Sensivity and stability of singular systems under proportional and derivative feedback

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Abstract:- We consider triples of matrices (E, A, B), representing singular linear time invariant systems in the form $E\dot{x}(t) = Ax(t) + Bu(t)$, with $E, A \in M_{p\times n}(C)$ and $B \in M_{n\times m}(C)$, under proportional and derivative feedback. Using geometrical techniques we obtain miniversal deformations that permit us to study sensivity and structural stability of singular systems.

Key-Words:- Singular linear systems, proportional and derivative feedback, canonical reduced form, structural invariants, structural stability.

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1 Introduction

We denote by $M_{r\times s}(C)$ the space of complex matrices having r rows and s columns, and in the case which $r = s$ we write $M_r(C)$.

We consider the set $\mathcal{M}$ of triples of matrices (E, A, B) representing families of generalized linear time invariant systems in the form $E\dot{x}(t) = Ax(t)+Bu(t)$, with $E, A \in M_{p\times n}(C)$, $B \in M_{n\times m}(C)$, $(n, m, p > 0)$ and an equivalence relation, which generalizes the block similarity equivalence between pairs of matrices representing standard systems $\dot{x}(t) = Ax(t) + Bu(t)$, with $E, A \in M_n(C)$, $B \in M_{n\times m}(C)$, $(n, m, > 0)$.

The concept of structural stability, in the qualitative theory of dynamical systems (structurally stable elements being those whose behavior does not change when applying small perturbations) has been widely studied by several authors in control theory (see [9], [10], [13] for example). The stability not only is studied for standard and singular systems, but for switched linear systems, for example Brás in [2], studies the diagonal stability of switched linear systems.

In this paper we will consider the concept of structural stability as appearing in [15]: a structurally stable element is an element having a neighborhood contained in its equivalence class that is to say a small perturbation of it gives rise to an element equivalent to it. The characterization of structurally stable triples of matrices can be deduced after viewing the equivalence relation considered as the equivalence relation defined by the action of a Lie group acting on the differentiable manifold of triples of matrices, giving rise to orbits which are also differentiable manifolds. Then, the Arnold's techniques of versal deformations [1], provide a special parametrization of matrix spaces, which can be effectively applied to perturbation an structural stability analysis. This technique was used by the author in [7], to study perturbations and bifurcation diagrams for regularisable singular systems. In this paper we generalize this result to all singular systems, therefore, the main results are Theorem 4, which describes the orthogonal miniversal deformation of a triple of matrices, and Theorem 5, which gives the desired characterization of the structurally stable triples of matrices.

Different useful and interesting equivalence relations between singular systems have been defined. We deal with the equivalence relation preserving the controllability character of a system. Remember that a system $(E, A, B) \in \mathcal{M}$ is controllable (see [4]), if and only if, matrices E and A are square matrices,

$$\begin{cases} \text{rank} \begin{pmatrix} E & B \end{pmatrix} = n \\ \text{rank} \begin{pmatrix} sE - A & B \end{pmatrix} = n; \quad \text{for all } s \in \mathbb{C} \end{cases}$$

Notice that, the first condition implies that, there exists a matrix $U \in M_{m\times n}(\mathbb{C})$ such that the matrix $E+BU$ has full rank, and the second condition implies that, there exists a matrix $V \in M_{m\times n}(\mathbb{C})$ such that the matrix $A + BV$ has full rank. Therefore the equivalence considered in this paper will be under feedback and derivative feedback.

The structure of the paper is as follows.

In section 2 we present some notations used in the paper. In Section 3, we recall the equivalence relation between triples, we present a canonical reduced form as well a complete system of structural invariants. In section 4, we see the equivalence relation defined as induced by the action of a Lie group, in order to prove that the equivalence classes are regular submanifolds and to obtain a description of their tangent spaces. Section 5, contains a explicit description of the miniversal deformation. Section 6, is devoted to main Theorem. First, we recall the definition of structural stability and we characterize it as any miniversal deformation being zero-dimensional, that is, as the nonexistence of nontrivial solutions of the system appearing in Theorem 4. Thus, the conditions in Theorem 5, are obtained.

2 Notations

In this paper we will use the following notations.

- I_n denotes the n-order identity matrix,
- N denotes a nilpotent matrix in its reduced form $N = \text{diag}(N_1, \ldots, N_\ell)$, $N_i = \begin{pmatrix} 0 & I_{n_i-1} \\ 0 & 0 \end{pmatrix} \in M_{n_i}(C)$,
- J denotes the Jordan matrix $J = \text{diag}(J_1, \ldots, J_t)$, $J_i = \text{diag}(J_{i_1}, \ldots, J_{i_s})$, $J_{i_j} = \lambda_i I + N$,
- $L = \text{diag} = (L_1, \ldots, L_q)$, $L_j = \begin{pmatrix} I_{n_j} & 0 \end{pmatrix} \in M_{n_j\times(n_j+1)}(C)$,
- $R = \text{diag}(R_1, \ldots, R_p)$, $R_{n_j} = \begin{pmatrix} 0 & I_{n_j} \end{pmatrix} \in M_{n_j\times(n_j+1)}(C)$.
- A^* denotes adjoint matrix of matrix A, that is, the complex conjugate of the transpose of A, ($A^* = \overline{A}^t$).

