{aligned} m \cdot (n \cdot (z_0, z_1)) &= m \cdot (e^{2\pi i \frac{n}{p}} z_0, e^{2\pi i \frac{nq}{p}} z_1) = (e^{2\pi i \frac{(m+n)}{p}} z_0, e^{2\pi i \frac{(m+n)q}{p}} z_1). \\ &= (m+n) \cdot (z_1, z_2) \end{aligned}$$

²Remember that a topological manifold is a locally euclidean Hausdorff space satisfying the second axiom of countability.

Moreover, the action consists in two rotations and hence, it is continuous when n is fixed. Since \mathbb{Z}_p has the discrete topology, it can be checked that the action is continuous in n , z_0 and z_1 .

On the other hand, the action is properly discontinuous. Sure enough, for any $x = (z_1, z_2) \in \mathbb{S}^3$ there exist an open sufficiently small set U_x such that

$$n \cdot (U_x) \cap m \cdot (U_x) = \emptyset.$$

The key point in the proper discontinuity is that the group \mathbb{Z}_p is finite and has p elements. This means that we perform a finite number of rotations and therefore we can take a sufficiently small set such that translations of this set do not intersect with the other sets. Note that if the group was \mathbb{Z} instead of \mathbb{Z}_p , the proper discontinuity property would not be verified. As an example, $0 \cdot (U_x) = p \cdot (U_x)$.

So, in all, $\mathbb{S}^3 \rightarrow \mathbb{S}^3/\mathbb{Z}_p$ is a covering. Since \mathbb{S}^3 is simply connected, we conclude that

$$\pi_1(L(p, q), x_0) = \pi_1(\mathbb{S}^3/\mathbb{Z}_p, x_0) \cong \mathbb{Z}_p.$$

Notice that the fundamental group depends on p and not on q . This means that $L(p, q)$ and $L(p, q')$ have the same fundamental group for any q' coprime with p . However, theorem (3.17) shows that they might not be homeomorphic.

THEOREM 3.17. *$L(p, q)$ is homeomorphic to $L(p, r)$ if either*

$$q \equiv \pm r \pmod{p} \quad \text{or} \quad qr \equiv \pm 1 \pmod{p}.$$

The converse of this theorem is also true.

We omit the proof because it needs cohomology techniques of algebraic topology, which are not the matter of this section. As a curiosity, the converse of the theorem is very difficult to prove. Finally, as we stated at the introduction of the section, the lens space is a compact connected 3-manifold.

PROPOSITION 3.18. *The lens space $L(p, q)$ is a compact connected 3-manifold.*

PROOF. We must check that the lens space is compact, locally euclidean, Hausdorff and that satisfies the second axiom of countability.

- (1) Consider the projection map or covering

$$\pi : \mathbb{S}^3 \rightarrow \mathbb{S}^3/\mathbb{Z}_p \cong L(p, q).$$

The set \mathbb{S}^3 is compact and the covering is surjective. Therefore, the lens space $L(p, q)$ is compact.

- (2) Consider again the previous covering. Since π is a covering, for all $x \in L(p, q)$ there exists a set $U_x \subset L(p, q)$ such that

$$\pi^{-1}(U_x) = \bigcup_{j \in J} U_j,$$

for some index set J such that

- (a) U_j are open sets of \mathbb{S}^3 for all $j \in J$.
- (b) Each U_j is homeomorphic to U_x .

Since \mathbb{S}^3 is locally euclidean, this means that U_j is locally euclidean for all $j \in J$. Thanks to the homeomorphism, we conclude that $L(p, q)$ is also locally euclidean.

- (3) Consider two different points $x, y \in L(p, q)$ and consider the sets

$$\begin{aligned}\pi^{-1}(x) &= \{x_1, x_2, \dots, x_n\}, \\ \pi^{-1}(y) &= \{y_1, y_2, \dots, y_m\}.\end{aligned}$$

Since π is a covering and \mathbb{S}^3 is Hausdorff, in \mathbb{S}^3 there exist open neighbourhoods U_{x_i} and V_{y_j} verifying

- (a) U_{x_i} and V_{y_j} are open sets for all $1 \leq i \leq n$ and $1 \leq j \leq m$.
 - (b) $x_i \in U_{x_i}$ for all i and $y_j \in V_{y_j}$ for all j .
 - (c) The sets U_{x_i} are disjoint for all i . The sets V_{y_j} are disjoint for all j .
- Moreover, $U_{x_i} \cap V_{y_j} = \emptyset$ for all i and j .

Now consider the two open sets

$$\begin{aligned}U &:= \bigcup_{i=1}^n U_{x_i}, \\ V &:= \bigcup_{j=1}^m V_{y_j}.\end{aligned}$$

They verify that $U \cap V = \emptyset$. By proposition (3.3), the covering π is an open map and therefore, $\pi(U)$ and $\pi(V)$ are open disjoint neighbourhoods of x and y respectively in $L(p, q)$. Thus, $L(p, q)$ is Hausdorff.

- (4) First, \mathbb{S}^3 satisfies the second axiom of countability. This means that there exists a countable basis

$$\mathcal{B} = \{U_n\}_{n \in \mathbb{N}}.$$

Consider now the surjective projection $\pi : \mathbb{S}^3 \rightarrow L(p, q)$. We want to check that

$$\pi(\mathcal{B}) = \{\pi(U_n)\}_{n \in \mathbb{N}}$$

is a basis of $L(p, q)$. Consider $y \in L(p, q)$ and $V_y \subset L(p, q)$ an open set that contains y . Then, $\pi^{-1}(V_y)$ is an open set of \mathbb{S}^3 . More specifically,

$$\pi^{-1}(V_y) = V_1 \cup \dots \cup V_p,$$

with $V_i \subset \mathbb{S}^3$ open sets, each one homeomorphic to V_y via the mapping π . Let $x \in \mathbb{S}^3$ be a point such that $\pi(x) = y$. Therefore, x belongs to a set V_i for some $1 \leq i \leq p$. On the other hand, since \mathcal{B} is a base, there exists an $n \in \mathbb{N}$ such that $x \in U_n \subset V_i$. Therefore,

$$y = \pi(x) \in \pi(U_n) \subset \pi(V_i) = V.$$

Thus, $\pi(\mathcal{B})$ is a basis of $L(p, q)$ and the lens space is 2AN.

□

Chapter 4

The Seifert-Van Kampen theorem and some applications

1. Presentations and free products of groups

Consider a set S and consider its elements to be just symbols. With such symbols we can generate finite non-commutative *words* of the form

$$W = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_k^{\varepsilon_k},$$

with $x_i \in S$, repetitions being allowed, and $\varepsilon_i = \pm 1$. A word is said to be *reduced* if there is not any symbol x^1 followed by a x^{-1} or x^{-1} followed by an x^1 . For example, $x_1^1 x_1^1 x_1^1$ and $x_1^1 x_2^{-1}$ are reduced words but $x_1^1 x_1^1 x_1^{-1}$ is not. Any non-reduced word can be *reduced* just by deleting the symbols $x^1 x^{-1}$ or $x^{-1} x^1$. In this sense, $x_1^{-1} x_2^1 x_2^{-1} x_1^1 x_3^1$ is reduced to $x_1^{-1} x_1^1 x_3^1$, which is again reduced to x_3^1 . For our convenience, we will consider the *empty* word, which is written with no symbols. Finally, two words can be juxtaposed to form another word. Notice that this new word may not be reduced. Hence, we can consider an operation $*$ in the set of words which consists in a juxtaposition of two words followed by a reduction.

PROPOSITION 4.1. *Let $\langle S \rangle$ be the set of all reduced words generated by S . Then, $(\langle S \rangle, *)$ is a group.*

PROOF. The proof consists only in checking the axioms of a group.

- (1) The operation $*$ is closed.
- (2) The product is associative.
- (3) The neutral element is the empty word.
- (4) The inverse of a word $x_1^{\varepsilon_1} x_2^{\varepsilon_2} \cdots x_n^{\varepsilon_n}$ is the word

$$x_n^{-\varepsilon_n} x_{n-1}^{-\varepsilon_{n-1}} \cdots x_1^{-\varepsilon_1}$$

□

DEFINITION 4.2. *The group $\langle S \rangle$ is called the free group generated by S .*

Notice that the set S could be a group (S, \cdot) . In presentations, the operation \cdot plays no role in the set $\langle S \rangle$. This means that if x_1, x_2, g are elements of S such that

$x_1 \cdot x_2 = g$, then we can not say that the word $x_1x_2x_3$ of $\langle S \rangle$ is equal to the word gx_3 . In our case, we are only interested in the elements of S as symbols and we forget any possible additional structures of S .

From now on, it is convenient to introduce some notation. We will write x^2 instead of x^1x^1 , x^{-2} instead of $x^{-1}x^{-1}$, etc. As an example,

$$\langle \{x\} \rangle = \{1, x^1, x^{-1}, x^2, x^{-2}, x^3, x^{-3}, \dots\} \cong \mathbb{Z}.$$

Notice that here the element 1 is the neutral element of the free group, i.e. it represents the empty word. Until now, if S has at least one element, the group $\langle S \rangle$ will always be infinite. In order to obtain finite groups and a wider variety of infinite groups, we must introduce equivalence relations. Therefore, our aim is to define a new group. Consider $R \subseteq \langle S \rangle$ a subset of the set of reduced words and consider W, W' two words of $\langle S \rangle$. We say that W is equivalent to W' (and we write $W \sim W'$) if and only if the word W' can be obtained from W by performing a finite number of operations of the following form

- (1) Insert xx^{-1} or $x^{-1}x$ in a word, where $x \in S$.
- (2) Delete xx^{-1} or $x^{-1}x$ in a word, where $x \in S$.
- (3) Insert r or r^{-1} in a word, where $r \in R$.
- (4) Delete r or r^{-1} in a word, where $r \in R$.

By inserting r (or xx^{-1}) in a word W we mean write W as W_1W_2 and then insert r to form the word W_1rW_2 . Of course, W_1 or W_2 can be empty, which would mean that you prepend or append r . It is easy to check that this relation is an equivalence relation and that the set of equivalence classes is a group with the juxtaposition as a law of composition.

DEFINITION 4.3. *We denote by $\langle S; R \rangle$ the group $\langle S \rangle / \sim$. This group is said to be the group with presentations $(S; R)$. The set S is called the generators and the set R is called the set of relations.*

Notice that now, the elements of $\langle S; R \rangle$ are equivalence classes. We will denote the equivalence class containing the word W by W itself and it will not cause any confusion. Since W can represent a word and an equivalence class interchangeably, if W and W' belong to the same equivalence class, we will say that $W \sim W'$ or simply $W = W'$. This notation will not cause any confusion at all. Let's go through some examples.

- (1) The group $\langle S; \emptyset \rangle$ is the free group generated by S . It is called the free group because it is *free from relations*.
- (2) Consider $n \in \mathbb{N}$, $n > 1$. Then, the group $\langle \{x\}; \{x^n\} \rangle$ consists of the words

$$1, x, x^2, \dots, x^{n-1}.$$

It can be readily verified that this group is isomorphic to the cyclic group \mathbb{Z}_n .

- (3) Consider the group $\langle \{x, y\}; \{xyx^{-1}y^{-1}\} \rangle$. This is a good example since it shows how to work with presentations in order to find an isomorphic group. For example, we shall prove that $xy = yx$.

$$xy = (xyx^{-1}y^{-1})^{-1}xy = yxy^{-1}x^{-1}xy = yxy^{-1}y = yx.$$

Notice that this procedure can be carried out in the middle of a word and hence,

$$x^2y^2 = xxyy = xyxy = yxxy = yxyx = yyxx = y^2x^2.$$

So, by induction, it is easy to see that $x^ay^b = y^bx^a$. Finally, using these properties, we deduce that any word W satisfies

$$W = x^{a_1}y^{b_1}x^{a_2}y^{b_2}\dots x^{a_k}y^{b_k} = x^ay^b,$$

where

$$a = \sum_{i=1}^k a_i \quad \text{and} \quad b = \sum_{i=1}^k b_i.$$

Therefore, it is easy to prove that $\langle \{x, y\}; \{xyx^{-1}y^{-1}\} \rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

- (4) In practice, different presentations may lead to isomorphic groups as we saw in the previous point. More examples are given in increasing order of difficulty, being the first one easy to prove and the last one extremely hard to prove:

$$\begin{aligned} \langle \{x, y\}; \{y\} \rangle &\cong \langle \{x\}, \emptyset \rangle, \\ \langle \{a, b\}; \{baba^{-1}\} \rangle &\cong \langle \{a, c\}; \{a^2c^2\} \rangle, \\ \langle \{x, y\}; \{xy^2x^{-1}y^{-3}, yx^2y^{-1}x^{-3}\} \rangle. \end{aligned}$$

In all, proving that two presentations are isomorphic is usually really hard, even if we are sure that they are. Luckily, there are some results that help with this task. However, we will not use them since we will not deepen in this area.

Before concluding the section, we must see that any group is isomorphic to a certain presentation and hence, we can study interchangeably groups and presentations.

DEFINITION 4.4. *Given a group G , we will say that G has a presentation $(S; R)$ if G is isomorphic to $\langle S; R \rangle$.*

PROPOSITION 4.5. *Every group G has a presentation $(S_G; R_G)$.*

PROOF. We must determine who S_G and R_G are. It is natural to think of $S_G = \{g \in G\}$. With this choice, there is a surjective group morphism between the free group $\langle S_G \rangle$ and G

$$\begin{aligned} f : \langle S_G \rangle &\longrightarrow G \\ g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n} &\longrightarrow g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n}. \end{aligned}$$

Notice that the left hand side element is just a word and its image is the element that results from operating in G all elements that appear in the word. By the first isomorphism theorem, we obtain

$$\langle S_G \rangle / \ker(f) \cong G.$$

The set $\ker(f)$ consists of all words W such that $f(W) = 1_G$. Since f is a morphism, for all $r \in \ker(f)$ and for all words,

$$f(g_1^{\varepsilon_1} \cdots g_n^{\varepsilon_n}) = f(g_1^{\varepsilon_1} \cdots g_{i-1}^{\varepsilon_{i-1}} r g_{i+1}^{\varepsilon_{i+1}} \cdots g_n^{\varepsilon_n}).$$

If we define

$$R_G = \ker(f) = \{W \in \langle S_G \rangle \mid f(W) = 1\} \subseteq \langle S_G \rangle,$$

then

$$G \cong \langle S_G \rangle / \ker(f) \cong \langle S_G; R_G \rangle$$

□

In this proof we have seen that the elements of R_G can be thought as words or as elements of a kernel. Hence, the set $\{r \mid r \in R\}$ is usually rewritten as $\{r = 1 \mid r \in R\}$ with a clear abuse of notation. For example, if $r = xy^{-1} \in R$, then instead of mentioning a relation xy^{-1} we will mention a relation $x = y$. Again, this abuse of notation will not be confusing at all.

Finally, it is important to introduce the concept of *free product* of groups to understand further results.

DEFINITION 4.6. *If $G = \langle S_G; R_G \rangle$ and $H = \langle S_H; R_H \rangle$ are two presentations, the free product of G and H is defined as*

$$G * H := \langle S_G \cup S_H; R_G \cup R_H \rangle.$$

Notice here that the unions are, in fact, disjoint unions.

In all, the elements of the free product of G and H are words generated by elements of two different groups, which are subject to the relations that each separate presentation has. For example, if $G = \langle \{x\}; x^4 = 1 \rangle \cong \mathbb{Z}_4$ and $H = \langle \{y\}; y^{-7} = 1 \rangle$, then

$$G * H = \langle \{x, y\}; \{x^4 = y^{-7} = 1\} \rangle.$$

Hence, the word yx^5y^8x can be reduced to xyx . From now on, we will use the following notation

$$F_n := \mathbb{Z} * \underbrace{!}_{(n)} * \mathbb{Z}.$$

2. The Seifert-Van Kampen theorem

The Seifert-Van Kampen theorem is a useful theorem to calculate the fundamental group of topological spaces in terms of presentations and relations. Before introducing the theorem, we need a technical result.

PROPOSITION 4.7. *Consider a topological space X of the form $X = U_1 \cup U_2$ with U_1, U_2 and $U_1 \cap U_2$ being subsets of X . Let's denote $\varphi_1, \varphi_2, \psi_1, \psi_2$ the inclusions of figure (1). Consider a base point $x_0 \in U_1 \cap U_2$. Then, the diagram in figure (2)*

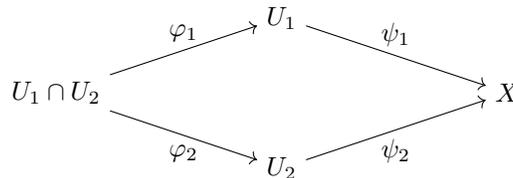


FIG. 1.

is commutative.

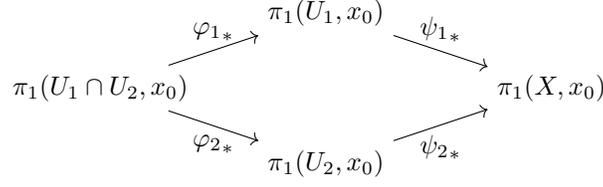


FIG. 2.

PROOF. Since all functions of the diagram in figure (1) are inclusions, the diagram is commutative. This means $\psi_1 \circ \varphi_1 = \psi_2 \circ \varphi_2$ and hence, $(\psi_1 \circ \varphi_1)_* = (\psi_2 \circ \varphi_2)_*$. Finally, using the first result (1) in the proof of theorem (2.12) we obtain that

$$\psi_{1*} \circ \varphi_{1*} = \psi_{2*} \circ \varphi_{2*}.$$

Hence, the diagram (2) is commutative. \square

THEOREM 4.8 (Seifert-Van Kampen (SVK)). Consider a topological space X of the form $X = U_1 \cup U_2$ with

- (1) U_1 and U_2 being non-empty open path-connected spaces of X ,
- (2) $U_1 \cap U_2$ being a non-empty path-connected space of X .

Consider the notation of the previous proposition (4.7) and consider the situation in figures (1) and (2). Consider also the presentations of the following fundamental groups

- (1) $\pi_1(U_1 \cap U_2, x_0) \cong \langle S; R \rangle$.
- (2) $\pi_1(U_1, x_0) \cong \langle S_1; R_1 \rangle$.
- (3) $\pi_1(U_2, x_0) \cong \langle S_2; R_2 \rangle$.

