

ON THE BICANONICAL MAP OF IRREGULAR VARIETIES

MIGUEL ANGEL BARJA, MARTÍ LAHOZ,
JUAN CARLOS NARANJO, AND GIUSEPPE PARESCHI

ABSTRACT. From the point of view of uniform bounds for the birationality of pluricanonical maps, irregular varieties of general type and maximal Albanese dimension behave similarly to curves. In fact Chen-Hacon showed that, at least when their holomorphic Euler characteristic is positive, the tricanonical map of such varieties is always birational. In this paper we study the bicanonical map. We consider the natural subclass of varieties of maximal Albanese dimension formed by primitive varieties of Albanese general type. We prove that the only such varieties with non-birational bicanonical map are the natural higher-dimensional generalization to this context of curves of genus 2: varieties birationally equivalent to the theta-divisor of an indecomposable principally polarized abelian variety. The proof is based on the (generalized) Fourier-Mukai transform.

1. INTRODUCTION

Pluricanonical maps are an essential tool for understanding varieties of general type. In particular, given a variety of general type X , it is important to know, or at least bound, the minimal integer $m_0(X)$ such that the pluricanonical maps

$$\phi_m : X \dashrightarrow \mathbf{P}(H^0(X, \omega_X^m)^*)$$

are birational onto their image for each $m \geq m_0(X)$. In this paper we will deal with this sort of problems for complex *irregular* varieties (i.e. varieties such that $q(X) := h^1(\mathcal{O}_X) > 0$) of general type, mostly of *maximal Albanese dimension* (*m.A.d.* for short), i.e. such that their Albanese map $alb : X \rightarrow \text{Alb } X$ is generically finite. In a sense these are the most basic irregular varieties since the Stein factorization of the Albanese map provides a canonical fibration onto a normal variety of the same irregularity, whose smooth models have m.A.d.. From the point of view of the birationality of pluricanonical maps, Chen-Hacon ([CH]) showed that m.A.d. varieties of general type and $\chi(\omega_X) > 0$ behave like curves: their tricanonical map is always birational¹. Such result have been made more precise in [PP5], where it is proved that if X is *non-special* (see below) then the tricanonical map is an embedding outside the exceptional locus of the Albanese map. It is worth to note that ample line bundles on abelian varieties have the same behavior, since the third power of an ample line bundle is always very ample. In fact the analogy has been explained in [PP3] and references therein, where it is showed that the two facts can be proved in the same way.

This work was completed while GP was visiting member of MSRI during the Program “Algebraic Geometry” (April 2009) and JCN was visiting the University of Pavia (Spring 2009, granted by the Spanish Ministerio de Ciencia e Innovación), and MAB was visiting the Centre de Recerca Matemàtica. GP thanks MSRI for great hospitality and excellent research atmosphere. JCN thanks the University of Pavia for his warm hospitality. MAB, ML and JCN were partially supported by the Proyecto de Investigación MTM2006-14234. MAB and ML were also partially supported by 2005SGR-557 and JCN was partially supported by 2005SGR-787.

¹ m.A.d. varieties of general type with $\chi(\omega_X) = 0$ were discovered by Ein-Lazarsfeld in [EL] and do not exist in dimension ≤ 2 . However one knows, again by [CH], that their 6-canonical map is birational.

In this paper we address the next natural problem: *which are the m.A.d. varieties of general type whose bicanonical map is not birational?* To describe our view of the question, some preliminary remarks about irregular varieties are in order. According to the seminal work of Green-Lazarsfeld ([GL1, GL2]) and Ein-Lazarsfeld ([EL]), key invariants of irregular varieties are their *cohomological support loci*

$$V^i(\omega_X) = \{\alpha \in \text{Pic}^0 X \mid h^i(\omega_X \otimes \alpha) > 0\}$$

These are intimately related to the geometry of X since, by the main theorem of [GL2], the positive dimensional components of $V^i(\omega_X)$ are translates of subtori arising, for $i > 0$, only in presence of a morphism with connected fibres to a lower dimensional m.A.d. normal variety. It follows that, up to birational equivalence, a variety such that $\dim V^i(\omega_X) > 0$ for some $i > 0$ has a morphism onto a smooth lower-dimensional m.A.d. variety Y such that $\dim V^i(\omega_Y) = 0$ for all $i > 0$. This is the reason of the following

Definition (Catanese, [Ca, Def. 1.24]). *A smooth projective irregular variety such that $\dim V^i(\omega_X) = 0$ for all $i > 0$ is called primitive.*

Moreover in a recent paper Pareschi-Popa introduced a single numerical invariant measuring the *codimension* of the various support loci, namely the *generic vanishing index*

$$gv(X) := \min_{i>0} \{\text{codim}_{\text{Pic}^0 X} V^i(\omega_X) - i\}.$$

In this terminology Green-Lazarsfeld's Generic Vanishing Theorem of [GL1] can be rephrased as follows: *if X has m.A.d. then $gv(\omega_X) \geq 0$.* The main feature of the invariant $gv(X)$ is that it governs the local sheaf-theoretic properties of the Fourier-Mukai transform of the structure sheaf ([PP6] Def.3.1 and Cor.3.2). A variety verifying the extremal case $gv(X) = 0$ has quite special properties. From the Fourier-Mukai point of view, $gv(X) = 0$ means that the transform of the structure sheaf of X has torsion (see Theorem 2.6 below). From the geometric side, it means that the image X via the Albanese map is fibered by subtori of the Albanese variety ([EL], Proof of Theorem 3). For these reasons, m.A.d. varieties with $gv(\omega_X) = 0$ will be referred to as *special*.

Consequently, we are lead to divide m.A.d. varieties into four disjoint subclasses: the *non-special* varieties are further distinguished into *primitive* and *non-primitive* ones and similarly for the *special* varieties. An immediate computation shows that for *primitive* varieties, non-special (resp. special) mean simply $\dim X < q(X)$ (resp. $\dim X = q(X)$, i.e. the Albanese map is surjective). We recall that in the literature there is a specific definition also for m.A.d. varieties with $\dim X < q(X)$: they are called *of Albanese general type* (Catanese, [Ca, Def. 1.7]). Therefore *primitive non-special* is equivalent to *primitive of Albanese general type*.

Let us go back to the problem of describing m.A.d. varieties of general type such that their bicanonical map is not surjective. In dimension 1 these are the curves of genus 2. The problem of classifying surfaces of general type whose bicanonical map is not birational has attracted considerable interest, starting from Du Val, later on Bombieri and more recently, Catanese, Ciliberto, Francia, Mendes Lopes, Pardini, Xiao Gang and others. We refer to the surveys [Ci] and (more recent) [BCP] for an account on this work. Although the general classification is still not fully achieved, things are much better behaved for surfaces of maximal Albanese dimension. In fact, from Theorems 8, 9 and 10 of [BCP] one extracts the following result, due to the combined efforts of Catanese, Ciliberto and Mendes Lopes ([CCM, CM]): minimal m.A.d. surfaces of general type whose bicanonical map is not surjective are either fibered by curves of genus 2 (this is usually referred to as *the standard case*) or

(a) the symmetric product of a curve of genus 3;

(b) the double cover of principally polarized abelian surface branched at a divisor $D \in |2\Theta|$.

Interestingly enough, the analogy with curves persists, as both cases (a) and (b) can be seen as 2-dimensional generalizations of curves of genus 2:

(a) curves of genus 2 are the theta-divisors of indecomposable principally polarized abelian surfaces. Symmetric products of curves of genus 3 are precisely the minimal surfaces which are birational to theta divisors of indecomposable principally polarized 3-folds. This generalizes to arbitrary dimension: the bicanonical map of a variety which is birational to a theta-divisor of an indecomposable p.p.a.v. has degree 2, as it factors (up to birational equivalence) through the Kummer map.

(b) A surface as in (b) is the 2-dimensional analogue of the double cover of an elliptic curve branched at two points, i.e. a bielliptic curves of genus 2. Again, in arbitrary dimension a variety X birational to the double cover of p.p.a.v A , branched at a (say smooth, for simplicity) divisor $B \in |2\Theta|$ has degree 2, as it factors (up to birational equivalence) through the lifting to X of the natural involution of A .

An immediate computation shows that the varieties described in (a) and (b) above are *primitive* (more precisely, $V^i(\omega_X) = \{\hat{0}\}$ for $i > 0$, where $\hat{0}$ denotes the identity point of $\text{Pic}^0 X$). The ones of (a) are *non-special* while the ones of (b) are *special*. As for curves, in both cases their holomorphic Euler characteristic is $\chi(\omega_X) = 1$, the minimal value². Concerning the standard case, m.A.d surfaces fibered in curves of genus 2 are *non-primitive*. Products of curves, $C \times D$, such that $g(C) \geq g(D) = 2$ are *non-special* examples. The surface of [CCM] Th.3.23 is an example of a *special* m.A.d. surface presenting the standard case. For m.A.d varieties of arbitrary dimension, the natural generalization of the notion of *standard case* is a variety fibered by *primitive* varieties whose bicanonical map is not surjective.

In this paper we will focus on non-special primitive varieties. The somewhat surprising result is that, in arbitrary dimension, the picture is the same:

Theorem A. *Let X be a smooth complex variety of maximal Albanese dimension and of general type. Assume moreover that X is primitive and that $\dim X < q(X)$. The following are equivalent*

- (a) *the bicanonical map of X is non-birational,*
- (b) *X is birationally equivalent to a theta-divisor of an indecomposable p.p.a.v..*

Observe that the hypothesis of being primitive and that $\dim X < q(X)$ could also be phrased as being primitive and non-special or primitive and of Albanese general type.

As a consequence of the above quoted main theorem of [GL2], it follows

Corollary B. *Let X be a smooth complex variety of general type with $\dim X < q(X)$. If the bicanonical map of X is non-birational then either X has a morphism onto a lower dimensional irregular normal variety or X is birational to the theta-divisor of an indecomposable p.p.a.v..*

Theorem A leaves open the classification of m.A.d varieties of general type with non-birational bicanonical map of the other three types. We conjecture that:

- if *primitive and special* such varieties should have $\chi(\omega_X) = 1$ (but we don't have a clear idea about the possibility of other examples besides (b) above, and what they should be);
- the *non-primitive* ones should all present the *standard case* (see the above discussion). Among these the *non-special* ones should be birational to the product of a non-special m.A.d. varieties and a theta-divisor.

² m.A.d. varieties of general type satisfy the sharp lower bound $\chi(\omega_X) \geq 0$ ([EL]), but if X is, in addition, *primitive* then $\chi(\omega_X) \geq 1$ (see Proposition 4.10 below for a more general statement).

The method of proof of Theorem A is different from the one of [CCM] and [CM], even if restricted to the 2-dimensional case. The basic framework is the (generalized) Fourier-Mukai transform and its relation with generic vanishing (the necessary background material is reviewed in §2). In particular, the equivalence

$$gv(\mathcal{F}) \geq 1 \quad \Leftrightarrow \quad \widehat{R\Delta\mathcal{F}} \text{ torsion-free}$$

([PP5, PP6], see also Theorem 2.6 below) is repeatedly used.

In §3 we prove a slight improvement of Hacon-Pardini's cohomological characterization of theta-divisors [HP, Thm. 2]. The result is Proposition 3.1 below, stating that: *primitive non-special varieties with $\chi(\omega_X) = 1$ are birational to theta-divisors*³. Interestingly, a slightly weaker version of the above result holds in any characteristic (we refer to Corollary 3.2 for the precise statement).