In the sequel we identify triples of matrices (E, A, B) with rectangular matrices $\begin{pmatrix} E & A & B \end{pmatrix}$ in order to use matrix expressions.

3 Equivalence relation

A manner to understand the properties of the system is treating it by purely algebraic techniques. The main aspect of this approach is defining an equivalence relation preserving these properties.

The standard transformations in state and input spaces $x(t) = Px_1(t)$, $u(t) = Ru_1(t)$ premultiplication by an invertible matrix $QE\dot{x}(t) = QAx(t)+Qu(t)$, as well as feedback $u(t) = u_1(t) - Vx(t)$ and derivative feedback $u(t) = u_1(t) - U\dot{x}(t)$, realized over generalized systems relate them in the following manner, two systems are related when one can be obtained from the other by means of one, or more, of the transformations considered. In fact, this transformations define an equivalence relation in the corresponding space of triples of matrices in the following manner.

Definition 1 *Let (E_i, A_i, B_i), $i = 1, 2$ be two triples in $\mathcal{M}$. Then, (E_1, A_1, B_1) is equivalent to (E_2, A_2, B_2) if and only if there exist invertible matrices $Q \in Gl(p; C)$, $P \in$*

$Gl(n;C)$, $R \in Gl(m;C)$, and matrices $U, V \in M_{m\times n}(C)$, such that

$$\begin{pmatrix} E_2 & A_2 & B_2 \end{pmatrix} = Q\begin{pmatrix} E_1 & A_1 & B_1 \end{pmatrix}\begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ U & V & R \end{pmatrix}. \quad (1)$$

It is easy to check that this relation is an equivalence relation.

Let $(E, A, B) \in \mathcal{M}$ be a triple of matrices, we can associate the following pencil $\lambda\begin{pmatrix} E & B & 0 \end{pmatrix} + \begin{pmatrix} A & 0 & B \end{pmatrix}$.

Proposition 1 *Two triples of matrices in $\mathcal{M}$ are equivalent with respect the equivalence relation considered, if and only if the associate pencils are strictly equivalent.*

Having defined an equivalence relation, the standard procedure then is to look for a canonical form, that is to say to look for a quadruple of matrices which is equivalent to a given quadruple and which has a simple form from which we can directly read off the properties and invariants of the corresponding singular system. In this case we have the following Theorem.

Theorem 1 ([8]) *Let (E, A, B) be a triple. Then, it is equivalent to*

$$\left(\begin{pmatrix} E_1 & \\ & S_E \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_1 & \\ & S_A \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_2 \end{pmatrix}\right), \quad (2)$$

where (E_1, A_1, B_1) is a regular triple in its Kronecker reduced form (see [5]), concretely

$$(E_1, A_1, B_1) = \left(\begin{pmatrix} I_1 & & \\ & I_2 & \\ & & N_2 \end{pmatrix}, \begin{pmatrix} N_1 & & \\ & J & \\ & & I_3 \end{pmatrix}, \begin{pmatrix} B_1 \\ 0 \\ 0 \end{pmatrix}\right)$$

The triple (I_1, N_1, B_1), is a controllable system in its Kronecker reduced form, $(I_2, J, 0)$ corresponds to the finite zeros of the triple and J in its Jordan reduced form, $(N_2, I_3, 0)$ corresponds to the infinite zeros of the triple and N_2 in its Jordan reduced form. The triple $(S_E, S_A, 0)$ is the strictly singular part of the system in its Kronecker reduced form:

$$\left(\begin{pmatrix} L_1 & \\ & L_2^t \end{pmatrix}, \begin{pmatrix} R_1 & \\ & R_2^t \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)$$

Remark 1 Not all parts necessarily appear in the canonical reduced form.

A complete system of invariants to obtain the canonical reduced form can be fond in [8].

Definition 2 *For each triple $(E, A, B) \in \mathcal{M}$, we define the collection of the discrete numbers in the following manner*

1- $r_1 = (r_1^0, r_1^1, \ldots, r_1^\ell, \ldots)$, *where*

0) $r_1^0 = \operatorname{rk} B$

ℓ) $$r_1^\ell = \operatorname{rk}\begin{pmatrix} E & 0 & \dots & 0 & B & 0 & & & \\ A & E & & & 0 & B & & & \\ 0 & A & \ddots & & & \ddots & \ddots & & \\ & & \ddots & & & & & & \\ & & & E & & & & B & 0 \\ & & & A & & & & 0 & B \end{pmatrix}$$
$\in M_{(\ell+1)p\times(\ell n+(\ell+1)m)}(C)$

2- $r_2 = (r_2^1, \ldots, r_2^\ell, \ldots)$, *where*

1) $r_2^1 = \operatorname{rank}\begin{pmatrix} E & A & B \end{pmatrix} \in M_{p\times(2n+m)}(C)$

ℓ) $$r_2^\ell = \operatorname{rk}\begin{pmatrix} E & A & & & 0 & B & & \\ 0 & E & A & & & 0 & B & \\ & & \ddots & \ddots & & & & \ddots \\ & & & E & A & & 0 & B \end{pmatrix}$$
$\in M_{\ell n\times \ell(n+m)}(C)$

3) $r_3 = (r_3^1, \ldots, r_3^\ell, \ldots)$, *where*

$$r_3^j = \operatorname{rk}\begin{pmatrix} E & B & & & C_n & 0 \\ M & 0 & & & C_{n-1} & 0 \\ 0 & 0 & & & C_{n-2} & 0 \\ & & \ddots & & & \\ & & E & B & C_2 & 0 \\ & & M & 0 & C_1 & B \end{pmatrix}$$

with $M = -\lambda_0 E + A$, $C_i = \begin{cases} M & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, *for* $j = 1, 2, \ldots$.

