Some Applications of Orthogonal Polynomials of a Discrete Variable to Graphs and Codes *

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Abstract

We consider some related families of orthogonal polynomials of a discrete variable, and discuss their applications in the study of (distance-regular) graphs and codes. One of the main peculiarities of such orthogonal systems is their non-standard normalization condition, requiring that the square norm of each polynomial must equal its value at a given point of the mesh.

1 On orthogonal polynomials of a discrete variable

In this section we survey some old and some news results about polynomials of a discrete variable. In order to do this paper more accesible to readeres not familiarized with this topic, we have included all the proofs.

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Let $\mathcal{M} := \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$ be a mesh of real numbers. A real function of a discrete variable $f : \mathcal{M} \longrightarrow \mathbb{R}$ can be seen as the restriction on $\mathcal{M}$ of a number of functions of real variable. Moreover, if we only consider polynomial functions, the class of possible extensions of one discrete function on $\mathcal{M}$ constitute an element of the quotient algebra $\mathbb{R}[x]/(Z)$ where (Z) is the ideal generated by the polynomial $Z := \prod_{l=0}^{d}(x - \lambda_l)$. Each class has a unique canonical representative of degree at most d. Denoting by $\mathcal{F}(\mathcal{M})$ the set of functions on the mesh, we then have the following natural identifications:

$$\mathcal{F}(\mathcal{M}) \longleftrightarrow \mathbb{R}[x]/(Z) \longleftrightarrow \mathbb{R}_d[x]. \tag{1}$$

For simplicity, we represent by the same symbol, say p, any of the three objects identified in (1). When we need to point out some of the above three sets, we will make it explicit.

A positive function $g : \mathcal{M} \longrightarrow \mathbb{R}$ will be called a *weight function* on $\mathcal{M}$. We say that it is *normalized* when $g(\lambda_0) + g(\lambda_1) + \cdots + g(\lambda_d) = 1$. We shall write, for short, $g_l := g(\lambda_l)$. From the pair $(\mathcal{M}, g)$ we can define an inner product in $\mathbb{R}_d[x]$ (indistinctly in $\mathcal{F}(\mathcal{M})$ or in $\mathbb{R}[x]/(Z)$) as

$$\langle p, q \rangle := \sum_{l=0}^{d} g_l p(\lambda_l) q(\lambda_l) \qquad p, q \in \mathbb{R}_d[x], \tag{2}$$

with corresponding norm $\|\cdot\|$. From now on, this will be referred to as *the scalar product associated to* $(\mathcal{M}, g)$. Note that $\langle 1, 1 \rangle = 1$ is equivalent to the normalized character of the weight function g, which will be hereafter assumed.

In order to simplify some expressions, it is useful to introduce the following moment-like parameters, computed from the points of the mesh $\mathcal{M}$,

$$\pi_k := \prod_{l=0\,(l\neq k)}^{d} |\lambda_k - \lambda_l| = (-1)^k \prod_{l=0\,(l\neq k)}^{d} (\lambda_k - \lambda_l) \qquad (0 \leq k \leq d); \tag{3}$$

and the family of interpolating polynomials (with degree d):

$$Z_k := \frac{(-1)^k}{\pi_k} \prod_{l=0\,(l\neq k)}^{d} (x - \lambda_l) \qquad (0 \leq k \leq d)\,, \tag{4}$$

which satisfy:

$$Z_k(\lambda_h) = \delta_{hk}\ , \qquad \langle Z_h, Z_k \rangle = \delta_{hk} g_k\,. \tag{5}$$

Then, using Lagrange interpolation, $p = \sum_{k=0}^{d} p(\lambda_k) Z_k$ for any $p \in \mathbb{R}_d[x]$. In particular, when $p = x^i$, $i = 0, 1, \ldots, d$, we get $x^i = \sum_{k=0}^{d} \lambda_k^i Z_k$ whence, equating the terms with degree d,

$$\sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} \lambda_k^i = 0 \quad (0 \le i \le d-1)\,, \qquad \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} \lambda_k^d = 1\,. \tag{6}$$

1.1 Forms and orthogonal systems

Each real number λ induces a linear form on $\mathbb{R}_d[x]$, defined by $[\lambda](p) := p(\lambda)$. Then, equality (5) can be interpreted by saying that the forms $[\lambda_0]$, $[\lambda_1]$, $\ldots$, $[\lambda_d]$ are the dual basis of the polynomials Z_0, Z_1, $\ldots$, Z_d. The scalar product associated to $(\mathcal{M}, g)$ induces an isomorphism between the space $\mathbb{R}_d[x]$ and its dual, where each polynomial p corresponds to the form ω_p, defined as $\omega_p(q) := \langle p, q\rangle$ and, reciprocally, each form ω is associated to a polynomial p_ω through $\langle p_\omega, q\rangle = \omega(q)$. By observing how the isomorphism acts on the bases $\{[\lambda_l]\}_{0\le l\le d}$, $\{Z_l\}_{0\le l\le d}$, we get the expressions:

$$\omega_p = \sum_{l=0}^{d} g_l p(\lambda_l)[\lambda_l]\,, \qquad p_\omega = \sum_{l=0}^{d} \frac{1}{g_l}\omega(Z_l) Z_l \tag{7}$$

In particular, the polynomial corresponding to $[\lambda_k]$ is

$$\begin{aligned} H_k \;&:=\; p_{[\lambda_k]} = \sum_{l=0}^{d} \frac{1}{g_l}[\lambda_k](Z_l) Z_l = \sum_{l=0}^{d} \frac{1}{g_l}\delta_{lk} Z_l \\ &=\; \frac{1}{g_k} Z_k = \frac{(-1)^k}{g_k \pi_k}(x-\lambda_0)\cdots\widehat{(x-\lambda_k)}\cdots(x-\lambda_d)\,, \end{aligned}$$

and their scalar products are:

$$\langle H_h, H_k\rangle = [\lambda_h](H_k) = H_k(\lambda_h) = \frac{1}{g_h}\delta_{hk} \tag{8}$$

Moreover, property (6) is equivalent to stating that the form $\sum_{k=0}^{d} \frac{(-1)^k}{\pi_k}[\lambda_k]$ annihilates on the space $\mathbb{R}_{d-1}[x]$.

Lemma 1.1 *In the space $\mathbb{R}_d[x]$, let us consider the scalar product associated to $(\mathcal{M}, g)$. Then, the polynomial*

$$T := \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} H_k = \sum_{k=0}^{d} \frac{1}{g_k \pi_k^2}(x-\lambda_0)\cdots\widehat{(x-\lambda_k)}\cdots(x-\lambda_d)$$

verify the following.

(*a*) *T is orthogonal to $\mathbb{R}_{d-1}[x]$;*

(*b*) $\|T\|^2 = \sum_{k=0}^{d} \frac{1}{g_k \pi_k^2}$;

(*c*) $T(\lambda_0) = \frac{1}{g_0 \pi_0}$.

Proof. The proof of (*a*) is straightforward by considering the form ω_T associated to T, whereas (b) and (c) are proved by simple computations:

$\|T\|^2 = \sum_{h,k=0}^{d} \frac{(-1)^{h+k}}{\pi_h \pi_k} \langle H_h, H_k \rangle = \sum_{k=0}^{d} \frac{1}{g_k \pi_k^2}$;

$T(\lambda_0) = \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} H_k(\lambda_0) = \frac{1}{g_0 \pi_0}$. $\square$

A family of polynomials $r_0, r_1, \ldots, r_d$ is said to be an *orthogonal system* when each polynomial r_k is of degree k and $\langle r_h, r_k \rangle = 0$ for any $h \neq k$.

Proposition 1.2 *Every orthogonal system $r_0, r_1, \ldots, r_d$ satisfies the following properties:*

(*a*) *There exists a tridiagonal matrix $\boldsymbol{R}$ (called the recurrence matrix of the system) such that, in $\mathbb{R}[x]/(Z)$:*

$$x\boldsymbol{r} := x \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \begin{pmatrix} a_0 & c_1 & 0 & & & & \\ b_0 & a_1 & c_2 & 0 & & & \\ 0 & b_1 & a_2 & \ddots & \ddots & & \\ & 0 & \ddots & \ddots & \ddots & 0 & \\ & & \ddots & \ddots & a_{d-2} & c_{d-1} & 0 \\ & & & 0 & b_{d-2} & a_{d-1} & c_d \\ & & & & 0 & b_{d-1} & a_d \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ \vdots \\ r_{d-2} \\ r_{d-1} \\ r_d \end{pmatrix} = \boldsymbol{R}\boldsymbol{r} \,,$$

and this equality, in $\mathbb{R}[x]$, reads:

$$x\boldsymbol{r} = \boldsymbol{R}\boldsymbol{r} + \begin{pmatrix} 0 & 0 & \cdots & 0 & \frac{\|r_d\|^2}{r_d(\lambda_0)} \frac{1}{g_0 \pi_0} Z \end{pmatrix}^{\top}$$

(*b*) *All the entries b_k, c_k, of matrix $\boldsymbol{R}$ are nonnull and satisfy $b_k c_{k+1} > 0$.*

(*c*) *The matrix $\boldsymbol{R}$ diagonalizes with eigenvalues the elements of $\mathcal{M}$. An eigenvector associated to λ_k is $(r_0(\lambda_k), r_1(\lambda_k), \ldots, r_{d-1}(\lambda_k), r_d(\lambda_k))^{\top}$.*

(*d*) *For every* $k = 1, \ldots, d$ *the polynomial* r_k *has real simple roots. If* $\mathcal{M}_k$ *denotes the mesh of the ordered roots of* r_k*, then (the points of) the mesh* $\mathcal{M}_d$ *interlaces* $\mathcal{M}$ *and, for each* $k = 1, 2, \ldots, d-1$*,* $\mathcal{M}_k$ *interlaces* $\mathcal{M}_{k+1}$*.*