In §4, via the notion of *continuous global generation* ([PP1, PP3] and references therein), we prove a birationality criterion asserting, roughly speaking, that *the non-birationality of the bicanonical map implies that, for general $\alpha \in \text{Pic}^0 X$, the linear series $|\omega_X \otimes \alpha|$ has a base divisor* (we refer to Theorem 4.13 and Corollary 4.14 for the precise statements). This works under hypotheses which are more general than those of Theorem A. For example, it works also for primitive special varieties. Moreover, it is worth to note that the above mentioned analogy with ample line bundles on abelian varieties persists. In fact our birationality criterion, and its proof, are similar to Ohbuchi's theorem, asserting that twice an ample line bundle L on an abelian variety is ample unless L has a base divisor, as proved in [PP2].

The above birationality criterion is used in §5 where, by means of a geometric analysis of the paracanonical system, combined with the Fourier-Mukai transform, we show that a variety X satisfying (a) of Theorem A has $\chi(\omega_X) = 1$. At this point Theorem A is proved by applying the above mentioned Hacon-Pardini's type characterization of theta-divisors.

Finally, in the Appendix we provide the proof of an useful technical fact about Fourier-Mukai transform that we couldn't find in the literature. It is our hope that some of the steps of the argument (namely, the birationality criterion of §4, the decomposition of the Picard torus and the explicit description of the Poincaré line bundle of §5, the result of the Appendix) will be of independent interest.

Unless otherwise stated, throughout the paper the word *variety* will mean *smooth projective complex variety* (but the careful reader will notice that at most steps of the arguments are completely algebraic, and work in any characteristic). Given a line bundle α on a variety X , by abuse of language we will also denote α the point of the Picard variety of X parametrizing α .

Acknowledgments. We thank Mihnea Popa for numerous conversations. In fact his contribution in shaping many tools used in this paper doesn't need to be acknowledged. More recently a question of his lead us to discover a mistaken argument in a previous version of this work. We thank also Rob Lazarsfeld and Roberto Pignatelli.

2. (GENERALIZED) FOURIER-MUKAI TRANSFORM

Here we review from [Mu2] and [PP4, PP5, PP6] the material about (generalized) Fourier-Mukai transforms and generic vanishing that will be needed in the sequel.

³This result was proved independently (with a different proof) by Lazarsfeld-Popa [LP], building on the ideas of [HP].

2.1. (Generalized) Fourier-Mukai transforms and generic vanishing.

Terminology/Notation 2.1 (Fourier-Mukai and generic vanishing). Let X be a variety of dimension d , equipped with a morphism to a n -dimensional abelian variety

$$a : X \rightarrow A.$$

Let \mathcal{P} be a Poincaré line bundle on $A \times \text{Pic}^0 A$. We will denote

$$P_a = (a \times \text{id}_{\text{Pic}^0 A})^*(\mathcal{P})$$

When $a = \text{alb}$, the Albanese map of X , then the map alb^* identifies $\text{Pic}^0(\text{Alb} X)$ to $\text{Pic}^0 X$ and the line bundle P_{alb} is identified to the Poincaré line bundle of X . We will denote $P = P_{\text{alb}}$. Letting p and q the two projections of $X \times \text{Pic}^0 A$, we consider the left-exact functor

$$\Phi_{P_a}(\mathcal{F}) = q_*(p^*(\mathcal{F}) \otimes P_a)$$

and its derived functor

$$R\Phi_{P_a} : \mathbf{D}(X) \rightarrow \mathbf{D}(\text{Pic}^0 A).$$

Sometimes we will have to consider the analogous derived functor $R\Phi_{P_a^\vee} : \mathbf{D}(X) \rightarrow \mathbf{D}(\text{Pic}^0 A)$ as well. Since $\mathcal{P}^{-1} \cong (1_A \times (-1)_{\text{Pic}^0 A})^*\mathcal{P}$, we have that

$$R\Phi_{P_a^\vee} = (-1_{\text{Pic}^0 A})^* R\Phi_{P_a}$$

Finally, given a coherent sheaf \mathcal{F} on X , its i -th cohomological support locus with respect to a is

$$V_a^i(\mathcal{F}) = \{\alpha \in \text{Pic}^0 A \mid h^i(\mathcal{F} \otimes a^*\alpha) > 0\}$$

Again, when a is the Albanese map of X , we will omit the subscript, simply writing $V^i(\mathcal{F})$. A natural measure of the size of the full package of the $V_a^i(\mathcal{F})$'s is provided by the *generic vanishing index* of \mathcal{F} (with respect to a) (see [PP6, Def. 3.1])

$$gv_a(\mathcal{F}) := \min_{i>0} \{\text{codim}_{\text{Pic}^0 A} V_a^i(\omega_X) - i\}.$$

A first basic result relates the generic vanishing index to the fact that the F-M transform of its Grothendieck dual is a sheaf (in cohomological degree d). In what follows, we will adopt the following notation for the dualizing functor

$$R\Delta(\mathcal{F}) = R\mathcal{H}om(\mathcal{F}, \omega_X).$$

Theorem 2.2 ([PP4, Thm. A],[PP6, Thm. 2.2]). *The following are equivalent*

- (a) $gv_a(\mathcal{F}) \geq 0$;
- (b) $R^i\Phi_{P_a}(R\Delta\mathcal{F}) = 0$ for all $i \neq d$.

Terminology/Notation 2.3 (GV-sheaves). If $gv_a(\mathcal{F}) \geq 0$ the sheaf \mathcal{F} is said to be a *GV-sheaf* (generic vanishing sheaf). If this is the case, Theorem 2.2 says that the full transform $R\Phi_{P_a}(R\Delta(\mathcal{F}))$ is a sheaf concentrated in degree $\dim X$:

$$R\Phi_{P_a}(R\Delta\mathcal{F}) = R^d\Phi_{P_a}(R\Delta\mathcal{F})[-d]$$

Then one usually denotes

$$R^d\Phi_{P_a}(R\Delta\mathcal{F}) = \widehat{R\Delta\mathcal{F}}$$

Note that, by (a) of Theorem 2.2, $H^i(\mathcal{F} \otimes a^*\alpha) = 0$ for all $i > 0$ and general $\alpha \in \text{Pic}^0 A$. Therefore, by deformation-invariance of χ , the generic value of $H^0(\mathcal{F} \otimes a^*\alpha)$ equals $\chi(\mathcal{F})$. Since, by base-change, the fiber of $\widehat{R\Delta\mathcal{F}}$ at a general point $\alpha \in \text{Pic}^0 A$ is isomorphic to $H^d(R\Delta(\mathcal{F}) \otimes a^*\alpha) \cong H^0(\mathcal{F} \otimes a^*\alpha^{-1})^*$, the (generic) rank of $\widehat{R\Delta(\mathcal{F})}$ is

$$(1) \quad \text{rk}(\widehat{R\Delta\mathcal{F}}) = \chi(\mathcal{F})$$

Via base-change Theorem 2.2 yields

Corollary 2.4 ([Ha, Thm. 1.2], [PP4, Prop. 3.13]). *If $gv_a(\mathcal{F}) \geq 0$ then*

$$V_a^d(\mathcal{F}) \subseteq \cdots \subseteq V_a^1(\mathcal{F}) \subseteq V_a^0(\mathcal{F})$$

From Grothendieck duality and Theorem 2.2 it follows

Corollary 2.5 ([PP4, Rem. 3.12], [PP6, Pf. of Cor. 3.2]). *If $gv_a(\mathcal{F}) \geq 0$ then*

$$\mathcal{E}xt_{\mathcal{O}_{\text{Pic}^0 A}}^i(\widehat{R\Delta\mathcal{F}}, \mathcal{O}_{\text{Pic}^0 A}) \cong R^i\Phi_{P_a^\vee}(\mathcal{F}) \cong (-1_{\text{Pic}^0 A})^* R^i\Phi_{P_a}(\mathcal{F})$$

In [PP6] Pareschi-Popa established a dictionary between the value of $gv_a(\mathcal{F})$ and the local properties of the transform $\widehat{R\Delta(\mathcal{F})}$. Its first instance is the following

Theorem 2.6 ([PP6, Cor. 3.2]). *Assume that \mathcal{F} is GV (with respect to a). Then the following are equivalent*

- (a) $gv_a(\mathcal{F}) \geq 1$;
- (b) $\widehat{R\Delta\mathcal{F}}$ is a torsion-free sheaf.

In the above terminology, the *Generic Vanishing Theorem* of Green-Lazarsfeld ([GL1], see also [EL, Rem. 1.6]) asserts, in particular, that $gv_a(\omega_X) \geq 0$ if a is generically finite. The following Proposition shows that also the converse is true

Proposition 2.7. *Assume that $gv_a(\omega_X) \geq 0$. Then a is generically finite.*

More generally, one can prove in the same way the converse of the full Generic Vanishing theorem of [GL1], namely that if $gv(\omega_X) \geq -k$ then $\dim a(X) \geq d - k$. Such statement was proved independently by Lazarsfeld-Popa [LP, Prop. 1.5].

Proof. Let $e = \dim X - \dim a(X)$. Let

$$X \xrightarrow{b} Y \xrightarrow{c} A$$

be the Stein factorization of a . Since the $V_a^i(\omega_X)$ are birational invariants, we can assume that Y is smooth. By Kollár's theorem [Ko2, Thm. 3.1], the Leray spectral sequence of b splits:

$$(2) \quad H^i(\omega_X \otimes b^*(c^*\alpha)) \cong \bigoplus_{k=0}^i H^k(R^{i-k}b_*(\omega_X) \otimes c^*\alpha)$$

Moreover, again by a result of Kollár [Ko1, Prop. 7.6],

$$(3) \quad R^e b_*(\omega_X) = \omega_Y$$

Assume that $\dim Y < \dim X$, i.e. $e > 0$. From (2) for $i = e$ and (3) it follows that $V_a^e(\omega_X)$ contains $V_c^0(\omega_Y)$. The fact that $gv_a(\omega_X) \geq 0$ implies that $\text{codim } V_c^0(\omega_Y) \geq e > 0$. Since $c : Y \rightarrow A$ is generically finite, the above mentioned Green-Lazarsfeld Generic Vanishing Theorem yields that $gv_c(\omega_Y) \geq 0$. Hence, by Theorem 2.2, $\Phi_{P_c}(\mathcal{O}_Y)$ is a sheaf (in cohomological degree equal to $\dim Y$), denoted $\widehat{\mathcal{O}}_Y$. Since $V_c^0(\omega_Y)$ is a *proper* subvariety of $\text{Pic}^0 A$, $\widehat{\mathcal{O}}_Y$ must be a torsion sheaf. Therefore, by Theorem 2.6, there is a $i > 0$ such that $\text{codim}_{\text{Pic}^0 A} V_c^i(\omega_Y) = i$. Since, again by (2) and (3), $V_c^i(\omega_Y)$ is contained in $V_a^{e+i}(\omega_X)$, it follows that $\text{codim}_{\text{Pic}^0 A} V_a^{e+i}(\omega_X) < e + i$, a contradiction. \square

