Descent for blow-up's on smooth schemes

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Abstract

These are the notes of a talk in which we translate from the $\mathbb{A}^1$-homotopy theoretic context an argument from [Mor99] showing that a homotopy invariant presheaf of spectra on the category of smooth schemes that satisfies Nisnevich descent automatically satisfies descent for abstract blow-ups.

More concretely, Theorem 3.3.1 in [Mor99] roughly says that a Nisnevich distinguished square of schemes is homotopy cartesian when seen in the stable $\mathbb{A}^1$-homotopy category. We complete the details of the proof of this theorem and write it in the language of presheaves of spectra. This is theorem 3.7 in this notes.

Contents

1 Preliminaries

In this section we recall some preliminary notions we need. We collect some known results on different Grothendieck topologies we use.

In these notes $\mathbf{Sch}_k$ will denote the category of schemes of finite type over $\operatorname{Spec} k$ where k is a field. Note that in particular, every scheme $X \in \mathbf{Sch}_k$ is Noetherian. The subcategory $\mathbf{Sch}_k$ consisting of smooth schemes will be denoted by $\mathbf{Sm}_k$.

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When talking about coverings in a Grothendieck site $\mathcal{C}$, we will refer either to the collection of morphisms $\{U_i \to X\}$ which cover X or to the coproduct of them, which is a single morphism $U = \coprod U_i \to X$. The context will make clear what we mean. Note that if the covering is not finite, the morphism $U \to X$ is not of finite type.

1.1 cd-structures and topologies defined by squares

Here we set some terminology and sketch a couple of important ideas related to a class of Grothendieck topologies defined by generating commutative squares in an abstract way as done by Voevodsky in [Voe00a]. The main examples of such topologies we will need are the Nisnevich and cdh topologies. They will be discussed later.

Definition 1.1. Let $\mathcal{C}$ be a category with an initial object 0. A *cd-structure* on $\mathcal{C}$ is a class P of commutative squares in $\mathcal{C}$ closed under isomorphism of squares. The squares in P will often be called *distinguished squares*.

A cd-structure P on $\mathcal{C}$ gives an associated Grothendieck topology t_P, the coarsest one for which every commutative square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

in P gives a covering $\{A \to X, Y \to X\}$ for t_P.

Definition 1.2. Let P be a cd-structure on $\mathcal{C}$ and t_P the associated Grothendieck topology. The class of simple coverings is the smallest class of coverings of t_P such that

i) The isomorphisms are simple coverings.

ii) Every distinguished square

$$\begin{tikzcd} B \arrow[r] \arrow[d] & Y \arrow[d] \\ A \arrow[r] & X \end{tikzcd}$$

in P gives a simple covering $\{A \to X, Y \to X\}$.

iii) The composition of simple coverings is simple.

Observation 1.3. The finite simple coverings are the coverings obtained composing a finite number of coverings coming from distinguished squares.

Voevodsky defines three important properties for the topologies coming from cd-structures: completeness, regularity and boundedness. What matters to our discussion is that complete and regular cd-topologies are nice because the sheaf condition can be checked only on distinguished squares. If in addition the cd-topology is bounded the sheaves have finite cohomological dimension. The topologies that we are interested in, the Nisnevich and cdh ones, are complete, regular and bounded, as shown by Voevodsky in [Voe00b].

1.2 Splitting sequences

Now we restrict to the category $\mathbf{Sch}_k$.

Definition 1.4. A *splitting sequence* for a morphism of schemes $f\colon U \to X$ is a finite filtration

$$\emptyset = X_0 \subset X_1 \subset \cdots \subset X_r = X$$

of X such that

i) Every morphism $X_i \to X_{i+1}$ is a closed embedding

ii) The morphism $U \times_X X_i \to X_i$ admits a section over $X_i \setminus X_{i-1}$ for every i, i.e. there are morphisms $\sigma_i \colon X_i \setminus X_{i-1} \to U \times_X X_i$ such that $f\sigma_i = \mathrm{id}_{X_i \setminus X_{i-1}}$.

Definition 1.5. A morphism $f \colon U \to X$ has the *rational point lifting property* if every point $x \colon \operatorname{Spec} k \to X$ admits a lifting $\tilde{x} \colon \operatorname{Spec} k \to U$ such that

i) The diagram

$$\begin{tikzcd} & U \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[ur, "\tilde{x}"] \arrow[r, "x"] & X \end{tikzcd}$$

is commutative,

ii) f induces an isomorphism on residue fields $k(x) \to k(\tilde{x})$.

Proposition 1.6. *A morphism of schemes $f \colon U \to X$ has a splitting sequence if, and only it has the rational point lifting property.*

Proof. See [Voe00b] lemma 2.15. □

1.3 The Nisnevich topology

The Nisnevich topology is a Grothendieck topology (strictly) in between the Zariski and the étale topologies. In some sense has the good properties of both topologies avoiding the bad ones. For example, We will see that the Nisnevich topology is generated by distinguished squares, like the Zariski topology, and is fine enough to allow an easy local description of closed embeddings of smooth schemes. They are, Nisnevich locally, like the zero section embedding of a vector bundle.