Proof. (*a*) Working in $\mathbb{R}[x]/(Z)$, we have $\langle xr_k, r_h\rangle = 0$ provided that $k < h-1$ and, by symmetry, the result is also zero when $h < k-1$. Therefore we can write, for any $k = 0, 1, \ldots, d$,

$$xr_k = \sum_{h=0}^{d} \frac{\langle xr_k, r_h\rangle}{\|r_h\|^2} r_h = \sum_{h=\max\{0,k-1\}}^{\min\{k+1,d\}} \frac{\langle xr_k, r_h\rangle}{\|r_h\|^2} r_h = b_{k-1}r_{k-1} + a_k r_k + c_{k+1} r_{k+1}\,,$$

where, in order to uniform the notation, we have introduced, the null formal terms $b_{-1}r_{-1}$, and $c_{d+1}r_{d+1}$. Then, for any $k = 0, 1, \ldots, d$ the parameters b_k, a_k, c_k are defined by:

$$\begin{aligned} b_k &= \frac{\langle xr_{k+1}, r_k\rangle}{\|r_k\|^2} \quad (0 \le k \le d-1)\,, \qquad b_d = 0\,, \\ a_k &= \frac{\langle xr_k, r_k\rangle}{\|r_k\|^2} \quad (0 \le k \le d)\,, \\ c_0 &= 0\,, \qquad c_k = \frac{\langle xr_{k-1}, r_k\rangle}{\|r_k\|^2} \quad (1 \le k \le d)\,. \end{aligned}$$

Given any $k = 0, 1, \ldots, d$ let $Z_k := \prod_{l=0, l\ne k}^{d}(x - \lambda_l) = \xi_0 r_d + \xi_1 r_{d-1} + \cdots$, where we note that ξ_0 does not depend on k. Thus,

$$\langle r_d, Z_k\rangle = g_k r_d(\lambda_k)(-1)^k \pi_k = \xi_0 \|r_d\|^2 = g_0 r_d(\lambda_0)\pi_0 \ne 0 \tag{9}$$

Moreover, for any $k = 0, 1, \ldots, d$, we get:

$$r_d - \frac{\|r_d\|^2}{r_d(\lambda_0)} \frac{1}{g_0\pi_0} Z_k \in \mathbb{R}_{d-1}[x]\,, \qquad \frac{r_d(\lambda_k)}{r_d(\lambda_0)} = (-1)^k \frac{g_0\pi_0}{g_k\pi_k} \tag{10}$$

Then, the equality $xr_d = b_{d-1}p_{d-1} + a_d r_d$, holding in $\mathbb{R}[x]/(Z)$, and the comparison of the degrees allows us to stablish the existence of $\psi \in \mathbb{R}$ such that $xr_d = b_{d-1}p_{d-1} + a_d r_d + \psi Z$ en $\mathbb{R}[x]$. Noticing that ψ is the first coefficient of r_d, we get, from (10), $\psi = \frac{\|r_d\|^2}{r_d(\lambda_0)} \frac{1}{g_0\pi_0}$.

(*b*) By looking again to the degrees, we realize that $c_1, c_2, \ldots, c_d$ are nonzero. For $k = 0, 1, \ldots, d-1$, we have, from the equality

$$b_k = \frac{\langle xr_{k+1}, r_k\rangle}{\|r_k\|^2} = \frac{\langle xr_k, r_{k+1}\rangle}{\|r_{k+1}\|^2} \frac{\|r_{k+1}\|^2}{\|r_k\|^2} = \frac{\|r_{k+1}\|^2}{\|r_k\|^2} c_{k+1}\,,$$

that the parameters $b_0, b_1, \ldots, b_{d-1}$ are also nonnull and, moreover, $b_k c_{k+1} > 0$ for any $k = 0, 1, \ldots, d-1$.

(c) This result follows immediately if we evaluate, in $\mathbb{R}[x]$ and for each λ_k, the matrix equation obtained in (a).

(d) From (10) we observe that r_d takes alternating signs on the points of $\mathcal{M}$. Hence, this polynomial has d simple roots whose mesh $\mathcal{M}_d$ interlaces $\mathcal{M}$. Noticing that Z takes alternating signs over the elements of $\mathcal{M}_d$, from the equality $b_{d-1}r_{d-1} = (x - a_d)r_d - \psi Z$ it turns out that r_{d-1} takes alternating signs on the elements of $\mathcal{M}_d$; whence $\mathcal{M}_{d-1}$ interlaces $\mathcal{M}_d$ and r_d has alternating signs on $\mathcal{M}_{d-1}$. Recursively, suppose that, for $k = 1, \ldots, d-2$, the polynomials r_{k+1} and r_{k+2} have simple real roots and that $\mathcal{M}_{k+1}$ interlaces $\mathcal{M}_{k+2}$, so that r_{k+2} takes alternating signs on $\mathcal{M}_{k+1}$. Then, the result follows by just evaluating the equality $b_k r_k = (x - a_{k+1})r_{k+1} - c_{k+2}r_{k+2}$ at the points of $\mathcal{M}_{k+1}$. $\square$

1.2 The canonical orthogonal system

Consider the space $\mathbb{R}_d[x]$ with the scalar product associated to $(\mathcal{M}, g)$. From the identification of such a space with its dual by contraction of the scalar product, the form $[\lambda_0] : p \to p(\lambda_0)$ is represented by the polynomial $H_0 = \frac{1}{g_0 \pi_0}(x - \lambda_1) \cdots (x - \lambda_d)$ through $\langle H_0, p \rangle = p(\lambda_0)$.

For any given $0 \leq k \leq d-1$, let $q_k \in R_k[x]$ denote the orthogonal projection of H_0 over $R_k[x]$. Alternatively, the polynomial q_k can be defined as the unique polynomial in $\mathbb{R}_k[x]$ such that

$$\|H_0 - q_k\| = \min\{\|H_0 - q\| : q \in \mathbb{R}_k[x]\}.$$

(See Fig. 1.) Let $\mathcal{S}$ denote the sphere in $\mathbb{R}_d[x]$ such that 0 and H_0 are antipodal points on it; that is, the sphere with center $\frac{1}{2}H_0$ and radius $\frac{1}{2}\|H_0\|$. Notice that its equation $\|p - \frac{1}{2}H_0\|^2 = \frac{1}{4}\|H_0\|^2$ can also be written as $\|p\|^2 = \langle H_0, p \rangle = p(\lambda_0)$. Consequently,

$$\mathcal{S} = \{p \in \mathbb{R}_d[x] : \|p\|^2 = p(\lambda_0)\} = \{p \in \mathbb{R}_d[x] : \langle H_0 - p, p \rangle = 0\}.$$

Note also that the projection q_k is on the sphere $\mathcal{S}_k := \mathcal{S} \cap \mathbb{R}_k[x]$ since, in particular, $\langle H_0 - q_k, q_k \rangle = 0$.

Proposition 1.3 *The polynomial q_k, which is the orthogonal projection of H_0 on $\mathbb{R}_k[x]$, can be defined as the unique polynomial of $\mathbb{R}_k[x]$ satisfying*

$$\langle H_0, q_k \rangle = q_k(\lambda_0) = \max\{q(\lambda_0) : \text{ per a tot } q \in \mathcal{S}_k\},$$

where $\mathcal{S}_k$ is the sphere $\{q \in R_k[x] : \|q\|^2 = q(\lambda_0)\}$. Equivalently, q_k is the antipodal point of the origin in $\mathcal{S}_k$.

Proof. Since q_k is orthogonal to $H_0 - q_k$, we have $\|q_k\|^2 + \|H_0 - q_k\|^2 = \|H_0\|^2 = \frac{1}{g_0}$. Then, as $q_k \in \mathcal{S}_k$ we get

$$\begin{aligned} q_k(\lambda_0) &= \|q_k\|^2 = \frac{1}{g_0} - \|H_0 - q_k\|^2 = \frac{1}{g_0} - \min\{\|H_0 - q\|^2 : \forall q \in \mathcal{S}_k\} = \\ &= \max\{\|q\|^2 : \forall q \in \mathcal{S}_k\} = \max\{q(\lambda_0) : \forall q \in \mathcal{S}_k\}. \end{aligned}$$

Considering the equivalent form $\|q_k\| = \max\{\|q\| : \forall q \in \mathcal{S}_k\}$, the proof is complete. □

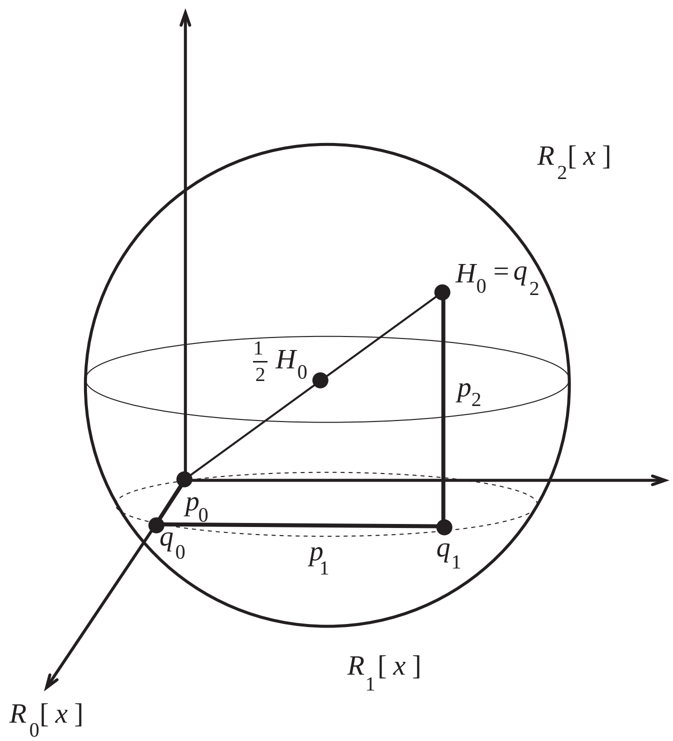

Figure 1: Obtaining the q's and the p's by projecting H_0.

With the notation $q_d := H_0$, we obtain the family of polynomials $q_0, q_1, \dots, q_{d-1}, q_d$. Let us remark some of their properties.

Corollary 1.4 *The polynomials $q_0, q_1, \dots, q_{d-1}, q_d$, satisfy the following.*

(*a*) *Each* q_k *has degree exactly* k.