2.2. Mukai's equivalence of derived categories of abelian varieties. Assume that X coincides with the abelian variety A (and the map a is the identity). In this special case, according to Notation 2.1, \mathcal{P} denotes the Poincaré line bundle on $A \times \text{Pic}^0 A$. Then Mukai's theorem asserts that $R\Phi_{\mathcal{P}}$ is an equivalence of categories. More precisely, denoting $R\Psi_{\mathcal{P}} : \mathbf{D}(\text{Pic}^0 A) \rightarrow \mathbf{D}(A)$ and $n = \dim A$:

Theorem 2.8 ([Mu2, Thm. 2.2]).

$$R\Psi_{\mathcal{P}} \circ R\Phi_{\mathcal{P}} = (-1)_A^*[n], \quad R\Phi_{\mathcal{P}} \circ R\Psi_{\mathcal{P}} = (-1)_{\text{Pic}^0 A}^*[n].$$

Given a morphism $a : X \rightarrow A$, the functors $R\Phi_{P_a}$ and $R\Phi_{\mathcal{P}}$ (see Notation 2.1) are related by the following formula

Proposition 2.9.

$$R\Phi_{P_a} \cong R\Phi_{\mathcal{P}} \circ Ra_*$$

$$\begin{aligned} \text{Proof. } R\Phi_{P_a}(\cdot) &= Rq_*(p_X^*(\cdot) \otimes (a \times \text{id})^*\mathcal{P}) \stackrel{L+PF}{\cong} Rq_*(R(a \times \text{id})_*(p_X^*(\cdot)) \otimes \mathcal{P}) \cong \\ &\stackrel{BC}{\cong} Rq_*(p_A^*(Ra_*(\cdot)) \otimes \mathcal{P}) = R\Phi_{\mathcal{P}} \circ Ra_*(\cdot) \end{aligned}$$

where: L = Leray, PF = projection formula and BC = Base Change. □

Remark 2.10. Notice that, with the exception of Proposition 2.7 (which uses Kollár theorems on higher direct images of dualizing sheaves), all the results of the present section are algebraic, and work on algebraically closed fields of any characteristic.

3. ON A COHOMOLOGICAL CHARACTERIZATION OF THETA-DIVISORS

As a simple but instructive application of the results reviewed in the previous section, we prove the following slight improvement of Hacon-Pardini's cohomological characterization of theta-divisors [HP, Prop. 4.2]. Proposition 3.1 below is proved independently, with a different proof, also in [LP, Prop. 3.13]. An algebraic version, valid in any characteristic, is provided by Corollary 3.2 below.

Proposition 3.1. *Let X be a d -dimensional smooth projective variety (or a compact Kähler manifold) such that: (a) $\dim V^i(\omega_X) = 0$ for all $i > 0$; (b) $d = \dim X < q(X)$ (i.e., in the terminology of the introduction, X is primitive and non-special), and (c) $\chi(\omega_X) = 1$. Then $\text{Alb } X$ is a principally polarized abelian variety and the Albanese map $\text{alb} : X \rightarrow \text{Alb } X$ maps X birationally onto a theta-divisor.*

Proof. We denote $q = q(X)$. By hypothesis (a) and (b) $gv(\omega_X) \geq 1$. Therefore, by Theorem 2.6, $\widehat{\mathcal{O}}_X$ is torsion-free. Since, by (1), $\text{rk } \widehat{\mathcal{O}}_X = \chi(\omega_X) \stackrel{(c)}{=} 1$, we get that $\widehat{\mathcal{O}}_X$ is an ideal sheaf twisted by a line bundle of $\text{Pic}^0 X$:

$$\widehat{\mathcal{O}}_X = \mathcal{I}_Z \otimes L$$

By base change, the support of Z is contained in the union of the $V^i(\omega_X)$ for $i > 0$ which are assumed, by (a), to be finite sets. Therefore $\mathcal{E}xt^i(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 X}) = \mathcal{E}xt^{i+1}(\mathcal{O}_Z, \mathcal{O}_{\text{Pic}^0 X}) = 0$ for $i + 1 \neq q$. On the other hand, by Proposition 6.1 of the Appendix, and Corollary 2.5 it follows that

$$(4) \quad \mathcal{E}xt^d(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 X}) \cong (-1_{\text{Pic}^0 X})^* R^d \Phi_{\mathcal{P}}(\omega_X) \cong \mathbb{C}(\hat{0}).$$

This implies:

(a) $d = q - 1$.

(b) $\mathcal{O}_Z = \mathbb{C}(\hat{0})$ (Indeed $\mathcal{E}xt^i(\mathbb{C}(\hat{0}), \mathcal{O}_{\text{Pic}^0 X})$ is zero for $i < q(X)$ and equal to $\mathbb{C}(\hat{0})$ for $i = q(X)$). Since $R\mathcal{H}om(\cdot, \mathcal{O}_{\text{Pic}^0 X})$ is an involution, (b) follows from (4)). In conclusion

$$\widehat{\mathcal{O}}_X = \mathcal{I}_{\hat{0}} \otimes L,$$

where L is a line bundle on $\text{Pic}^0 X$ and $\mathcal{I}_{\hat{0}}$ is the ideal sheaf of the (reduced) point $\hat{0}$. By Proposition 2.9

$$R\Phi_{\mathcal{P}}(\mathcal{O}_X) = R\Phi_{\mathcal{P}}(R\text{alb}_*\mathcal{O}_X) = \mathcal{I}_{\hat{0}} \otimes L[-q+1]$$

Therefore, by Mukai's Inversion Theorem 2.8

$$(5) \quad R\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = (-1)_{\text{Pic}^0 X}^* R\text{alb}_*\mathcal{O}_X[-1]$$

In particular:

$$(6) \quad R^0\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0 \quad \text{and} \quad R^1\Phi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) \cong \text{alb}_*\mathcal{O}_X$$

Applying $\psi_{\mathcal{P}}$ to the standard exact sequence

$$(7) \quad 0 \rightarrow \mathcal{I}_{\hat{0}} \otimes L \rightarrow L \rightarrow \mathcal{O}_{\hat{0}} \otimes L \rightarrow 0,$$

and using (6) we get,

$$(8) \quad 0 \rightarrow R^0\Psi_{\mathcal{P}}(L) \rightarrow \mathcal{O}_{\text{Alb} X} \rightarrow \text{alb}_*\mathcal{O}_X$$

whence $R^0\Psi_{\mathcal{P}}(L)$ is supported everywhere (since $\text{alb}_*\mathcal{O}_X$ is supported on a divisor). It is well known that this implies that L is *ample*. Therefore $R^i\Psi_{\mathcal{P}}(L) = 0$ for $i > 0$. Therefore, by sequence (7), $R^i\Psi_{\mathcal{P}}(\mathcal{I}_{\hat{0}} \otimes L) = 0$ for $i > 1$. By (5) and (6), this implies that $R^i\text{alb}_*(\mathcal{O}_X) = 0$ for $i > 0$. Furthermore, (8) implies easily that $h^0(L) = 1$, i.e. L is a *principal* polarization. Therefore, via the identification $\text{Alb}(X) \cong \text{Pic}^0(X)$ provided by L , we have $R^0\Psi_{\mathcal{P}}(L) \cong L^{-1}$ (see [Mu2, Prop. 3.11(1)]). Since the arrow on the right in (8) is onto, it follows that $\text{alb}_*\mathcal{O}_X = \mathcal{O}_D$, where D is a divisor in $|L|$. Since we already know that alb is generically finite (Prop. 2.7), this implies that alb is a birational morphism onto D . \square

Note that the proof is entirely algebraic, except for the use of Prop. 2.7, (see Remark 2.10). Therefore, the following statement holds

Corollary 3.2. *let X be a smooth projective variety (over any algebraically closed field) such that: (a) $\dim V^i(\omega_X) = 0$ for all $i > 0$; (b) $d = \dim X < \dim \text{Alb} X$ and the Albanese map of X is generically finite, and (c) $\chi(\omega_X) = 1$. Then $\text{Alb} X$ is a principally polarized abelian variety and the Albanese map $\text{alb} : X \rightarrow \text{Alb} X$ maps X birationally onto a theta-divisor.*

4. BIRATIONALITY CRITERION

First we review the necessary background about the notion of *continuous global generation*. Then we apply such machinery to the canonical bundle of a variety, under some hypothesis which apply to the main theorem of the present paper. Finally, by applying the same machinery to the ideal sheaf of a point twisted by the canonical line bundle, we prove the birationality criterion we are aiming for, i.e. Theorem 4.13 and its Corollary 4.14.

4.1. Continuous global generation. The notion of *continuous global generation* is a useful notion related to global generation. It was introduced by Pareschi-Popa in [PP1] and applied to various geometric problems in [PP1, PP2, PP5]. See also the survey [PP3].

Let \mathcal{F} be a coherent sheaf on X , and let T be a subset of $\text{Pic}^0 A$. Then we have the *continuous evaluation map associated to the pair* (\mathcal{F}, T) :

$$ev_{T, \mathcal{F}} : \bigoplus_{\alpha \in T} H^0(X, \mathcal{F} \otimes a^* \alpha^{-1}) \otimes a^* \alpha \rightarrow \mathcal{F}$$

Definition 4.1. Let p be a point of X . The sheaf \mathcal{F} is said to be *continuously globally generated at p* (*CGG at p for short*), with respect to the morphism a , if the map ev_U is surjective at p , for all non-empty Zariski open subsets $U \subseteq \text{Pic}^0 A$. When possible, we will omit the reference to the morphism a , and say simply *CGG at p* .

Remark 4.2. If L is a line bundle, then L is not *CGG at p* if and only if there is a non-empty Zariski open set $V \subseteq \text{Pic}^0 A$ such that p is a base point of $L \otimes a^* \alpha^{-1}$ for all $\alpha \in V$. This is equivalent to the fact that p is a base point of $L \otimes a^* \alpha^{-1}$ for all $\alpha \in \text{Pic}^0 A$ such that $h^0(L \otimes a^* \alpha^{-1})$ is minimal.

We will also need the following weaker version. This is in fact a variant of the *weak continuous global generation* of [PP2]. However, for technical reasons we prefer to give the following definition, which is natural in view of Proposition 4.4(b) and Theorem 4.5(b) below.

Definition 4.3. Let p be a point of X and let \mathcal{F} be a GV-sheaf. Let $\widehat{R\Delta}(\mathcal{F})$ be the transform of the dual of \mathcal{F} and let $\tau = \tau(\widehat{R\Delta}(\mathcal{F}))$ be its torsion sheaf. Then \mathcal{F} is said to be *essentially continuously globally generated at p* (*ECGG at p for short*), with respect to the morphism a , if the map ev_T is surjective at p , for all subsets of the form $T = U \cup S$, where U is a non-empty Zariski open subset of $\text{Pic}^0 A$ and S is the underlying subset of $\text{supp } \tau$. As above, when possible, we will omit the reference to the morphism a , and say simply *ECGG at p* .

Obviously *CGG at p* implies *ECGG at p* . Moreover, we will say simply that a sheaf is *CGG* (resp. *ECGG*), when it is *CGG* (resp. *ECGG*) for all p .