4) $r_4 = (r_4^1, \ldots, r_4^\ell, \ldots)$, *where*

$$r_4^j = \operatorname{rk}\begin{pmatrix} A & B & \dots & 0 & 0 & C_n & 0 \\ E & 0 & & & & & \\ & & \ddots & & & & \\ & & & A & B & C_1 & 0 \\ & & & E & 0 & C_0 & B \end{pmatrix}$$

where $C_i = \begin{cases} E & i = j, \\ 0 & i \neq j \end{cases}$, *for* $j = 1, 2, \ldots$.

And the set of continuous invariants in the following manner:

$\sigma(E,A,B) =$
$\{\lambda_0 \in C \mid \mathrm{rk}\,(\lambda_0 E - A\ B) < \mathrm{rk}\,(\lambda E - A\ B)\}.$

Proposition 2 In the set $\mathcal{M}$ of singular systems, the r_i numbers as well all $\lambda_0 \in \sigma(E,A,B)$, are invariant under the equivalence relation considered.

Proposition 3 In the set $\mathcal{M}$ of singular systems, the r_i numbers as well all $\lambda_0 \in \sigma(E,A,B)$, constitutes a complete system of invariants characterizing the equivalence classes of triples under equivalence relation considered.

Remark 2 Controllable blocks in controllable part of the system are obtained joining one block L of size one with one among of the blocks L of biggest size in the corresponding associate pencil $\lambda\begin{pmatrix}E & B & 0\end{pmatrix}+\begin{pmatrix}A & 0 & B\end{pmatrix}$, (see [8]).

4 Lie group action

Equivalence relation given in definition (1) may be seen as induced by the action of the Lie group $\mathcal{G} = \{(Q,P,R,U,V) \mid Q \in Gl(p;C), P \in Gl(n;C), R \in Gl(m;C), U,V \in M_{m\times n}(C)\}$. Using short notations $g = (Q,P,R,U,V) \in \mathcal{G}$ and $x = (E,A,B) \in \mathcal{M}$, we define multiplication in $\mathcal{G}$, action of the group $\mathcal{G}$, and equivalence condition (1) as follows

$$\begin{array}{l} g_1g_2 = \\ (Q_2Q_1, P_1P_2, R_1R_2, U_1P_2 + R_1U_2, V_1P_2 + R_1V_2), \\ g\circ x = Q\begin{pmatrix}E_1 & A_1 & B_1\end{pmatrix}\begin{pmatrix}P & 0 & 0\\ 0 & P & 0\\ U & V & R\end{pmatrix}, \\ x_2 = g\circ x_1. \end{array} \tag{3}$$

Multiplication in the group corresponds to successive equivalence transformations: $g_2 \circ (g_1 \circ x) = (g_1g_2)\circ x$. Unit element of $\mathcal{G}$ has the form $e = (I_p, I_n, I_m, 0, 0)$, where I_p, I_n and I_m are the identity matrices.

Let us fix a triple $x_0 = (E_0, A_0, B_0) \in \mathcal{M}$ and define the mapping

$$\alpha_{x_0}(g) = g\circ x_0. \tag{4}$$

The equivalence class of the triple x_0 under equivalence relation considered coincides with the equivalence class of the triple with respect to the action of $\mathcal{G}$; that is, the equivalence class is the range of the function α_{x_0} and it is called the *orbit* of x_0 and denoted by

$$\mathcal{O}(x_0) = \mathrm{Im}\,\alpha_{x_0} = \{g\circ x_0 \mid g \in \mathcal{G}\}. \tag{5}$$

The *stabilizer* of x_0 under the $\mathcal{G}$-action is a null-space of the function $\alpha_{x_0} - x_0$. We denote it by

$$\mathcal{S}(x_0) = \mathrm{Ker}\,(\alpha_{x_0} - x_0) = \{g \in \mathcal{G} \mid g\circ x_0 = x_0\}. \tag{6}$$

The mapping α_{x_0} is differentiable, and $\mathcal{O}(x_0)$ and $\mathcal{S}(x_0)$ are smooth submanifolds of $\mathcal{M}$ and $\mathcal{G}$ respectively.