If $s \in S$, then $\varphi_{1*}s \in \pi_1(U_1, x_0)$ and $\varphi_{2*}s \in \pi_1(U_2, x_0)$, so we can represent these elements as words generated by S_1 and S_2 respectively. Let $'\varphi_{1*}s'$ and $'\varphi_{2*}s'$ be such words. We must define one last relation. We denote R_s the set of relations

$$R_s := \{ '\varphi_{1*}s' = '\varphi_{2*}s', s \in S \}.$$

With the previous notation, $\pi(X, x_0)$ is isomorphic to $\langle S_1 \cup S_2; R_1 \cup R_2 \cup R_s \rangle$.

Notice that the set of relations R of $\pi_1(U_1 \cap U_2, x_0)$ does not show up in the presentation of $\pi_1(X, x_0)$. The last comment is that this theorem is really powerful since it permits to calculate fundamental groups of a wide variety of topological spaces. However, we should bear in mind that working with presentations may not be pleasant at all in terms of finding isomorphic groups.

3. The fundamental group of a join of spaces

A particular case of SVK is when $U_1 \cap U_2$ is simply connected since it simplifies calculations. We already saw that its fundamental group is $\pi_1(U_1 \cap U_2, x_0) = 1$. Since the fundamental group only has one element, it is isomorphic to the presentation $\langle \{\}, \emptyset \rangle$, which is the free group generated only by the empty word.

DEFINITION 4.9 (Join of two spaces). *Consider two topological spaces X and Y and two points $x_0 \in X$ and $y_0 \in Y$. We define the join of X and Y as*

$$X \vee Y := (X \sqcup Y) / \{x_0 \sim y_0\}.$$

Intuitively, the join of two spaces consists in bringing spaces X and Y together so they coincide in a single point. The contact point in X is x_0 and the contact point in Y is y_0 . Figure (3) shows an example with X and Y being \mathbb{S}^1 . Consider that X

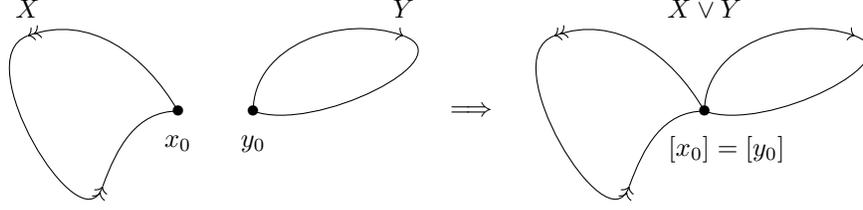


FIG. 3.

and Y are pathconnected spaces. With the particular case of a join of X and Y , we could be tempted to say that $X \vee Y = p(X) \cup p(Y)$ ¹ and that $p(X) \cap p(Y) = \{[x_0]\}$ is simply connected. Hence, by the SVK theorem, we would get the following result:

If $\pi_1(X, x_0) \cong \langle S_1; R_1 \rangle$ and $\pi_1(Y, y_0) \cong \langle S_2; R_2 \rangle$, then

$$\pi_1(X \vee Y, [x_0]) \cong \langle S_1 \cup S_2; R_1 \cup R_2 \rangle.$$

However, this is not true in general because the projections of X and Y may not be open sets of $X \vee Y$, and hence, the hypothesis of the SVK theorem are not fulfilled. So, in all, the previous calculation is true if we ask $p(X)$ and $p(Y)$ to be open sets in $X \vee Y$.

Luckily, the previous calculation is also true in some particular cases where at least one of the projections is not an open set in $X \vee Y$. Let's consider the case of figure (3), in which X and Y are homeomorphic to \mathbb{S}^1 . Consider X to be the \mathbb{S}^1 on the left and $\tilde{x} \in X$ a point. Consider also Y to be the \mathbb{S}^1 on the right and $\tilde{y} \in Y$ a point. Now, let U_1 be the open path connected space $(X \vee Y) \setminus \{\tilde{x}\}$ and let U_2 be the open path connected subspace $(X \vee Y) \setminus \{\tilde{y}\}$. Therefore, $X \vee Y = U_1 \cup U_2$ and $U_1 \cap U_2 \simeq [x_0]$. The SVK theorem says that

$$\pi_1(X \vee Y, x_0) = \langle \{x, y\}, \emptyset \rangle,$$

which is the free group generated by two elements. Other examples in which we can proceed in exactly the same way are the ones in figure (5).

It is clear by the SVK theorem that their fundamental groups are

$$\pi_1(\mathbb{S}^1 \vee \mathbb{T}^2, p) = \langle \{a, b, c\}; \{aba^{-1}b^{-1}\} \rangle \cong \pi_1(\mathbb{S}^1, x_0) * \pi_1(\mathbb{T}^2, y_0)$$

¹We denote the projections of X and Y to the quotient space as $p(X)$ and $p(Y)$. Recall that X and $p(X)$ (and Y and $p(Y)$ respectively) are homeomorphic via the projection $p : X \rightarrow p(X) \subset X \vee Y$ (and $p : Y \rightarrow p(Y) \subset X \vee Y$ respectively).

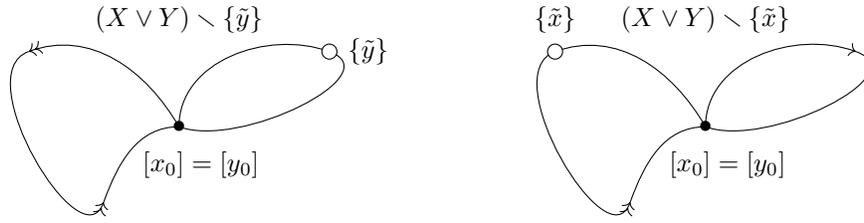


FIG. 4.

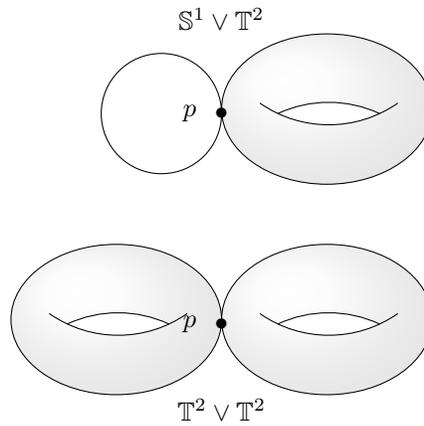


FIG. 5.

and

$$\pi_1(\mathbb{T}^2 \vee \mathbb{T}^2, p) = \langle \{a, b, c, d\}; \{aba^{-1}b^{-1}, cdc^{-1}d^{-1}\} \rangle \cong \pi_1(\mathbb{T}^2, x_0) * \pi_1(\mathbb{T}^2, y_0)$$

respectively. In all, let's formalise these calculations in a proposition

PROPOSITION 4.10. *Let X, Y be two spaces and consider $x_0 \in X, y_0 \in Y$. Consider also that x_0 and y_0 have neighbourhoods $U_{x_0} \subset X$ and $V_{y_0} \subset Y$ such that x_0 and y_0 are deformation retracts of U_{x_0} and V_{y_0} respectively. Then*

$$\pi_1(X \vee Y, [x_0] = [y_0]) = \pi_1(X, x_0) * \pi_1(Y, y_0).$$

Here, the symbol $*$ denotes the free product.

4. The fundamental group of a graph

As we saw in the previous section, the fundamental group of the example in figure (3) can be readily calculated. Since X and Y are circles and the fundamental group of a circle is $\mathbb{Z} \cong \langle \{x\}; \emptyset \rangle$, we obtain that the fundamental group of two circles that intersect in a single point is the free product generated by two elements

$$\langle \{x\} \cup \{y\}; \emptyset \rangle.$$

Moreover, by induction, the fundamental group of n circles with only one point x_0 in common is the free group generated by n elements. Many more spaces are straightforward to compute now. Take, for example, the graph of figure (6), which we will call G . Notice that segments $\overline{p_0p_1}$, $\overline{p_0p_3}$, $\overline{p_7p_8}$, $\overline{p_7p_9}$ and $\overline{p_7p_{10}}$ are con-

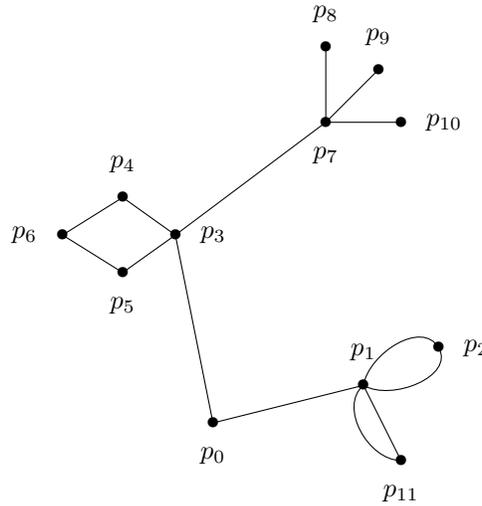


FIG. 6.

tractible, so G is of the same homotopy type of the space in figure (7), which we will call \tilde{G} . Since the fundamental group is invariant by equivalence homotopies,

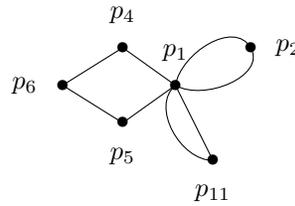


FIG. 7.

we obtain

$$\pi_1(G, p_1) \cong \pi_1(\tilde{G}, p_1) \cong \langle \{x, y, z\}; \emptyset \rangle \cong \mathbb{Z} * \mathbb{Z} * \mathbb{Z} := F_3.$$

These examples shows us that what really matters in a graph is the number of cycles that it has. However, graphs can become too difficult to perform calculations by inspection. Therefore, the necessity of an algorithm to determine the fundamental group of a graph arises. Let's first translate the idea of a graph into a topological concept.

DEFINITION 4.11 (Graph). *A graph is a pair (X, X^0) with X a Hausdorff space and X^0 a subspace of X (called the set of vertices of X) such that:*

- (1) X^0 is a discrete² closed subspace of X . Points of X^0 are called vertices.
- (2) $X \setminus X^0$ is a disjoint union of open subsets e_i , where e_i is homeomorphic to an open interval of the real line. These sets e_i are called edges.
- (3) For each edge e_i , its boundary $\bar{e}_i \setminus e_i$ is a subset of X^0 consisting of one or two points. If $\bar{e}_i \setminus e_i$ consists of two points, then the pair (\bar{e}_i, e_i) is homeomorphic to the pair $([0, 1], (0, 1))$. On the other hand, if $\bar{e}_i \setminus e_i$ consists of one point, then the pair (\bar{e}_i, e_i) is homeomorphic to the pair $(\mathbb{S}^1, \mathbb{S}^1 \setminus \{(1, 0)\})$, where \mathbb{S}^1 is the unit circle in the plane.
- (4) X has the weak topology: A subset $A \subset X$ is closed (or open) if and only if $A \cap \bar{e}_i$ is closed (or open) in \bar{e}_i for all edges e_i .

A graph is said to be finite if it has only a the set of vertices and edges is finite. Otherwise, it is said to be infinite.

The fourth property is always verified if X has a finite number of edges. If X has an infinite number of edges, this property may no be true and hence, it is demanded axiomatically. We will discuss this point later on in section (5). To work comfortably, we need to introduce further notations and definitions.

DEFINITION 4.12. Given a graph (X, X^0) consider A a subset of X and $A^0 = A \cap X^0$. A pair (A, A^0) is called a subgraph if it is a graph on its own.

Now we must define what is an *orientation*. The definition of orientation of an edge pretends to inform about how we move along an edge. Roughly speaking, an edge in a graph is oriented by placing an arrow on it to indicate a choice of positive direction along the edge. Every edge has two possible orientations. Let's formalise this idea.

DEFINITION 4.13. By definition, an edge e is homeomorphic to $(0, 1)$, so let h_0 and h_1 be two such homeomorphisms from e to $(0, 1)$. In particular they are bijective and hence, $h_0 \circ h_1^{-1}$ is also bijective. Any bijective function from I to itself is either monotone increasing or monotone decreasing. Therefore, we can consider an equivalence relation as follows: We say that two homeomorphisms

$$h_0, h_1 : e \longrightarrow (0, 1)$$

are equivalent if $h_0 \circ h_1^{-1} : (0, 1) \longrightarrow (0, 1)$ is a monotone increasing map. By the previous reasoning, if we denote by C the set of all homeomorphisms from e to $(0, 1)$, then the quotient set

$$C / \sim = \{[h], [\tilde{h}]\}$$

given by this equivalence relation consists of only two elements. Therefore, an oriented edge is a pair consisting of the proper edge and a choice of any of the two elements of the quotient set.

DEFINITION 4.14 (Edge path). An edge path is a path in the graph given by a sequence of linked edges (e_1, \dots, e_n) with $n \geq 1$, i.e., a sequence of edges such that the ending of the edge e_{i-1} coincide with the beginning of the edge e_i for $1 \leq i \leq n$. Given an edge path (e_1, \dots, e_n) , the initial vertex of e_1 is called the initial vertex of the edge path and the ending vertex of e_n is called the terminal vertex of the edge

²Recall that a set A in a topological space X is discrete if every point $x \in A$ has a neighbourhood U of X such that $A \cap U = \{x\}$. The points of A are said to be isolated.

path. If the initial vertex and the terminal vertex of the edge path are the same, the edge path is said to be closed. Moreover, an edge path is called reduced if in the sequence there is not any pair (e_{i-1}, e_i) such that e_{i-1} and e_i are the same edge with opposite orientation. We are mainly interested in reduced paths.

Notice that a graph can be either connected or disconnected. Although we only know how to compute the fundamental group of path connected spaces, it would not be difficult to compute it for disconnected spaces. However we will consider only connected (and therefore path connected) spaces.

DEFINITION 4.15 (Tree). *A tree is a connected graph that contains no closed reduced edge paths.*

To avoid the usual problems with definitions in trivial cases, we will consider that a graph consisting of a single vertex and no edges is a tree. For example, a single edge with two vertices is a tree whereas an edge with only one vertex (i.e., \bar{e} is homeomorphic to \mathbb{S}^1) is not. In addition, the graph in figure (6) is not a tree. The calculation of the fundamental group of a graph deals with trees, so we need to prove some properties before we go on with the general theorem. The following results are true in graph theory, so by proving them in a topology sense will mean that our definition of (topologic) graph is accurate. However, the goal of this section is to find a result involving fundamental groups and graphs. For this reason, we will skip proofs of some intuitive statements such as propositions (4.16) and (4.17) and some statements inside proofs.

PROPOSITION 4.16. *In a tree, any two distinct vertices can be joined by a unique reduced edge path.*

PROPOSITION 4.17. *Any connected subgraph of a tree is also a tree.*

PROPOSITION 4.18. *Any tree (X, X^0) is contractible.*

PROOF. Let's prove first the result for finite trees, using induction on the number of edges. If the graph has no edges, then X consists only of a single vertex and thus is automatically contractible. The case with one edge is similar. A graph that consists only of a single edge e_1 can have either one or two vertices. As we discussed before, if it has only one vertex, it is not a tree. If it has two vertices, then \bar{e}_1 is homeomorphic to $[0, 1]$ and thus $X = \{\bar{e}_1\}$ is contractible. So assume that the proposition is true for all trees with strictly less than n edges with $n > 1$, and let T be a tree with n edges. The proof will be divided into two parts.

In the first part we must prove that there exists a vertex v that is incident to only one edge. It can be proven by contradiction; if all vertices in a connected graph are incident with two or more edges, then we can find a closed edge path. So consider e the unique edge with which v is incident and consider $\tilde{T} = T \setminus (e \cup \{v\})$. In this case, \tilde{T} is a connected subgraph of T . Therefore, \tilde{T} has less than n edges and we can use the inductive hypothesis to deduce that \tilde{T} is contractible.

The second part of the proof consists in noticing that \tilde{T} is a deformation retract of T . Therefore, T is contractible.

It remains to prove the proposition for a general (not necessarily finite) tree. Let T be an arbitrary tree, v_0 be a vertex of T and x be a point of T . We define an homotopy

$$f : T \times I \longrightarrow T$$

such that $f(x, 0) = x$, $f(x, 1) = v_0$ and $f(v_0, t) = v_0$ for any $x \in T$ and for any $t \in I$. This is an homotopy that contracts a path from v_0 to x relatively to v_0 . Notice that such a path always exists because T is connected by assumption. With the same reasoning, for each vertex $v \in T$ there exists a path from v to v_0 . Therefore, for each vertex $v \in T$ we can consider a finite connected subgraph $T(v)$ that contains the vertices v_0 and v ³. For each such v , choose a path $\sigma_v(t) = f(v, t)$ in $T(v)$ (with $t \in [0, 1]$) with initial point v and terminal point v_0 . We also define the constant path $\sigma_{v_0} = f(v_0, t) = v_0$ for all $t \in [0, 1]$. Now let e be any edge of T . We want see that we can extend the map f over the set $\bar{e} \times I$. Let v_1 and v_2 be the vertices incident with e . Notice that since T is a tree, v_1 and v_2 must be different. Consider now the graph $T(v_1) \cup T(v_2) \cup e$. This graph is a tree because the union of subtrees of trees is a tree⁴. Moreover, it is connected and finite and therefore, by the first part of the proof, it is contractible and hence, simply connected. On the other hand, \bar{e} is homeomorphic to $[0, 1]$, so the set $\bar{e} \times I$ is homeomorphic to a square. The map f is defined on the two boundaries of the square $\{v_1\} \times I$ and $\{v_2\} \times I$. On the other two boundaries we have no choice as to the definition. One of the boundaries is $\bar{e} \times \{0\}$ and f is defined there as

$$f(x, 0) = x.$$

The other boundary is $\bar{e} \times \{1\}$ and f is defined there as

$$f(x, 1) = v_0.$$

In all we have a square and a function defined in the boundary with its image in $T(v_1) \cup T(v_2) \cup e$, which is simply connected. To finish the proof we need the following lemma from topology, which we will not prove:

LEMMA 4.19. *Let D denote a closed disk, that is, a topological space homeomorphic to a disk of \mathbb{R}^2 . Let B denote the boundary of D , which is a circle and let $g : I \longrightarrow B$ denote the continuous map which rounds the circle only once. More precisely, $g(0) = g(1) = d_0 \in B$ and g maps the interval $(0, 1)$ homeomorphically into $B \setminus \{x_0\}$. Finally, consider a topological space X . A continuous map $f : B \longrightarrow X$ can be extended to a map $D \longrightarrow X$ if and only if the closed loop $f \circ g : I \longrightarrow X$ is equivalent to the constant loop at the base point $f(d_0)$.*

We now invoke this lemma (4.19) to conclude that the map f can be extended over the interior of the square. By this process, we can extend the map $f : T \times I \longrightarrow T$ to a map $f : T \times I \longrightarrow I$. It remains to prove some technical issues, such as that the map thus defined is continuous. The key point is that the map is continuous over $\bar{e} \times I$ for every edge e and that T has the weak topology. We will skip this last technical lemma. \square

PROPOSITION 4.20. *Let X be a graph. Then, there exist maximal a tree contained in X and any tree contained in X is contained in this maximal tree in X .*

³Recall that this path is contained in T . If we denote this path by σ , then the set $\sigma(I) \subset T$ is a tree indeed. Hence, we can always find such a subtree.