The structure sheaf of an abelian variety X is an easy example of a line bundle which is not *CGG* at any point, since for any open subset U of $\text{Pic}^0 X$ not containing the identity point $\hat{0}$, the map ev_{U, \mathcal{O}_X} is zero, but it is *ECGG*. In fact it is well known that in this case $\widehat{\mathcal{O}_X} = \mathbb{C}(\hat{0})$ ([Mu2]). Hence the underlying subset of the support of $\tau(\widehat{\mathcal{O}_X}) = \mathbb{C}(\hat{0})$ is the identity point $\{\hat{0}\}$, and all evaluation maps $ev_{U \cup \{\hat{0}\}, \mathcal{O}_X}$ are trivially surjective at all points. A generalization of this example is provided by Corollary 4.11(b) below.

A useful relation between continuous global generation and the usual *global generation* is provided by the following, where, given a line bundle M , $\text{Bs}(M)$ will denote its base locus.

Proposition 4.4 ([PP2, Prop. 2.4]). *Let \mathcal{F} and L be respectively a coherent sheaf and a line bundle on X , and let p be a point of X .*

(a) *If both \mathcal{F} and L are continuously globally generated at p then $\mathcal{F} \otimes L \otimes a^* \beta$ is globally generated at p for any $\beta \in \text{Pic}^0 A$.*

(b) *Assume that L is CGG at p and that \mathcal{F} is ECGG at p . Let $\tau(\widehat{R\Delta}(\mathcal{F}))$ be the torsion sheaf of $\widehat{R\Delta}(\mathcal{F})$, and assume that the underlying set S of the support of $\tau(\widehat{R\Delta}(\mathcal{F}))$ is finite. For any $\beta \in \text{Pic}^0 A$, if $p \notin \bigcup_{\alpha \in S_{+\beta}} \text{Bs}(L \otimes a^* \alpha)$, then $\mathcal{F} \otimes L \otimes a^* \beta$ is globally generated at p .*

Proof. (a) By Remark 4.2 there is an open subset V_p of $\text{Pic}^0 A$ such that p is not a base point of $L \otimes a^* \alpha$ for all $\alpha \in V_p$, i.e. the evaluation map at p : $H^0(L \otimes a^* \alpha) \rightarrow (L \otimes a^* \alpha)_p$ is surjective for all $\alpha \in V_p$. Since \mathcal{F} is *CGG* at p , it follows that the map

$$ev_{V_p} : \bigoplus_{\alpha \in V_p} H^0(\mathcal{F} \otimes a^* \beta \otimes a^* \alpha^{-1}) \otimes H^0(L \otimes a^* \alpha) \rightarrow (\mathcal{F} \otimes L \otimes a^* \beta)_p$$

is surjective. This proves the assertion, since the above map factors through $H^0(\mathcal{F} \otimes L \otimes a^* \beta)$.

(b) The proof is the same with the difference that now if we use the continuous evaluation map ev_{T_p} with $T_p = V_p \cup (S_{+\beta})$. If $p \notin \bigcup_{\alpha \in S_{+\beta}} \text{Bs}(L \otimes a^* \alpha)$, then the evaluation map $H^0(L \otimes a^* \alpha) \rightarrow (L \otimes a^* \alpha)_p$ is surjective also for all $\alpha \in S_{+\beta}$. \square

As for the usual global generation, in many applications it is useful to have a criterion ensuring that, if the higher cohomology of a given sheaf \mathcal{F} satisfies certain vanishing conditions, then \mathcal{F} is *CGG* or *ECGG*. The following criterion, which applies to *sheaves on abelian varieties*, is due to Pareschi-Popa. In the reference a sheaf \mathcal{F} on an abelian variety A such that $gv(\mathcal{F}) \geq 1$ is called *M-regular*. The first part of the following theorem is [PP1, Prop. 2.13] or [PP5, Cor. 5.3]. The proof of the second part essentially follows the proof of [PP2, Thm. 4.1].

Theorem 4.5. *Let \mathcal{F} be a sheaf on an abelian variety A .*

(a) *If $gv(\mathcal{F}) \geq 1$ then \mathcal{F} is *CGG*.*

(b) *If \mathcal{F} is a *GV-sheaf* and $\text{supp } \tau(\widehat{R\Delta}(\mathcal{F}))$ is a reduced scheme⁴, then \mathcal{F} is *ECGG*.*

The main point is the following

Lemma 4.6. *Let \mathcal{F} be a *GV-sheaf* on an abelian variety A . Let $\widehat{R\Delta}(\mathcal{F})$ be the transform of the dual of \mathcal{F} and let $\tau(\widehat{R\Delta}(\mathcal{F}))$ be its torsion sheaf. Let L be an ample line bundle on A . Then, for all sufficiently high $n \in \mathbb{N}$, and for any subset $T \subseteq \text{Pic}^0 A$, the Fourier-Mukai transform $\Phi_{\mathcal{P}}$ induces a canonical isomorphism*

$$H^0(A, \text{coker } ev_{T, \mathcal{F}} \otimes L^n) \cong (\ker \psi_{T, \mathcal{F}})^*,$$

where ψ is the natural evaluation map,

$$(9) \quad \psi_{T, \mathcal{F}} : \text{Hom}(\widehat{L}^n, \widehat{R\Delta}(\mathcal{F})) \rightarrow \prod_{\alpha \in T} \text{Hom}(\widehat{L}^n, \widehat{R\Delta}(\mathcal{F})) \otimes \mathbb{C}(\alpha).$$

Proof. Let $T \subseteq \text{Pic}^0 A$ be any subset. The map $H^0(ev_T \otimes L^n)$ is the ‘‘continuous multiplication map of global sections’’:

$$m_{\mathcal{F}, L^n}^T : \bigoplus_{\alpha \in T} H^0(\mathcal{F} \otimes a^* \alpha^{-1}) \otimes H^0(L^n \otimes a^* \alpha) \rightarrow H^0(\mathcal{F} \otimes L^n).$$

A standard argument with Serre vanishing shows that, if n is big enough,

$$H^0(\text{coker}(ev_{T, \mathcal{F}}) \otimes L^n) \cong \text{coker}(m_{\mathcal{F}, L^n}^T)$$

By Serre duality, the dual of $m_{\mathcal{F}, L^n}^T$ is

$$(10) \quad \text{Ext}^q(L^n, R\Delta\mathcal{F}) \rightarrow \prod_{\alpha \in T} \text{Hom}_{\mathbb{C}}(H^0(L^n \otimes a^* \alpha), H^q((R\Delta\mathcal{F}) \otimes a^* \alpha))$$

Let us interpret such map via the Fourier-Mukai transform. Concerning the source, Mukai’s Theorem 2.8 provides the isomorphism

$$\text{Hom}_{\mathbf{D}(A)}(L^n, R\Delta\mathcal{F}[q]) \cong \text{Hom}_{\mathbf{D}(\text{Pic}^0 A)}(R\Phi_{\mathcal{P}}(L^n), R\Phi_{\mathcal{P}}(R\Delta\mathcal{F})[q]) \cong \text{Hom}_{\mathbf{D}(\text{Pic}^0 A)}(\widehat{L}^n, \widehat{R\Delta}(\mathcal{F}))$$

⁴We consider the annihilator support, instead of the Fitting support [E].

(here $q = \dim A$) i.e., since \widehat{L}^n is locally free,

$$(11) \quad \text{Ext}_A^q(L^n, R\Delta\mathcal{F}) \cong \text{Hom}_{\text{Pic}^0 A}(\widehat{L}^n, \widehat{R\Delta\mathcal{F}}).$$

Note that, besides \widehat{L}^n , also $\widehat{R\Delta(\mathcal{F})}$ has the base change property, since $H^{d+1}(\mathcal{F} \otimes a^*\alpha) = 0$ for all $\alpha \in \text{Pic}^0 A$, see [M, Cor. 3, p. 53]. This means that, in the target of the map (10), we have that $H^0(L^n \otimes a^*\alpha)$ (respectively $H^q((R\Delta\mathcal{F}) \otimes a^*\alpha)$) are isomorphic to the fiber at the point α of \widehat{L}^n (resp. of $\widehat{R\Delta\mathcal{F}}$). Hence also the sheaf $\mathcal{H}om(\widehat{L}^n, \widehat{R\Delta\mathcal{F}})$ has the base change property and the Fourier-Mukai isomorphism (11) identifies the map (10) to the evaluation map of the sheaf $\mathcal{H}om(\widehat{L}^n, \widehat{R\Delta\mathcal{F}})$ at points in T :

$$\text{Hom}(\widehat{L}^n, \widehat{R\Delta\mathcal{F}}) \rightarrow \prod_{\alpha \in T} \mathcal{H}om(\widehat{L}^n, \widehat{R\Delta\mathcal{F}}) \otimes \mathbb{C}(\alpha).$$

□

Now we are ready to proof the Theorem

Proof of Theorem 4.5. (a) By Theorem 2.2 we can assume that $\widehat{R\Delta\mathcal{F}}$ is torsion free. Then the evaluation map $\psi_{U,\mathcal{F}}$ is injective for *all* open subsets $U \subseteq \text{Pic}^0 A$, so $H^0(\text{coker } ev_{U,\mathcal{F}} \otimes L^n) = 0$ for $n \gg 0$. From Serre's theorem it follows that $\text{coker } ev_{U,\mathcal{F}} = 0$.

(b) By Nakayama's Lemma, given a non-zero global section $s \in \text{Hom}(\widehat{L}^n, \widehat{R\Delta\mathcal{F}})$, we have that $s(\alpha) \in \mathcal{H}om(\widehat{L}^n, \widehat{R\Delta\mathcal{F}}) \otimes \mathbb{C}(\alpha)$ vanishes for all α in an dense subset $T \subset \text{Pic}^0 A$, only if T does not meet a component of the support of the torsion part $\tau(\widehat{R\Delta(\mathcal{F})})$ or this support is non-reduced. The second possibility is excluded by hypothesis. And the first possibility is excluded if we consider subsets of the form $T = U \cup S$, where S is the underlying set of $\text{supp } \tau(\widehat{R\Delta(\mathcal{F})})$. Hence, the map $\psi_{T,\mathcal{F}}$ is injective. Therefore $H^0(\text{coker } ev_{T,\mathcal{F}} \otimes L^n) = 0$ and, by Serre's theorem, $\text{coker } ev_{T,\mathcal{F}} = 0$. □

4.2. The canonical bundle. Throughout the rest of the present section we will work under the following

Hypothesis 4.7. Let X be a variety of dimension d , equipped with a morphism to an abelian variety $a : X \rightarrow A$ such that:

- (a) $\text{codim } V_a^i(\omega_X) \geq i + 1$ for all i such that $0 < i < d$ (note that this hypothesis is weaker than $gv(\omega_X) \geq 1$);
- (b) the map $a^* \text{Pic}^0 A \rightarrow \text{Pic}^0 X$ is an embedding.

Hypothesis (b) yields that $V_a^d(\omega_X) = \{\hat{0}\}$, where $\hat{0}$ denotes the identity point of $\text{Pic}^0 A$. Moreover Hypothesis (a) is more than enough to ensure that $gv_a(\omega_X) \geq 0$. Therefore, by Theorem 2.2, the transform of $R\Delta(\omega_X) = \mathcal{O}_X$ is the sheaf $\widehat{\mathcal{O}_X}$ concentrated in degree d . Moreover the morphism a is generically finite by Proposition 2.7. We remark that Hypothesis (a) is equivalent to $gv_a(\omega_X) \geq 1$ unless $\dim X = \dim A$, i.e. the morphism a is surjective. Therefore, by Theorem 2.6, the sheaf $\widehat{\mathcal{O}_X}$ has torsion if and only if $\dim X = \dim A$, i.e. the map $a : X \rightarrow A$ is surjective.