Let us use the notation $T_e\mathcal{G}$ for a tangent space to the manifold $\mathcal{G}$ at the unit element e. Since $\mathcal{G}$ is an open subset of $M_n(C)\times M_m(C)\times M_{m\times n}(C)\times M_{m\times n}(C)$, we have

$$T_e\mathcal{G} = M_p(C)\times M_n(C)\times M_m(C)\times (M_{m\times n}(C))^2$$

and, since $\mathcal{M}$ is a linear space,

$$T_{x_0}\mathcal{M} = \mathcal{M}.$$

The Euclidean scalar products in the spaces $\mathcal{M}$ and $T_e\mathcal{G}$ considered in this paper are defined as follows

$$\begin{array}{l} \langle x_1, x_2\rangle_1 = \mathrm{tr}(E_1E_2^*) + \mathrm{tr}(A_1A_2^*) + \mathrm{tr}(B_1B_2^*), \\ \text{where } x_i = (E_i, A_i, B_i) \in \mathcal{M}, \\ \langle y_1, y_2\rangle_2 = \\ \quad \mathrm{tr}(Q_1Q_2^*) + \mathrm{tr}(P_1P_2^*) + \mathrm{tr}(R_1R_2^*) + \\ \qquad \mathrm{tr}(U_1U_2^*) + \mathrm{tr}(V_1V_2^*), \\ \text{where } y_i = (Q_i, P_i, R_i, U_i, V_i) \in T_e\mathcal{G}, \end{array} \tag{7}$$

Let $d\alpha_{x_0} : T_e\mathcal{G} \longrightarrow \mathcal{M}$ be the differential of α_{x_0} at the unit element e. Using expressions (3) and (4), we find,

$$\begin{array}{l} d\alpha_{x_0}(y) = (X,Y,Z) \in \mathcal{M},, \text{ with} \\ X = E_0P + QE_0 + B_0U, \\ Y = A_0P + QA_0 + B_0V, \\ Z = B_0R + QB_0) \\ y = (Q,P,R,U,V) \in T_e\mathcal{G}. \end{array} \quad (8)$$

The adjoint linear mapping $d\alpha^*_{x_0} : \mathcal{M} \longrightarrow T_e\mathcal{G}$ is defined by the relation

$$\langle d\alpha_{x_0}(y), z\rangle_1 = \langle y, d\alpha^*_{x_0}(z)\rangle_2, \; y \in T_e\mathcal{G}, \; z \in \mathcal{M}. \quad (9)$$

The mappings $d\alpha_{x_0}$ and $d\alpha^*_{x_0}$ provide a simple description of the tangent spaces $T_{x_0}\mathcal{O}(x_0)$, $T_e\mathcal{S}(x_0)$ and their normal complements $(T_{x_0}\mathcal{O}(x_0))^\perp$, $(T_e\mathcal{S}(x_0))^\perp$.

Theorem 2 *The tangent spaces to the orbit and stabilizer of the triple of matrices x_0 and the corresponding normal complementary subspaces with respect to $\mathcal{M}$ and $T_e\mathcal{G}$ can be found in the following form*

i) $T_{x_0}\mathcal{O}(x_0) = \operatorname{Im} d\alpha_{x_0} \subset \mathcal{M}$,

ii) $(T_{x_0}\mathcal{O}(x_0))^\perp = \operatorname{Ker} d\alpha^*_{x_0} \subset \mathcal{M}$,

iii) $T_e\mathcal{S}(x_0) = \operatorname{Ker} d\alpha_{x_0} \subset T_e\mathcal{G}$,

iv) $(T_e\mathcal{S}(x_0))^\perp = \operatorname{Im} d\alpha^*_{x_0} \subset T_e\mathcal{G}$.

Proof.
Assertions i and iii follow from (8). Then assertions ii and iv follow from properties of the adjoint function $d\alpha^*_{x_0}$ (see [6] for example). □

Corollary 1 *The mappings $d\alpha_{x_0}$ and $d\alpha^*_{x_0}$ define one-to-one correspondences between the subspaces $T_{x_0}\mathcal{O}(x_0)$ and $(T_e\mathcal{S}(x_0))^\perp$.*

Proposition 4 *Let $x_0 = (E,A,B) \in \mathcal{M}$ be a triple of matrices. Then,*

$$\begin{array}{l} T_{x_0}\mathcal{O}(x_0) = \\ \{(QE+EP+BU, QA+AP+BV, QB+BR) \mid \\ \qquad \forall (Q,P,R,U,V) \in \mathcal{G}\}. \\ (T_{x_0}\mathcal{O}(x_0))^\perp = \\ \{(X,Y,Z) \mid EX^* + AY^* + BZ^* = 0, \\ X^*E + Y^*A = 0, X^*B = 0, Y^*B = 0, Z^*B = 0\}. \end{array}$$

5 Miniversal deformation

Let $\mathcal{U}_0$ be a neighborhood of the origin of C^ℓ. A *deformation* $x(\gamma)$ of x_0 is a smooth mapping

$$x : \mathcal{U}_0 \longrightarrow \mathcal{M}$$

such that $x(0) = x_0$. The vector $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \mathcal{U}_0$ is called the parameter vector. The deformation $x(\gamma)$ is also called the *family* of triple of matrices. The deformation $x(\gamma)$ of x_0 is called *versal* if any deformation $z(\xi)$ of x_0, where $\xi = (\xi_1, \ldots, \xi_k) \in \mathcal{U}'_0 \subset C^k$ is the parameter vector, can be represented in some neighborhood of the origin in the following form

$$z(\xi) = g(\xi) \circ x(\phi(\xi)), \quad \xi \in \mathcal{U}''_0 \subset \mathcal{U}'_0, \quad (10)$$

where $\phi : \mathcal{U}''_0 \longrightarrow F^\ell$ and $g : \mathcal{U}''_0 \longrightarrow \mathcal{G}$ are differentiable mappings such that $\phi(0) = 0$ and $g(0) = e$. The versal deformation with minimal possible number of parameters ℓ is called *miniversal.*

The following result, proved by Arnold [1] for $\mathrm{Gl}(n;C)$ acting on $M_{n\times n}(C)$, provides the relation between the versal deformation of x_0 and the local structure of the orbit and stabilizer of x_0.