⁴It is immediate to prove it by contradiction using the definition of a tree.

PROOF. If X is finite, then there are only a finite number of subgraphs of X and therefore the proposition becomes trivial. To proof the infinite case, we must use the Zorn's lemma. Consider a subgraph T which is a tree and consider $\{T_\lambda | T \subset T_\lambda, \lambda \in \Delta\}$ be a family of trees of X containing T which is linearly ordered by inclusion. Notice that

$$\tilde{T} := \bigcup_{\lambda \in \Delta} T_\lambda$$

is a subgraph of X which is a tree. Therefore, by the Zorn's lemma, \tilde{T} is a maximal tree (so we have proven the existence of a maximal tree) and $T \subset \tilde{T}$. \square

PROPOSITION 4.21. *Let X be a connected graph and T be a subgraph of X which is a tree. Then, T is a maximal tree if and only if T contains all the vertices of X .*

PROOF. We will prove the direct proposition by contradiction. Suppose that T is a maximal tree which does not contain all the vertices of X . Since X is connected, there exists always an edge path (e_1, \dots, e_n) in X such that the initial vertex is contained in T and the terminal vertex is not in T . Using the same argument we can find one edge e_i in the edge path such that its initial vertex is in T and its terminal vertex is not. Therefore, \bar{e}_i is not contained in T . However, $T \cup \bar{e}_i$ is a connected subgraph of X which is a tree. This contradicts the maximality of T . To prove the converse, assume that T is a tree that contains all vertices of X . Consider e to be an edge of X not contained in T . By hypothesis, all vertex of e are contained in T and hence, $T \cup e$ is a connected subgraph of X . However, $T \cup e$ must contain a closed edge path and therefore, $T \cup e$ is not a tree. Since this argument can be applied for any edge of $X \setminus T$, we conclude that we can not add any edge to T such that the union is a tree and hence, T is a maximal tree. \square

Finally, we are due to show that the fundamental group of a connected graph X is a free group. We already saw that if X is a tree, then the fundamental group is trivial. To avoid special cases, we will consider the trivial group as a free group with an empty set of generators. If X is not finite, there is some work to do.

First of all, let's do some remarks. Although an edge path in a graph and a path in a topological space are two different concepts, it seems intuitive that there is a relation between these two concepts. An edge path (e_1, \dots, e_n) in the graph joining two vertices v_0 and v_1 determines a unique equivalence class of paths in the topological space X joining v_0 and v_1 as follows. For each oriented edge e_i , choose a functions $f_i : I \rightarrow \bar{e}_i$ such that f_i restricted to $(0, 1)$ is a homeomorphism from $(0, 1)$ to e_i and such that the inverse belongs to the the same class determined by the orientation of e_i . If we denote by α_i the equivalence class of the path f_i , then the product $\alpha_1 \alpha_2 \dots \alpha_n$ is uniquely determined by the edge path (e_1, \dots, e_n) . It seems intuitive to mix the concepts of edge path and topological paths. For this reason we will relax the notation and we will use both concepts interchangeably. The meaning will be clear by the context.

Let X be a connected graph, let v_0 be a vertex of X and let T be a maximal tree in X containing v_0 . We know that such a tree exists by proposition (4.20) and we know that it must contain v_0 by proposition (4.21). Consider the set $\{e_\lambda | \lambda \in \Delta\}$, which denotes the set of edges of X not contained in T . For each edge, choose a

definite orientation and let a_λ and b_λ denote the initial and terminal vertices of e_λ . Notice that a_λ and b_λ can be equal. Now we want to associate to each e_λ an element $\alpha_\lambda \in \pi_1(X, v_0)$. Notice that since T is a tree, then there is a unique reduced edge path A_λ from v_0 to a_λ and a unique reduced edge path B_λ from b_λ back to v_0 . Therefore $(A_\lambda, e_\lambda, B_\lambda)$ is an edge path, so we consider α_λ the path class associated with this edge path. With this notation, we can introduce the last theorem of the section.

THEOREM 4.22. *With the previous notations, the fundamental group $\pi_1(X, v_0)$ is a free group on the set of generators $\{\alpha_\lambda \mid \lambda \in \Delta\}$.*

PROOF. First we will prove the statement for the case which $\{\alpha_\lambda \mid \lambda \in \Delta\}$ has only one element. This is, there is only one edge of X not contained in T . Let e_1 be such edge. Since X is not a tree, there exist closed edge paths in X and since T is a maximal tree, any such closed edge path must contain the edge e_1 . Consider a definite orientation for e_1 . In this case, there exist reduced closed edge paths in X starting in e_1 , i.e., reduced closed edge paths of the form (e_1, \dots, e_n) . From all these reduced closed edge paths, there exist at least one which no vertex or edge occurs twice⁵. Denote this reduced closed edge path by (e_1, \dots, e_m) and consider

$$C = \bigcup_{i=1}^m \bar{e}_i.$$

Then, C is a subgraph of X homeomorphic to a circle. Consider now the complementary set $X \setminus C$ and let $\{Y_i\}$ be the set of components of $X \setminus C$. Each \bar{Y}_i is a subgraph of the tree T and therefore it is a tree. Moreover, each \bar{Y}_i has exactly one vertex in common with C . Since a tree can be contracted to any of its vertices, we can contract each tree to their unique contact point with C . In all, what we have proven is that X is of the same homotopy type of C (more concretely, C is a deformation retract of X). Therefore, since C is homeomorphic to a circle,

$$\pi_1(X, v_0) \cong \pi_1(C, v_0) \cong \mathbb{Z} \cong \langle \{e_1\}; \emptyset \rangle.$$

Now we must proof the general case using the SVK theorem. For each $\lambda \in \Delta$, choose a point $x_\lambda \in e_\lambda$. The set $\{x_\lambda \mid \lambda \in \Delta\}$ is closed and discrete because X has the weak topology. Consider U the complement of $\{x_\lambda \mid \lambda \in \Delta\}$ in X . Then, U is contractible because T is a deformation retract of U . For each index λ , define

$$V_\lambda := U \cup \{x_\lambda\}.$$

In this case, $U \subset V_\lambda$ for all λ and if $\lambda \neq \mu$,

$$V_\lambda \cap V_\mu = U.$$

It can be readily seen that $T \cup e_\lambda$ is a deformation retract of V_λ and therefore, the fundamental group $\pi_1(V_\lambda, v_0)$ is infinite cyclic and is generated by α_λ , i.e., is $\langle \{\alpha_\lambda\}; \emptyset \rangle$. In all, now we have a set $\{V_\lambda \mid \lambda \in \Delta\}$ satisfying

⁵Since the edge path is closed, one could argue that the initial and terminal vertices are the same, there is a vertex that occurs twice. However, we will consider that only this specific vertex occurs only once.

(1) X is the union of all V_λ , i.e.,

$$X = \bigcup_{\lambda \in \Delta} V_\lambda.$$

(2) The fundamental group of each V_λ in the point v_0 is \mathbb{Z} .

(3) For any $\lambda, \mu \in \Delta$, $V_\lambda \cap V_\mu$ is simply connected, i.e.,

$$\pi_1(V_\lambda \cap V_\mu, v_0) = 1.$$

In this case, it is straightforward to compute the fundamental group of X using inductively the SFK theorem. The result is that the fundamental group of X is a free group on the set of generators $\{\alpha_\lambda \mid \lambda \in \Delta\}$. \square

An immediate corollary is the following:

COROLLARY 4.23. *If Δ is countable, then the fundamental group is countable.*

A final remark is that for all free group F_n , there always exists a graph whose fundamental group is F_n . For a general group, this result also holds.

PROPOSITION 4.24. *If G is a group, there exists a space X whose fundamental group is isomorphic to G , i.e., $\pi_1(X, x_0) \cong G$.*

We will not need this proposition in the future, so we will skip the proof. To conclude the study of graphs, let's see a final discussion about the definition of a graph in the following section.

5. The Hawaiian earring

In this section we discuss the importance of the fourth point in the definition of a graph. Moreover, we will see that infinite graphs behave a little bit different from expected with respect to finite graphs.

Consider the topological space called *Hawaiian earring*, which consist of a countable set of circles that intersect at a common single point,

$$E := \bigcup_{n \in \mathbb{N}} \left\{ (x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \frac{1}{n^2} \right\}.$$

The space E has the induced topology of \mathbb{R}^2 and graphically, it looks like figure (8). Is the Hawaiian earring a graph? There are many ways of checking whether it is a graph or not. One way is looking at the definition of a graph. In the definition, the controversial point is the fourth one. Has the earring with the subspace topology the weak topology? Another approach is by using the calculations that we developed in the previous section.

If we have a graph that consists of a finite join of n circles that intersect in only one point, the fundamental group will be $\mathbb{Z} * \dots * \mathbb{Z}$. Moreover, according to corollary (4.23), if we have a countable union of circles that intersect in only one point, then fundamental group will be countable as well. So, in all, if E was a graph, its

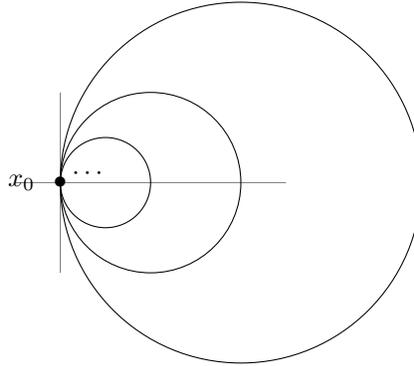


FIG. 8.

fundamental group should be countable. However, the fundamental group of the Hawaiian earring is not countable. Hence, although the earring may look like a regular infinite graph, it can not be a graph.

Let's prove that the fundamental group of the earring is not countable. For a fixed n , consider the circle $C_n \subset E$. We can define a map f as follows:

$$f_n : E \longrightarrow C_n$$

$$x \longrightarrow \begin{cases} x_0 & \text{if } x \notin C_n, \\ id(x) = x & \text{if } x \in C_n. \end{cases}$$

This map induces an application between the fundamental groups

$$f_{n*} : \pi_1(E, x_0) \longrightarrow \pi_1(C_n, x_0) \cong \mathbb{Z},$$

which is clearly surjective. Using the same principle, we can define a surjective map

$$\begin{aligned} \pi_1(E, x_0) &\longrightarrow \pi_1\left(\bigcup_{n \in \mathbb{N}} C_n, x_0\right) = \prod_{n \in \mathbb{N}} \pi_1(C_n, x_0) = \prod_{n \in \mathbb{N}} \mathbb{Z}. \\ x &\longrightarrow (f_1(x), f_2(x), \dots, f_n(x), \dots). \end{aligned}$$

Here, the big product symbol \prod denotes the cartesian product. Now, the set $\prod_{n \in \mathbb{N}} \mathbb{Z}$ is not countable. Since the map is clearly surjective, we conclude that $\pi_1(E, x_0)$ is not countable. However, if E was a graph, its fundamental group should be countable. Therefore, E can not be a graph.

In this example we see that the fourth point in the definition of a graph is non trivial at all. Asking the space to have the weak topology exclude some spaces that we could think intuitively as graphs.

6. The fundamental group of a compact connected surface

Another particular case of the SVK theorem is when one of the covering sets is simply connected, i.e, $X = U_1 \cup U_2$ with U_2 simply connected. With the notation of the SVK theorem, we obtain that

$$\pi_1(X, x_0) \cong \langle S_1; R_1 \cup R_s \rangle,$$

with

$$R_s = \{ \varphi_{1*} s' \mid s \in \pi_1(U_1 \cap U_2, x_0) \}.$$

This particular case will be extremely useful to compute the fundamental group of connected compact surfaces.

Since we will only study the case of connected compact surfaces, we will refer to them simply as surfaces. In a general course of topology it is usually introduced the classification of connected compact surfaces. The classification theorem says that any surface can be represented as a regular polygon with identifications on its edges. Moreover, any surface is homeomorphic to either a sphere, a connected sum of tori or a connected sum of projective planes. The way to represent surfaces is by words so a word is assigned to every surface. From the classification theorem it can be deduced that we can assign to any surface a *standard* word. A standard word is a word of the following form

- (1) $A_0 = aa^{-1}$ (the sphere).
- (2) $A_g = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ (The connected sum of g tori).
- (3) $B_g = a_1 a_1 a_2 a_2 \cdots a_g a_g$ (The connected sum of g projective planes).

Our aim is to compute the fundamental group of a surface given its word, also called its *representation*. Before we proceed with the general case, we will immerse ourselves into two simpler examples in full detail. Let's start with the fundamental group of a torus.

We already saw that its fundamental group is \mathbb{Z}^2 . Consider a general torus \mathbb{T} as a square with the corresponding identifications like in figure (9 (left)). Figure (9

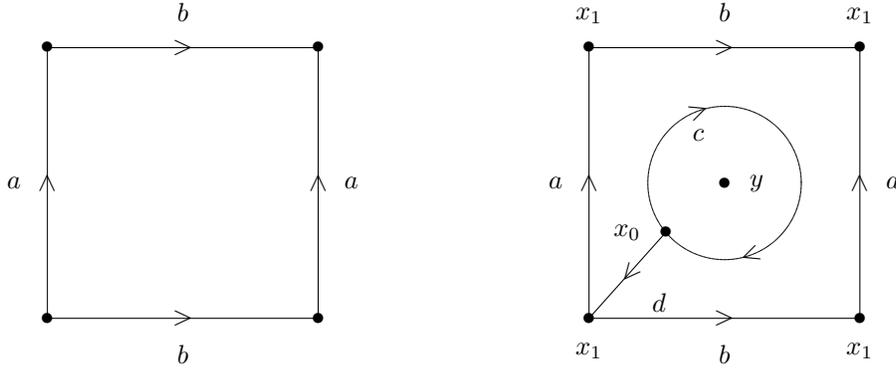


FIG. 9.

(right)) shows the notation that we are due to use. Let y be a point in the interior of the square and let us denote a and b the identified edges of the torus. We define

$$\begin{aligned} U_1 &:= \mathbb{T} \setminus \{y\}, \\ U_2 &:= \mathbb{T} \setminus (a \cup b). \end{aligned}$$

The set U_1 is the torus without an interior point and U_2 is the interior of the square. Both sets are path connected and opened and their intersection is non-empty and path connected. Hence, we can apply the SVK theorem. Consider x_0 and x_1 two

points in \mathbb{T} different from y and let d be the segment from x_0 to x_1 . Finally, let c be the circle with center y that contains x_0 . First of all, U_1 is of the same homotopy type of a closed square, with no interior, defined by the four identified edges, which is also of the same homotopy type of the two circles with a single contact point x_1 (figure 10). Then, we can state that the fundamental group of U_1 is isomorphic

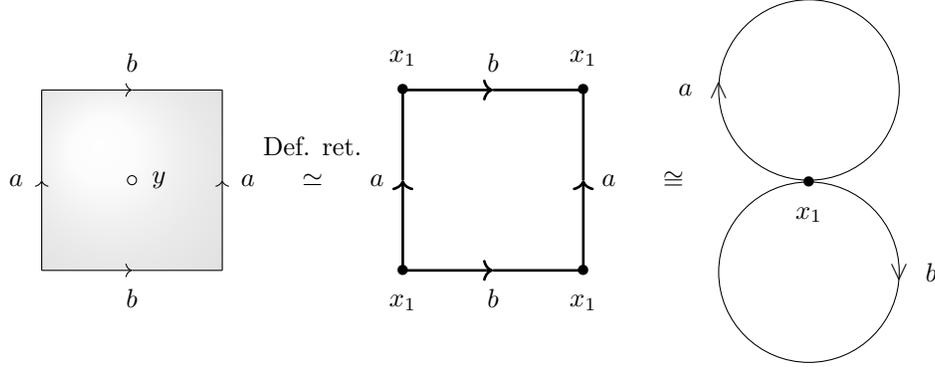


FIG. 10.

to $\mathbb{Z} * \mathbb{Z}$. More specifically, consider α_1 and α_2 closed paths in U_1 based at x_1 that round the circle only once through a and b respectively with the indicated direction in the figure. Then, $\pi_1(U_1, x_1) \cong \langle \{[\alpha_1], [\alpha_2]\}; \emptyset \rangle$. If we want to compute the fundamental group on x_0 , we only have to consider a path δ that corresponds to d that goes from x_0 to x_1 (that is, that $\delta : I \rightarrow d$ is a homeomorphism). In this case

$$\pi_1(U_1, x_0) = \langle \{[\delta * \alpha_1 * \bar{\delta}], [\delta * \alpha_2 * \bar{\delta}]\}; \emptyset \rangle.$$

To make it more handy, we will denote the first generator by A_1 and the second generator by A_2 .