(*b*) $1 = q_0(\lambda_0) < q_1(\lambda_0) < \cdots < q_{d-1}(\lambda_0) < q_d(\lambda_0) = \frac{1}{g_0}$.

(*c*) *The polynomials* $q_0, q_1, \ldots, q_{d-1}$ *constitute an orthogonal system with respect to the scalar product associated to the mesh* $\{\lambda_1 > \lambda_2 > \cdots > \lambda_d\}$ *and the weight function* $\lambda_k \mapsto (\lambda_0 - \lambda_k)g_k$, $k = 1, \ldots, d$.

Proof. (*a*) Notice that $\mathcal{S}_0 = \{0, 1\}$. Consequently, $q_0 = 1$. Assume that q_{k-1} has degree $k-1$, but q_k has degree lesser than k. Because of the uniqueness of the projection, $q_k = q_{k-1}$ and $H_0 - q_{k-1}$ would be orthogonal to $\mathbb{R}_k[x]$. In particular,

$$\begin{aligned} 0 &= \langle H_0 - q_{k-1}, (x - \lambda_0)q_{k-1}\rangle = \langle (x-\lambda_0)H_0 - (x-\lambda_0)q_{k-1}, q_{k-1}\rangle \\ &= \langle (\lambda_0 - x)q_{k-1}, q_{k-1}\rangle = \sum_{l=0}^{d} g_l(\lambda_0 - \lambda_l)q_{k-1}^2(\lambda_l). \end{aligned}$$

Hence, $q_{k-1}(\lambda_l) = 0$ for any $1 \le l \le d$ and q_{k-1} would be null.

(*b*) If $q_{k-1}(\lambda_0) = q_k(\lambda_0)$, from Proposició 1.3 we would get $q_{k-1} = q_k$, which is not possible because of (*a*).

(*c*) Let $0 \le h < k \le d-1$. Since $H_0 - q_k$ is orthogonal to $\mathbb{R}_k[x]$ we have, in particular, that

$$\begin{aligned} 0 &= \langle H_0 - q_k, (x-\lambda_0)q_h\rangle = \langle (x-\lambda_0)H_0 - (x-\lambda_0)q_k, q_h\rangle = \langle (\lambda_0 - x)q_k, q_h\rangle = \\ &= \sum_{l=0}^{d} g_l(\lambda_0 - \lambda_l)q_k(\lambda_l)q_h(\lambda_l) = \sum_{l=1}^{d}(\lambda_0 - \lambda_l)g_l q_k(\lambda_l)q_h(\lambda_l), \end{aligned}$$

stablishing the claimed orthogonality. □

The polynomial q_k, as the orthogonal projection of H_0 over $\mathbb{R}_k[x]$, can also be seen as the orthogonal projection of q_{k+1} over $\mathbb{R}_k[x]$, as $q_{k+1} - q_k = H_0 - q_k - (H_0 - q_{k+1})$ is orthogonal to $\mathbb{R}_k[x]$. Consider the family of polynomials defined as

$$\begin{aligned} p_0 := q_0 = 1,\ p_1 := q_1 - q_0,\ p_2 := q_2 - q_1, \ldots, \\ p_{d-1} := q_{d-1} - q_{d-2},\ p_d := q_d - q_{d-1} = H_0 - q_{d-1} \end{aligned} \tag{11}$$

Note that, then, $q_k = p_0 + p_1 + \cdots + p_k$ $(0 \le k \le d)$, and, in particular, $p_0 + p_1 + \cdots + p_d = H_0$. Let us now begin the study of the polynomials $(p_k)_{0 \le k \le d}$.

Proposition 1.5 *The polynomials* $p_0, p_1, \ldots, p_{d-1}, p_d$ *constitute an orthogonal system with respect to the scalar product associated to* $(\mathcal{M}, g)$.

Proof. From $p_k = q_k - q_{k-1}$ we see that p_k has degree k. Moreover, we have already seen that $p_k = q_k - q_{k-1}$ is orthogonal to $\mathbb{R}_{k-1}[x]$, whence the polynomials p_k form an orthogonal system. $\square$

The sequence of polynomials $(p_k)_{0 \leq k \leq d}$, defined in (11), will be called the *canonical orthogonal system* associated to $(\mathcal{M}, g)$. The next result gives three different characterizations of such systems.

Proposition 1.6 *Let $r_0, r_1, \ldots, r_{d-1}, r_d$ an orthogonal system with respect to the scalar product associated to $(\mathcal{M}, g)$. Then the following assertions are all equivalent.*

(a) $(r_k)_{0 \leq k \leq d}$ is the canonical orthogonal system associated to $(\mathcal{M}, g)$;

(b) $r_0 = 1$ and the entries of the recurrence matrix $\boldsymbol{R}$ associated to $(r_k)_{0 \leq k \leq d}$, satisfy $a_k + b_k + c_k = \lambda_0$, for any $k = 0, 1, \ldots, d$;

(c) $r_0 + r_1 + \cdots + r_d = H_0$;

(d) $\|r_k\|^2 = r_k(\lambda_0)$ for any $k = 0, 1, \ldots, d$.

Proof. Let $(p_k)_{0 \leq k \leq d}$ be the canonical orthogonal system el sistema ortogonal associated to $(\mathcal{M}, g)$. The espace $\mathbb{R}_k[x] \cap \mathbb{R}_{k-1}^{\perp}[x]$ has dimension one, and hence the polynomials r_k, p_k are proporcional: $r_k = \xi_k p_k$. Let $\boldsymbol{j} := \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\top$.

$(a) \Rightarrow (b)$: We have $r_0 = p_0 = 1$. Consider the recurrence matrix $\boldsymbol{R}$, Proposition 1.2, associated to the canonical orthogonal system $(r_k)_{0 \leq k \leq d} = (p_k)_{0 \leq k \leq d}$. Then, computing xq_d en $\mathbb{R}[x]/(Z)$ in two different ways we get:

$$
\begin{aligned}
xq_d &= x \sum_{k=0}^{d} p_k = x\boldsymbol{j}^\top \boldsymbol{p} = \boldsymbol{j}^\top \boldsymbol{R}\boldsymbol{p} = \begin{pmatrix} a_0 + b_0 & c_1 + a_1 + b_1 & \cdots & c_d + a_d \end{pmatrix}^\top \boldsymbol{p} = \\
&= \sum_{k=0}^{d} (a_k + b_k + c_k) p_k\,; \\
xq_d &= xH_0 = \lambda_0 H_0 = \sum_{k=0}^{d} \lambda_0 p_k\,,
\end{aligned}
$$

and, from the linear independence of the polynomials p_k, we get $a_k + b_k + c_k = \lambda_0$.

$(b) \Rightarrow (c)$: Working in $\mathbb{R}[x]/(Z)$ and from $x\boldsymbol{r} = \boldsymbol{R}\boldsymbol{r}$, we have:

$$
0 = \boldsymbol{j}^\top (x\boldsymbol{r} - \boldsymbol{R}\boldsymbol{r}) = x \sum_{k=0}^{d} r_k - \boldsymbol{j}^\top \boldsymbol{R}\boldsymbol{r} = x \sum_{k=0}^{d} r_k - \lambda_0 \boldsymbol{j}^\top \boldsymbol{r} = (x - \lambda_0) \sum_{k=0}^{d} r_k\,.
$$

Therefore there exists ξ such that $\sum_{k=0}^{d} r_k = \xi H_0 = \sum_{k=0}^{d} \xi p_k$. Since, also, $\sum_{k=0}^{d} r_k = \sum_{k=0}^{d} \xi_k p_k$, on $\xi_0 = 1$, it turns out that $\xi_0 = \xi_1 = \cdots = \xi_d = \xi = 1$. Consequently, $\sum_{k=0}^{d} r_k = H_0$.

$(c) \Rightarrow (d)$: $\|r_k\|^2 = \langle r_k, r_0 + r_1 + \cdots + r_d \rangle = \langle r_k, H_0 \rangle = r_k(\lambda_0)$.

$(d) \Rightarrow (a)$: From $r_k = \xi_k p_k$, we have $\xi_k^2 \|p_k\|^2 = \|r_k\|^2 = r_k(\lambda_0) = \xi_k p_k(\lambda_0) = \xi_k \|p_k\|^2$. Whence $\xi_k = 1$ and $r_k = p_k$. $\square$

Corollary 1.7 *The highest degree polynomial p_d of the canonical orthogonal system associatet to $(\mathcal{M}, g)$ satisfies the following:*

- $$p_d = \left(\sum_{l=0}^{d} \frac{g_0 \pi_0}{g_l \pi_l^2} \right)^{-1} \sum_{k=0}^{d} \frac{1}{g_k \pi_k^2} (x - \lambda_0) \cdots \widehat{(x - \lambda_k)} \cdots (x - \lambda_d)\,;$$
- $$p_d(\lambda_0) = \frac{1}{g_0} \left(\sum_{l=0}^{d} \frac{g_0 \pi_0^2}{g_l \pi_l^2} \right)^{-1}; \qquad p_d(\lambda_k) = (-1)^k \frac{g_0 \pi_0}{g_k \pi_k} p_d(\lambda_0) \quad (1 \le k \le d)\,.$$

Proof. Recalling that the polynomial $T = \sum_{k=0}^{d} \frac{(-1)^k}{\pi_k} H_k$, introduced in Lemma 1.1, is orthogonal to $\mathbb{R}_{d-1}[x]$, the exists a constant ξ such that $p_d = \xi T$. From $\|p_d\|^2 = p_d(\lambda_0)$, we then obtain $p_d = \frac{T(\lambda_0)}{\|T\|^2} T$. Substituting into this formula the values of $T(\lambda_0)$ and $\|T\|^2$, given also in Lemma 1.1, we obtain the claimed expressions for p_d, $p_d(\lambda_0)$ and $p_d(\lambda_k)$. $\square$

From the last equality of Corollary 1.7 we get:

$$g_k = g_0 \frac{\pi_0}{\pi_k} \frac{p_d(\lambda_0)}{(-1)^k p_d(\lambda_k)} \qquad (0 \le k \le d) \tag{12}$$

which, together with the normalization of g, implies that, given $\mathcal{M}$, the knowledge of p_d allows us to reconstruct the weight function.