Theorem 3 *i) A deformation $x(\gamma)$ of x_0 is versal if and only if it is transversal to the orbit $\mathcal{O}(x_0)$ at x_0.*
ii) Minimal number of parameters of a versal deformation is equal to the codimension of the orbit of x_0 in $\mathcal{M}$, $\ell = \operatorname{codim} \mathcal{O}(x_0)$.
iii) If $x(\gamma)$ is a miniversal deformation and values of the mapping $g(\xi)$ are restricted to belong to a smooth submanifold $\mathcal{R} \subset \mathcal{G}$, which is transversal to $\mathcal{S}(x_0)$ at e and has the minimal dimension $\dim \mathcal{R} = \operatorname{codim} \mathcal{S}(x_0)$, then the mappings $\phi(\xi)$ and $g(\xi)$ in representation (17) are uniquely determined by $z(\xi)$.

Let us denote by $\{t_1, \ldots, t_d\}$, $d = \dim T_{x_0}\mathcal{O}(x_0)$, a basis of the tangent space $T_{x_0}\mathcal{O}(x_0)$; by $\{n_1, \ldots, n_\ell\}$, $\ell = \operatorname{codim} T_{x_0}\mathcal{O}(x_0)$, a basis the normal complement $(T_{x_0}\mathcal{O}(x_0))^\perp$; by $\{c_1, \ldots, c_\ell\}$ a basis of an arbitrary complementary subspace

$(T_{x_0}\mathcal{O}(x_0))^c$ to $T_{x_0}\mathcal{O}(x_0)$; and by $\{r_1, \dots, r_d\}$ a basis of $(T_e\mathcal{S}(x_0))^{\perp}$. By Corollary (2.1.1), if we have the basis $\{t_1, \dots, t_d\}$, then the basis $\{r_1, \dots, r_d\}$ can be chosen in the form $\{d\alpha^*_{x_0}(t_1), \dots, d\alpha^*_{x_0}(t_d)\}$, and vice versa, if the basis $\{r_1, \dots, r_d\}$ is known, then we can choose the basis $\{t_1, \dots, t_d\}$ in the form $\{d\alpha_{x_0}(r_1), \dots, d\alpha_{x_0}(r_d)\}$.

Corollary 2 *The deformation*

$$x(\gamma) = x_0 + \sum_{i=1}^{\ell} c_i \gamma_i \tag{11}$$

is a miniversal deformation. The functions $\phi(\xi)$ and $g(\xi)$ in the versal deformation reduction (16) are uniquely determined, if the mapping $g(\xi)$ is taken in the form

$$g(\xi) = e + \sum_{j=1}^{d} r_j \mu_j(\xi), \tag{12}$$

where $\mu_j(\xi)$ are smooth functions in C such that $\mu_j(0) = 0$, $j = 1, \dots, d$.

If we take $c_i = n_i$, $i = 1, \dots, \ell$, in (18), then the corresponding miniversal deformation is called *orthogonal*.

5.1 Explicit miniversal deformation

Solving the system defining $(T_{x_0}\mathcal{O})^{\perp}$ we deduce and explicit miniversal deformation. For that we partition the system in six independent subsystems, corresponding to the partition of the triple in the following manner

$$\left(\begin{pmatrix} E_1 & \\ & S_E \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A_1 & \\ & S_A \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B_1 & 0 \\ 0 & 0 \\ 0 & B_2 \end{pmatrix}\right),$$

(E_1, A_1, B_1) being the regular subsystem, (S_E, S_A) the completely singular part, and the matrices $X^* = \begin{pmatrix} X_1 & X_2 & X_5 \\ X_3 & X_4 & X_6 \end{pmatrix}$, $Y^* = \begin{pmatrix} Y_1 & Y_2 & Y_5 \\ Y_3 & Y_4 & Y_6 \end{pmatrix}$, $Z^* = \begin{pmatrix} Z_1 & Z_2 & Z_5 \\ Z_3 & Z_4 & Z_6 \end{pmatrix}$ corresponding to the partition of the triple.

$$\left.\begin{aligned} E_1X_1 + A_1Y_1 + B_1Z_1 &= 0 \\ X_1E_1 + Y_1A_1 &= 0 \\ X_1B_1 &= 0 \\ Y_1B_1 &= 0 \\ Z_1B_1 &= 0 \end{aligned}\right\} \quad \text{(I)}$$

$$\left.\begin{aligned} E_1X_2 + A_1Y_2 + B_1Z_2 &= 0 \\ X_2S_E + Y_2S_A &= 0 \end{aligned}\right\} \quad \text{(II)}$$

$$\left.\begin{aligned} S_EX_3 + S_AY_3 &= 0 \\ X_3E_1 + Y_3A_1 &= 0 \\ X_3B_1 &= 0 \\ Y_3B_1 &= 0 \end{aligned}\right\} \quad \text{(III)}$$