Definition 1.7. The *Nisnevich topology* on $\mathbf{Sch}_k$ is the Grothendieck topology given by the covering families $\{U_i \to X\}$ such that

i) The morphisms $U_i \to X$ are étale.

ii) The morphism $U = \coprod U_i \to X$ is an epimorphism.

iii) The morphism $U = \coprod U_i \to X$ has the rational point lifting propery.

Definition 1.8. We say that a diagram of schemes

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is a *Nisnevich distinguished square* if

i) it is pull-back diagram,

ii) the map $U \to X$ is an open embedding,

iii) the map $Z \to X$ is an étale morphism which induces an isomorphism $(Z \setminus W)_{\mathrm{red}} \to (X \setminus U)_{\mathrm{red}}$ with the reduced scheme structures.

Observation 1.9. The Nisnevich distinguished squares form a cd-structure on $\mathbf{Sch}_k$. Moreover, a Nisnevich distinguished square gives a covering $\{U \to X, Z \to X\}$ which is a covering for the Nisnevich topology.

Proposition 1.10. *Every Nisnevich covering in $\mathbf{Sch}_k$ admits a finite simple refinement for the Nisnevich cd-structure.*

Proof. See [Voe00b] proposition 2.16. □

Corollary 1.11. *The Nisnevich topology on* $\mathbf{Sch}_k$ *is generated by the Nisnevich distinguished squares, in the sense that the Nisnevich topology is the coarsest topology in which the Nisnevich distinguished squares give coverings.*

Proof. The fact that Nisnevich distinguished squares give Nisnevich covers, says that the Nisnevich topology is finer than the one generated by the distinguished squares. The converse comes from the proposition. □

1.4 The abstract blow up topology

Definition 1.12. The *abstract blow-up topology* is the Grothendieck topology on $\mathbf{Sch}_k$ given by the coverings $\{U_i \to X\}$ such that $U = \coprod U_i \to X$ is a proper epimorphism with the rational point lifting property.

Definition 1.13. Let

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

be a diagram of schemes. We say it is an *abstract blow-up square* if

i) it is a pullback square,

ii) the map $Y \to X$ is a closed embedding,

iii) the map $Z \to X$ is a proper map that induces an isomorphism on $Z \backslash W \to X \backslash Y$.

We say that it is a *blow-up square* if

i) it is a pullback square,

ii) the map $Y \to X$ is a closed embedding,

iii) Z is obtained by blowing up X with center Y.

Both, abstract blow-up squares and blow-up squares give cd-structures on $\mathbf{Sch}_k$.

Observation 1.14. Every abstract blow-up

$$\begin{tikzcd} W \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

gives a covering $\{Y \to X, Z \to X\}$ for the abstract blow-up topology.

Observation 1.15. Note that a blow-up square is an abstract blow-up, but the converse does not hold in general for singular schemes.

Proposition 1.16. *Every abstract blow-up covering* $\{U_i \to X\}$ *admits a finite simple refinement in the abstract blow-up cd-structure.*

Proof. See [Voe00b] proposition 2.17. □

Proposition 1.17. *If* k *admits resolution of singularities (for example if* $\operatorname{char} k = 0$*), every abstract blow-up covering* $\{U_i \to X\}$ *of a smooth scheme* X *by smooth schemes admits a finite simple refinement in the blow-up cd-structure.*

Proof. By the proposition 1.16 it only remains to check that a covering $\{Y \to X, Z \to X\}$ coming from an abstract blow-up square can be refined to a covering obtained by composition of blow-up's of smooth schemes with smooth centers. But this follows from resolution of singularities. □

Corollary 1.18. *The abstract blow-up topology on* $\mathbf{Sch}_k$ *is generated by the abstract blow-up squares. Moreover, if k admits resolution of singularities, the abstract blow-up topology on* $\mathbf{Sm}_k$ *is generated by actual blow-up squares.*

1.5 The cdh topology

The cdh topology is the combination of the Nisnevich and abstract blow-up topology.

Definition 1.19. The *cdh topology* on $\mathbf{Sch}_k$ is the coarsest Grothendieck topology such that every Nisnevich and abstract blow-up covering is also a cdh covering.

Proposition 1.20. *Let* $\{U_i \to X\}$ *be a cdh covering. Then there is a factorization*

$$U = \coprod U_i \to X' \to X$$

such that $U \to X'$ *is a Nisnevich covering and* $X' \to X$ *an abstract blow-up covering.*

Proof. See [SV00] proposition 5.9. □

Now we will consider the *combined cd-structure* in which the distinguished squares are both, Nisnevich distinguished squares and abstract blow-ups. Then we have

Corollary 1.21. *Every cdh covering admits a finite simple refinement for the combined cd structure.*

Proof. This follows from proposition 1.20 and the fact that the Nisnevich and abstract blow-up coverings admit simple refinements. □

2 Presheaves of spectra

In this section we recall some facts about presheaves of spectra. We work on a fixed category **Spt** of spectra, for example the Bousfield-Kan spectra with its model structure as described in [BF78].