1.3 The conjugate canonical orthogonal system

Consider a given mesh $\mathcal{M} = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$. As we have seen, each normalized weight function $g : \mathcal{M} \to \mathbb{R}$ induces a scalar product and its corresponding canonical orthogonal system $(p_k)_{0 \le k \le d}$. Moreover, we know that its recurrence matrix $\boldsymbol{R}$, given in Proposition 1.2, satisfies:

$$x\boldsymbol{p} = \boldsymbol{R}\boldsymbol{p}\,, \qquad \boldsymbol{j}^\top \boldsymbol{R} = \lambda_0 \boldsymbol{j}^\top \tag{13}$$

where $\boldsymbol{p}$ and $\boldsymbol{j}$ are the column matrices $\begin{pmatrix} p_0 & p_1 & \cdots & p_d \end{pmatrix}^\top$ and $\begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}^\top$, respectively.

Given an $n \times m$ matrix $\boldsymbol{A} = (a_{ij})$ we denote by $\boldsymbol{A}^\star$ the $n \times m$ matrix with (i,j)-entry $a_{n-i+1,m-j+1}$, which results when applying a central symmetry to $\boldsymbol{A}$. It is immediate to check that $(\lambda \boldsymbol{A} + \mu \boldsymbol{B})^\star = \lambda \boldsymbol{A}^\star + \mu \boldsymbol{B}^\star$ and $(\boldsymbol{AB})^\star = \boldsymbol{A}^\star \boldsymbol{B}^\star$. The square $(d+1)$-matrix $\boldsymbol{S}$ with null entries excepting those on the principal antidiagonal which are 1's, satisfies $\boldsymbol{S}^\top = \boldsymbol{S}^{-1} = \boldsymbol{S}$ and, when $\boldsymbol{A}$ is any square $(d+1)$-matrix, $\boldsymbol{A}^\star = \boldsymbol{SAS}$.

The polynomial p_d has an inverse in $\mathbb{R}[x]/(Z)$ and, therefore, we can define $\tilde{p}_k := p_d^{-1} p_{d-k}$ for any $k = 0, 1, \ldots, d$. Then, with the notation

$$\tilde{\boldsymbol{p}} := \begin{pmatrix} \tilde{p}_0 & \tilde{p}_1 & \cdots & \tilde{p}_d \end{pmatrix}^\top = p_d^{-1} \boldsymbol{p}^\star = \begin{pmatrix} p_d^{-1} p_d & p_d^{-1} p_{d-1} & \cdots & p_d^{-1} p_0 \end{pmatrix}^\top$$

we obtain, from (13),

$$x \tilde{\boldsymbol{p}} = \boldsymbol{R}^\star \tilde{\boldsymbol{p}}\ , \qquad \boldsymbol{j}^\top \boldsymbol{R}^\star = \lambda_0 \boldsymbol{j}^\top \tag{14}$$

The entries of the tridiagonal matrix $\boldsymbol{R}^\star$, which, according to the notation of Proposition 1.2, are denoted by $\tilde{a}_k, \tilde{b}_k, \tilde{c}_k$, are defined by: $\tilde{a}_k = a_{d-k}$, $\tilde{b}_k = c_{d-k}$, $\tilde{c}_k = b_{d-k}$ and $\tilde{a}_k + \tilde{b}_k + \tilde{c}_k = \lambda_0$, per a $k = 0, 1, \ldots, d$. Since $\tilde{b}_k \tilde{c}_{k+1} = c_{d-k} b_{d-k-1} > 0$ and $\tilde{p}_0 = 1$ it turns out that each $\tilde{p}_k$ has degree k.

Let $\boldsymbol{P}$, respectively $\tilde{\boldsymbol{P}}$, denote the square $(d+1)$-matrix with (i,j)-entry $p_i(\lambda_j)$, respectively $\tilde{p}_i(\lambda_j)$, $0 \le i, j \le d$. Also, let us consider the following diagonal matrices $\boldsymbol{D} := \operatorname{diag}(\|p_0\|^2, \|p_1\|^2, \ldots, \|p_d\|^2)$, $\tilde{\boldsymbol{D}} := \operatorname{diag}(\tilde{p}_0(\lambda_0), \tilde{p}_1(\lambda_0), \ldots, \tilde{p}_d(\lambda_0))$, $\boldsymbol{P}_d := \operatorname{diag}(p_d(\lambda_0), p_d(\lambda_1), \ldots, p_d(\lambda_d))$, and $\boldsymbol{G} := \operatorname{diag}(g_0, g_1, \ldots, g_d)$. Now, we have the following facts:

(a) The sequence $(p_k)_{0 \le k \le d}$ is the canonical orthogonal system with respect to the inner product associated to $(\mathcal{M}, g)$ if and only if $\operatorname{dgr} p_k = k$, $0 \le k \le d$, and

$$\boldsymbol{PGP}^\top = \boldsymbol{D}; \tag{15}$$

(b) By the definition of $\tilde{\boldsymbol{P}}$ we immediately have $\tilde{\boldsymbol{P}} = \boldsymbol{SPP}_d^{-1}$;

(c) Similarly, from the definition of $\tilde{\boldsymbol{D}}$, we get $\tilde{\boldsymbol{D}} = p_d^{-1}(\lambda_0) \boldsymbol{SDS}$.

Then, the following computation

$$\begin{aligned} \tilde{\boldsymbol{P}} \left(p_d^{-1}(\lambda_0) \boldsymbol{P}_d \boldsymbol{G} \boldsymbol{P}_d \right) \tilde{\boldsymbol{P}}^\top &= p_d^{-1}(\lambda_0) \tilde{\boldsymbol{P}} \boldsymbol{P}_d \boldsymbol{G} \boldsymbol{P}_d \tilde{\boldsymbol{P}}^\top \\ &= p_d^{-1}(\lambda_0) \boldsymbol{SPP}_d^{-1} \boldsymbol{P}_d \boldsymbol{G} \boldsymbol{P}_d \boldsymbol{P}_d^{-1} \boldsymbol{P}^\top \boldsymbol{S} \\ &= p_d^{-1}(\lambda_0) \boldsymbol{SPGP}^\top \boldsymbol{S} = p_d^{-1}(\lambda_0) \boldsymbol{SDS} = \tilde{\boldsymbol{D}}\,, \end{aligned}$$

stablishes that the family of polynomials $(\tilde{p}_k)_{0\le k\le d}$, with $\mathrm{dgr}\,\tilde{p}_k = k$, $0 \le k \le d$, is the canonical orthogonal system with respect to the product $(\mathcal{M}, \tilde{g})$, where, using Corollary 1.7, $\tilde{g}_k := \tilde{g}(\lambda_k)$ corresponds to the expression:

$$\begin{aligned} \tilde{g}_k &= p_d^{-1}(\lambda_0)\,(\boldsymbol{P}_d\boldsymbol{G}\boldsymbol{P}_d)_{kk} = p_d^{-1}(\lambda_0) g_k p_d^2(\lambda_k) \\ &= g_0 \frac{g_0\pi_0^2}{g_k\pi_k^2} p_d(\lambda_0) = \frac{g_0\pi_0^2}{g_k\pi_k^2}\left(\sum_{l=0}^{d} \frac{g_0\pi_0^2}{g_l\pi_l^2}\right)^{-1}. \end{aligned} \tag{16}$$

Note that, in particular, $\tilde{g} : \mathcal{M} \to \mathbb{R}$ is a normalized weight function on $\mathcal{M}$. All the above facts are summarized in the following result:

Proposition 1.8 *Given a mesh $\mathcal{M} = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$, we associate, to each normalized weight function $g : \mathcal{M} \to \mathbb{R}$, a new weight function $\tilde{g} : \mathcal{M} \to \mathbb{R}$, which is also normalized, defined as:*

$$\tilde{g}_k = \frac{g_0\pi_0^2}{g_k\pi_k^2}\left(\sum_{l=0}^{d} \frac{g_0\pi_0^2}{g_l\pi_l^2}\right)^{-1} \qquad (0 \le k \le d).$$

Then the respective canonical orthogonal systems:

$(p_k)_{0\le k\le d}$ with respect to $\langle p, q\rangle = \sum_{l=0}^{d} g_l p(\lambda_l)q(\lambda_l)$; and

$(\tilde{p}_k)_{0\le k\le d}$ with respect to $\langle p, q\rangle^{\tilde{}} := \sum_{l=0}^{d} \tilde{g}_l p(\lambda_l)q(\lambda_l)$;

are related by $\tilde{p}_k = p_d^{-1}p_{d-k}$, $0 \le k \le d$, and the respective recurrence matrices, $\boldsymbol{R}$ and $\tilde{\boldsymbol{R}}$, coincide up to a central symmetry.

Corollary 1.9 *The mapping $g \mapsto \tilde{g}$, defined on the set of normalized weight functions on $\mathcal{M}$, is involutive.*

Proof. The result follows immediately from the fact that $\tilde{\boldsymbol{R}}$ is the matrix obtained from by applying a central symmetry to $\boldsymbol{R}$. Or, alternatively, since $\tilde{p}_d = p_d^{-1}$ we have: $\tilde{p}_d^{-1}\tilde{p}_{d-k} = p_d p_d^{-1} p_k = p_k$. Whence, using (12), it turns out that the conjugate weight function of $\tilde{g}$ is g itself. $\square$

We shall say that the weight functions g and $\tilde{g}$, the respective scalar products, and the corresponding canonical orthogonal systems are mutually *conjugate*.