Secondly, U_2 is contractible and hence, $\pi_1(U_2, x_0) = 1$. Moreover, the set $U_1 \cap U_2$ is of the same homotopy type that the circle c . So, if γ is a closed path based at x_0 that rounds the circle only once in the direction shown in figure (9 (right)), then

$$\pi_1(U_1 \cap U_2, x_0) \cong \langle \{[\gamma]\}, \emptyset \rangle \cong \mathbb{Z}.$$

Seifert-Van Kampen tells us that the fundamental group of the torus is generated by A_1 and A_2 and is subject to the relation

$$'\varphi_{1*}[\gamma]' = '\varphi_{2*}[\gamma]'.$$

In U_1 we have

$$\begin{aligned} \varphi_{1*}[\gamma] &= [\varphi_1 \circ \gamma] = [\delta * \alpha_1 * \alpha_2 * \bar{\alpha}_1 * \bar{\alpha}_2 * \bar{\delta}] \\ &= [\delta * \alpha_1 * \bar{\delta}][\delta * \alpha_2 * \bar{\delta}][\delta * \bar{\alpha}_1 * \bar{\delta}][\delta * \bar{\alpha}_2 * \bar{\delta}]. \end{aligned}$$

The proof of this equality is given at the end of this proof. For now, let's assume that this is true. Therefore

$$'\varphi_{1*}[\gamma]' = A_1 A_2 A_1^{-1} A_2^{-1}.$$

On the other hand, $'\varphi_{2*}[\gamma]' = 1$ because U_2 is simply connected. Notice that here 1 means the empty word. So, in all,

$$\pi_1(\mathbb{T}, x_0) \cong \langle \{A_1, A_2\}; \{A_1 A_2 A_1^{-1} A_2^{-1}\} \rangle \cong \mathbb{Z}^2.$$

There is one last think to discuss. We will prove now that $[\varphi_1 \circ \gamma] = [\delta * \alpha_1 * \alpha_2 * \bar{\alpha}_1 * \bar{\alpha}_2 * \bar{\delta}]$ in U_1 . We will prove an equivalent statement, which is that $[\bar{\delta} * (\varphi_1 \circ \gamma) * \delta] = [\alpha_1 * \alpha_2 * \bar{\alpha}_1 * \bar{\alpha}_2]$ in U_1 . If we denote

$$\beta := \alpha_1 * \alpha_2 * \bar{\alpha}_1 * \bar{\alpha}_2,$$

then we must show an homotopy from $\bar{\delta} * (\varphi_1 \circ \gamma) * \delta$ to β relatively to $\{0, 1\}$ in U_1 . Let's consider that the path β is as follows:

- i It starts at $\varphi_1(x_1)$ at time $t_0 = 0$ and reaches $\varphi_1(x_0)$ at a certain time t_1 , moving along d^{-1} .
- ii It moves from $\varphi_1(x_0)$ to $\varphi_1(x_0)$ round the circle c from time t_1 to a certain time t_2 .
- iii It moves from $\varphi_1(x_0)$ to $\varphi_1(x_1)$ from time t_2 to $t_3 = 1$, moving along d .

Recall that the edges of the square are a deformation retract of U_1 . Therefore, the path $\bar{\delta} * (\varphi_1 \circ \gamma) * \delta$ is of the same homotopy type as β , and the homotopy relative to $\{0, 1\}$ is constructed as follows:

- i The path described in (i) is homotopic to the path described by the edge a on the left hand side of the square.
- ii The path described in (ii) is homotopic to the path described by the edges b on top of the square followed by a^{-1} on the right hand side of the square.
- iii The path described in (iii) is homotopic to the path described by the edge b^{-1} at the bottom of the square.

Therefore, the equality of the statement holds.

The last example is the real projective plain. We already saw that its fundamental group is \mathbb{Z}_2 . Our *modus operandi* will be exactly the same as in the previous example, so we will see it quickly. Let's take a look at figure (11) to see what notation we will use. Consider $U_1 = \mathbb{P}^2 \setminus \{y\}$ and $U_2 = \mathbb{P}^2 \setminus \{a\}$. The sets U_1, U_2 and

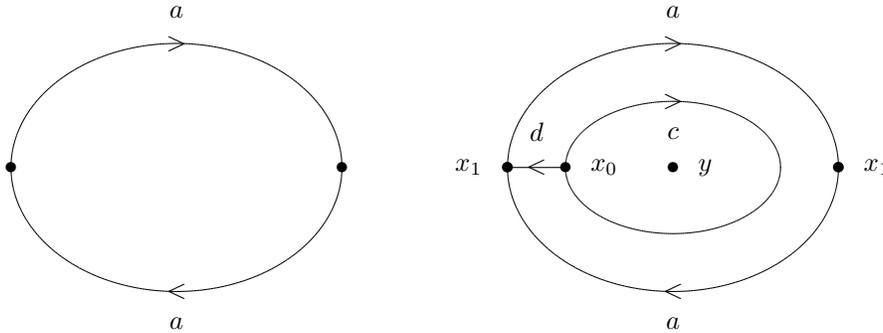


FIG. 11.

$U_1 \cap U_2$ verify the conditions of SVK. On the one hand, the circle a is a deformation retract of U_1 . So, if we denote by α the closed path based at x_1 that corresponds to a , then the fundamental group of $\pi_1(U_1, x_1) \cong \langle [\alpha]; \emptyset \rangle$. Moreover, if δ represents the path from x_0 to x_1 that corresponds to d , then $\pi_1(U_1, x_0) \cong \langle [\delta * \alpha * \bar{\delta}]; \emptyset \rangle$. Let's denote $A = [\delta * \alpha * \bar{\delta}]$. On the other hand, U_2 is contractible, so $\pi_1(U_2, x_0) = 1$. The circle c is homotopy equivalent to $U_1 \cap U_2$. More precisely, c is a deformation retract of $U_1 \cap U_2$. Hence, $\pi_1(U_1 \cap U_2, x_0) \cong \langle \{[\gamma]\}; \emptyset \rangle$, with γ being the closed path in $U_1 \cap U_2$ based at x_0 corresponding to c , i.e. a path that rounds the circle c only once in the direction shown in figure (11). The SVK theorem says that

$$\pi_1(X, x_0) \cong \langle \{A\}; R_s \rangle.$$

Let's compute the relations R_s

$$' \varphi_{1*}[\gamma]' = ' \varphi_{2*}[\gamma]'.$$

On the one hand,

$$\varphi_{1*}[\gamma] = [\varphi_1 \circ \gamma] = [\delta * \alpha * \alpha * \bar{\delta}] = [\delta * \alpha * \bar{\delta}][\delta * \alpha * \bar{\delta}].$$

Therefore, $' \varphi_{1*}[\gamma]' = A^2$. On the other hand, $' \varphi_{2*}[\gamma]' = 1$, so, in all,

$$\pi_1(X, x_0) \cong \langle \{A\}; \{A^2\} \rangle \cong \mathbb{Z}_2.$$

Until here, we have seen all the ideas that are needed to compute the fundamental group of a surface. The proof of the general case is exactly the same.

THEOREM 4.25. *Let X be a connected compact surface. Let's denote the possible standard words $A_g = a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$, $B_g = a_1 a_1 a_2 a_2 \dots a_g a_g$ and $A_0 = a a^{-1}$.*

(1) *If the standard word of X is A_g , then*

$$\pi_1(X, x_0) \cong \langle \{a_1, b_1, a_2, b_2, \dots, a_g, b_g\}; \{a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}\} \rangle.$$

(2) *If the standard word of X is B_g , then*

$$\pi_1(X, x_0) \cong \langle \{a_1, a_2, \dots, a_g\}; \{a_1^2 a_2^2 \dots a_g^2\} \rangle.$$

(3) *If the standard word of X is A_0 , then $\pi_1(X, x_0) \cong 1$.*

PROOF. Proof of (1). Consider a connected sum of n tori as in figure (12). This figure also shows the so common used notation. Consider $U_1 = \mathbb{T}^n \setminus \{y\}$ and $U_2 = \mathbb{T}^n \setminus (a_1 \cup b_1 \cup \dots \cup a_n \cup b_n)$. All sets U_1 , U_2 and $U_1 \cap U_2$ verify the hypothesis of the SVK theorem. The set U_1 is of the same homotopy type of $2n$ circles with common intersection $\{x_1\}$. If we denote by α_i and β_i the closed paths based at x_1 that rounds the i th circle (and $i + 1$ th circle respectively) only one through a_i and b_i respectively, then

$$\pi_1(U_1, x_1) \cong \langle \{[\alpha_1], [\beta_1], \dots, [\alpha_n], [\beta_n]\}; \emptyset \rangle.$$

Moreover, if we denote by δ the path that corresponds to d that goes from x_0 to x_1 , then

$$\pi_1(U_1, x_1) \cong \langle \{A_1, B_1, \dots, A_n, B_n\}; \emptyset \rangle,$$

where $A_i := [\delta * \alpha_i * \bar{\delta}]$ $B_i := [\delta * \beta_i * \bar{\delta}]$. On the other hand, U_2 is simply connected and hence $\pi_1(U_2, x_0) = 1$. In addition, the set $U_1 \cap U_2$ is of the same homotopy

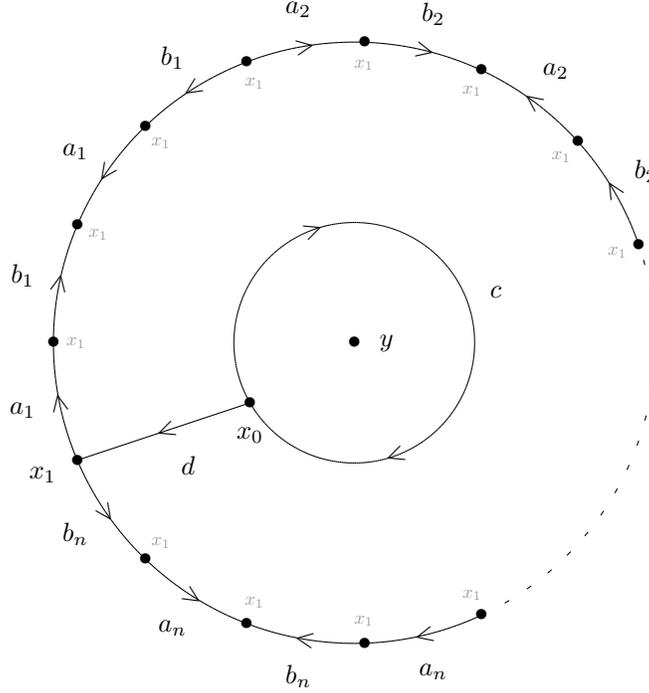


FIG. 12.

type that the circle c . So, if γ is a closed path based at x_0 that rounds the circle only once in the direction of the figure, then

$$\pi_1(U_1 \cap U_2, x_0) \cong \langle [\gamma]; \emptyset \rangle.$$

Let's compute now the set R_s by computing the relations

$$' \varphi_{1*}[\gamma]' = ' \varphi_{2*}[\gamma]'.$$

The left hand side of the equality is

$$\begin{aligned} \varphi_{1*}[\gamma] &= [\varphi_1 \circ \gamma] \\ &= [\delta * \alpha_1 * \beta_1 * \overline{\alpha_1} * \overline{\beta_1} * \cdots * \alpha_n * \beta_n * \overline{\alpha_n} * \overline{\beta_n} * \delta] \\ &= A_1 B_1 A_2 B_2 \dots A_n B_n. \end{aligned}$$

The right hand side of the equality is $' \varphi_{2*}[\gamma]' = 1$ because U_2 is simply connected. So, in all,

$$\pi_1(\mathbb{T}^n, x_0) \cong \langle \{A_1, B_1, \dots, A_n, B_n\}; \{A_1 B_1 A_2 B_2 \dots A_n B_n\} \rangle,$$

as we wanted to prove.

Proof of (2). Consider a connected sum of n projective plains as in figure (13). This figure also shows the notation that we will use. Consider $U_1 = \mathbb{P}^n \setminus \{y\}$ and $U_2 = \mathbb{P}^n \setminus (a_1 \cup \cdots \cup a_n)$. All sets U_1 , U_2 and $U_1 \cap U_2$ verify the hypothesis of the SVK theorem. The set U_1 is of the same homotopy type of n circles with common intersection $\{x_1\}$. If we denote by α_i the closed paths based at x_1 that round the

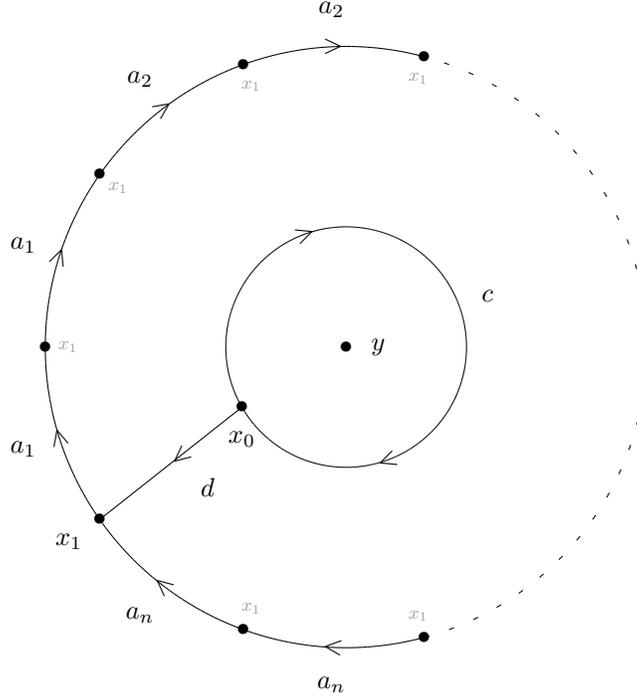


FIG. 13.

i th circle only once through a_i , then

$$\pi_1(U_1, x_1) \cong \langle \{[\alpha_1], \dots, [\alpha_n]\}; \emptyset \rangle.$$

Moreover, if we denote by δ the path that corresponds to d that goes from x_0 to x_1 , then

$$\pi_1(U_1, x_0) \cong \langle \{A_1, \dots, A_n\}; \emptyset \rangle,$$

where $A_i := [\delta * \alpha_i * \bar{\delta}]$. On the other hand, U_2 is simply connected and hence $\pi_1(U_2, x_0) = 1$. In addition, the set $U_1 \cap U_2$ is of the same homotopy type that the circle c . So, if γ is a closed path based at x_0 that rounds the circle only once in the direction of the figure, then

$$\pi_1(U_1 \cap U_2, x_0) \cong \langle [\gamma]; \emptyset \rangle.$$

Let's compute now the set R_s by computing the relations

$$' \varphi_{1*}[\gamma]' = ' \varphi_{2*}[\gamma]'.$$

The left hand side of the equality is

$$\begin{aligned} \varphi_{1*}[\gamma] &= [\varphi_1 \circ \gamma] \\ &= [\delta * \alpha_1 * \alpha_1 * \dots * \alpha_n * \alpha_n * \bar{\delta}] \\ &= A_1 A_1 A_2 A_2 \dots A_n A_n. \end{aligned}$$

The right hand side of the equality is $' \varphi_{2*}[\gamma]' = 1$ because U_2 is simply connected. So, in all,

$$\pi_1(\mathbb{T}^n, x_0) \cong \langle \{A_1, \dots, A_n\}; \{A_1 A_1 A_2 A_2 \dots A_n A_n\} \rangle,$$

as we wanted to prove.

Proof of (3). A surface whose reduced word is A_0 is homeomorphic to a sphere. Performing the same steps as before, it is straightforward to see that the fundamental group of such surface is the trivial one. Thus, here is the proof that the sphere is simply connected. \square

There is one last think to discuss. This technique shown here can be applied in many more cases, not just the surfaces whose word is in the standard form. For example,

- (1) The Klein bottle, which we will denote by \mathbb{K} (it can be thought as a square with the edges identified as $A_1A_2A_1^{-1}A_2$). Its fundamental group is

$$\pi_1(\mathbb{K}, x_0) \cong \langle \{A_1, A_2\}; \{A_1A_2A_1^{-1}A_2\} \rangle.$$

- (2) An n -th poligon whose edges are identified as $A_1A_1 \dots A_1$. Its fundamental group is

$$\langle \{A_1\}; \{A_1^n\} \rangle \cong \mathbb{Z}_n.$$

The key point in the surfaces of theorem (4.25) and surfaces of these last comment is that all *vertices* in the n -th poligon are the same. So, in all, the main technique of this chapter can be applied to any surface satisfying that all vertices are the same and therefore there is no need to write the surface into the standard form.

7. Abelianization of the fundamental group of compact connected surfaces

At this stage we can ask ourselves if there exist some g and \tilde{g} such that $g\mathbb{T} \cong \tilde{g}\mathbb{T}$, $g\mathbb{P}^2 \cong \tilde{g}\mathbb{P}^2$ or even $g\mathbb{T} \cong \tilde{g}\mathbb{P}^2$. If any of theses cases were true, then their fundamental groups would be isomorphic. However, intuition says that this should not be possible. In the first case, g copies of a torus glued together has a different ammount of *holes* than \tilde{g} copies of a torus glued together. In the last case, for example, if the fundamental groups of \mathbb{T} and \mathbb{P}^2 are not isomorphic (we already proved it), it should happen the same with g copies of a torus glued together and \tilde{g} copies of a projective plain glued together.

Trying to answer this question using the representation of their fundamental groups seen in proposition (4.25) is extremely difficult. As we discussed in the beginning of the chapter, trying to determine if two presentations $\langle S_1; R_1 \rangle$ and $\langle S_2; R_2 \rangle$ lead to isomorphic groups can be extremely difficult even in simple cases. For this reason, we should introduce a new concept.

DEFINITION 4.26. *Given a group $G = \langle S; R \rangle$, the abelianization of G is the group*

$$G_{ab} := \langle S; R \cup \{xyx^{-1}y^{-1}\} \rangle.$$

The abelianization of the group is the *closest* group to G which is abelian. Demanding that the elements of G must satisfy $xyx^{-1}y^{-1}$ means that $xy = yx$. So, what are the abelianizations of $\pi_1(g\mathbb{T}, x_0)$ and $\pi_1(\tilde{g}\mathbb{P}^2, x_0)$?