Proposition 4.8. *Under Hypotheses 4.7, assume moreover that $\dim X = \dim A$, i.e. that the morphism a is surjective. Then the torsion $\tau(\widehat{\mathcal{O}_X})$ is isomorphic to the one-dimensional skyscraper sheaf $\mathbb{C}(\hat{0})$*

Proof. The argument is similar to the proof of Proposition 2.8 of [PP5]. Since $\text{Pic}^0 A$ is smooth, the functor $R\mathcal{H}om(\cdot, \mathcal{O}_X)$ is an involution on $\mathbf{D}(X)$. Thus there is a spectral sequence

$$E_2^{i,-j} := \mathcal{E}xt^i\left(\mathcal{E}xt^j(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 A}), \mathcal{O}_{\text{Pic}^0 A}\right) \Rightarrow H^{i-j} = \mathcal{H}^{i-j}\widehat{\mathcal{O}}_X = \begin{cases} \widehat{\mathcal{O}}_X & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

By duality (Corollary 2.5), $\dim \text{supp}(\mathcal{E}xt^i(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 A})) = \dim \text{supp} R^i \Phi_{P_a}(\omega_X)$ and, by base-change, $\text{supp} R^i \Phi_{P_a}(\omega_X) \subseteq V_a^i(\omega_X)$. Thus Hypothesis 4.7(a) yields that for all i such that $0 < i < d$ we have $\text{codim} \text{supp}(\mathcal{E}xt^i(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 A})) > i$. Therefore $\mathcal{E}xt^i(\mathcal{E}xt^j(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 A}), \mathcal{O}_{\text{Pic}^0 A}) = 0$ for all (i, j) such that $i - j \leq 0$, except for $(i, j) = (0, 0)$ and $(i, j) = (d, d)$. Hence the only non-zero $E_\infty^{i,-i}$ terms are $E_\infty^{d,-d}$ and $E_\infty^{0,0}$, and we have the exact sequence

$$(12) \quad 0 \rightarrow E_\infty^{d,-d} \rightarrow H^0 = \widehat{\mathcal{O}}_X \rightarrow E_\infty^{0,0} \rightarrow 0$$

The differentials coming into $E_p^{0,0}$ are always zero, so we get an isomorphism,

$$E_\infty^{0,0} \subseteq E_2^{0,0} = \widehat{\mathcal{O}}_X^{**}$$

and (12) is identified to the canonical exact sequence

$$0 \rightarrow \tau(\widehat{\mathcal{O}}_X) \rightarrow \widehat{\mathcal{O}}_X \rightarrow \widehat{\mathcal{O}}_X^{**}$$

Hence the torsion $\tau(\widehat{\mathcal{O}}_X)$ is canonically isomorphic to $E_\infty^{d,-d}$. By Proposition 6.1 of the Appendix, $R^d \Phi_{P_a}(\omega_X) \cong \mathbb{C}(\hat{0})$. Hence $E_2^{d,-d} = \mathcal{E}xt^d(R^d \Phi_{P_a}(\omega_X), \mathcal{O}_{\text{Pic}^0 A}) \cong \mathcal{E}xt^d(\mathbb{C}(\hat{0}), \mathcal{O}_{\text{Pic}^0 A}) = \mathbb{C}(\hat{0})$. Since the differentials going out of $E_p^{d,-d}$ are always zero, we get a surjection

$$E_2^{d,-d} \cong \mathbb{C}(\hat{0}) \rightarrow E_\infty^{d,-d} \cong \tau(\widehat{\mathcal{O}}_X) \rightarrow 0$$

As we already know that $\tau(\widehat{\mathcal{O}}_X) \neq 0$, it follows that $\tau(\widehat{\mathcal{O}}_X) \cong \mathbb{C}(\hat{0})$. \square

Remark 4.9. It is easily seen that the injection $\mathbb{C}(\hat{0}) \hookrightarrow \widehat{a_* \mathcal{O}}_X$ is $R^q \Phi_{\mathcal{P}}$ of the natural injection $\mathcal{O}_A \rightarrow a_* \mathcal{O}_X$.

As a first consequence we get

Proposition 4.10. *The following are equivalent:*

- (a) Hypothesis 4.7 holds, and X is not of general type;
- (b) Hypothesis 4.7 holds, and $\chi(\omega_X) = 0$;
- (c) the morphism $a : X \rightarrow A$ is birational.

Proof. (a) \Rightarrow (b): The morphism a is generically finite (Prop. 2.7). In this case it is well known that $\chi(\omega_X) \geq 1$ implies that X is of general type (by [CH], one knows even that the tricanonical map of X is birational).

(b) \Rightarrow (c): since $\chi(\omega_X) = 0$ then, by (1), $\widehat{\mathcal{O}}_X$ is a torsion sheaf. By Theorem 2.6 it follows that $gv(\omega_X) = 0$. We already know that this, together with Hypothesis 4.7, is equivalent to the fact that the morphism a is surjective. Therefore, by Proposition 4.8, $\widehat{\mathcal{O}}_X = \mathbb{C}(\hat{0})$. By Proposition 2.9

$$\mathbb{C}(\hat{0}) = \widehat{\mathcal{O}}_X = \widehat{Ra_* \mathcal{O}}_X$$

where the hat on the left is the transform $R\Phi_{P_a}$ (from X to $\text{Pic}^0 A$) and the hat on the right is the transform $R\Phi_{\mathcal{P}}$ (from A to $\text{Pic}^0 A$). By Mukai's Theorem 2.8

$$\mathcal{O}_A = R\Psi_{\mathcal{P}}(\mathbb{C}(\hat{0})) = R\Psi_{\mathcal{P}}(R\Phi_{\mathcal{P}}(\widehat{Ra_* \mathcal{O}}_X)) \stackrel{2.8}{=} (-1)_A^* Ra_* \mathcal{O}_X$$

whence $Ra_* \mathcal{O}_X = \mathcal{O}_A$. In particular, a has degree 1. \square

Another useful consequence of the previous analysis is the following. We recall that in the first case ($gv(\omega_X) \geq 1$) everything is essentially already known [PP5, Prop. 5.5].

Corollary 4.11. *Assume Hypothesis 4.7 and that X is of general type.*

(a) *If $\dim X < \dim A$ then $a_*\omega_X$ is CGG.*

(b) *If $\dim X = \dim A$, i.e. a is surjective, then $a_*\omega_X$ is ECGG, with $\{\hat{0}\}$ as underlying subset of $\text{supp } \tau(R\Delta(a_*\omega_X))$.*

Proof. By Grauert-Riemenschneider vanishing (and projection formula), $R^i a_*(\omega_X \otimes a^*\alpha) = 0$ for $i > 0$. Therefore the Leray spectral sequence degenerates giving

$$(13) \quad V_a^i(\omega_X) = V^i(a_*\omega_X).$$

(a) If $\dim X < \dim A$ then $gv(a_*\omega_X) \geq 1$. Therefore it is CGG by Theorem 4.5(a). As remarked before, this part of the result is well known [PP5, Prop. 5.5].

(b) If $\dim X = \dim A$ then $\tau(\widehat{\mathcal{O}_X}) \cong \mathbb{C}(\hat{0})$ (Proposition 4.8). Hence, by Theorem 4.5(b), $a_*\omega_X$ is ECGG. \square

4.3. Birationality criterion. We introduce a last piece of notation.

Terminology/Notation 4.12. (a) Given a line bundle L , we denote $\text{Bs}(L)$ its base locus.

(b) We denote U_0 the complement in $\text{Pic}^0 A$ of the closed subset $V_a^1(\omega_X)$. Since, by Corollary 2.4, $V_a^1(\omega_X) \supseteq \dots \supseteq V_a^d(\omega_X)$, it follows that, for all $\alpha \in U_0$, $h^0(\omega_X \otimes a^*\alpha)$ takes the minimal value, i.e. $\chi(\omega_X)$.

(c) Given a point $p \in X$, we denote $\mathcal{B}_a(p)$ the subset (closed in U_0)

$$\mathcal{B}_a(p) = \{\alpha \in U_0 \mid p \in \text{Bs}(\omega_X \otimes a^*\alpha)\}$$

(d) We will say that a line bundle “is birational” to mean that the associated rational map to projective space is birational.

The statements we are aiming at are

Theorem 4.13. *Let X be a variety of general type satisfying Hypothesis 4.7. If, for general p in X , $\text{codim}_{\text{Pic}^0 A} \mathcal{B}_a(p) \geq 2$ then $\omega_X^2 \otimes a^*\alpha$ is birational for all $\alpha \in \text{Pic}^0 A$. In particular, ω_X^2 is birational.*

Corollary 4.14. *Let X be a variety of general type satisfying Hypothesis 4.7, and assume that there exists a $\alpha \in \text{Pic}^0 A$ such that $\omega_X^2 \otimes a^*\alpha$ is not birational. Then, for general $\beta \in U_0$, $\text{Bs}(\omega_X \otimes a^*\beta)$ has codimension one. Moreover X is covered by the divisorial components of $\text{Bs}(\omega_X \otimes a^*\beta)$, for β varying in U_0 .*

Note that it makes sense to speak of the base locus of $\omega_X \otimes a^*\alpha$ since, by Proposition 4.10, the hypotheses of both the Theorem and the Corollary imply that $\chi(\omega_X) > 0$, whence $h^0(\omega_X \otimes a^*\alpha) > 0$ for all $\alpha \in \text{Pic}^0 A$.

Proof of Corollary 4.14. Let \mathcal{B}_a the closed subvariety of $X \times U_0$ defined as

$$\mathcal{B}_a = \{(p, \alpha) \in X \times U_0 \mid p \text{ is a base point of } \omega_X \otimes a^*\alpha\}.$$

If, for some $\alpha \in \text{Pic}^0 X$, $\omega_X^2 \otimes a^*\alpha$ is not birational then, by Theorem 4.13, for general $p \in X$ we have that $\text{codim } \mathcal{B}_a(p) = 1$. Since $\mathcal{B}_a(p)$ is the fiber of the projection $p : \mathcal{B}_a \rightarrow X$ it follows that $\text{codim}_{X \times U_0} \mathcal{B}_a = 1$. This implies that the fibers of the other projection $q : \mathcal{B}_a \rightarrow \text{Pic}^0 A$ have codimension 1 in X . \square

Proof of Theorem 4.13. We first recall some basic facts about the Fourier-Mukai transform of the sheaves $\mathcal{I}_p \otimes \omega_X$. In the first place, if p does not belong to $\text{exc}(a)$ then

$$(14) \quad R^i a_*(\mathcal{I}_p \otimes \omega_X \otimes a^* \alpha) = 0 \quad \text{for } i > 0$$

This follows immediately from the exact sequence

$$(15) \quad 0 \rightarrow \mathcal{I}_p \otimes \omega_X \rightarrow \omega_X \rightarrow \mathcal{O}_p \otimes \omega_X \rightarrow 0$$

and the Grauert-Riemenschneider vanishing theorem. Hence, as for the canonical bundle (see the proof of Prop. 4.8), the Leray spectral sequence yields that

$$(16) \quad V_a^i(\mathcal{I}_p \otimes \omega_X) = V^i(a_*(\mathcal{I}_p \otimes \omega_X))$$