1.4 The dual canonical polynomials

Associated to the canonical polynomials, there is another set of orthogonal polynomials, which are called the "dual (canonical) polynomials". In order to introduce them, notice that the orthogonality property in (15) can also be written as $\boldsymbol{P}^{\top}\boldsymbol{D}^{-1}\boldsymbol{P} = \boldsymbol{G}^{-1}$. This may be rewritten, in turn, as

$$\widehat{\boldsymbol{P}}\boldsymbol{D}\widehat{\boldsymbol{P}}^{\top} = \boldsymbol{G}^{-1}, \tag{17}$$

where we have introduced the new matrix $\widehat{\boldsymbol{P}} := \boldsymbol{P}^{\top}\boldsymbol{D}^{-1}$. Then, note that (17) can also be interpreted as an orthogonality property, with respect to the scalar product

$$\langle p, q\rangle^* := \sum_{l=0}^{d} \|p_l\|^2 p(\lambda_l)q(\lambda_l) \tag{18}$$

for the new polynomials $\hat{p}_k$, $0 \le k \le d$, defined as

$$\hat{p}_k(\lambda_l) := (\widehat{\boldsymbol{P}})_{kl} = \frac{p_l(\lambda_k)}{\|p_l\|^2} = \frac{p_l(\lambda_k)}{p_l(\lambda_0)} \qquad (0 \le l \le d), \tag{19}$$

and which will be called the *dual polynomials* of the p_k. Thus, (17) reads

$$\langle \hat{p}_k, \hat{p}_l\rangle^* = \delta_{kl} g_l^{-1} \qquad (0 \le k, l \le d), \tag{20}$$

whence, using (18), the values of the weight function can be computed from the polynomials $(p_k)_{0\le k\le d}$ as:

$$g_l = \frac{1}{(\|\hat{p}_l\|^*)^2} = \left(\sum_{k=0}^{d} \frac{p_k(\lambda_l)^2}{p_k(\lambda_0)}\right)^{-1}. \tag{21}$$

This is an alternative formula to (12).

Moreover, we have already seen, in Proposition 1.2(c) that the l-th column of $\boldsymbol{P}$, namely $(p_0(\lambda_l), p_1(\lambda_l), \ldots, p_d(\lambda_l))^{\top}$, is an eigenvector of the tridiagonal recurrence matrix $\boldsymbol{R}$, with eigenvalue λ_l. That is,

$$\boldsymbol{R}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{D}_{\lambda} \tag{22}$$

where $\boldsymbol{D}_{\lambda} := \mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_d)$. Similarly, from (15) and the definition of $\widehat{\boldsymbol{P}}$ we see that $\boldsymbol{P}^{-1} = \boldsymbol{G}\boldsymbol{P}^{\top}\boldsymbol{D}^{-1} = \boldsymbol{G}\widehat{\boldsymbol{P}}$. Then, (22) yields

$$\widehat{\boldsymbol{P}}\boldsymbol{R} = \boldsymbol{D}_{\lambda}\widehat{\boldsymbol{P}}. \tag{23}$$

That is, the k-th row of $\widehat{\boldsymbol{P}}$,

$$(\hat{p}_k(\lambda_0), \hat{p}_k(\lambda_1), \ldots, \hat{p}_k(\lambda_d)) = \left(\frac{p_0(\lambda_k)}{p_0(\lambda_0)}, \frac{p_1(\lambda_k)}{p_1(\lambda_0)}, \ldots, \frac{p_d(\lambda_k)}{p_d(\lambda_0)}\right),$$

is a left eigenvector of $\boldsymbol{R}$ with eigenvalue λ_k.

The *number of sign-changes* in a given sequence of real numbers is the number of times that consecutive terms (after removing the null ones) have distinct sign. Thus, if $(p_k)_{0\le k\le d}$ is a (canonical) orthogonal system, the fact that $\operatorname{dgr} p_k = k$ implies that the sequence $p_k(\lambda_0), p_k(\lambda_1), \ldots, p_k(\lambda_d)$ has exactly k sign-changes. Although the degrees of the dual polynomials $(\hat{p}_k)_{0\le k\le d}$ does not necessarily coincide with their indexes; they keep the above property and the sequence $\hat{p}_l(\lambda_0), \hat{p}_l(\lambda_1), \ldots, \hat{p}_l(\lambda_d)$ also has exactly l sign-changes. This is a direct consequence of a known result about orthogonal polynomials (see e.g. [15, 12]), which we formally state in the the next lemma, and prove it by considering the "equivalent" sequence $p_0(\lambda_l), p_1(\lambda_l), \ldots, p_d(\lambda_l)$ (since $p_k(\lambda_0) > 0$ for any $0 \le k \le d$)).

Lemma 1.10 *Let $(p_k)_{0\le k\le d}$ be a sequence of orthogonal polynomials and let $\lambda_0 > \lambda_1 > \cdots > \lambda_d$ be the zeros of p_{d+1} Then, for any given $0 \le l \le d$, the sequence $p_0(\lambda_l)$, $p_1(\lambda_l)$, $\ldots$, $p_d(\lambda_l)$ has exactly l sign-changes.*

Proof. We know that, between any two consecutive zeros of p_{k+1}, there lies one zero of p_k. With this in mind, this could be seen as a "proof without words"; consider Fig. 2: The number of sign-changes coincide with the crossed "staircases". $\square$

A "non-visual" proof of this result can be found in Godsil [12]. Moreover, since each column of the recurrence matrix sums to λ_0, we also have the following corollary:

Corollary 1.11 *Let us consider a recurrence with coefficients satisfying $a_k + b_k + c_k = \mu_0$, $0 \le k \le d$. Then, for any $1 \le l \le d$, the sequence $\hat{p}_l(\lambda_0) - \hat{p}_l(\lambda_1)$, $\ldots$,$\hat{p}_l(\lambda_{d-1}) - \hat{p}_l(\lambda_d)$ has exactly $l-1$ sign-changes.*

Proof. Let $\boldsymbol{C}$ be the $(d+1)\times(d+1)$ matrix with 1's on the principal diagonal, -1's on the diagonal below the principal one, and 0's elsewhere. We know that $\widehat{\boldsymbol{P}}_l := (\hat{p}_l(\lambda_0), \hat{p}_l(\lambda_1), \ldots, \hat{p}_l(\lambda_d))$ is a (left) eigenvector of the recurrence matrix $\boldsymbol{R}$, so that $\widehat{\boldsymbol{P}}_l\boldsymbol{C}$ is an eigenvector of the (also tridiagonal) matrix $\boldsymbol{R}' := \boldsymbol{C}^{-1}\boldsymbol{R}\boldsymbol{C}$. From this, one deduces that $(\hat{p}_l(\lambda_0) - \hat{p}_l(\lambda_1), \ldots, \hat{p}_l(\lambda_{d-1}) - \hat{p}_l(\lambda_d))$ is a left eigenvector of the $d \times d$ principal submatrix of $\boldsymbol{R}'$:

$$\begin{pmatrix} \lambda_0 - b_0 - c_1 & c_1 & & & \\ b_1 & \lambda_0 - b_1 - c_2 & c_2 & & \\ & b_2 & \cdot & & \\ & & \cdot \quad \cdot & & \\ & & & \cdot \quad \cdot & c_{d-1} \\ & & & b_{d-1} & \lambda_0 - b_{d-1} - c_d \end{pmatrix}$$

with corresponding eigenvalue $\lambda_l \in \mathrm{ev}\, \boldsymbol{R}\backslash\{\lambda_0\}$. Then the result follows from Lemma 1.10. $\square$

2 Applications to graphs

Some graph concepts

- $G = (V, E)$: *simple graph*, with vertex set $V = \{i, j, \ldots, n\}$ and set of edges (unordered pairs of vertices) E
- *Adjacency* $\{i, j\} \in E$: $\quad i \sim j$
- *Distance* between vertices i and j: $\quad \partial(i, j)$
- Set k-*apart* from vertex i:
$$\Gamma_k(i) = \{j : \partial(i, j) = k\}$$
- The k-*Neighbourhood* of vertex i is the set of vertices at distance at most k from i:
$$N_k(i) = \Gamma_0(i) \cup \Gamma_1(i) \cup \cdots \Gamma_k(i).$$
- *Degree* of vertex i: $\quad \delta_i = |\Gamma_1(i)| \equiv |\Gamma(i)|$.
- *Eccentricity* of vertex i:
$$\varepsilon = \mathrm{ecc}(i) = \max_{1 \le j \le n} \partial(i, j)$$
- *Diameter* of G:
$$D = D(G) = \max_{1 \le i \le n} \mathrm{ecc}(i)$$
- *Radius* of G:
$$r = r(G) = \min_{1 \le i \le n} \mathrm{ecc}(i)$$

Some algebraic-graph concepts

- *Adjacency matrix* of G, $\boldsymbol{A} = \boldsymbol{A}(G)$:
$$(\boldsymbol{A})_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

- *Characteristic polynomial* of G:

$$\phi_G(x) = \det(x\boldsymbol{I} - \boldsymbol{A}) = \prod_{l=0}^{d}(x-\lambda_l)^{m(\lambda_l)}$$

- *Spectrum* of G:

$$\operatorname{sp} G := \operatorname{sp} \boldsymbol{A} = \{\lambda_0^{m(\lambda_0)}, \lambda_1^{m(\lambda_1)}, \dots, \lambda_d^{m(\lambda_d)}\}.$$

- Different eigenvalues of G:

$$\operatorname{ev} G = \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}.$$

- "Moment-like" parameters defined as in (3) from the eigenvalues mesh $calM = \operatorname{ev} G$:

$$\pi_l := \prod_{h=0,h\neq l}^{m} |\lambda_l - \lambda_h| \qquad (0 \leq l \leq d)$$

  and satisfying (6):

$$\sum_{l=0}^{m}(-1)^l \frac{\lambda_l^k}{\pi_l} = \begin{cases} 0 & \text{if } 0 \leq k < d \\ 1 & \text{if } k = d. \end{cases}$$

Some algebraic-graph results

- Since G is supposed to be connected ($D < \infty$) we have, by the Perron-Frobenius theorem for nonnegative matrices, that $\lambda_0 = \rho(A) > 0$ with eigenvector $\boldsymbol{v} > 0$ (let $\boldsymbol{v}$ s.t. $\min_{1\leq i\leq n} v_i = 1$). Throughout this paper, we suppose, for simplicity, that G is *(δ-)regular*; that is $\delta_i = \delta$ for every $i \in V$. In this case, $\lambda_0 = \delta$ and $\boldsymbol{v} = \boldsymbol{j}$, the all-1 vector.