PROPOSITION 4.27. *The abelianization of the fundamental group of $g\mathbb{T}$ is \mathbb{Z}^{2g} . The abelianization of the fundamental group of $\tilde{g}\mathbb{P}^2$ is $\mathbb{Z}^{\tilde{g}-1} \times \mathbb{Z}_2$.*

PROOF. Consider the space $g\mathbb{T}$. Its fundamental group is

$$G := \pi_1(g\mathbb{T}, x_0) \cong \langle S_G; R_G \rangle,$$

where

$$S_G = A_1 B_1 \dots A_g B_g, \quad R_G = A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1}.$$

The abelianization of G is

$$G_{ab} := \langle S_G; R_G \cup \{s_1 s_2 s_1^{-1} s_2^{-1} \mid s_1, s_2 \in S_G\} \rangle.$$

Thanks to this last set of relations, the relation R_G becomes

$$A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g B_g A_g^{-1} B_g^{-1} = A_1 A_1^{-1} B_1 B_1^{-1} \dots A_g A_g^{-1} B_g B_g^{-1} = 1.$$

In fact, R_G is a consequence of the relation $\{s_1 s_2 s_1^{-1} s_2^{-1} \mid s_1, s_2 \in S_G\}$. Hence, the abelianization of G is

$$G_{ab} = \langle S_G; \{s_1 s_2 s_1^{-1} s_2^{-1} \mid s_1, s_2 \in S_G\} \rangle \cong \mathbb{Z}^{2g}.$$

Consider now the space $\tilde{g}\mathbb{P}^2$. Its fundamental group is

$$\tilde{G} := \pi_1(\tilde{g}\mathbb{P}^2, x_0) \cong \langle S_{\tilde{G}}; R_{\tilde{G}} \rangle,$$

where

$$S_{\tilde{G}} = A_1 A_2 \dots A_{\tilde{g}}; \quad R_{\tilde{G}} = (A_1)^2 A_2^2 \dots A_{\tilde{g}}^2.$$

The abelianization of \tilde{G} is

$$\tilde{G}_{ab} := \langle S_{\tilde{G}}; R_{\tilde{G}} \cup \{s_1 s_2 s_1^{-1} s_2^{-1} \mid s_1, s_2 \in S_{\tilde{G}}\} \rangle.$$

Due to this new set of relations, \tilde{G}_{ab} can be rewritten as

$$\tilde{G}_{ab} := \langle S_{\tilde{G}}; (A_1 A_2 \dots A_{\tilde{g}})^2 \cup \{s_1 s_2 s_1^{-1} s_2^{-1} \mid s_1, s_2 \in S_{\tilde{G}}\} \rangle.$$

Now, any element of this group can be rewritten as

$$A_1^{j_1} A_2^{j_2} \dots A_{\tilde{g}}^{j_{\tilde{g}}},$$

with $j_k \in \mathbb{Z}$ for $k = 1, \dots, \tilde{g}$. But

$$A_1^{j_1} A_2^{j_2} \dots A_{\tilde{g}}^{j_{\tilde{g}}} = A_1^{j_1 - j_{\tilde{g}}} A_2^{j_2 - j_{\tilde{g}}} \dots A_{\tilde{g}-1}^{j_{\tilde{g}-1} - j_{\tilde{g}}} (A_1 A_2 \dots A_{\tilde{g}})^{j_{\tilde{g}}}.$$

Hence,

$$\tilde{G}_{ab} \cong \mathbb{Z}^{\tilde{g}-1} \times \mathbb{Z}_2.$$

□

PROPOSITION 4.28. *If $g, \tilde{g} \in \mathbb{N}$ are different natural numbers, then*

- (1) $\mathbb{Z}^g \not\cong \mathbb{Z}^{\tilde{g}}$.
- (2) $\mathbb{Z}^{g-1} \times \mathbb{Z}_2 \not\cong \mathbb{Z}^{\tilde{g}-1} \times \mathbb{Z}_2$.
- (3) $\mathbb{Z}^g \not\cong \mathbb{Z}^{\tilde{g}-1} \times \mathbb{Z}_2$.

PROOF. Without loss of generality, consider that $g \leq \tilde{g}$.

(1) In \mathbb{Z}^g , the set

$$\{e_i := (0, \dots, 0, 1, 0, \dots, 0) \mid 1 \leq i \leq g\}$$

generates the group \mathbb{Z}^g but it does not generate $\mathbb{Z}^{\tilde{g}}$. Therefore, they can not be isomorphic.

(2) It is immediate due to the previous point.

(3) The group \mathbb{Z}^g is torsion free but $\mathbb{Z}^{\tilde{g}-1} \times \mathbb{Z}_2$ is not. Therefore, they can not be isomorphic.

□

Now we can answer the question at the beginning of the section. It is impossible to find g and \tilde{g} different such that $g\mathbb{T}$ and $\tilde{g}\mathbb{T}$ are homeomorphic because \mathbb{Z}^{2g} and $\mathbb{Z}^{2\tilde{g}}$ are not isomorphic. Moreover, for the same reason, it is impossible to find g and \tilde{g} different such that $g\mathbb{P}^2$ and $\tilde{g}\mathbb{P}^2$ are homeomorphic. Finally, the groups \mathbb{Z}^{2g} and $\mathbb{Z}^{\tilde{g}-1} \times \mathbb{Z}_2$ are not isomorphic for any integers g and \tilde{g} . This leads to the conclusion that it is impossible to find g and \tilde{g} such that $g\mathbb{T}$ and $\tilde{g}\mathbb{P}^2$ are homeomorphic.

Although we have already achieved our goal, it is interesting to notice some corollaries of the discussion in this section.

COROLLARY 4.29. *Two compact connected surfaces are homeomorphic if and only if the abelianization of their fundamental groups are isomorphic.*

PROOF. The direct implications is always true. For the converse, it can be readily deduced from the classification theorem of surfaces. Any compact connected surface is either a sphere, a connected sum of tori or a connected sum of projective planes. So, if X and Y are connected compact surfaces, there are three options:

- (1) The case in which X and Y are spheres is trivial.
- (2) $X = g\mathbb{T}$ and $\tilde{g}\mathbb{T}$. If the abelianization of their fundamental groups are isomorphic, then $g = \tilde{g}$ and hence $X = Y$.
- (3) $X = g\mathbb{P}^2$ and $\tilde{g}\mathbb{P}^2$. With the same reasoning, the only way that the abelianization of their fundamental groups are isomorphic is with $g = \tilde{g}$ and hence $X = Y$.
- (4) The case $X = g\mathbb{T}$ and $\tilde{g}\mathbb{P}^2$ does not verify the hypothesis of the statement because the abelianization of their fundamental groups are never isomorphic. The case where X is a sphere and Y is a connected sum of tori or projective planes neither verifies the hypothesis of the statement.

□

COROLLARY 4.30. *A compact surface is simply connected if and only if it is homeomorphic to a sphere.*

COROLLARY 4.31. *Let X be a space and $x_0 \in X$. If $\pi_1(X, x_0)_{ab}$ is not of the form \mathbb{Z}^{2m} or $\mathbb{Z}^{m-1} \times \mathbb{Z}_2$, then X is not a connected compact surface.*

The last corollary gives us a *test* to determine whether a space is a compact connected surface or not.

8. Torus knots

In this section we are interested in studying a special case in which the fundamental group is a useful tool to obtain a strictly topological result. In other words, we have a statement in which the fundamental group does not appear in the statement. More precisely, we will use the fundamental group to prove that two certain topological spaces called *torus knots* are equivalent if and only if their fundamental groups are isomorphic.

DEFINITION 4.32. A *knot* is a subspace of $K \subset \mathbb{R}^3$ homeomorphic to a circle \mathbb{S}^1 .

In figure (14) we show some popular knots. We are interested in torus knots, which

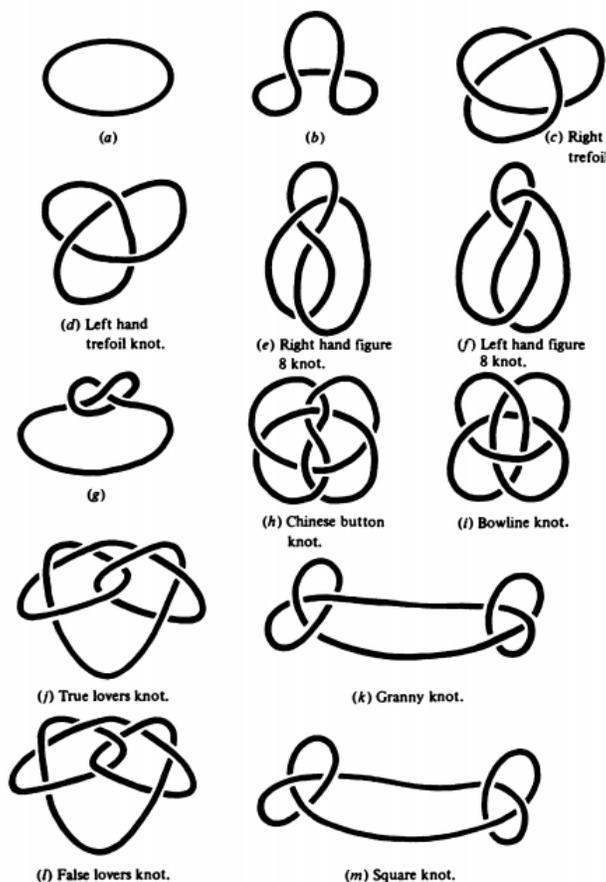


FIG. 14.

are knots arranged in a certain way on the surface of a torus, thought as a subspace of \mathbb{R}^3 . Let us consider a torus \mathbb{T} as $\mathbb{S}^1 \times \mathbb{S}^1$, in which every point is of the form $(e^{i\varphi}, e^{i\theta})$ with $0 \leq \varphi, \theta \leq 2\pi$. We will think \mathbb{R}^3 as $\mathbb{C} \times \mathbb{R}$ and we will use polar coordinates $(r, \theta) \equiv re^{i\theta}$ in \mathbb{C} . So, in all, every point in \mathbb{R}^3 is represented by (r, θ, z) .

We can define now the parametrization of a torus of radius 1 and $1/2$ in cylindrical coordinates $f : \mathbb{S}^1 \times \mathbb{S}^1 \rightarrow \mathbb{R}^3$ as

$$f(e^{i\varphi}, e^{i\theta}) = \left(1 + \frac{1}{2} \cos(\varphi), \theta, \frac{1}{2} \sin(\varphi) \right).$$

Since f is a parametrization, $f(\mathbb{S}^1 \times \mathbb{S}^1)$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^1$. We shall now define a closed curve contained in the torus.

DEFINITION 4.33. *Let n, m be two positive coprime integers. We define the curve $K_{n,m}$ as the subset of the torus in \mathbb{R}^3*

$$K_{n,m} := \{f(e^{2\pi i n t}, e^{2\pi i m t}) \mid t \in I\}.$$

The curve $K_{n,m}$ is called the torus knot of type (n, m) .

For example, for $n = 2$ and $m = 3$, we obtain a knot called *the trefoil knot*, which can be seen in figure (15). Notice that $K_{n,m}$ is a closed curve indeed for all values

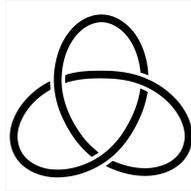


FIG. 15.

of n and m described above. We could have defined the torus knot of type (n, m) as follows: First consider the torus as the orbit space $\mathbb{R}^2/\mathbb{Z}^2$. Consider an infinite straight line in the square that starts at $(0, 0)$ with slope n/m . The projection of this straight line to the quotient space is the torus knot of type (n, m) .

DEFINITION 4.34. *We say that two torus knots $K_{n,m}, K_{n',m'}$ are similar if there exists an homeomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $h(K_{n,m}) = K_{n',m'}$.*

If two torus knots are similar, then the homeomorphism h of definition (4.34) restricts to an homeomorphism between $\mathbb{R}^3 \setminus K_{n,m}$ and $\mathbb{R}^3 \setminus K_{n',m'}$. Then, $\pi_1(\mathbb{R}^3 \setminus K_{n,m}, x_0)$ and $\pi_1(\mathbb{R}^3 \setminus K_{n',m'}, x_0)$ are isomorphic.

DEFINITION 4.35. *The group $\pi_1(\mathbb{R}^3 \setminus K_{n,m}, x_0)$ is called the group of the knot $K_{n,m}$.*

THEOREM 4.36. *The group of the knot $K_{n,m}$ is*

$$\pi_1(\mathbb{R}^3 \setminus K_{n,m}, x_0) \cong \langle \{a, b\}; \{a^n = b^m\} \rangle.$$

The proof of this theorem is essentially an application of the SFV theorem. Since the proof requires some additional concepts and we have already seen many applications, we will skip it.

At this point we could proceed as we did with surfaces. Will the abelianization of this fundamental group lead us to a characterization of all torus knots? In this case, the answer is negative.

PROPOSITION 4.37. *Let G be the group*

$$G := \langle \{a, b\}; \{a^n = b^m\} \rangle.$$

Then, the abelianization G_{ab} is \mathbb{Z} for all n and m .

PROOF. It is clear that the elements of G_{ab} are of the form $a^p b^q$, with $p, q \in \mathbb{Z}$. Consider the following application

$$\begin{aligned} \varphi : G_{ab} &\longrightarrow \mathbb{Z} \\ a^p b^q &\longrightarrow pm + qn \end{aligned}$$

It can be readily verified that φ is an isomorphism. On the one hand, the map is surjective thanks to the Bézout's identity. On the other hand, $\ker \varphi$ is the set of words generated by $a^n b^{-m}$. Hence, the map is injective. \square

In all, the abelianization of the fundamental group of the torus knot does not give us any characterization of the knot. Therefore, we must abroad the problem from another perspective. In this case, the answer is purely algebraic.

THEOREM 4.38. *Let $n, m, n', m' > 1$. Two torus knots $K_{n,m}$ and $K_{n',m'}$ are similar iff $n = n'$ and $m = m'$, or $n = m'$ and $m = n'$.*

PROOF. The proof only purely algebraic. Consider G a group of the form

$$G := \langle \{a, b\} \mid \{a^n = b^m\} \rangle.$$

Firstly, notice that a^n commute both with a and b ,

$$\begin{aligned} aa^n &= a^{n+1} = a^n a, \\ ba^n &= bb^m = b^{m+1} = b^m b = a^n b. \end{aligned}$$

Therefore, a^n commute also with every element of G . Consider now N to be the subgroup generated by a^n . Due to the previous discussion, N is a normal subgroup in G and hence, the quotient group G/N is well defined. Let now g be an element of G of the form

$$g = a^{\alpha(1)} b^{\beta(1)} \dots a^{\alpha(k)} b^{\beta(k)},$$

where $\alpha(i)$ and $\beta(j)$ are integer numbers. In the quotient space, the class of g , which we will denote as $[g]$ ($[g] = gN$), can be written as

$$[g] = [a^{\alpha(1)}][b^{\beta(1)}] \dots [a^{\alpha(k)}][b^{\beta(k)}] = [a]^{\alpha(1)}[b]^{\beta(1)} \dots [a]^{\alpha(k)}[b]^{\beta(k)}.$$

This last equality proves that the quotient set G/N is generated by $[a]$ and $[b]$. Now, any relation in G/N will be of the form $[g] = 1$: In this case, $g \in N$ and hence, $g = (a^n)^l = (b^m)^l$ for some integer l . Therefore, the relations in G/N are defined by

$$[a]^n = [g] = 1 \quad \text{and} \quad [b]^m = [g] = 1.$$

In all, what we have is that

$$\begin{aligned} G/N &= \langle \{[a], [b]\}; [a] = 1, [b] = 1 \rangle \\ &\cong \langle \{c, d\}; c^n = 1, d^m = 1 \rangle. \end{aligned}$$

First of all, we want to prove that the center of G/N , $Z(G/N)$ is trivial. Let's state it in a lemma.

LEMMA 4.39. *The center of G/N is trivial.*

PROOF. To see it, consider an element x of the center. Therefore, $cx = xc$ and $dx = xd$. If x is of the form

$$x := c^{\alpha(1)}d^{\beta(1)} \dots c^{\alpha(j)}d^{\beta(j)},$$

then

$$\begin{aligned} cx &= c^{\alpha(1)+1}d^{\beta(1)} \dots c^{\alpha(j)}d^{\beta(j)}, \\ xc &= c^{\alpha(1)}d^{\beta(1)} \dots c^{\alpha(j)}d^{\beta(j)}c. \end{aligned}$$

So, both words are equal if and only if

$$cd^{\beta(1)} \dots c^{\alpha(j)}d^{\beta(j)} = d^{\beta(1)} \dots c^{\alpha(j)}d^{\beta(j)}c.$$

Since c and d don't commute and since there are no relations involving c and d at the same time, we deduce that this is true if and only if $d^{\beta(1)} \dots c^{\alpha(j)}d^{\beta(j)} = 1$ is the neutral element. Hence, $x = c^\alpha$ for some integer α . Using the same reasoning with the equation $dx = xd$, we obtain that $x = d^\beta$ for some integer β . In all, $x = c^\alpha = d^\beta$. This is impossible unless if α is a multiple of n and β is a multiple of m . In this case, $x = 1$, as we wanted to prove. \square

Let's proceed with the rest of the proof. It is immediate to see that the projection of $Z(G)$ to the quotient space is a subset of $Z(G/N)$. For this reason and due to the fact that $Z(G/N)$ is trivial, we deduce that $Z(G) = N$. So, in all,

$$G/Z(G) \cong \langle \{c, d\}; c^n = 1, d^m = 1 \rangle.$$

If the two torus knots of type (n, m) and (n', m') were similar, then their fundamental groups G and G' would be isomorphic. Moreover, the groups $G/Z(G)$ and $G'/Z(G')$ would also be isomorphic. Let f be such isomorphism. Firstly, since the order of c is n , the order of $f(c)$ is also n . Secondly, it is easy to notice that the only elements of finite order in $G'/Z(G')$ are powers of c' , powers of d' and their conjugates. Therefore, it must hold that

$$f(c) = x(c')^\alpha x^{-1},$$

for some $\alpha \in \mathbb{Z}$ and $x \in G'/Z(G')$. Since $1 = f(c)^n = x(c')^{n\alpha}x^{-1}$, we deduce that $n\alpha = kn'$. In this case n and k are coprime. Sure enough, if they had a common factor $p > 1$, then we could write $n = pq$ for some $q \in \mathbb{Z}$ and hence $f(c)^q = 1$. Therefore, n divides n' . Consider now $f^{-1}(c')$, which has also finite order. With the same reasoning, it can happen only two things, $f^{-1}(c') = yc^\beta y^{-1}$ or $f^{-1}(c') = yd^\beta y^{-1}$ for some $\beta \in \mathbb{Z}$ and $y \in G/Z(G)$. In this latter case, we are due to see that we get a contradiction,

$$f(d^{\alpha\beta}) = f(y)^{-1}(c')^\alpha f(y) = f(y)^{-1}x^{-1}f(c)xf(y) = f(z^{-1}cz)$$

with $z = f^{-1}(x)y$. However, this is impossible because f is injective. Therefore, $f^{-1}(c') = yc^\beta y^{-1}$ and by the same reasoning as before, n' divides n . This leads us to conclude that $n = n'$. Finally, by considering $f(d)$ and $f^{-1}(d')$ we obtain $m = m'$. \square

Chapter 5

Proof of the Seifert-Van Kampen theorem

In the previous chapter we became familiar with the theorem and we saw how powerful it is with some applications. In this chapter we give the rigorous proof of the theorem. The proof is divided in two parts. The first one deals with the generators and the second one deals with the set of relations.