By sequence (15), tensored by $a^* \alpha$, it follows that

$$(17) \quad V_a^i(\mathcal{I}_p \otimes \omega_X) = V_a^i(\omega_X) \quad \text{for all } i \geq 2$$

Concerning the case $i = 1$ we have the surjection

$$H^1(\mathcal{I}_p \otimes \omega_X \otimes a^* \alpha) \rightarrow H^1(\omega_X \otimes a^* \alpha),$$

which is an isomorphism if and only if p is not a base point of $\omega_X \otimes \alpha$. In other words

$$V_a^1(\mathcal{I}_p \otimes \omega_X) = \mathcal{B}_a(p) \cup V_a^1(\omega_X)$$

Therefore the hypothesis about $\mathcal{B}_a(p)$ ensures that

$$(18) \quad \text{codim } V_a^1(\mathcal{I}_p \otimes \omega_X) \geq 2$$

Now we distinguish two cases:

(a) $\dim X < \dim A$. In this case, by Hypothesis 4.7, together with (16), (17) and (18), $gv(a_*(\mathcal{I}_p \otimes \omega_X)) \geq 1$. Hence, by Theorem 4.5(a), $a_*(\mathcal{I}_p \otimes \omega_X)$ is CGG. Therefore $\mathcal{I}_p \otimes \omega_X$ itself is CGG outside $\text{exc}(a)$ (with respect to a). Since the same is true for ω_X (Corollary 4.11(a)), it follows from Proposition 4.4(a) that, for p outside $\text{exc}(a)$ and for all $\alpha \in \text{Pic}^0 A$, $\mathcal{I}_p \otimes \omega_X^2 \otimes a^* \alpha$ is globally generated outside $\text{exc}(a)$. This means that the projective map associated to $\omega_X^2 \otimes a^* \alpha$ is birational (in fact an isomorphism on the complement of $\text{exc}(a)$).

(b) $\dim X = \dim A$, i.e. the map a is surjective. Again by Hypotheses 4.7, (17) and (18), the sheaf $\mathcal{I}_p \otimes \omega_X$ satisfies,

$$\text{codim } V^i(\mathcal{I}_p \otimes \omega_X) \geq i + 1 \quad \text{for all } i \text{ such that } 0 < i < d$$

while

$$V^d(\mathcal{I}_p \otimes \omega_X) = \{\hat{0}\} \quad \text{and} \quad R^d \Phi_{P_a}(\mathcal{I}_p \otimes \omega_X) = R^d \Phi_{P_a}(\omega_X) = \mathbb{C}(\hat{0}).$$

Therefore the exact same arguments of Proposition 4.8 apply, proving that the torsion of $R\mathcal{H}om(\widehat{a_*(\mathcal{I}_p)}, \mathcal{O}_A)$ is $\mathbb{C}(\hat{0})$. Hence, by Theorem 4.5(b), $a_*(\mathcal{I}_p \otimes \omega_X)$ is ECGG, with $\{\hat{0}\}$ as underlying subset of $\text{supp } \tau(R\Delta(\widehat{\mathcal{I}_p \otimes \omega_X}))$. It follows that, for p not belonging to $\text{exc}(a)$, the sheaf $\mathcal{I}_p \otimes \omega_X$ is ECGG away of $\text{exc}(a)$. Let W be the non-empty open set of points $p \in X$ such that ω_X is CGG at p . In view of Remark 4.2, W is the complement of the intersection of all base loci $\text{Bs}(\omega_X \otimes a^* \alpha^{-1})$, for $\alpha \in \text{Pic}^0 A$ such that $h^0(\omega_X \otimes a^* \alpha^{-1})$ is minimal, i.e. equal to $\chi(\omega_X)$. Now let $\alpha \in \text{Pic}^0 A$. It follows from Proposition 4.4(b) that, if q is not a base point of $\omega_X \otimes a^* \alpha$ (and does not lie in $\text{exc}(a)$), then $\mathcal{I}_p \otimes \omega_X^2 \otimes a^* \alpha$ is globally generated at q . Denoting U_α the complement of $\text{exc}(a) \cup \text{Bs}(\omega_X \otimes a^* \alpha)$, we conclude that for all $p \in U_\alpha \cap W$ the sheaf $\mathcal{I}_p \otimes \omega_X^2 \otimes a^* \alpha$ is globally generated at all points of $U_\alpha \cap W$. As above, this means that the projective map associated to $\omega_X^2 \otimes a^* \alpha$ is an isomorphism on $U_\alpha \cap W$. \square

5. PROOF OF THE THEOREM

Here we prove Theorem A of the Introduction.

(b) \Rightarrow (a) Let (A, Θ) be an indecomposable p.p.a.v., and let $X \rightarrow \Theta$ be a desingularization. Without loss of generality, we can assume that Θ is *symmetric*, i.e. $\Theta = (-1)^*\Theta$. The restriction map $H^0(A, \mathcal{O}_A(2\Theta)) \rightarrow H^0(\Theta, \mathcal{O}_\Theta(2\Theta)) = H^0(\Theta, \omega_X^2)$ is surjective. Hence the projective map associated to ω_X^2 has degree two, since x and $-x$ have the same image. By a result of Ein-Lazarsfeld, Θ is normal and has rational singularities [EL, Thm. 1]. Hence Θ has canonical singularities by a result of Kollár [Ko3, Thm. 11.1(1)], and therefore $H^0(X, \omega_X^2) \cong H^0(\Theta, \omega_\Theta^2)$. It follows that the bicanonical map of X has degree 2.

The rest of the section is devoted to the proof of the implication (a) \Rightarrow (b). The argument is composed of various steps.

5.1. Decomposition. The hypotheses of Theorem A are more than enough to imply that $alb : X \rightarrow \text{Alb } X$, the Albanese map of X , satisfy Hypothesis 4.7. Keeping the Terminology/Notation 4.12, it follows that, by Corollary 4.14, $|\omega_X \otimes \alpha|$ has a base divisor for all $\alpha \in U_0$. More precisely, as in the Proof of Corollary 4.14, let us consider the relative base locus

$$\mathcal{B} = \{(p, \alpha) \in X \times U_0 \mid p \text{ is a base point of } \omega_X \otimes \alpha\}.$$

equipped with the projections on the two factors, p and q . \mathcal{B} has a natural subscheme structure given by the image of the relative evaluation map $p^*(p_*\mathcal{L}) \otimes \mathcal{L}^{-1} \rightarrow \mathcal{O}_{X \times U_0}$, where $\mathcal{L} = p^*\omega_X \otimes P_{X \times U_0}$. We know from Theorem 4.13 that \mathcal{B} has codimension 1 and that its divisorial part is dominant on X and surjects on U_0 via p and q . Let

$$\mathcal{Y} \subseteq \mathcal{B}$$

be the union of the divisorial components of \mathcal{B} . Finally, for $\alpha \in U_0$, let F_α be the (scheme-theoretic) fiber of $q : \mathcal{Y} \rightarrow U_0$. Therefore, at a *general* point $\alpha \in U_0$, there are no other fixed divisors, so that F_α is the fixed divisor of $\omega_X \otimes \alpha$:

$$|\omega_X \otimes \alpha| = |M_\alpha| + F_\alpha$$

where $|M_\alpha|$ is the (possibly empty) mobile part. We consider the Abel-Jacobi map

$$f_{\alpha_0} : \text{Pic}^0 A \rightarrow \text{Pic}^0 X \quad \alpha \mapsto \mathcal{O}_X(F_\alpha - F_{\alpha_0})$$

where α_0 is fixed in U_0 . As it is a map between abelian varieties, it is a morphism. By rigidity

$$f := f_{\alpha_0} - f_{\alpha_0}(0) : \text{Pic}^0 A \rightarrow \text{Pic}^0 X$$

is a homomorphism. Note that f does not depend on α_0 since, given another suitably general $\alpha_1 \in \text{Pic}^0 A$, $f_{\alpha_1} - f_{\alpha_0}$ is a translation. We have the following strong constraint on the Albanese and Picard variety of X :

Lemma 5.1. $f^2 = f$ and $\text{Pic}^0 X$ decomposes as $\text{Pic}^0 X \cong \ker f \times \ker(\text{id} - f)$.

Proof. Let $\alpha, \beta \in \text{Pic}^0 A$. We have that

$$(19) \quad \mathcal{O}_X(M_\alpha) \otimes \mathcal{O}_X(F_\beta) = \omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1})$$

This follows by definition of f since the left hand side is isomorphic to

$$\mathcal{O}_X(M_\alpha + F_\alpha) \otimes \mathcal{O}_X(F_\beta - F_\alpha) \cong \omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1}).$$

Assuming that α and β are *general*, so that the fixed divisor of $\omega_X \otimes \alpha \otimes f(\beta \otimes \alpha^{-1})$ is $F_{\alpha \otimes f(\beta \otimes \alpha^{-1})}$ (i.e. $|M_{\alpha \otimes f(\beta \otimes \alpha^{-1})}|$ has no divisors), it follows from (19) that

$$F_\beta = F_{\alpha \otimes f(\beta \otimes \alpha^{-1})},$$

i.e. $f(\beta) = f(\alpha \otimes f(\beta \otimes \alpha^{-1}))$. This means that $f(f(\beta \otimes \alpha^{-1})) = f(\beta \otimes \alpha^{-1})$ for general α and β , hence $f^2 = f$. This gives the splitting of the exact sequence $\hat{0} \rightarrow \ker f \rightarrow \text{Pic}^0 A \rightarrow \text{Im} f \rightarrow \hat{0}$ and the identification $\text{Im} f \cong \ker(\text{id} - f)$. \square

5.2. Triviality of $\ker f$.

Lemma 5.2. *The morphism f is injective. i.e. $\text{Pic}^0 X = \ker(\text{id} - f)$.*

Proof. Assume that $\dim \ker f > 0$. Since the complement of U_0 is $V^1(\omega_X)$, which is a finite set, $U_0 \cap \ker f$ is a non-empty open subset of $\ker f$. By construction, for all $\beta \in U_0 \cap \ker f$ we have the decomposition

$$(20) \quad \omega_X \otimes \beta = \mathcal{O}_X(M_\beta + F)$$

where $F := F_\beta$ is constant for β varying in $U_0 \cap \ker f$. We make the following

Claim: *for general $\beta \in U_0 \cap \ker f$, F is the fixed divisor of $\omega_X \otimes \beta$ (in other words: $\mathcal{O}_X(M_\beta)$ has no fixed divisors).*

Admitting the Claim for the moment, we conclude the proof. By Lemma 5.1, $\text{Alb} X \cong B \times C$, where B and C are the dual abelian varieties of $\ker f$ and $\ker(\text{id} - f)$. Since $B \neq 0$, composing the Albanese map of X with the projection on B we get a morphism $b : X \rightarrow B$ satisfying Hypothesis 4.7. Indeed $V_b^i(\omega_X) = V^i(\omega_X) \cap \ker f$ and therefore the hypothesis of Theorem A yields that $\dim V_b^i(\omega_X) = 0$ for all $i > 0$. Moreover, $\text{Pic}^0 B = \ker f$ is embedded in $\text{Pic}^0 X$. By Proposition 2.7, b is generically finite. By the Claim, the divisorial part of the base locus of $\omega_X \otimes b^* \beta$ is constant on a non-empty Zariski open set of $\text{Pic}^0 B$. But, since ω_X^2 is not birational, this is in contrast with Corollary 4.14 (last part of the statement) applied to b . Hence $\ker f$ has to be trivial.