$$\left.\begin{aligned} S_EX_4 + S_AY_4 &= 0 \\ X_4S_E + Y_4S_A &= 0 \end{aligned}\right\} \quad \text{(IV)}$$

$$\left.\begin{aligned} E_1X_5 + A_1Y_5 + B_1Z_5 &= 0 \\ X_5B_2 &= 0 \\ Y_5B_2 &= 0 \\ Z_5B_2 &= 0 \end{aligned}\right\} \quad \text{(V)}$$

$$\left.\begin{aligned} S_EX_6 + S_AY_6 + B_2Z_6 &= 0 \\ X_6B_2 &= 0 \\ Y_6B_2 &= 0 \\ Z_6B_2 &= 0 \end{aligned}\right\} \quad \text{(VI)}$$

Theorem 4 *Let $(E, A, B) \in \mathcal{M}$ be a triple in its canonical reduced form. A explicit miniversal deformation is given as follows.*

The system (I) correspond to the miniversal deformation of the regularizable subsystem solved in [5] and system (IV) correspond to the miniversal deformation of a pencil containing only the singular part solved in [14].

Now, we solve systems II and III.

With respect system II, partitioning it following blocks in matrices:

$$E_1 = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & N_2 \end{pmatrix},\ A_1 = \begin{pmatrix} N_1 & & \\ & J & \\ & & I_3 \end{pmatrix},\ B_1 = \begin{pmatrix} \overline{B}_1 \\ 0 \\ 0 \end{pmatrix},$$

$$S_E = \begin{pmatrix} L_1 & \\ & L_2^t \end{pmatrix},\ S_A = \begin{pmatrix} R_1 & \\ & R_2^t \end{pmatrix},$$

and

$$X_2 = \begin{pmatrix} X_1^2 & X_2^2 \\ X_3^3 & X_4^2 \\ X_5^2 & X_6^2 \end{pmatrix},\ Y_2 = \begin{pmatrix} Y_1^2 & Y_2^2 \\ Y_3^2 & Y_4^2 \\ Y_5^2 & Y_6^2 \end{pmatrix},\ Z_2 = \begin{pmatrix} Z_1^2 & Z_2^2 \end{pmatrix},$$

and each subsystem partitioned into blocks corresponding to the partition of the matrices N, J, L, L^t into blocks of the same type, we obtain the following subsystems:

$$\left.\begin{array}{rl} X^2_{1i}+N_1Y^2_{1i}+\overline{B}_1Z^2_{1i} &= 0\\ X^2_{1i}L_1+Y^2_{1i}R_1 &= 0\end{array}\right\} \quad i)$$

$$\left.\begin{array}{rl} X^2_{2i}+N_1Y^2_{2i}+\overline{B}_1Z^2_{2i} &= 0\\ X^2_{2i}L^t_1+Y^2_{2i}R^t_1 &= 0\end{array}\right\} \quad ii)$$

$$\left.\begin{array}{rl} X^2_{3i}+JY^2_{3i} &= 0\\ X^2_{3i}L_1+Y^2_{3i}R_1 &= 0\end{array}\right\} \quad iii)$$

$$\left.\begin{array}{rl} X^2_{4i}+JY^2_{4i} &= 0\\ X^2_{4i}L^t_1+Y^2_{4i}R^t_1 &= 0\end{array}\right\} \quad iv)$$

$$\left.\begin{array}{rl} N_2X^2_{5i}+Y^2_{5i} &= 0\\ X^2_{5i}L_1+Y^2_{5i}R_1 &= 0\end{array}\right\} \quad v)$$

$$\left.\begin{array}{rl} N_2X^2_{6i}+Y^2_{6i} &= 0\\ X^2_{6i}L^t_1+Y^2_{6i}R^t_1 &= 0\end{array}\right\} \quad vi)$$

where matrices $L_{1i}, R_{1i} \in M_{q_{1i}\times(q_{1i}+1)}(\mathbb{C})$, $N_{1i} \in M_{p_{1i}}(\mathbb{C})$, $J = aI + N \in M_{\ell_i}(\mathbb{C})$.

Systems iii), v) have zero solution.

The solutions of systems i) are:

$X^2_{1i} = 0$, $Z^2_{1i} = 0$ and $Y^2_{1i} = 0$ if $p_{1i} \geq q_{1i}+1$ and

$$Y^2_{1i} = \begin{pmatrix} y_1 & y_2 & \dots & y_r & 0 & \dots & & \dots & 0\\ 0 & y_1 & \ddots & & y_r & \ddots & & & 0\\ \vdots & \vdots & \ddots & & & & \ddots & & \vdots\\ 0 & 0 & & y_1 & \dots & & y_r & 0\dots & 0\end{pmatrix}$$

$$Z^2_{ii} = \begin{pmatrix} 0 & 0 & 0 & y_1 & \dots & y_r & 0 & \dots & 0\end{pmatrix}$$

with $r = q_{1i} - p_{1i}$.