1. The Seifert-Van Kampen theorem

THEOREM 5.1 (Seifert-Van Kampen (SVK)). *Consider a topological space X of the form $X = U_1 \cup U_2$ with*

- (1) U_1 and U_2 being non-empty open path-connected spaces of X ,
- (2) $U_1 \cap U_2$ being a non-empty path-connected space of X .

Let's denote $\varphi_1, \varphi_2, \psi_1, \psi_2$ the inclusions of figure (1). Consider a base point

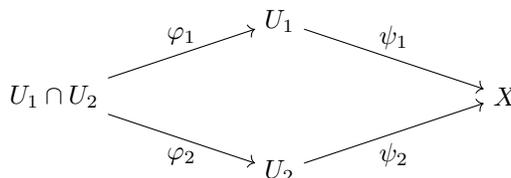


FIG. 1.

$x_0 \in U_1 \cap U_2$. We already know that the diagram (2) is commutative. Consider the presentations of the following fundamental groups

- (1) $\pi_1(U_1 \cap U_2, x_0) \cong \langle S; R \rangle$.
- (2) $\pi_1(U_1, x_0) \cong \langle S_1; R_1 \rangle$.
- (3) $\pi_1(U_2, x_0) \cong \langle S_2; R_2 \rangle$.

$$\begin{array}{ccccc}
 & & \pi_1(U_1, x_0) & & \\
 & \nearrow \varphi_{1*} & & \searrow \psi_{1*} & \\
 \pi_1(U_1 \cap U_2, x_0) & & & & \pi_1(X, x_0) \\
 & \searrow \varphi_{2*} & & \nearrow \psi_{2*} & \\
 & & \pi_1(U_2, x_0) & &
 \end{array}$$

FIG. 2.

If $s \in S$, then $\varphi_{1*}s \in \pi_1(U_1, x_0)$ and $\varphi_{2*}s \in \pi_1(U_2, x_0)$, so we can represent these elements as words generated by S_1 and S_2 respectively. Let $'\varphi_{1*}s'$ and $'\varphi_{2*}s'$ be such words. We must define one last relation. We denote R_s the set of relations

$$R_s := \{'\varphi_{1*}s' = '\varphi_{2*}s', s \in S\}.$$

With the previous hypothesis, the thesis of the theorem says that $\pi(X, x_0)$ is isomorphic to $\langle S_1 \cup S_2; R_1 \cup R_2 \cup R_s \rangle$.

2. Previous propositions and discussions

Before we deep into the proof, it would be interesting to discuss a little bit what does the theorem talk about.

The SVK theorem is mainly based on the first isomorphism theorem. The inclusions

$$\begin{aligned}
 \psi_1 : U_1 &\longrightarrow X \\
 \psi_2 : U_2 &\longrightarrow X
 \end{aligned}$$

induce two homomorphisms

$$\begin{aligned}
 \psi_{1*} : \langle S_1; R_1 \rangle &\cong \pi_1(U_1, x_0) \longrightarrow \pi_1(X, x_0), \\
 \psi_{2*} : \langle S_2; R_2 \rangle &\cong \pi_1(U_2, x_0) \longrightarrow \pi_1(X, x_0),
 \end{aligned}$$

which, at the same time, define a morphism given by the image of the generators as follows

$$\begin{aligned}
 \phi : \langle S_1 \cup S_2; R_1 \cup R_2 \rangle &\longrightarrow \pi_1(X, x_0) \\
 W &\longrightarrow \phi(W) = \begin{cases} \psi_{1*}(W) & \text{if } W \in S_1, \\ \psi_{2*}(W) & \text{if } W \in S_2. \end{cases}
 \end{aligned}$$

It may seem logic to think that the kernel of this function may not be trivial, which means that there are some words whose image is the neutral elements. Exactly those words are what constitute the set R_s . With another point of view, if we apply the first isomorphism theorem, (recall that ϕ is surjective) we obtain that

$$\langle S_1 \cup S_2; R_1 \cup R_2 \rangle / \ker \phi \cong \text{Im}(\phi) = \pi_1(X, x_0).$$

What this equations says is that we must *remove* some words in $\langle S_1 \cup S_2; R_1 \cup R_2 \rangle$ to obtain the fundamental group of X . The words that we remove belong to the kernel of ϕ . So, in all R_s is $\ker \phi$. Although the intuitive idea seems quite straight forward, we must recall that the SVK theorem has been stated in terms of presentations and not in terms of free products. Translating the above idea in presentations makes the proof more tedious, since it must introduce lots of new notations.

To finish this section, we recall a proposition that we will need later and that is usually introduced in an introductory topology course.

PROPOSITION 5.2 (Lebesgue's number). *Let X be a compact topological space obtained from a metric space with a distance d . Given an open cover $\{U_j \mid j \in J\}$ there exists a real number $\delta > 0$ (called the Lebesgue's number of the cover) such that any set with diameter lower than δ is strictly contained in one of the U_j , $j \in J$.*

3. Part I of the proof. Generators.

First we shall prove that $\pi_1(X, x_0)$ is generated by

$$\psi_{1*}(\pi_1(U_1, x_0)) \cup \psi_{2*}(\pi_1(U_2, x_0)).$$

In other words, if $\alpha \in \pi_1(X, x_0)$, then

$$\alpha = \prod_k \psi_{\lambda(k)*} \alpha_k,$$

with $\alpha_k \in \pi_1(U_{\lambda(k)}, x_0)$ and $\lambda(k) = 1$ or 2 .

PROOF. Consider $f : I \rightarrow X$ to be a closed path based at $x_0 \in X$. Since $X = U_1 \cup U_2$ with U_i open sets, the set $\{f^{-1}(U_1), f^{-1}(U_2)\}$ is an open cover of I . Let δ be the Lebesgue number of this cover. Therefore, if t_0, t_1, \dots, t_n are real numbers such that

$$0 = t_1 \leq t_2 \leq \dots \leq t_n = 1$$

and such that $t_i - t_{i-1} < \delta$, then $[t_{i-1}, t_i]$ is contained in either $f^{-1}(U_1)$ or $f^{-1}(U_2)$ for $i = 1, 2, \dots, n$. Hence, $f([t_{i-1}, t_i])$ is contained in either U_1 or U_2 . Moreover, we can consider that $f(t_i) \in U_1 \cap U_2$. Sure enough, if $f(t_i) \notin U_1 \cap U_2$, both $[t_{i-1}, t_i]$ and $[t_i, t_{i+1}]$ would be contained in either $f^{-1}(U_1)$ or $f^{-1}(U_2)$. Therefore, we could merge these two intervals into only one $[t_{i-1}, t_{i+1}]$ verifying that $f([t_{i-1}, t_{i+1}])$ is contained in either U_1 or U_2 . In such case we just rearrange the indexes and continue the proof. Now consider the paths $f_i : I \rightarrow X$, $i = 1, 2, \dots, n$ defined by

$$f_i(t) = f((1-t)t_i + tt_i).$$

By the previous reasoning, f_i is a path contained in either U_1 or U_2 , with origin in $f(t_{i-1})$ and ending in $f(t_i)$. We assert now that

$$[f] = [f_1 * f_2 * \dots * f_n].$$

Let's put this statement as a lemma.

LEMMA 5.3. *Let $f : I \rightarrow X$ a path and*

$$0 = t_0 \leq t_1 \leq t_2 \leq \dots \leq t_n = 1.$$

If $f_i : I \rightarrow X$ for $i = 1, 2, \dots, n$ is defined as

$$f_i(t) = f((1-t)t_{i-1} + tt_i),$$

then

$$[f] = [f_1 * f_2 * \dots * f_n].$$

PROOF. There is a direct proof and a proof by induction. The former can be deduced from the latter so we will prove it by induction. Consider the case $n = 2$. Therefore, $0 = t_0 \leq t_1 \leq t_2 = 1$ and

$$\begin{aligned} (f_1 * f_2)(t) &= \begin{cases} f_1(2t) & 0 \leq t \leq \frac{1}{2}, \\ f_2(2t - 1) & \frac{1}{2} \leq t \leq 1, \end{cases} \\ &= \begin{cases} f(2tt_1) & 0 \leq t \leq \frac{1}{2}, \\ f((1 - (2t - 1))t_1 + 2t - 1) & \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

Now we must check that $f_1 * f_2$ is homotopic to f by showing an homotopy $F : I \times I \rightarrow X$

$$F(t, s) = \begin{cases} f((1 - s)2tt_1 + st) & 0 \leq t \leq \frac{1}{2}, \\ f((1 - s)(t_1 + (2t - 1)(1 - t_1)) + st) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Assume now that $n > 2$ and that the result holds for all integers lower or equal than $n - 1$. Now we have

$$0 = t_0 \leq t_1 \leq \dots \leq t_n = 1.$$

In particular, we have $0 = t_0 \leq t_{n-1} \leq t_n = 1$, so we can use the previous case to deduce that f is homotopic to $g * f_n$, with $g(t) = f(tt_{n-1})$. On the other hand,

$$0 = \frac{t_1}{t_{n-1}} \leq \frac{t_2}{t_{n-1}} \dots \leq \frac{t_{n-1}}{t_{n-1}} = 1,$$

so using the induction hypothesis we get that

$$[g] = [g_1 * g_2 * \dots * g_{n-1}],$$

with

$$\begin{aligned} g_i(t) &= g((1 - t)\frac{t_i}{t_{n-1}} + t\frac{t_i}{t_{n-1}}) \\ &= f((1 - t)t_i + tt_i) \\ &= f_i(t). \end{aligned}$$

Therefore, $[f] = [f_1 * f_2 * \dots * f_n]$. \square

Back to the main proof, let's consider now, for $i = 1, 2, \dots, n - 1$, paths $q_i : I \rightarrow X$ such that $q_i(0) = x_0$ and $q_i(1) = f(t_i)$. Notice that $x_0, f(t_i) \in U_1 \cap U_2$ for all $i = 1, 2, \dots, n$ and that $U_1 \cap U_2$ is path connected. Therefore, such paths $q_i(t)$ exist and may be chosen to be contained in $U_1 \cap U_2$ for all $t \in I$. Analogously, let's define $q_0(t)$ and $q_n(t)$ as $q_0(t) = q_n(t) = x_0$ for all $t \in I$. Hence, we obtain

$$\begin{aligned} [f] &= [f_1 * f_2 * \dots * f_n] \\ &= [q_0 * f_1 * \overline{q_1} * q_1 * f_2 * \overline{q_2} * \dots * q_{n-1} * f_n * \overline{q_n}] \\ &= [q_0 * f_1 * \overline{q_1}] [q_1 * f_2 * \overline{q_2}] \dots [q_{n-1} * f_n * \overline{q_n}]. \end{aligned}$$

Each of $q_i * f_{i+1} * \overline{q_{i+1}}$ are closed paths based at x_0 contained either in U_1 or U_2 . Therefore $[q_i * f_{i+1} * \overline{q_{i+1}}]$ is an element of $\psi_{1*}(\pi_1(U_1, x_0))$ or $\psi_{2*}(\pi_1(U_2, x_0))$. Hence, any element of $\pi_1(X, x_0)$ can be written as a product of the images of elements of $\pi_1(U_1, x_0)$ and $\pi_1(U_2, x_0)$. \square

COROLLARY 5.4. *The fundamental group $\pi_1(X, x_0)$ is generated by the set $\psi_{1*}S_1 \cup \psi_{2*}S_2$, where S_1, S_2 are the generators of $\pi_1(U_1, x_0)$ and $\pi_1(U_2, x_0)$ respectively.*

From now on it will be convenient to write s instead of $\psi_{1*}s$ or $\psi_{1*}s$. In other words, if $f : I \rightarrow U_j$, we denote the function

$$I \xrightarrow{f} U_j \xrightarrow{\psi_j} X$$

also as f . With this notation, $\pi_1(X, x_0)$ is generated by $S_1 \cup S_2$ (instead of $\psi_{1*}(S_1) \cup \psi_{2*}(S_2)$). This new notation will not cause any confusion.

4. Part II of the proof. Relations.

In this section we prove that the relations of $\pi_1(X, x_0)$ are R_1 , R_2 and R_s as stated in the theorem. The proof is divided into two lemmas.

LEMMA 5.5. *The generators S_1 and S_2 of $\pi_1(X, x_0)$ satisfy the relations R_1 , R_2 and R_s .*

PROOF. Recall that

$$\psi_{j*} : \pi_1(U_j, x_0) \rightarrow \pi_1(X, x_0)$$

are homomorphisms for $j = 1, 2$. Therefore, any relation that the elements of S_j satisfy in $\pi_1(U_j, x_0)$ is also satisfied by the elements of $\psi_{j*}S_j$ in $\pi_1(X, x_0)$. Therefore, using the previous (abuse of) notation which drops the ψ_{1*} , we can say that the elements of $S_1 \cup S_2$ satisfy the relations R_1 and R_2 in $\pi_1(X, x_0)$. On the other hand, remember that if $s \in \pi_1(U_1 \cap U_2, x_0)$, then

$$\psi_{1*}\varphi_{1*}s = \psi_{2*}\varphi_{2*}s.$$

If a word in S_j represents the element $\varphi_{j*}s$, then the same word in S_j represents the element $\psi_{j*}\varphi_{j*}s$ in $\pi_1(X, x_0)$. Hence,

$$' \varphi_{1*}s' = ' \varphi_{2*}s',$$

with $s \in S$. Then, the lemma is proved. \square

To conclude the proof of the SVK theorem it remains to see that there are no other relations than R_1, R_2 and R_s , or equivalently, that any relation in $\pi_1(X, x_0)$ is a consequence of these three sets of relations.

LEMMA 5.6. *Any relation verified by the elements of $S_1 \cup S_2$ can be expressed as a combination of the relations of R_1, R_2 and R_s .*

PROOF. Although the proof of this lemma is not difficult, it is quite technical and demands to introduce some notation. Consider that $\alpha_1^{\varepsilon(1)}\alpha_2^{\varepsilon(2)}\dots\alpha_k^{\varepsilon(k)} = 1$ is a relation satisfied by $S_1 \cup S_2 \subset \pi_1(X, x_0)$, with $\varepsilon(i) = \pm 1$, $\alpha_i \in \lambda(i)$ for $i = 1, 2, \dots, k$ and $\lambda(i) = 1, 2$. For each $i = 1, 2, \dots, k$, consider a closed path candidate f_i in $U_{\lambda(i)}$ based at x_0 such that $[f_i] = \alpha_i^{\varepsilon(i)}$. In other words, $\alpha_i = [f_i]$ if $\varepsilon(i) = 1$ and $\alpha_i = [\overline{f_i}]$ if $\varepsilon(i) = -1$. We define a path $f : I \rightarrow X$ which consists in the concatenation of

all f_i . Hence, we need to scale the velocity in which we move along each f_i so that we move along all paths in unitary time.

$$f(t) = \begin{cases} f_1(kt) & \text{for } 0 \leq t \leq \frac{1}{k}, \\ \vdots & \\ f_i(kt - i + 1) & \text{for } \frac{i-1}{k} \leq t \leq \frac{i}{k}, \\ \vdots & \\ f_1(kt - k + 1) & \text{for } \frac{k-1}{k} \leq t \leq 1, \end{cases}$$

for $i = 1, 2, \dots, k$. We defined f as a function of f_i and conversely, we can now express f_i as a function of f as follows

$$f_i(t) = f\left(\frac{(i-1)}{k}(1-t) + \frac{j}{k}t\right).$$

Since

$$0 = \frac{0}{k} \leq \frac{1}{k} \leq \dots \leq \frac{k}{k} = 1,$$

we can invoke lemma (5.3) to deduce that

$$[f] = [f_1 * f_2 * \dots * f_k] = [f_1][f_2] \dots [f_k].$$

The last equality holds due to the fact that all f_i are closed paths based at x_0 . Since $\alpha_1^{\varepsilon(1)} \alpha_2^{\varepsilon(2)} \dots \alpha_k^{\varepsilon(k)} = 1$, then

$$[f] = [f_1] * [f_2] * \dots * [f_k] = \alpha_1^{\varepsilon(1)} \alpha_2^{\varepsilon(2)} \dots \alpha_k^{\varepsilon(k)} = 1.$$

Therefore, $[f] = 1$, which means that f is homotopic to the constant path ε_{x_0} relatively to $\{0, 1\}$. Let $F : I \times I \rightarrow X$ be such homotopy between f and ε_{x_0} , i.e., such that

$$\begin{aligned} F(t, 0) &= f(t) & \forall t \in I, \\ F(t, 1) &= x_0 & \forall t \in I, \\ F(t, s) &= F(0, s) = F(1, s) = x_0 & \forall t, s \in I. \end{aligned}$$

On the other hand, $X = U_1 \cup U_2$, so $\{F^{-1}(U_1), F^{-1}(U_2)\}$ is an open cover of $I \times I$. Let δ be the Lebesgue number of this open cover of $I \times I$ and let us choose a partition of $I \times I$ such that each part or subset has a diameter lower than δ , i.e., a partition

$$\begin{aligned} 0 &= t_0 < t_1 < t_2 < \dots < t_m = 1, \\ 0 &= s_0 < s_1 < s_2 < \dots < s_n = 1, \end{aligned}$$

such that

$$\{1/k, 2/k, \dots, (k-1)/k\} \subseteq \{t_1, t_2, \dots, t_{m-1}\} \quad \text{and} \\ (t_i - t_{i-1})^2 + (s_j - s_{j-1})^2 < \delta^2$$

for all $i, j \in \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ ¹. Notice that such a choice is always possible. Let's denote $R_{i,j}$ the rectangle $[t_{i-1}, t_i] \times [s_{j-1}, s_j] \subset I \times I$. Due to this choice, $F(R_{i,j})$ is contained in either U_1 or U_2 for all i, j . Now, for each i, j , let $a_{i,j} : I \rightarrow X$ be a path from x_0 to $F(t_i, s_j)$ such that if $F(t_i, s_j)$ is contained in U_1 (or U_2 or $U_1 \cap U_2$), then $a_{i,j}(I)$ is contained in U_1 (or U_2 or $U_1 \cap U_2$ respectively). Notice that $a_{i,j}$ is well defined because U_1, U_2 and $U_1 \cap U_2$ are path connected. In

¹The second inequality is simple to understand. If we consider the rectangle $[t_i - t_{i-1}] \times [s_j - s_{j-1}]$, the greater distance in this rectangle is the distance between $\{(t_{i-1}, s_{j-1})\}$ and $\{(t_i, s_j)\}$, which is $\sqrt{(t_i - t_{i-1})^2 + (s_j - s_{j-1})^2}$. So if we want the rectangles to have a diameter lower than the lebesgue number δ , we must impose $\sqrt{(t_i - t_{i-1})^2 + (s_j - s_{j-1})^2} < \delta$.

addition, if $F(t_i, s_j) = x_0$, then we impose $a_{i,j} = \varepsilon_{x_0}$. Figure (3) explains these definitions.