- Eigenvectors vs. charges:

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v} \quad\Longleftrightarrow\quad \sum_{j\sim i} v_j = \lambda v_i$$

- Number of k-walks between vertices i and j:

$$(\boldsymbol{A}^k)_{ij} = \sum_{l=0}^{d} m(\lambda_l)\lambda_l^k$$

- Diameter D:

$$D \leq d = |\operatorname{ev} G| - 1$$

Distance-regularity

We say that a (regular) graph G is *distance-regular* around a vertex i with eccentricity $\mathrm{ecc}(i) = \varepsilon$, whenever the numbers

$$c_k(j) := |\Gamma(j) \cap V_{k-1}|,$$

$$a_k(j) := |\Gamma(j) \cap V_k|,$$

$$b_k(j) := |\Gamma(j) \cap V_{k+1}|,$$

where $V_k := \Gamma_k(i)$, defined for any $j \in V_k$ and $0 \leq k \leq \varepsilon$ (where, by convention, $c_0(i) = 0$ and $b_\varepsilon(j) = 0$ for any $j \in V_\varepsilon$) do not depend on the considered vertex $j \in V_k$, but only on the value of k. In such a case, we denote them by c_k, a_k and b_k respectively (the *intersection numbers*). Then, the matrix

$$\mathcal{I}(i) := \begin{pmatrix} 0 & c_1 & \cdots & c_{\varepsilon-1} & c_\varepsilon \\ a_0 & a_1 & \cdots & a_{\varepsilon-1} & a_\varepsilon \\ b_0 & b_1 & \cdots & b_{\varepsilon-1} & 0 \end{pmatrix}$$

is called the *intersection array* around vertex i of G.

A graph G is called *distance-regular* when it is distance regular around each of its vertices and with the same intersection array.

The local spectrum

For each eigenvalue λ_l, let $\boldsymbol{E}_l$ be the matrix representing the orthogonal projections onto the eigenspace $\mathcal{E}_l := \mathrm{Ker}(\boldsymbol{A} - \lambda_l \boldsymbol{I})$. These are called the *(principal) idempotents* of $\boldsymbol{A}$.

- $\boldsymbol{E}_l = Z_l(\boldsymbol{A})$, $0 \leq l \leq d$, with Z_l being the interpolating polynomial in (4); that is $Z_l = \frac{(-1)^l}{\pi_l} \prod_{h=0\,(h \neq l)}^{d} (x - \lambda_h)$. In particular, $\boldsymbol{E}_0 = \frac{1}{n} \boldsymbol{v}\boldsymbol{v}^\top$.
- $\boldsymbol{E}_l \boldsymbol{E}_h = \begin{cases} \boldsymbol{E}_l & \text{if } l = h \\ \boldsymbol{0} & \text{otherwise;} \end{cases}$
- $\boldsymbol{A}\boldsymbol{E}_l = \lambda_l \boldsymbol{E}_l$;
- $p(\boldsymbol{A}) = \sum_{l=0}^{d} p(\lambda_l) \boldsymbol{E}_l$, for any $p \in R[x]$.

- Taking $p=1$, we have $\sum_{l=0}^{d} \boldsymbol{E}_l = \boldsymbol{I}$
- For $p=x$ we get the *spectral decomposition* theorem:

$$\boldsymbol{A} = \sum_{l=0}^{d} \lambda_l \boldsymbol{E}_l$$

- The *(i-)local multiplicity* of $\lambda_l \in \operatorname{ev} G$:

$$m_i(\lambda) := m_{ii}(\lambda_l) = \|\boldsymbol{E}_l \boldsymbol{e}_i\|^2 \geq 0$$

  (for instance, $m_i(\lambda_0) = 1/\|\boldsymbol{v}\|^2$)

- As if the graph G were "seen" from vertex i: $\sum_{l=0}^{d} m_i(\lambda_l) = 1$
- $m(\lambda_l) = \operatorname{tr} \boldsymbol{E}_l = \sum_{i=1}^{n} m_i(\lambda_l)$
- $\mathcal{C}_{ii}(k) = (\boldsymbol{A}^k)_{ii} = \sum_{l=0}^{d} \lambda_l^k m_i(\lambda_l)$
- The *(i-)local spectrum*:

$$\operatorname{sp} i := \{\lambda^{m_i(\mu_0)}, \mu_1^{m_i(\mu_1)}, \ldots, \mu_{d_i}^{m_i(\mu_{d_i})}\}.$$

  with $\mu_0 = \lambda_0$ and $m_i(\mu_l) \neq 0$, $0 \leq l \leq d_i$.

- Degree δ_i:

$$\delta_i = \sum_{l=0}^{d_i} m_i(\mu_l) \mu_l^2$$

- Eccentricity $\operatorname{ecc}(i)$:

$$\operatorname{ecc}(i) \leq d_i = |\operatorname{ev}_i G| - 1$$

- Characteristic polynomials:

$$\frac{\phi_{G\setminus i}(x)}{\phi_G(x)} = \frac{\phi_i'(x)}{\phi_i(x)}$$

- G is *spectrally regular* (i.e., $\mathrm{sp}\, i = \mathrm{sp}\, j$ for any $i, j \in V$)

  $\iff \quad \phi_i = \phi_j$ for any $i, j \in V$

  $\iff$ the local multiplicities only depend on λ_l: $m_i(\lambda_l) = \frac{m(\lambda_l)}{n}$ for any $\lambda_l \in \mathrm{ev}\, G$

  $\iff$ G is *walk regular* (i.e. $\mathcal{P}_{ii}(k)$ only depends on k)

  $\iff$ $\mathrm{sp}\, G \setminus i) = \mathrm{sp}\, G \setminus j)$ for any $i, j \in V$

The local predistance polynomials

Given a vertex i of a graph G, *(i-local) predistance polynomials* $(p_k^i)_{0 \leq k \leq d_i}$ are no more than the canonical orthogonal system associated to the mesh $\mathcal{M} = \mathrm{sp}\, i$ and weight function $g_l = m_i(\mu_l)$, $0 \leq l \leq d_i$:

- The *(i-)local* scalar product:

$$\langle f, g \rangle_i := (fg(\boldsymbol{A}))_{ii} = \sum_{l=0}^{d_i} m_i(\mu_l) f(\mu_l) g(\mu_l)$$

  with normalized weight function $g_l := m_i(\mu_l)$, $0 \leq l \leq d_i$, since $\sum_{l=0}^{d} g_l = 1$

- The *(i-local) predistance* polynomials: $(p_k^i)_{0 \leq k \leq d_i}$ such that $\mathrm{dgr}\, p_k^i = k$ and

$$\langle p_k^i, p_l^i \rangle_i = \begin{cases} 0 & \text{if } k \neq l \\ p_k^i(\lambda_0) & \text{if } k = l \end{cases}$$

- As we already know, such a system is unique and characterized by the conditions of Proposition 1.6.

2.1 Some "predistance" results

Let G be a (regular) graph on $n = |V|$ vertices. Let $i \in V$.

- The i-local multiplicities of G are given by

$$m_i(\mu_l) = \frac{\phi_0 p_{d_i}^i(\lambda_0)}{n \phi_l p_{d_i}^i(\mu_l)} \qquad (0 \leq l \leq d_i)$$

  where $\phi_l = \prod_{h=0 (h \neq l)}^{d_i} (\mu_l - \mu_h) = (-1)^l \pi_l$.

- The value at λ_0 of the highest degree polynomial is

$$p^i_{d_i}(\lambda_0) = \left(\sum_{l=0}^{d_i} \frac{m_i^2(\lambda_0)\pi_0^2}{m_i(\mu_l)\pi_l^2} \right)^{-1}.$$

- G is distance-regular around vertex i, $\mathrm{ecc}(i) = \varepsilon$, if and only if

$$p^i_k(\boldsymbol{A})\boldsymbol{e}_i = \boldsymbol{\rho} V_k = \sum_{j \in V_k} \boldsymbol{e}_j \qquad (0 \le k \le \varepsilon)$$

  where $V_k := \Gamma_k(i)$.

- Bounding the number of i-extremal vertices ($\partial(i,j) = d_i = |\operatorname{ev}_i G| - 1$):

$$|V_{d_i}| \le p^i_{d_i}(\lambda_0) = \left(\sum_{l=0}^{d_i} \frac{m_i^2(\lambda_0)\pi_0^2}{m_i(\mu_l)\pi_l^2} \right)^{-1}$$

  and equality occurs iff G is distance-regular around vertex i.

Characterizing distance-regularity

In [6], the authors gave the following characterization of distance-regularity

- A (connected) regular graph G with

$$\operatorname{sp} G = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \dots, \lambda_d^{m_d}\}$$

  is distance-regular iff for each vertex i the number of i-diametral vertices is

$$|V_d| = n \left(\sum_{l=0}^{d} \frac{\pi_0^2}{m_l \pi_l^2} \right)^{-1}$$

Using some results from [6, 9], the first author proved in [5] the following result, which gives another characterization of distance-regular graphs.

- Let G be a regular graph with n vertices and $d+1$ distinct eigenvalues. For every vertex $i \in V$, let $s_{d-1}(i) := |N_{d-1}(i)|$. Then, any polynomial $r \in \mathbb{R}_{d-1}[x]$ satisfies the bound

$$\frac{r(\lambda_0)^2}{\|r\|_G^2} \le \frac{n}{\sum_{i \in V} \frac{1}{s_{d-1}(i)}}, \tag{24}$$

and equality is attained if and only if G is a distance-regular graph. Moreover, in this case, we have

$$\frac{r(\lambda_0)}{\|r\|_G^2} r = q_{d-1} = \sum_{k=0}^{d-1} p_k,$$

where the p_k's are the distance polynomials of G.

Note that the above upper bound is, in fact, the harmonic mean of the numbers $s_{d-1}(i)$, $i \in V$, which is hereafter denoted by H. Moreover, in case of equality, $q_{d-1}(\boldsymbol{A}) = \sum_{k=0}^{d-1} \boldsymbol{A}_k = \boldsymbol{J} - \boldsymbol{A}_d$, and the distance-$d$ polynomial of G is just

$$p_d = q_d - q_{d-1} = q_d - \frac{r(\lambda_0)}{\|r\|_G^2} r \tag{25}$$

where q_d represents the *Hoffman polynomial*; that is, $q_d = H_0 = \frac{n}{\pi_0} \prod_{l=1}^{d} (x - \lambda_l)$ (see [13]).