The solutions of systems ii) are:

$$X^2_{2i} = -\begin{pmatrix} y_2 & y_3 & \dots & y_q\\ y_3 & y_4 & \dots & y_{q+1}\\ & \iddots & \iddots & \\ y_q & y_{q+1} & \dots & z_q\end{pmatrix}$$

$$Y^2_{2i} = \begin{pmatrix} y_1 & y_2 & \dots & y_{q+1}\\ y_2 & y_3 & \dots & y_{q+2}\\ & \iddots & \iddots & \\ y_p & y_{p+1} & \dots & y_{p+q}\end{pmatrix}$$

$$Z^2_{2i} = \begin{pmatrix} y_{p+1} & y_{p+2} & \dots & z_q\end{pmatrix}$$

The solutions of systems iv) are

$$Y^2_{4i}=\begin{pmatrix} y_1 & ay_1+y_2 & a^3y_1+3a^2y_2+3ay_3+y_4 & \dots & \\ y_2 & ay_2+y_3 & a^3y_2+3a^2y_3+3ay_4+y_5 & \dots & \\ \vdots & & & & \\ y_{p-1} & ay_{p-1}+y_p & a^3y_{p-1}+3a^2y_p & \dots & \\ y_p & ay_p & a^3y_p & \dots & a^{q-1}y_p\end{pmatrix}$$

and $X^2_{4i} = -JY^2_{4i}$ (a is the eigenvalue of the block J.

The solutions of systems vi) are

$$X^2_{6i} = \begin{pmatrix} 0 & \dots & 0 & x_p & x_{p-1} & \dots & x_1\\ & & & & \ddots & \ddots & \vdots\\ & & & & & x_p & x_{p-1}\\ 0 & \dots & 0 & 0 & \dots & 0 & x_p\end{pmatrix},$$

$$Y^2_{6i} = \begin{pmatrix} 0 & \dots & 0 & 0 & -x_p & \dots & -x_2\\ & & & & \ddots & \ddots & \vdots\\ & & & & & & -x_p\\ 0 & \dots & 0 & 0 & \dots & 0 & 0\end{pmatrix},$$

if $p_{2i} \leq q_{2i}$ and

$$X^2_{6i} = \begin{pmatrix} x_{p-q+1} & \dots & x_p\\ \vdots & & \vdots\\ x_1 & \dots & x_q\\ & \ddots & \vdots\\ 0 & \dots & x_1\end{pmatrix},$$

$$Y^2_{6i} = \begin{pmatrix} x_{p-q} & \dots & x_{p-1}\\ \vdots & & \vdots\\ x_1 & \dots & x_q\\ & \ddots & \vdots\\ 0 & \dots & x_1\\ 0 & \dots & 0\end{pmatrix},$$

otherwise.

With respect system III, as in the case of systems II, partitioning it following blocks in matrices:

$E_1 = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & N_2\end{pmatrix}$, $A_1 = \begin{pmatrix} N_1 & & \\ & J & \\ & & I_3\end{pmatrix}$, $B_1 = \begin{pmatrix} \overline{B}_1\\ 0\\ 0\end{pmatrix}$,

$S_E = \begin{pmatrix} L_1 & \\ & L^t_2\end{pmatrix}$, $S_A = \begin{pmatrix} R_1 & \\ & R^t_2\end{pmatrix}$,

and

$X_3 = \begin{pmatrix} X^3_1 & X^3_2 & X^3_3\\ X^3_4 & X^3_5 & X^3_6\end{pmatrix}$, $Y_3 = \begin{pmatrix} Y^3_1 & Y^3_2 & Y^3_3\\ Y^3_4 & Y^3_5 & Y^3_6\end{pmatrix}$,

and each subsystem partitioned into blocks corresponding to the partition of the matrices N, J, L, L^t into blocks of the same type, we obtain the following subsystems:

$$\left.\begin{array}{rl} L_{1i}X^3_{1i}+R_{1i}Y^3_{1i} &= 0\\ X^3_{1i}+Y^3_1N_{1i} &= 0\\ X^3_{1i}\overline{B}_1 &= 0\\ Y^3_{1i}\overline{B}_1 &= 0\end{array}\right\} \quad i)$$

$$\left.\begin{array}{rl} L_{1i}X^3_{2i}+R_{1i}Y^3_{2i} &= 0\\ X^3_{2i}+Y^3_1J_i &= 0\end{array}\right\} \quad ii)$$

$$\left.\begin{array}{rl} L_{1i}X^3_{3i}+R_{1i}Y^3_{3i} &= 0\\ X^3_{2i}N_{2i}+Y^3_{1i} &= 0\end{array}\right\} \quad iii)$$

$$\left.\begin{array}{rl} L^t_{2i}X^3_{4i}+R^t_{2i}Y^3_{4i} &= 0\\ X^3_{3i}+Y^3_4N_{1i} &= 0\\ X^3_{4i}\overline{B}_1 &= 0\\ Y^3_{4i}\overline{B}_1 &= 0\end{array}\right\} \quad iv)$$

$$\left.\begin{array}{rl} L^t_{2i}X^3_{5i}+R^t_{3i}Y^3_{5i} &= 0\\ X^3_{5i}+Y^3_{5i}J_i &= 0\end{array}\right\} \quad v)$$

$$\left.\begin{array}{rl} L^t_{2i}X^3_{6i}+R^t_{2i}Y^3_{6i} &= 0\\ X^3_{6i}N_{2i}+Y^3_{6i} &= 0\end{array}\right\} \quad vi)$$

where matrices $L_{1i}, R_{1i} \in M_{q_{1i}\times(q_{1i}+1)}(\mathbb{C})$, $N_{1i} \in M_{p_{1i}}(\mathbb{C})$, $J = aI + N \in M_{\ell_i}(\mathbb{C})$.