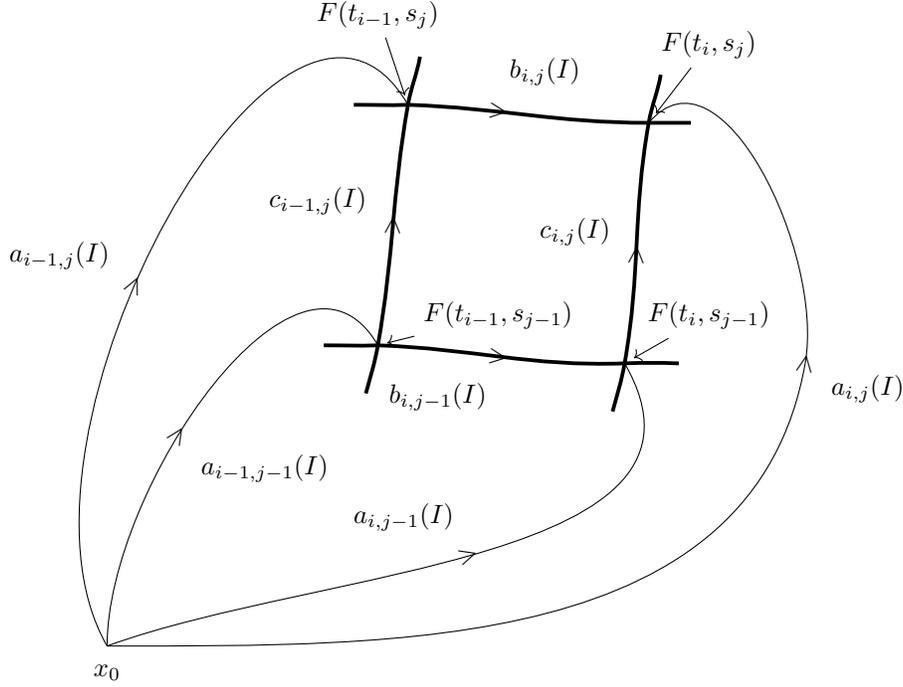


FIG. 3.

We also define paths $b_{i,j}$ from $F(t_{i-1}, s_j)$ to $F(t_i, s_j)$ and $c_{i,j}$ from $F(t_i, s_{j-1})$ to $F(t_i, s_j)$ as follows

$$\begin{aligned} b_{i,j}(t) &= F((1-t)t_{i-1} + tt_i, s_j), \\ c_{i,j}(t) &= F(t_i, (1-t)s_{j-1} + ts_j). \end{aligned}$$

With this notation,

$$\begin{aligned} [f] &= [F(t, 0)] = [b_{1,0} * b_{2,0} * \cdots * b_{m,0}], \\ [b_{i,n}] &= [\varepsilon_{x_0}] \quad \forall i = 1, 2, \dots, m \implies [\varepsilon_{x_0}] = [b_{1,n}][b_{2,n}] \cdots [b_{m,n}]. \end{aligned}$$

The paths $b_{i,j-1} * c_{i,j}$ and $c_{i-1,j} * b_{i,j}$ are homotopy equivalent in $I \times I$. Although it can be seen intuitively in figure (3), we can show an explicit (but complicated) homotopy $H : I \times I \rightarrow X$ between these two paths

$$H(t, s) = \begin{cases} F((1-s)((1-2t)t_{i-1} + 2tt_i) + st_{i-1}, \\ (1-s)s_{j-1} + s((1-2t)s_{j-1} + 2ts_j)) & 0 \leq t \leq \frac{1}{2}, \\ F((1-s)t_i + s((2-2t)t_{i-1} + (2t-1)t_i), \\ (1-s)((2-2t)s_{j-1} + (2t-1)s_j) + ss_j) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Notice that if $F(R_{i,j}) \subset U_1$ (or U_2 or $U_1 \cap U_2$), then $H(I \times I) \subset U_1$ (or U_2 or $U_1 \cap U_2$) respectively. We define now closed paths $f_{i,j}$ and $g_{i,j}$ based at x_0 as

$$\begin{aligned} f_{i,j} &= (a_{i-1,j} * b_{i,j}) * \overline{a_{i,j}}, \\ g_{i,j} &= (a_{i,j-1} * c_{i,j}) * \overline{a_{i,j}}. \end{aligned}$$

With this definition, we can consider the paths

$$\begin{aligned} f_{i,j-1} * g_{i,j} &= a_{i-1,j-1} * b_{i,j-1} * \overline{a_{i,j-1}} * a_{i,j-1} * c_{i,j} * \overline{a_{i,j}} \\ &= a_{i-1,j-1} * b_{i,j-1} * c_{i,j} * \overline{a_{i,j}}, \\ g_{i-1,j} * f_{i,j} &= a_{i-1,j-1} * c_{i-1,j} * \overline{a_{i-1,j}} * a_{i-1,j} * b_{i,j} * \overline{a_{i,j}} \\ &= a_{i-1,j-1} * c_{i-1,j} * b_{i,j} * \overline{a_{i,j}}. \end{aligned}$$

Since $b_{i,j-1} * c_{i,j}$ and $c_{i-1,j} * b_{i,j}$ are homotopy equivalent in $I \times I$, by proposition (2.2) we deduce that $f_{i,j-1} * g_{i,j}$ and $g_{i-1,j} * f_{i,j}$ are homotopy equivalent. In addition, if $F(R_{i,j}) \subset U_1$ (or U_2 or $U_1 \cap U_2$), then these two paths are homotopy equivalent in U_1 (or U_2 or $U_1 \cap U_2$) respectively. Therefore,

$$\begin{aligned} [f_{i,j-1} * g_{i,j}] &= [g_{i-1,j} * f_{i,j}] \implies \\ [f_{i,j-1}] [g_{i,j}] &= [g_{i-1,j}] [f_{i,j}], \end{aligned}$$

which means that

$$(1) \quad [f_{i,j-1}] = [g_{i-1,j}] [f_{i,j}] [\overline{g_{i,j}}],$$

in U_1 or U_2 or $U_1 \cap U_2$. Hence, we can consider now each of these elements as words in S_1 and S_2 and we obtain that

$$'[f_{i,j-1}]' = '[g_{i-1,j}]'' [f_{i,j}]'' [\overline{g_{i,j}}]',$$

which is a relation in either $\pi_1(U_1, x_0)$ or $\pi_1(U_2, x_0)$. Hence, this relation must be a consequence of R_1 or R_2 . Now we want to study all times t_i of the form j/k with $j = 1, 2, \dots, k-1$. So, consider the set of indexes

$$\{i(j) \mid \frac{j}{k} = t_{i(j)}, \quad j \in \{1, 2, \dots, k-1\}\}.$$

Let's start with the first index, this is, consider that $1/k = t_{i(1)}$. Therefore, it can be readily seen (by writing the definition or with the help of figure (3)) that

$$[f_1] = [f_{1,0}] [f_{2,0}] \dots [f_{i(1),0}].$$

Since f_1 is a closed path in $U_{\lambda(1)}$ based at x_0 , we can use the relations in R_{λ_1} to express $[f_{1,0}] [f_{2,0}] \dots [f_{i(1),0}]$ as words in $S_{\lambda(1)}$. Therefore, we obtain a relation

$$\alpha_1^{\varepsilon(1)} = [f_1] = '[f_{1,0}]'' [f_{2,0}]' \dots [f_{i(1),0}]',$$

which is a consequence of the relations $R_{\lambda(1)}$. Following the same procedure, we obtain similar relations for $\alpha_2^{\varepsilon(2)}, \alpha_3^{\varepsilon(3)}, \dots, \alpha_k^{\varepsilon(k)}$. In all, we obtain a relation

$$\alpha := \alpha_1^{\varepsilon(1)} \alpha_2^{\varepsilon(2)} \dots \alpha_k^{\varepsilon(k)} = '[f_{1,0}]'' [f_{2,0}]' \dots [f_{m,0}]',$$

which is a consequence of R_1 and R_2 . Moreover, using the relation (1) we can rewrite α as

$$\begin{aligned} \alpha &= ('[g_{0,1}]'' [f_{1,1}]'' [\overline{g_{1,1}}]') ('[g_{1,1}]'' [f_{2,1}]'' [\overline{g_{2,1}}]') \dots ('[g_{m-1,1}]'' [f_{m,1}]'' [\overline{g_{m,1}}]') \\ &= ('[g_{0,1}]')' [f_{1,1}]' ('[\overline{g_{1,1}}]'' [g_{1,1}]')' [f_{2,1}]' ('[\overline{g_{2,1}}]'' [g_{2,1}]')' \dots \\ &\quad \dots ('[\overline{g_{m-1,1}}]'' [g_{m-1,1}]')' [f_{m,1}]' ('[\overline{g_{m,1}}]')'. \end{aligned}$$

Recall that $g_{0,1} = g_{m,1} = \varepsilon_{x_0}$ and hence $'[g_{0,1}]' = 1$ and $'[g_{m,1}]' = 1$ are trivial relation. On the other hand, the relation

$$(2) \quad '[\overline{g_{i,1}}]''[g_{i,1}]' = 1$$

is also trivial if $'[\overline{g_{i,1}}]'$ and $'[g_{i,1}]'$ are both expressed as words in either S_1 or S_2 . However, if $g_{i,1}$ is a path in $U_1 \cap U_2$, then it is possible that one of them is expressed as a word in S_1 and the other as a word of S_2 . In such a case, relation (2) is a consequence of the relations in R_s . Hence, we obtain that

$$\alpha = '[f_{1,1}]''[f_{2,1}]' \dots '[f_{m,1}]'$$

is a consequence of R_1, R_2 and R_s .

Now, let's take a look at figure (4). This figure shows the homotopy F from f

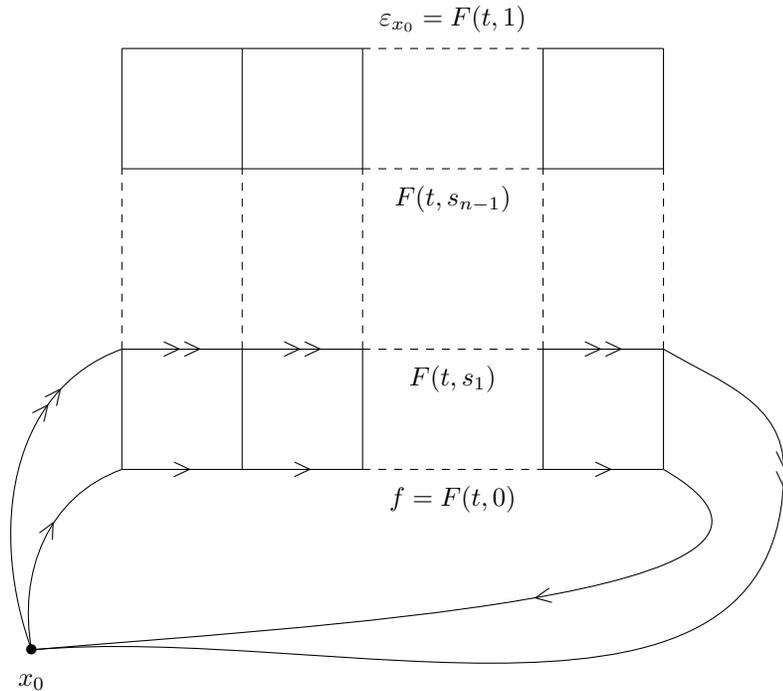


FIG. 4.

to ε_{x_0} , which has been *split into, let's say, many squares* according to notation in figure (3). Until here, we have proven that we can move from the closed path based at x_0 that goes along $f(t) = F(t, 0)$ (let's say in a vague way, *the bottom floor*) to the closed path based at x_0 that goes along $F(t, s_1)$ (let's say again in a vague way, *the first floor*). In this process we gain the relations R_s uniquely. In order to finish the proof, we must show that in this process of moving upwards we do not obtain more relations different from R_1, R_2 and R_s . It is just a matter of patience to see

that repeating the process we end up to the relation

$$\begin{aligned}\alpha &= ' [f_{1,n}]'' [f_{2,n}]' \dots ' [f_{m,n}]' \\ &= ' [\varepsilon_{x_0}]'' [\varepsilon_{x_0}]' \dots ' [\varepsilon_{x_0}]',\end{aligned}$$

which is also a consequence of the relations R_1 , R_2 and R_s . Therefore, the original relation $\alpha_1^{\varepsilon(1)} \alpha_2^{\varepsilon(2)} \dots \alpha_k^{\varepsilon(k)} = 1$ is a consequence of the relations in R_1 , R_2 and R_s . \square

Chapter 6

Finding topology in a factory

In this chapter we will find topology in a non-academic environment. Through an example of robots in a factory we will apply some of the concepts developed in the previous chapters, such as the fundamental group and orbit spaces. Moreover, we will show a brief introduction of what is braid theory. Finally, this chapter is more descriptive than the previous ones since the underlying proofs would need much more work.

1. Introduction to the problem

In this section we are interested in modelling a problem related with motion planning of robots. Consider that a factory has N robots, which must move in a certain area to carry out certain jobs. Moreover, we want to avoid collisions between robots since they cause many problems to the factory processes. For simplicity, we will assume that the robots move in \mathbb{R}^2 and that they can move in any direction in this area. At this point, it may be clear that if there is enough space, the problem of avoiding collisions is local. Just like it happens in a supermarket, if two people with their shopping carts are due to collide, they just move sideways and carry on walking. In all, there is no need to know where are all the shopping carts to avoid an specific collision. Although this strategy may not be optimal, it would do the job. However, this strategy and the whole problem itself becomes less trivial when the space available is too small, as figure (1) suggests.

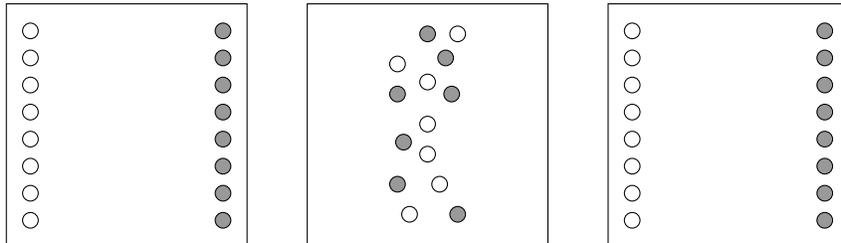


FIG. 1. White robots must get to the right hand side and grey robots must get to the left hand side. Collisions must be avoided.

So, in all, our goal is to define the *best* paths (in terms of efficiency) that the robots should follow such that there are no collisions among them. However, in this section we will only focus on the non-collision part of the problem.

With this goal in mind, we start by presenting the *configuration space*. The configuration space is the area in which the robots can move without bumping into each other. We will assume that each robot is just a point in \mathbb{R}^2 .

DEFINITION 6.1. *The configuration space of N robots is the set*

$$\mathcal{C}^N(\mathbb{R}^2) := \mathbb{R}^2 \times \cdots \times \mathbb{R}^2 \setminus \Delta,$$

where

$$\Delta := \{(x_1, \dots, x_N) \in \mathbb{R}^{2N} \mid x_i = x_j \ \forall 1 \leq i, j \leq N\}.$$

The set Δ is called the *generalised diagonal*.

Moreover, we could be more restrictive and think that the robot can only move in a rectangle, or in a rectangle with some obstacles, as figure (2) shows.

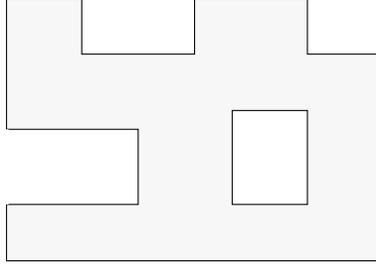


FIG. 2.

In the general case where the robots move in any space X , the definition of the configuration space is the same. For example, if X denotes the gray region of figure (2), then the configuration space of N robots in X would be $\mathcal{C}^N(X)$. If X is topological space, then $\mathcal{C}^N(X)$ is also a topological space with induced topology of the product topology.

Thus if we define a path in the configuration space

$$\sigma : I \longrightarrow \mathcal{C}^N(X) \quad \forall i \in \{1, 2, \dots, N\},$$

we will be sure that no collisions occur. So, now that we have a path in a topological space, what is fundamental group of this space? Before we show the answer, we need one more definition.

DEFINITION 6.2. *Consider the action*

$$\begin{aligned} \mathcal{C}^N(X) \times \mathcal{S}_N &\longrightarrow \mathcal{C}^N(X) \\ ((x_1, \dots, x_N), \sigma) &\longrightarrow (x_{\sigma(1)}, \dots, x_{\sigma(N)}), \end{aligned}$$

where \mathcal{S}_N denotes the symmetric group of N elements. The orbit space $\mathcal{C}^N(X)/\mathcal{S}_N$ is called the *unlabeled configuration space*, and it is denoted by $\mathcal{UC}^N(X)$.