Bounding special vertex sets

Let $i \in V$ be a vertex with eccentricity $\mathrm{ecc}(i) = \varepsilon$. Given the integers k, μ such that $0 \leq k \leq \varepsilon$ and $\mu \geq 0$, let $\Gamma_k^\mu(i)$ denote the set of vertices which are at distance at least k from $i \in V$ and there exist exactly μ (shortest) k-paths from i to each of such vertices. Note that $\Gamma_k^0(i) = V \setminus N_k(i)$, and if $\mu \neq 0$, then $\Gamma_k^\mu(i)$ contains only vertices at distance k from i, so that we get the partition $\Gamma_k(i) = \cup_{\mu \geq 1} \Gamma_k^\mu(i)$.

- Let i be a vertex of a (regular) graph G, with local spectrum $\mathrm{sp}\, i$, and let $(p_k^i)_{0 \leq k \leq d_i}$ be the local predistance polynomials. Let a_k denote the leading coefficient of p_k^i, and consider the sum polynomials $q_k^i = \sum_{l=0}^{k} p_l^i$. For any given integers $\mu > 0$ and $0 \leq k < d_i$, consider the spectral k-excess $e_k = n - v_i^2 q_k^i(\lambda_0)$, and define $\sigma_k(\mu) := a_k \mu - 1$. Then,

$$|\Gamma_k^\mu(i)| \leq \frac{p_k^i(\lambda_0) e_k}{p_k^i(\lambda_0) \sigma_k(\mu)^2 + a_k^2 \mu^2 e_k}, \tag{26}$$

  and equality is attained if and only if either

  (a) When $k = \varepsilon$:

$$P^* \boldsymbol{e}_i = \beta_\varepsilon^* \boldsymbol{j} + \gamma_\varepsilon^* \boldsymbol{\rho} \Gamma_\varepsilon^\mu(i), \tag{27}$$

with the polynomial

$$P^* := a_\varepsilon \mu e_\varepsilon p^i_\varepsilon + p^i_\varepsilon(\lambda_0)\sigma_\varepsilon(\mu) q^i_\varepsilon \tag{28}$$

and constants

$$\beta^*_\varepsilon := p^i_\varepsilon(\lambda_0)\sigma_\varepsilon(\mu), \quad \gamma^*_\varepsilon := p^i_\varepsilon(\lambda_0)\sigma_\varepsilon(\mu)^2 + a^2_\varepsilon \mu^2 e_\varepsilon; \tag{29}$$

(b) When $k < \varepsilon$:

$$p^i_k \boldsymbol{e}_i = \boldsymbol{\rho} V_k\,, \tag{30}$$

in which case

$$n^\mu_k = n_k = p^i_k(\lambda_0). \tag{31}$$

In the case of walk-regular graphs ($\Rightarrow p^i_k = p_k$ for every vertex i) and $k = d-1$, we have

$$e_{d-1} = n - q_{d-1}(\lambda_0) = q_d(\lambda_0) - q_{d-1}(\lambda_0) = p_d(\lambda_0)$$

and we get

$$n^\mu_{d-1} \le \frac{p_{d-1}(\lambda_0) p_d(\lambda_0)}{p_{d-1}(\lambda_0)\sigma_{d-1}(\mu)^2 + a^2_{d-1}\mu^2 p_d(\lambda_0)}. \tag{32}$$

When $d = 3$ the above result proves the following conjecture of Van Dam (1996).

- Let G be a regular graph with four distinct eigenvalues, and predistance polynomials p_k with leading coefficients a_k. Then, for any vertex $i \in V$, the number n^μ_2 of vertices non-adjacent to i, which have μ common neighbours with i, is upper-bounded by

$$n^\mu_2 \le \frac{p_2(\lambda_0) p_3(\lambda_0)}{p_2(\lambda_0)(a_2\mu - 1)^2 + a^2_2 \mu^2 p_3(\lambda_0)}. \tag{33}$$

Example. Let G be a regular graph with spectrum

$$\operatorname{sp} G = \{4^1, 2^3, 0^3, -2^5\}.$$

Then $n = 12$, and its proper polynomials and their values at $\lambda_0 = 4$ are:

- $p_0 = 1, \quad 1;$
- $p_1 = x, \quad 4;$

- $p_2 = \frac{2}{3}(x^2 - x - 4)$, $\frac{16}{3}$;
- $p_3 = \frac{1}{12}(3x^3 - 8x^2 - 16x + 20)$, $\frac{5}{3}$;

Then, (33) gives

$$n_2^\mu \leq \left\lfloor \frac{20}{7\mu^2 - 16\mu + 12} \right\rfloor$$

and hence

- $\mu = 0, 1, 2, 3, 4, \ldots \quad \Rightarrow \quad n_2^\mu \leq 1, 6, 2, 0, 0, \ldots$

An example of a graph with such a spectrum is the one given by Godsil [12] (as an example of walk-regular graph which is neither vertex-transitive nor distance-regular.) This graph can be constructed as follows: take two copies of the 8-cycle with vertex set $\mathbb{Z}_8$ and chords $\{1,5\}$, $\{3,7\}$; joint them by identifying vertices with the same even number and, finally, add edges between vertices labelled with equal odd number. The automorphism group of this graph has two orbits, formed by "even" and "odd" vertices respectively. Then,

- $n_2^\mu = 1, 4, 2, 0, 0, \ldots$ (for an even vertex);
- $n_2^\mu = 0, 6, 1, 0, 0, \ldots$ (for an odd vertex).

Representation theory

Given a graph G and $\lambda_l \in \operatorname{ev} G$, we define, for every pair of vertices i, j,

- The *ij-cosine* $w_{ij} = w_{ij}(\lambda_l)$:

$$w_{ij}(\lambda_l) = \frac{\langle \boldsymbol{E}_l \boldsymbol{e}_i, \boldsymbol{E}_l \boldsymbol{e}_j \rangle}{\|\boldsymbol{E}_l \boldsymbol{e}_i\| \|\boldsymbol{E}_l \boldsymbol{e}_j\|} = \frac{m_{ij}(\lambda_l)}{\sqrt{m_i(\lambda_l) m_j(\lambda_l)}}$$

  where $m_{ij}(\lambda_l) := m_{ij}(\lambda_l)$ is called the (ij-)crossed local multiplicity of λ_l.

When G is a distance-regular graph, $w_{ij}(\lambda_l)$ only depends on the distance $r := \partial(i, j)$, and it is referred to as the r-th cosine $w_r(\lambda_l)$.

In this case:

$$\lambda_l w_r = c_r w_{r-1} + a_r w_r + b_r w_{r+1} \quad (0 \leq r \leq d),$$

where c_r, a_r, and b_r are the intersection parameters of G; w_{-1} and w_{d+1} are irrelevant (since $c_0 = b_d = 0$); and $w_0 = 1$.

Some "cosine" results are the following:

- G distance-regular around vertex i and $\partial(i,j) = r$:

$$m_{ij}(\mu_l) = \frac{p_r^i(\mu_l)}{p_r^i(\mu_0)} m_i(\mu_l) \quad (0 \le l \le d_i)$$

  and hence

$$w_{ij}(\mu_l) = \frac{p_r^i(\mu_l)}{p_r^i(\mu_0)} \sqrt{\frac{m_i(\mu_l)}{m_j(\mu_l)}}$$

- G distance-regular and $\partial(i,j) = r$:

$$w_r(\lambda_l) = \frac{p_r(\lambda_l)}{p_r(\lambda_0)} \quad (0 \le l \le d)$$

Around a vertex set

Given a vertex subset $C \subset V$ of a graph G, we define the normalized vector

$$\boldsymbol{e}_C := \frac{1}{\sqrt{|C|}} \boldsymbol{\rho} C = \frac{1}{\sqrt{|C|}} \sum_{i \in C} \boldsymbol{e}_i,$$

- C*-multiplicity* of the eigenvalue λ_l

$$\begin{aligned} m_C(\lambda_l) &:= \|\boldsymbol{E}_l \boldsymbol{e}_C\|^2 = \langle \boldsymbol{E}_l \boldsymbol{e}_C, \boldsymbol{e}_C \rangle = \frac{1}{|C|} \sum_{i,j \in C} \langle \boldsymbol{E}_l \boldsymbol{e}_i, \boldsymbol{e}_j \rangle \\ &= \frac{1}{|C|} \sum_{i,j \in C} m_{ij}(\lambda_l) \qquad (0 \le l \le d). \end{aligned} \tag{34}$$

  The sequence of C-multiplicities $(m_C(\lambda_0), \ldots, m_C(\lambda_d))$ corresponds in fact to the so-called *"McWilliams transform"* of the vector $\boldsymbol{e}_C$. Since $\boldsymbol{e}_C$ is unitary, we have $\sum_{l=0}^{d} m_C(\lambda_l) = 1$.

- If $\mu_0 (= \lambda_0), \mu_1, \ldots, \mu_{d_C}$ represent the eigenvalues λ_i of G with non-zero C-multiplicities, then the (local) C*-spectrum* of C is

$$\operatorname{sp} C = \{\lambda_0^{m_C(\lambda_0)}, \mu_1^{m_C(\mu_1)}, \ldots, \mu_{d_C}^{m_C(\mu_{d_C})}\}$$

  where $d_C (\le d)$ is called the *dual degree* of C.