The solution of systems i) are:

$X^3_{1i} = 0$ and $Y^3_{1i} = 0$ if $p_{1i} \leq q_{1i} + 1$ and

$$Y^3_{1i} = \begin{pmatrix} y_1 & y_2 & \dots & y_r & 0 & \dots & & \dots & 0\\ 0 & y_1 & \ddots & & y_r & \ddots & & & 0\\ \vdots & \vdots & \ddots & & & \ddots & & & \vdots\\ 0 & 0 & & y_1 & \dots & & y_r & 0 \dots & 0\end{pmatrix}$$

with $r = p_{1i} - q_{1i}$.

The solution of systems ii) are

$$Y^3_{2i} = \begin{pmatrix} y_1 & y_2 & \dots & y_{\ell_i}\\ ay_1 & y_1+ay_2 & \dots & y_{\ell_i-1}+ay_{\ell_i}\\ a^2y_1 & 2ay_1+a^2y_2 & & \\ \vdots & & & \\ a^{q_i}y_1 & & & \end{pmatrix}$$

$(y_{ij} = y_{i-1\,j-1} + ay_{i-1\,j})$.

The solution of systems iii) are

$$X^3_{3i} = \begin{pmatrix} 0 & \dots & x_1 & \dots & x_{p_i-q_1+1}\\ \vdots & \iddots & \vdots & \iddots & \vdots\\ x_1 & \dots & x_{q_i+1} & \dots & x_{p_i}\end{pmatrix},$$

if $p_i \leq q_i$,

$$X^3_{3i} = \begin{pmatrix} 0 & \dots & 0\\ \vdots & & \vdots\\ 0 & \dots & 0\\ 0 & \dots & x_1\\ \vdots & \iddots & \vdots\\ x_1 & \dots & x_{p_i}\end{pmatrix}, \text{if } p_i < q_i.$$

The solution for systems iv), v) and vi) is $X = Y = 0$.

Analogously, we solve systems (V) and (VI).

Corollary 3 *The codimension of a triple $(E, A, B) \in \mathcal{M}$, is the number of parameters appearing in the miniversal deformation.*

6 Structural stability

As application of miniversal deformations we are going to analyze the structural stability of the triples. First of all we will recall the definition of structural stability, according to that appearing in the paper by Willems (see [15]).

Let X be a topological space and consider an equivalence relation defined on it.

Definition 3 *An element $x \in X$ is structurally stable if and only if there exists an open neighbourhood $U \subset X$ of x such that all the elements in it are equivalent to x.*

Applying this result a our particular setup we have the following Proposition.

Proposition 4 *A triple $(E, A, B) \in \mathcal{M}$ is structurally stable if and only if the miniversal deformation of the triple is zero.*

As a consequence we have the following Theorem.

Theorem 5 *A triple $(E, A, B) \in \mathcal{M}$ is structurally stable*

1. *for $m \geq n, p$ or $n \geq m > p$, if and only if $\operatorname{rank} B = p$.*

2. *for $n = p - m$ there are not stable triples.*

3. *for $n > p - m$, if and only if there are m blocks L_1, $\ell - s$ blocks L_{ℓ_1} and s L_{ℓ_1+1} where $n = (p - m)c + \ell$, and $p - m = \ell\ell_1 + s$*

4. *for $n < p - m$, if and only if there are m blocks L_1, $\ell - s$ blocks $L^t_{\ell_1}$ and s $L^t_{\ell_1+1}$ where $p - m = nc + \ell$, and $n = \ell\ell_1 + s$*

5. *for $n = p$, $m > 0$, if and only if there are not continuous invariants, nor infinite zeroes, and row minimal indices, rank$B = r = min\{p, m\}$, there are r column minimal indices of order 1, and r column minimal indices equals or differing in only one unity.*

Proof.

A system is structurally stable if and only if, the miniversal deformation is zero, so the system defining $T_{x_0}\mathcal{O}^\perp$ has only the zero solution. So, it suffices to analyze when all the systems (I), (II), (III), (IV), (V), (VI) have only the zero solution.

□

Examples

1. Let (E, A, B) be a triple in $\mathcal{M}$ with $E = I_2$, $A = N \in M_2(\mathbb{C})$ and $B = e_2^t$. In this case we have that $n = p > m$ and there are two blocks L_1 verifying condition 5 in theorem 4, (notice that the associate pencil is equivalent to $\lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$). So the triple is structurally stable.

2. Let (E, A, B) be a triple in $\mathcal{M}$ with $E = I_2$, $A = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $B = e_2^t$. In this case we have that $n = p > m$ and there is one block L_1, and continuous invariant $\lambda = 1$, it does not verify condition 5 in theorem 4, (notice that the associate pencil is equivalent to $\lambda \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$). So the triple is not structurally stable.

7 Conclusion

The knowledge of a canonical reduced form, permit us to deduce explicit miniversal deformations for triples of matrices under feedback and derivative feedback equivalence. Then the structural stability of triples of matrices can be easily analyzed.

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