To prove the Claim, we notice that, for $\gamma \in \ker(\text{id} - f) \cap U_0$, we have the decomposition $\omega_X \otimes \gamma = \mathcal{O}_X(M + F_\gamma)$, where F_γ is fixed and $\mathcal{O}_X(M) := \mathcal{O}_X(M_\gamma)$ is constant on $U_0 \cap \ker(\text{id} - f)$. From this, (20) and (19) it follows that $|\omega_X \otimes \beta \otimes \gamma| = |M_\beta| + F_\gamma$, with $\beta \in \ker f \cap U_0$ and $\gamma \in \ker(\text{id} - f) \cap U_0$. In other words $M_{\beta \otimes \gamma} = M_\beta$ and $F_{\beta \otimes \gamma} = F_\gamma$. For β and γ general, $\beta \otimes \gamma$ is general in $\text{Pic}^0 A$. The Claim follows since we know that $|M_{\beta \otimes \gamma}|$ has no fixed divisors. \square

5.3. The Poincaré line bundle on $X \times \text{Pic}^0 X$. It follows from Lemma 5.2 that, for all $\alpha \in U_0$,

$$(21) \quad |\omega_X \otimes \alpha| = |M| + F_\alpha,$$

where F_α (the fiber of the projection $\mathcal{Y} \rightarrow \text{Pic}^0 X$) is the fixed divisor, hence $\chi(\omega_X) = h^0(M)$.

Let $\overline{\mathcal{Y}}$ be the closure of \mathcal{Y} in $X \times \text{Pic}^0 X$. Moreover, for $p \in X$ let \mathcal{D}_p be the fiber of the projection $\overline{\mathcal{Y}} \rightarrow X$. For general $p \in X$, we have that \mathcal{D}_p is the closure of the union of the divisorial components of the locus of $\alpha \in U_0$ such that $p \in \text{Bs}(\omega \otimes \alpha)$. Since the Albanese map is defined up to a translation in $\text{Alb} X$, we can assume that there is one such point, say \overline{p} , such that $\text{alb}(\overline{p}) = 0$ in $\text{Alb} X$. Finally, we recall from §1 that P denotes the Poincaré line bundle on $X \times \text{Pic}^0 X$.

Lemma 5.3. $P \cong \mathcal{O}_{X \times \text{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^*(\omega_X^{-1} \otimes M) \otimes q^* \mathcal{O}_{\text{Pic}^0 X}(-\mathcal{D}_{\overline{p}})$

Proof. By the definition of $\overline{\mathcal{Y}}$ and (21) we have that

$$\mathcal{O}_{X \times \text{Pic}^0 X}(\overline{\mathcal{Y}}) \otimes p^*(\omega_X^{-1} \otimes M) \otimes q^*\mathcal{O}_{\text{Pic}^0 X}(-\mathcal{D}_{\overline{p}}),$$

- restricted to $X \times \{\alpha\}$ is isomorphic to $\mathcal{O}_X(F_\alpha) \otimes \mathcal{O}_X(F_\alpha)^{-1} \otimes \alpha = \alpha = P|_{X \times \{\alpha\}}$, for all $\alpha \in U_0$;

- restricted to $\{\overline{p}\} \times \text{Pic}^0 X$ is isomorphic to $\mathcal{O}_X(\mathcal{D}_{\overline{p}}) \otimes \mathcal{O}_X(-\mathcal{D}_{\overline{p}})$, i.e. trivial.

The Lemma follows from the see-saw principle. \square

5.4. Conclusion of the proof of Theorem A.

Lemma 5.4. $M = \mathcal{O}_X$, i.e. $\chi(\omega_X) = 1$

Having proved this, the implication (a) \Rightarrow (b) of Theorem A follows from Proposition 3.1.

Proof of Lemma 5.4. Lemma 5.3 yields the standard short exact sequence on $X \times \text{Pic}^0 X$

$$0 \rightarrow P^{-1} \xrightarrow{\overline{\mathcal{Y}}} p^*(\omega_X \otimes M^{-1}) \otimes q^*(\mathcal{O}_{\text{Pic}^0 X}(\mathcal{D}_{\overline{p}})) \rightarrow p^*(\omega_X \otimes M^{-1}) \otimes q^*(\mathcal{O}_{\text{Pic}^0 X}(\mathcal{D}_{\overline{p}}))|_{\overline{\mathcal{Y}}} \rightarrow 0$$

Applying $R^d q_*$, and recalling that $R\Phi_{P^{-1}} \cong (-1)_{\text{Pic}^0 X}^* R\Phi_P$ (see 2.1) we get

$$(22) \quad 0 \rightarrow (-1)_{\text{Pic}^0 X}^* \widehat{\mathcal{O}}_X \xrightarrow{\mu} \mathcal{O}_{\text{Pic}^0 X}(\mathcal{D}_{\overline{p}})^{\oplus \chi(\omega_X)} \rightarrow \tau \rightarrow 0$$

where:

- (a) The rank $\chi(\omega_X)$ in the middle appears because $h^d(\omega_X \otimes M^{-1}) = h^0(M) \stackrel{(21)}{=} \chi(\omega_X)$;
- (b) $\tau = R^d q_* p^*(\omega_X \otimes M^{-1}) \otimes q^*(\mathcal{O}_{\text{Pic}^0 X}(\mathcal{D}_{\overline{p}}))|_{\overline{\mathcal{Y}}}$ is supported at the locus of the $\alpha \in \text{Pic}^0 X$ such that the fiber of the projection $q : \overline{\mathcal{Y}} \rightarrow \text{Pic}^0 X$ has dimension d , i.e. it coincides with X . Such locus is contained in the union of the $V^i(\omega_X)$ for $i > 0$, hence it is finite set;
- (c) the map μ is injective since it is a generically surjective map of sheaves of the same rank (recall that $\text{rk } \widehat{\mathcal{O}}_X = \chi(\omega_X)$), and, as $gv(\omega_X) \geq 1$, the source $\widehat{\mathcal{O}}_X$ is torsion free (Theorem 2.6).
- (d) Since μ is $R^d q_*(m_s)$, where m_s is the multiplication for the section defining $\overline{\mathcal{Y}}$, μ is zero on the locus where the fiber of q has dimension d . Therefore τ is a (trivial) sheaf of rank equal to $\chi(\omega_X)$, supported at a finite set of points contained in the finite set $\bigcup_{i>0} V^i(\omega_X) = V^1(\omega_X)$.

From (22) and the fact that $\text{supp } \tau$ is a finite scheme it follows that $\mathcal{E}xt^i(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 X}) \cong \mathcal{E}xt^{i+1}(\tau, \mathcal{O}_{\text{Pic}^0 X}) = 0$ if $i \neq q(X) - 1$. On the other hand, by Corollary 2.5 and Proposition 6.1 of the Appendix, $\mathcal{E}xt^d(\widehat{\mathcal{O}}_X, \mathcal{O}_{\text{Pic}^0 X}) \cong \mathbb{C}(\hat{0})$. It follows that $d = q(X) - 1$ and that $\tau = \mathbb{C}(\hat{0})$. The assertion follows since the length of τ is equal to $\chi(\omega_X)$. \square

6. APPENDIX: A USEFUL LEMMA ON THE GENERALIZED FOURIER-MUKAI TRANSFORM OF THE CANONICAL SHEAF

As usual, let $a : X \rightarrow A$ be a morphism from a d -dimensional smooth projective variety to an abelian variety (over any algebraically closed field k). Let $P_a = (a \times \text{id})^* \mathcal{P}$, where \mathcal{P} is the Poincaré line bundle on $A \times \text{Pic}^0 A$. Assume furthermore that the induced morphism $a^* : \text{Pic}^0 A \rightarrow \text{Pic}^0 X$ is an *embedding*. Then the top cohomological support locus is $V^d(\omega_X) = \{\hat{0}\}$. By base change [M, Cor. 3, p. 53], it follows that, for $\alpha \in \text{Pic}^0 A$,

$$(23) \quad R^d \Phi_{P_a}(\omega_X) \otimes k(\hat{\alpha}) \cong \begin{cases} k(\hat{0}) & \text{if } \alpha = \hat{0} \\ 0 & \text{otherwise} \end{cases}$$

It follows that $R^d \Phi_{P_a}(\omega_X)$ is a sheaf supported at $\hat{0}$. The result we are aiming at is

Proposition 6.1. $R^d\Phi_{P_a}(\omega_X) \cong k(\hat{0})$ ⁵

Proof. Let $B = \mathcal{O}_{\text{Pic}^0 A, \hat{0}}$ and \mathfrak{m} its maximal ideal. By Nakayama's lemma, (23) implies that $R^d\Phi_{P_a}(\omega_X)$ is supported only at $\hat{0}$ and that $R^d\Phi_{P_a}(\omega_X) \cong B/J$, where J is a \mathfrak{m} -primary ideal.

Claim. $P_{a|X \times \text{Spec } B/J}$ is trivial.

Proof of the Claim. Let \mathcal{E} be a sheaf on $\text{Pic}^0 A$. By Grothendieck duality

$$(24) \quad R\mathcal{H}om_{\text{Pic}^0 A}(R\Phi_{P_a}(\omega_X), \mathcal{E}) \cong Rq_*(\mathcal{H}om_{X \times \text{Pic}^0 A}(P_a, q^*\mathcal{E}))[d]$$

Indeed,

$$\begin{aligned} R\mathcal{H}om_{\text{Pic}^0 A}(R\Phi_{P_a}(\omega_X), \mathcal{E}) &= R\mathcal{H}om_{\text{Pic}^0 A}(Rq_*(p^*\omega_X \otimes P_a), \mathcal{E}) \\ &\stackrel{GD}{\cong} Rq_*(R\mathcal{H}om_{X \times \text{Pic}^0 A}(p^*\omega_X \otimes P_a, p^*\omega_X \otimes q^*\mathcal{E}[d])) \\ &\cong Rq_*(\mathcal{H}om_{X \times \text{Pic}^0 A}(P_a, q^*\mathcal{E}))[d] \end{aligned}$$

Therefore we have a fourth quadrant spectral sequence

$$E_2^{i,-j} = \mathcal{E}xt_{\text{Pic}^0 A}^i(R^j\Phi_{P_a}(\omega_X), \mathcal{E}) \Rightarrow R^{i-j+d}q_*(\mathcal{H}om_{X \times \text{Pic}^0 A}(P_a, q^*\mathcal{E}))$$

Clearly the term $E_2^{i,-j}$ is non-zero only if $i \geq 0$. Assuming $i \geq 0$, in the case $i - j + d = 0$, i.e. $j = i + d$ we have that $R^j\Phi_{P_a}(\omega_X)$ is non zero if and only if $j = d$, i.e. $i = 0$. In conclusion for $i - j + d = 0$ the only non-zero E_2 -term is $E_2^{0,-d} = \mathcal{H}om_{\text{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E})$. Since the differentials from and to $E_2^{0,-d}$ are zero, we get that

$$\mathcal{H}om_{\text{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E}) = E_2^{0,-d} = E_\infty^{0,-d} \cong q_*(\mathcal{H}om_{X \times \text{Pic}^0 A}(P_a, q^*\mathcal{E})).$$