Some results involving the C-spectrum are the following:

- For any polynomial p,

$$\begin{aligned}\langle p(\boldsymbol{A})\boldsymbol{e}_C, \boldsymbol{e}_C\rangle &= \left\langle \sum_{l=0}^{d} p(\lambda_l)\boldsymbol{E}_l\boldsymbol{e}_C, \boldsymbol{e}_C \right\rangle = \sum_{l=0}^{d} p(\lambda_l)\langle \boldsymbol{E}_l\boldsymbol{e}_C, \boldsymbol{e}_C\rangle \\ &= \sum_{l=0}^{d} m_C(\lambda_l)p(\lambda_l)\end{aligned}$$

- Number of walks of length ℓ from (the vertices of) C to itself:

$$a_{CC}^{(\ell)} := \sum_{u,v\in C}(\boldsymbol{A}^\ell)_{uv} = |C|\langle \boldsymbol{A}^\ell\boldsymbol{e}_C, \boldsymbol{e}_C\rangle = |C|\sum_{l=0}^{d} m_C(\lambda_l)\lambda_l^\ell \qquad (\ell \geq 0).$$

- The *(C-local) predistance polynomials* $(p_k^C)_{0\leq k\leq d_C}$ are the canonical orthogonal system with respect to

$$\langle f, g\rangle_C := \sum_{l=0}^{d_C} m_C(\mu_l)f(\mu_l)g(\mu_l)$$

  (weight function $g_l = m_C(\mu_l)$), with $\|p_k^C\|_C^2 = p_k^C(\lambda_0)$.

- Sum polynomials $q_k^C := \sum_{h=0}^{k} p_h^C$, $0 \leq k \leq d_C$

In this context, our basic result reads as follows:

- Let C be a vertex subset of a (regular) graph G, with (C-local) predistance polynomials $(p_k^C)_{0\leq k\leq d}$. Then, for any polynomial $q \in \mathbb{R}_k[x]$,

$$\frac{q(\lambda_0)^2}{\|q\|_C^2} \leq \frac{|N_k(C)|}{|C|} \tag{35}$$

  and equality is attained if and only if

$$\frac{1}{\|q\|_C}q(\boldsymbol{A})\boldsymbol{e}_C = \boldsymbol{e}_{N_k}. \tag{36}$$

  where $\boldsymbol{e}_{N_k}$ represents the unitary characteristic vector of $N_k(C)$. Moreover, if this is the case, q is any multiple of q_k^C, whence (35) and (36) become

$$\begin{aligned}q_k^C(\lambda_0) &= \frac{|N_k(C)|}{|C|} \\ q_k^C(\boldsymbol{A})\boldsymbol{\rho}C &= \boldsymbol{\rho}N_k(C).\end{aligned}$$

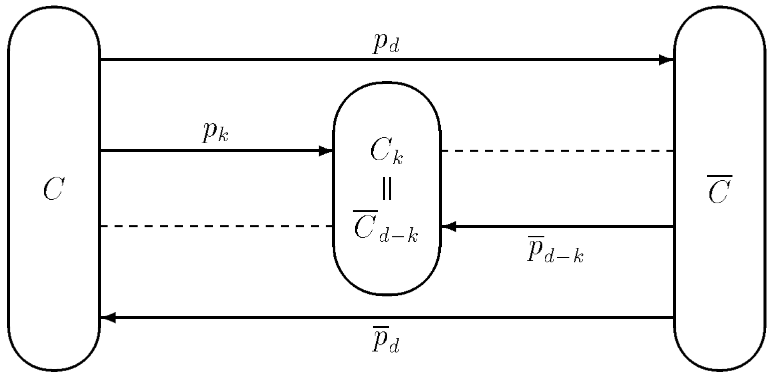

Figure 2: Antipodal tight sets and their distance partition

From the above result we consider the following definition:

- C is called *tight* when $q_k^C(\boldsymbol{A})\boldsymbol{\rho}C = \boldsymbol{\rho}N_k(C)$

- **Tight sets come in pairs:** Let C be a tight set of vertices with predistance orthogonal system $(p_k)_{0\le k\le d_C}$, and let $(\overline{p}_k)_{0\le k\le d_C}$ be the predistance orthogonal system associated to its antipodal set $\overline{C}$. Then,

  (a) The polynomials $(\overline{p}_k)_{0\le k\le d_C}$ are just the conjugate polynomials of the $(p_k)_{0\le k\le d_C}$:

$$\overline{p}_k = p_d^{-1}p_{d-k} \qquad p_k = p_d\overline{p}_{d-k} \quad (0 \le k \le d_C). \tag{37}$$

  (b) $\overline{C}$ is also tight. (If the family $\mathcal{T}$ of tight vertex sets of a graph G is not empty, then the application which maps every set to its antipodal fixes $\mathcal{T}$ and is involutive).

- The tight character of a set C—or, equivalently, the existence of the distance polynomial p_d^C— leads to the existence of all the distance polynomials with respect to both sets C and $\overline{C}$, and that they are just the members of their associated predistance orthogonal systems. What is more, for every $0 \le k \le d$, the action of the polynomial p_k on $\boldsymbol{\rho}C$ coincides with the action of $\overline{p}_{d-k}$ on $\boldsymbol{\rho}\overline{C}$, so revealing the symmetry between the roles of C and $\overline{C}$. (See Fig. 2.)

- Let C be a tight set, with antipodal set $\overline{C}$, and let $(p_k)_{0\le k\le d}$, $(\overline{p}_k)_{0\le k\le d}$ be the corresponding predistance orthogonal systems. Then,

$$p_k\boldsymbol{\rho}C = \boldsymbol{\rho}C_k = \boldsymbol{\rho}\overline{C}_{d-k} = \overline{p}_{d-k}\boldsymbol{\rho}\overline{C} \quad (0 \le k \le d_C).$$

Completely regular codes in distance-regular graphs

- Let $C \subset V$ be a vertex set with eccentricity ε. Then, we say that G is *distance-regular around* C if the distance partition $V = C_0 \cup C_1 \cup \cdots \cup C_\varepsilon$ is regular; that is the numbers

$$c_k(i) := |\Gamma(i) \cap C_{k-1}|, \quad a_k(i) := |\Gamma(i) \cap C_k|, \quad b_k(i) := |\Gamma(i) \cap C_{k+1}|$$

where $u \in C_k$, $0 \leq k \leq \varepsilon$, depend only on the values of k and h, but not on the chosen vertex u. The set C is also referred to as a *completely regular set* or *completely regular code.*

- A graph $G = (V, E)$ is distance-regular around a set $C \subset V$, with eccentricity ε, if and only if the predistance polynomials $p_0^C, p_1^C, \ldots, p_\varepsilon^C$ satisfy

$$\boldsymbol{\rho} C_k = p_k^C(\boldsymbol{A})\boldsymbol{\rho} C \qquad (0 \leq k \leq \varepsilon).$$

- Let $G = (V, E)$ be a regular graph. A vertex subset $C \subset V$, with r vertices and local spectrum $\operatorname{sp} C = \{\lambda_0^{m_C(\lambda_0)}, \mu_1^{m_C(\mu_1)}, \ldots, \mu_{d_C}^{m_C(\mu_{d_C})}\}$, is a completely regular code if and only if the number of vertices at distance d_C from C; that is, $n_{d_C}(C) := |C_{d_C}|$ satisfies

$$\begin{aligned} n_{d_C}(C) = p_{d_C}^C(\lambda_0) &= r\left(\sum_{i=0}^{d_C} \frac{m_C(\lambda_0)^2 \pi_0^2}{m_C(\mu_l)\pi_l^2}\right)^{-1} \\ &= \frac{n^2}{r}\left(\sum_{i=0}^{d_C} \frac{\pi_0^2}{m_C(\mu_l)\pi_l^2}\right)^{-1}. \end{aligned}$$

In a distance regular graph, the local multiplicities of a vertex subset C can be easily computed from the distance polynomials of G, its spectrum, and the inner distribution of C.

- Number of ordered pairs (i, j) of vertices from C which are k apart in a distance-regular graph G is given by

$$\sum_{i,j \in C} (\boldsymbol{A}_k)_{ij} = |C| \langle p_k(\boldsymbol{A})\boldsymbol{e}_C, \boldsymbol{e}_C \rangle = |C| \sum_{l=0}^{d} m_C(\lambda_l) p_k(\lambda_l)$$

- From this we see that, within C, the *mean number* of vertices at distance k in G is

$$\overline{n}_k := \frac{1}{|C|} \sum_{u \in C} |\Gamma_k(i) \cap C| = \sum_{i=0}^{d} m_C(\lambda_l) p_k(\lambda_l) \qquad (0 \leq k \leq d). \tag{38}$$

The numbers $\overline{n}_k$, $0 \leq k \leq d$ are called the *inner distribution* of C and, as commented by Godsil [12]), they represent the probability that a randomly chosen pair of vertices from C are at distance k. Notice that always $\overline{n}_0 = 1$ and $\sum_{k=0}^{d} \overline{n}_k = |C|$.

- Let G be a distance-regular graph G with a given subset C of r vertices. Then there exist nonnegative numbers $r_0(= 1), r_1, \ldots, r_k$, such that $\overline{n}_k = r_k$ for every $0 \leq k \leq d$, if and only if the C-multiplicities satisfy

$$m_C(\lambda_l) = \frac{m(\lambda_l)}{n} \sum_{k=0}^{d} r_k \frac{p_k(\lambda_l)}{p_k(\lambda_0)} \qquad (0 \leq l \leq d). \tag{39}$$

- Some interesting particular cases:
  $l = 0 \Rightarrow m_C(\lambda_0) = \frac{m(\lambda_0)}{n} \sum_{k=0}^{d} r_k = \frac{r}{n}$
  $C = \{u\}$ $(r_0 = 1, r_1 = \cdots = r_d = 0) \Rightarrow m_C(\lambda_l) = \frac{m(\lambda_l)}{n} \frac{p_0(\lambda_l)}{p_0(\lambda_0)} = \frac{m(\lambda_l)}{n}$
  In the case when all vertices are at distance k from each other, (39) gives:

$$m_C(\lambda_l) = \frac{m(\lambda_l)}{n} \left(1 + (r-1) \frac{p_k(\lambda_l)}{p_k(\lambda_0)} \right) \qquad (0 \leq i \leq d).$$

  In particular, when $k = d$ we know that $p_d(\lambda_l) < 0$ for every odd i. This, together with $m_C(\lambda_l) \geq 0$, yields the following upper bound for the maximum number of vertices mutually at maximum d:

$$r \leq 1 + \min_{l=1,3,\ldots} \frac{p_d(\lambda_0)}{|p_d(\lambda_l)|}$$

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