With these definitions, it can be seen that $B_N(X) := \pi_1(\mathcal{UC}^N(X), x_0)$ is the n -strand braid group and that $P_N(X) := \pi_1(\mathcal{C}^N(X), x_0)$ is the n -strand pure braid group.

This result shows that although the configuration space is quite easy to understand, its fundamental group can turn out to be a really complicated object. In the following section we introduce what is the braid group and we show an important result.

2. A brief introduction to braid theory

In this section we show an intuitive introduction to the braid group. The aim of this section is to get in touch with this field and to see an important result, which gives a lot of information of the configuration space.

DEFINITION 6.3. *Consider a unit cube and consider also n points on the top face which lie in a straight line. We place n strands of string, one at each point, verifying*

- (1) *Each strand begins on the top face of the cube and ends on the bottom face.*
- (2) *No two strands intersect.*
- (3) *As we move along any strand from the top face to the bottom face, we are always moving downwards, i.e., there is no horizontal segment or any segment that loops up.*

Such construction is called an n -braid.

Figure (3) shows examples of 3-braids whereas figure (4) shows examples of what are not braids. At this point it may be clear that not only is important to know

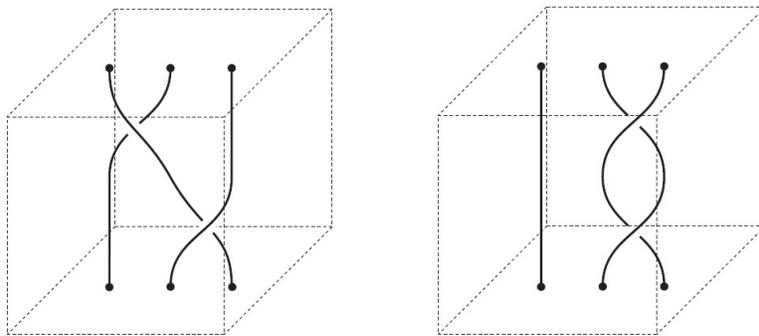


FIG. 3.

which is the beginning and the ending of a strand but also how this strand is placed with respect to the others. Moreover, we want to get rid of redundant or unnecessary information. For example, it is clear that the two braids of figure (5) are essentially *the same*, but the ones of figure (3) are not.

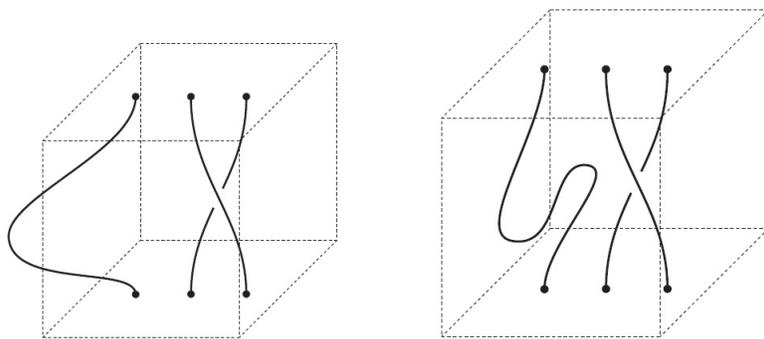


FIG. 4.

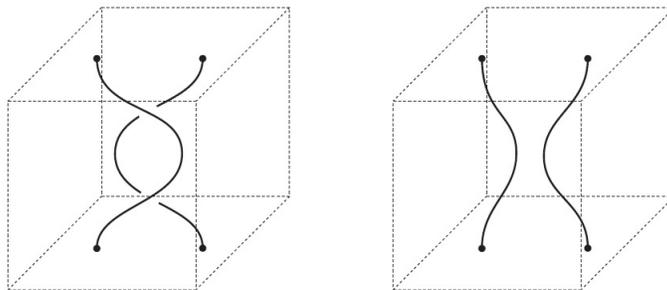


FIG. 5.

The idea is that two strands β and β' are the same if we can obtain the latest from the former only by *shaking the box*. More formally, two n -braids β and β' are said to be equivalent if the former can be transformed into the last one only using the so called *elementary moves*. Figure (6) shows an elementary move. If we have a strand AC-AB and no other strand crosses the region defined by ACB, we can replace the strand AC-AB by the strand AB. Of course, this movement also applies to strands which are not *straigh lines* or *poligonals*. This move don't change

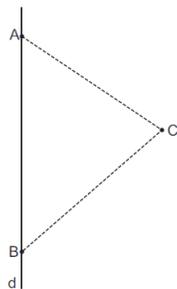


FIG. 6.

the relative position between the strands. As expected, the previous relation is an equivalence relation in the set of n -braids.

With this construction of the braids, construction, we can immediately *concatenate* two braids β and β' simply by joining the bottom face of the cube of β with the top face of the cube of β' and considering the resulting braid. We will denote this resulting braid as $\beta\beta'$. Just as a technicality, in order to obtain again a braid, we should shrink the resulting rectangle into a square with a simple homeomorphism. Figure (7) exemplifies the concatenation of two strands. So, in all, the set of n -

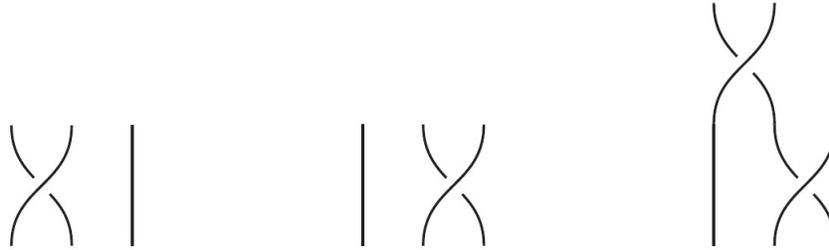


FIG. 7.

braids with the operation of concatenation form a group. Moreover, the operation of concatenation is inherited to the group of equivalence classes formed with the previous relation.

DEFINITION 6.4. *The set of equivalence classes of n -braids with the operation of concatenation form a group called the n -braid group B_n .*

Our goal now is to translate this geometric construction into algebra. The idea is very simple. Consider that the n points are labeled from 1 to n in this order. For $i \in \{1, \dots, n-1\}$, consider σ_i to be the strand indicated in figure (8) (consequently, σ_i^{-1} is the strand indicated in figure (9)).

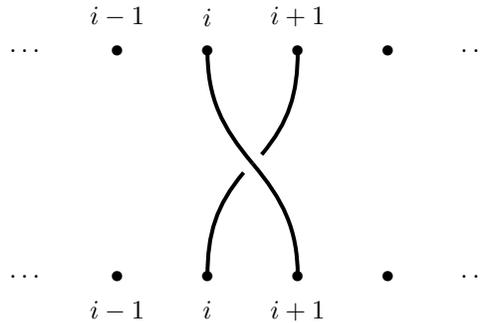


FIG. 8.

Clearly, the set

$$(3) \quad \{\sigma_i \mid 1 \leq i \leq n-1\}$$

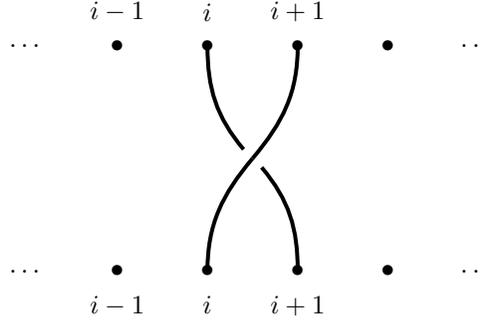


FIG. 9.

generates all possible strands. Moreover, it can be readily checked that the following relations are satisfied

$$(4) \quad \sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $i, j = 1, 2, \dots, n-1$ with $|i-j| \geq 2$, and

$$(5) \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i = 1, 2, \dots, n-2$. So, in all, all the strands verify this set of relations. However, any group satisfying these relations is a braid group? It can be proven that the answer is affirmative.

THEOREM 6.5. *Consider \mathcal{B}_n the group generated by (3) and that satisfies relations (4) and (5). Then, the n -braid group and \mathcal{B}_n are isomorphic groups.*

Now that we have an algebraic definition of the braid group, we can use all algebraic theory to study this set. For example, geometrically, the braid σ_i connects the i -th point with the $(i+1)$ -th point and the $(i+1)$ -th point with the i -th point. Therefore, it makes sense to think of the following homomorphism, defined by the image of the generators of \mathcal{B}_n

$$\begin{array}{ccc} \mathcal{B}_n & \xrightarrow{\varphi} & \mathcal{S}_n \\ \sigma_i & \longrightarrow & (i, i+1). \end{array}$$

Here, \mathcal{S}_n denotes the symmetric group of n elements. Of course, this homomorphism is surjective since transpositions generate the symmetric group. However, it is not injective because σ_i^2 is not the identity in the braid group. Finally, we introduce the *pure braid group*.

DEFINITION 6.6. *The set $\ker \varphi$ is called the pure braid group.*

To illustrate the importance of braids, let us consider the set \mathcal{B}_3 of 3-braids. Consider also torus knot $K_{2,3}$, which is the trefoil knot of figure (15). In this case,

$$\mathbb{B}_3 \cong \pi_1(\mathbb{R}^3 \setminus K_{2,3}, x_0) \cong \langle \{a, b\}; \{a^2 = b^3\} \rangle.$$

The isomorphism is easy to define by looking how \mathcal{B}_3 is defined.

$$\mathcal{B}_3 := \{ \sigma_1, \sigma_2 \mid \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \}$$

Therefore, it can be readily checked that

$$\begin{aligned} \langle \{a, b\}; \{a^2 = b^3\} \rangle &\longrightarrow \mathcal{B}_3 \\ a &\longrightarrow \sigma_2 \sigma_1 \sigma_2, \\ b &\longrightarrow \sigma_1 \sigma_2, \end{aligned}$$

is an isomorphism.

3. Study of a particular case: Graphs

At this point, there is an important remark about the modelization of the problem. The idea that robots are points in \mathbb{R}^2 may not be enough accurate in most situations. Therefore, we should change a little bit our definitions to model more realistic cases.

For example, if we need to take into account that the robots are not just points, but objects with a certain size on the plane, then the configuration space should change a little bit. In this case, we should remove not only the *generalized diagonal* but also a *generalized tube*, i.e., an area around the generalized diagonal, which would depend on the size of the robots. Moreover, if there are obstacles and we don't want robots to get too close to them, we should also remove a *safety area* around each obstacle.

Among all the ways of simplifying the problem, we are interested in the one which involve graphs. Let's see two approaches which motivates the study of graphs.

One idea is to *summarize the information* of the area in which the robots can move. In this case, we want to get rid of redundant information and we want to keep only the essential information of the area. We could deformation retract the space into a subset which looks like a graph, such as figure (10) shows. The important information here is that there is only one obstacle that *generates* an \mathbb{S}^1 . At this point, we could force robots to move near this graph.

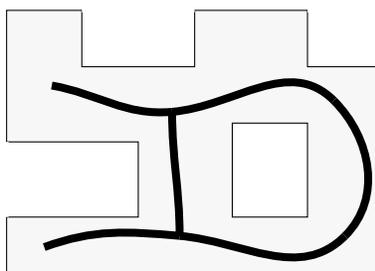


FIG. 10.

Another approach is to consider that the movement of robots in \mathbb{R}^2 can be simplified since there is too much freedom. For instance, if robots are attached to some tracks (or guides) on the floor, then the movement is more limited and it may be easier to control the robots. In this case, we could consider N robots that only move along the edges a graph.

In any case, if we force the robots to move along a graph, then the problem of avoiding collisions is not local, but global. We need to know where are all the robots in order to decide how to avoid a single collision.

Now, our goal is to compute the configuration space of a graph¹. As we already discussed in section (1), the calculations are usually very difficult. However, in some particular cases with a few robots, we can visualize this set. This is the case of two robots moving in a graph which looks like a T (see figure (11)). We will denote this

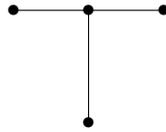


FIG. 11.

graph as Γ . To compute $\mathcal{C}^2(\Gamma)$ we must compute first the product graph $\Gamma \times \Gamma$, which looks like figure (12) Notice that this product graph defines nine squares with

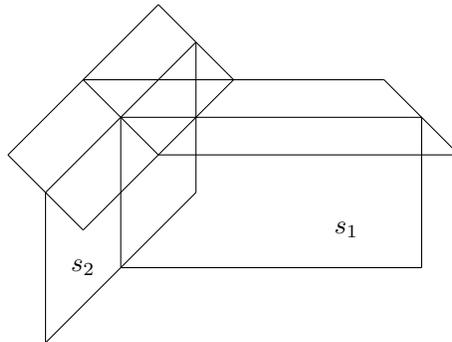


FIG. 12.

a point in common. Such squares are in bijection with the possible positions of the robots in the graph. For example, the squares marked as s_1 and s_2 correspond to the relative positions of the robots shown in figure (13) left and right respectively. Of these nine squares, six correspond to configurations in which robots lie on the same edge and three to configurations in which robots lie on different edges. To get the configuration space of Γ , we must remove from these three latest squares a diagonal from one vertex to another, which corresponds to Δ . In figure (14) there are marked these three squares and the diagonal that must be removed. Finally, figure (14), i.e., the product graph with the dashed lines removed, can be *unfolded* and we obtain that the configuration space $\mathcal{C}^N(\Gamma)$ looks like figure (15) So, in all, the configuration space of the graph which looks like a T is the space in figure (15). From this figure, one can immediately deduce that its fundamental group is isomorphic to \mathbb{Z} .

¹In the following examples, we will consider that robots are just dots in the space with no width.

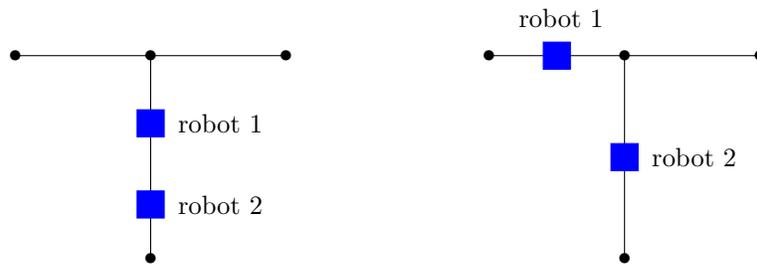
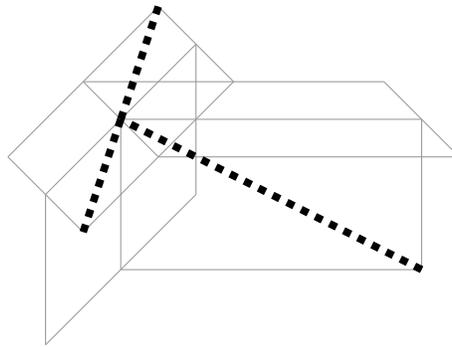
FIG. 13. Relative position of the robots in squares s_1 and s_2 respectively.

FIG. 14.

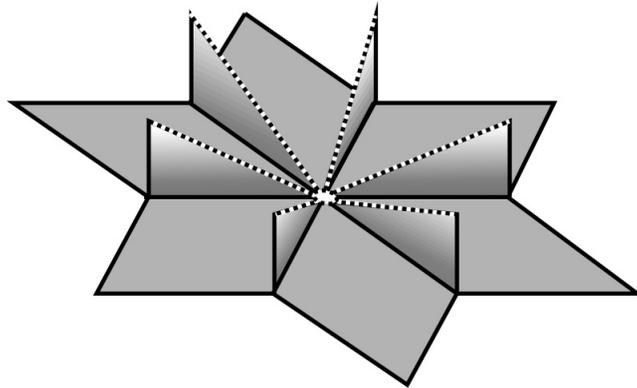


FIG. 15.

4. Final comments on configuration spaces

To finish the section we will introduce some interesting results. Moreover, we will extend the concept of fundamental group and see an important result. This chapter is illustrative, which means that we will not show any proof.

In general it is hopeless to calculate $\pi_1(\mathcal{C}^N(\Gamma), x_0)$. However, we can state some properties of $\mathcal{C}^N(\Gamma)$ and its fundamental group. The first result is that the fundamental group of a configuration space of a graph Γ is torsion-free.

PROPOSITION 6.7. *For any tree T and for any $N > 0$, the fundamental group $\pi_1(\mathcal{C}^N(T), x_0)$ is torsion-free.*

Before we show the last result, we need to introduce two more concepts. The first one is a generalisation of the fundamental group.

DEFINITION 6.8. *Let X be a topological space and $x_0 \in X$. For $k \geq 0$, consider the sphere \mathbb{S}^k and a point $s \in \mathbb{S}^k$. We define*

$$\pi_k(X, x_0) := [(\mathbb{S}^k, s); (X, x_0)].$$

In other words, $\pi_k(X, x_0)$ is the set of homotopy classes, relative to s , of maps $f : \mathbb{S}^k \rightarrow X$ such that $f(s) = x_0$.

Notice that if $k = 0$, then $\pi_0(X, x_0) = 1$ and if $k = 1$, $\pi_1(X, x_0)$ is the fundamental group of X at x_0 . As an important result, it turns out that $\pi_k(X, x_0)$ is an abelian group for all $k \geq 2$.

Some particular topological spaces are those called *Eilenberg-McLaine* spaces. The interest of the following spaces is that there is a powerful theorem that characterises the homotopy type of these spaces via the fundamental group.

DEFINITION 6.9. *Consider G to be a group. An arcwise connected space X is an Eilenberg-McLaine space of type $K(G, 1)$ if*

- (1) $\pi_1(X, x_0) = G$.
- (2) $\pi_k(X, x_0) = 1$ for all $k \geq 2$.

For example, it can be proved that the topological graphs are Eilenberg-McLaine spaces of type $K(F_n, 1)$. Notice the difference with other surfaces such as compact connected surfaces. It can be proven that compact connected surfaces are not Eilenberg-McLaine. The importance of EM spaces is that they give a really powerful result.

THEOREM 6.10. *If X and Y are Eilenberg-McLaine spaces of type $K(G, 1)$, then they are homotopy equivalent.*

Finally, Robert Ghrist proved the following important theorem.

THEOREM 6.11. *If Γ is a graph, then the configuration space $\mathcal{C}^n(\Gamma)$ is an Eilenberg-McLaine space of type $K(\pi_1, 1)$. By writing $K(\pi_1, 1)$ we mean that the first fundamental group (i.e., for $k = 1$) is not the trivial one and the rest of groups (π_k for $k \geq 2$) are trivial.*

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