Taking global sections, we get the isomorphism (functorial in \mathcal{E})

$$(25) \quad \text{Hom}_{\text{Pic}^0 A}(R^d\Phi_{P_a}(\omega_X), \mathcal{E}) \cong \text{Hom}_{X \times \text{Pic}^0 A}(P_a, q^*\mathcal{E})$$

Using the previous isomorphism twice, once for $\mathcal{E} = B/J$ and the other for $\mathcal{E} = k(\hat{0})$, by functoriality we get the commutative diagram

$$(26) \quad \begin{array}{ccccc} B/J & \xlongequal{\quad} & \text{Hom}(B/J, B/J) & \xlongequal{\quad} & \text{Hom}(P_{a|X \times \text{Spec } B/J}, \mathcal{O}_{X \times \text{Spec } B/J}) \\ \downarrow & & \downarrow & & \downarrow \\ k(\hat{0}) & \xlongequal{\quad} & \text{Hom}(B/J, k(\hat{0})) & \xlongequal{\quad} & \text{Hom}(P_{a|X \times \{\hat{0}\}}, \mathcal{O}_{X \times \{\hat{0}\}}) \end{array}$$

Since $P_{a|X \times \{\hat{0}\}}$ is trivial, we can take an isomorphism $h \in \text{Hom}(P_{a|X \times \{\hat{0}\}}, \mathcal{O}_{X \times \{\hat{0}\}})$. By the diagram above, h lifts to a morphism $\bar{h} : P_{a|X \times \text{Spec } B/J} \rightarrow \mathcal{O}_{X \times \text{Spec } B/J}$. Since \bar{h} is a map between invertible sheaves on $X \times \text{Spec } B/J$ which is an isomorphism when restricted to $X \times \{\hat{0}\}$, \bar{h} is an isomorphism. Therefore $P_{a|X \times \text{Spec } B/J}$ is trivial. The Claim is proved.

At this point the Proposition follows since *the smooth point $\hat{0}$ of $\text{Pic}^0 A$ is the maximal subscheme Z of $\text{Pic}^0 A$ such that $P_{a|X \times Z}$ is trivial* ([M] §10). This in turn follows from the well known fact ([M] §13) that the smooth point $\hat{0}$ of $\text{Pic}^0 A$ is the maximal subscheme Z of $\text{Pic}^0 A$ such that $\mathcal{P}_{|A \times Z}$ is trivial, combined with the fact that $a^* \text{Pic}^0 A \rightarrow \text{Pic}^0 X$ is an embedding. However, we provide an equivalent but self-contained argument. Let us consider, in analogy to

⁵When X itself is an abelian variety (or a complex torus) this statement is well known (see [BL, Cor. 14.1.6] and [Ke] for an elementary proof in the complex case and [Hu, p. 202], [M, p. 128] for arbitrary characteristic). This fact is crucial in the proof of Mukai Inversion Theorem 2.8

Subsection 2.3, the functor $R\Psi_a : \mathbf{D}(\mathrm{Pic}^0 A) \rightarrow \mathbf{D}(X)$, defined by $R\Psi_a(\cdot) = Rp_*(P_a \otimes q^*(\cdot))$. Since $P_a = (a \times \mathrm{id}_{\mathrm{Pic}^0 A})^* \mathcal{P}$, it follows that

$$(27) \quad R\Psi_a \cong La^* \circ R\Psi$$

The Claim implies, by the Künneth formula, that

$$(28) \quad R\Psi_{P_a}(B/J) = R^0\Psi_{P_a}(B/J) = \mathcal{O}_X^{\oplus r}$$

where $r = \mathit{length} B/J$. On the other hand, by [Mu1, Lemma 4.8], $R\Psi_{\mathcal{P}}(B/J) = R^0\Psi_{\mathcal{P}}(B/J) := U$, where U is a *unipotent* vector bundle on A of rank r , i.e. a vector bundle having a filtration $0 = U_0 \subset U_1 \subset \cdots \subset U_{r-1} \subset U_r = U$, such that $U_i/U_{i-1} \cong \mathcal{O}_A$. By (27) and (28) it follows that a^*U is trivial. The filtration of U , pulled back via a , induces the filtration of the trivial bundle:

$$0 \subset a^*U_1 \subset \cdots \subset a^*U_{r-1} \subset a^*U_r = \mathcal{O}_X^{\oplus r},$$

where $a^*U_i/a^*U_{i-1} \cong \mathcal{O}_X$. Since $h^0(X, a^*U_i) \leq i$ for all i , the fact that a^*U_r is trivial implies easily, by descending induction on i , that

$$(29) \quad h^0(X, a^*U_i) = i \quad \text{for all } i$$

This in turn implies that the sequence

$$0 \rightarrow a^*U_{i-1} \rightarrow a^*U_i \rightarrow \mathcal{O}_X \rightarrow 0$$

splits for all i (the coboundary map $H^0(\mathcal{O}_A) \rightarrow H^1(U_{i-1})$ is zero). In particular the extension

$$0 \rightarrow \mathcal{O}_X \rightarrow a^*U_2 \rightarrow \mathcal{O}_X \rightarrow 0$$

is split. But the natural pullback map

$$(30) \quad H^1(\mathcal{O}_A) \cong \mathrm{Ext}^1(\mathcal{O}_A, \mathcal{O}_A) \rightarrow \mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) \cong H^1(\mathcal{O}_X)$$

is identified with the differential at $\hat{0}$ of the map $a^* : \mathrm{Pic}^0 A \rightarrow \mathrm{Pic}^0 X$. Since a^* is assumed to be an embedding, (30) is injective. Hence also the extension

$$0 \rightarrow \mathcal{O}_A \rightarrow U_2 \rightarrow \mathcal{O}_A \rightarrow 0$$

is split. This yields that $h^n(U) \geq 2$. But this is impossible since, by Mukai's inversion, $R\Phi_{\mathcal{P}}(U) = \hat{U}[-n] \cong (-1_{\mathrm{Pic}^0 A})^* B/J[-n]$, and therefore, by base change, $H^n(U) \cong (B/J) \otimes k(\hat{0}) \cong k(\hat{0})$. In conclusion $r = \mathit{length}(B/J) = 1$, i.e. $B/J \cong k(\hat{0})$. \square

REFERENCES

- [BCP] I. Bauer, F. Catanese, R. Pignatelli, Complex surfaces of general type: some recent progress, In: *Global aspects of complex geometry*, 1–58, Springer, Berlin (2006)
- [BL] C. Birkenhake and H. Lange, *Complex abelian varieties*, 2nd edition, Springer 2004
- [Ca] F. Catanese, Moduli and classification of irregular Kaehler manifolds (and algebraic varieties) with Albanese general type fibrations, *Invent. Math.* **104** (1991), 389–407.
- [CCM] F. Catanese, C. Ciliberto, M. Mendes Lopes, On the classification of irregular surfaces of general type with non birational bicanonical map, *Trans. Amer. Math. Soc.* **350** (1998) 275–308
- [CH] J. A. Chen, C. Hacon, Linear series on irregular varieties In *Algebraic geometry in East Asia (Kyoto, 2001)*, 143–153, World Sci. Publ., River Edge, NJ (2002)
- [Ci] C. Ciliberto, The bicanonical map for surfaces of general type, In: *Algebraic Geometry - Santa Cruz 1995*, Proc. Sympos. Pure Math., 62, Part I, 57–84, Amer. Math. Soc., Providence (1997),
- [CM] C. Ciliberto, M. Mendes Lopes, On surfaces with $p_g = q = 2$ and non-birational bicanonical map, *Adv. Geom.* **2** (2002) 281–300,
- [EL] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, *J. Amer. Math. Soc.* **10** (1997), 243–258.
- [E] D. Eisenbud, *Commutative algebra with a view towards algebraic geometry*, Springer-Verlag GTM **150**, New York, 1995.

- [GL1] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, *Invent. Math.* **90** (1987), 389–407.
- [GL2] ———, Higher obstructions to deforming cohomology groups of line bundles, *J. Amer. Math. Soc.* **1** (1991), no.4, 87–103.
- [Ha] C. Hacon, A derived category approach to generic vanishing, *J. Reine Angew. Math.* **575** (2004), 173–187.
- [HP] C. Hacon and R. Pardini, On the birational geometry of varieties of maximal Albanese dimension, *J. Reine Angew. Math.* **546** (2002), 177–199.
- [Hu] D. Huybrechts, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford University Press 2006
- [Ke] G. Kempf, *Complex abelian varieties and theta functions*, Springer 1991
- [Ko1] J. Kollár, Higher direct images of dualizing sheaves I, *Ann. of Math.* **123** (1986), 11–42.
- [Ko2] ———, Higher direct images of dualizing sheaves II, *Ann. of Math.* **124** (1986), 171–202.
- [Ko3] ———, Singularities of pairs, *Algebraic geometry—Santa Cruz 1995*, 221–287, *Proc. Sympos. Pure Math.*, **62**, Part 1, Amer. Math. Soc., Providence, RI, 1997.
- [LP] R. Lazarsfeld and M. Popa, BGG correspondence of cohomology of compact Kähler manifolds, and numerical invariants, preprint
- [Mu1] S. Mukai, Semi-homogeneous vector bundles on an abelian variety, *J. Math. Kyoto Univ.*, **18** (1978) 239–272
- [Mu2] ———, Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves, *Nagoya Math. J.* **81** (1981), 153–175
- [M] D. Mumford, *Abelian varieties*, Second edition, Oxford University Press 1970
- [PP1] G. Pareschi and M. Popa, Regularity on abelian varieties, I, *J. Amer. Math. Soc.*, **16** (2003) 285–302.
- [PP2] ———, Regularity on abelian varieties, II. Basic results on linear series and defining equations, *J Algebraic Geom.* **13** (2004), 167–193.
- [PP3] ———, M -regularity and the Fourier-Mukai transform, *Pure and Applied Math. Quarterly*, F. Bogomolov issue II, **4** no.3 (2008).
- [PP4] ———, GV -sheaves, Fourier-Mukai transform, and Generic Vanishing, preprint arXiv:math/0608127.
- [PP5] ———, Regularity on abelian varieties III: relationship with Generic Vanishing and applications, preprint arXiv:0802.1021, to appear in the Proceedings of the Clay Mathematics Institute.
- [PP6] ———, Strong generic vanishing and a higher dimensional Castelnuovo-de Franchis inequality, preprint arXiv:math/0808.2444, to appear on *Duke Mathematical Journal*.

DEPARTAMENT DE MATEMÀTICA APLICADA I, UNIVERSITAT POLITÈCNICA DE CATALUNYA, ETSEIB AVDA. DIAGONAL 647, 08028 BARCELONA, SPAIN

E-mail address: miguel.angel.barja@upc.edu

DEPARTAMENT D'ÀLGEBRA I GEOMETRIA, FACULTAT DE MATEMÀTIQUES, UNIVERSITAT DE BARCELONA, GRAN VIA, 585, 08007 BARCELONA, SPAIN

E-mail address: marti.lahoz@ub.edu

DEPARTAMENT D'ÀLGEBRA I GEOMETRIA, FACULTAT DE MATEMÀTIQUES, UNIVERSITAT DE BARCELONA, GRAN VIA, 585, 08007 BARCELONA, SPAIN

E-mail address: jcnaranjo@ub.edu

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI ROMA, TOR VERGATA, V.LE DELLA RICERCA SCIENTIFICA, I-00133 ROMA, ITALY

E-mail address: pareschi@mat.uniroma2.it