Definition 2.1. A *presheaf of spectra* on a category $\mathcal{C}$ is a contravariant functor $F\colon \mathcal{C}^{op} \to \mathbf{Spt}$ from $\mathcal{C}$ to the category of spectra. A morphism of presheaves of spectra is a natural transformation. We denote the category of presheaves of spectra by $\mathbf{PreSh}(\mathcal{C}, \mathbf{Spt})$.

Definition 2.2. Let F be a presheaf of spectra. The *homotopy presheaf* $\pi_n F$ associated to F is the presheaf of abelian groups defined by

$$\pi_n F(U) = \pi_n(F(U)).$$

The category $\mathcal{C}$ will often come with a Grothendieck topology. Then, on can form the associated sheaf of the homotopy presheaves $\pi_n F$.

Definition 2.3. Let F be a presheaf of spectra on a site $\mathcal{C}$. The *homotopy sheaves* $\tilde{\pi}_n F$ are the sheaves associated to the presheaf $\pi_n F$ with respect to the topology on $\mathcal{C}$.

2.1 Model category structure on presheaves of spectra

Definition 2.4. Let $\mathcal{C}$ be a site, and $F, G \in \mathbf{PreSh}(\mathcal{C}, \mathbf{Spt})$. A map $f\colon F \to G$ of presheaves of spectra is called

- An *injective cofibration* if it is a level-wise cofibration of spectra.
- A *global weak equivalence* if it induces isomorphisms $f\colon \pi_n F \to \pi_n G$ on homotopy presheaves
- A *local weak equivalence* if it induces isomorphisms $f\colon \hat{\pi}_n F \to \hat{\pi}_n G$ on homotopy sheaves.
- A *local injective fibration* if it has the right lifting property with respect to cofibrations that are local weak equivalences.

Observation 2.5. A morphism $f\colon F \to G$ is a global weak equivalence if and only it induces degree-wise stable weak equivalences.

If the site $\mathcal{C}$ has enough points, a morphism $f\colon F \to G$ is a local weak equivalence if and only if it induces stalk-wise stable weak equivalences (see [Jar87b]).

Now we describe the Jardine model structure on presheaves of spectra.

Theorem 2.6. *The category of presheaves of spectra together with the injective cofibrations, local weak equivalences and local injective fibrations described above is a proper simplicial model category.*

Proof. See [Jar87b]. □

2.2 Hypercohomology and descent

We can use presheaves of spectra as coefficients for a cohomology, generalising at the same time the sheaf cohomology of abelian sheaves and the generalized cohomologies associated to a spectrum.

First of all, let's chose a fibrant replacement functor for the Jardine model structure on presheaves of spectra. We will denote the fibrant replacement of a presheaf of spectra F by $\mathbb{H}(\cdot, F)$. The homotopy groups of this fibrant replacement are called the *hypercohomology groups* of F,

$$\mathbb{H}^n(U, F) = \pi_{-n}\mathbb{H}(U, F).$$

Observation 2.7. Given a sheaf of abelian groups A, we can form a presheaf of Eilenberg-MacLane spectra as follows

$$KA(U) = K(A(U)),$$

which assigns as sections over U the Eilenberg-MacLane spectrum associated to the abelian group $A(U)$. An easy computation shows that the hypercohomology of the presheaf of spectra KA coincides with the sheaf cohomology of A, i.e.

$$\mathbb{H}^i(U, KA) \simeq H^i(U, A).$$

Observation 2.8. Consider the category **Top** of topological spaces. A spectrum T gives rise to a generalized cohomology $h^n_T\colon \mathbf{Top}^{\mathrm{op}} \to \mathbf{Ab}$ given by

$$h^n_T(U) = [\Sigma^\infty U, \Omega^n T].$$

We can define a presheaf of spectra on **Top** associated to the spectrum T taking the internal hom spectrum instead of the homotopy classes of maps.

$$F_T(U) = \mathbf{Hom}(\Sigma^\infty U, T)$$

Now it is easy to see that

$$h^n_T(U) \simeq \mathbb{H}^n(U, F_T)$$

Definition 2.9. A presheaf of spectra $F\colon \mathcal{C}^{op} \to \mathbf{Spt}$ is said to satisfy *descent* if the fibrant replacement map $F \to \mathbb{H}(\cdot, F)$, which, a priori, is a local weak equivalence, is in fact a global weak equivalence, i.e. for every $U \in \mathcal{C}$ the map

$$F(U) \to \mathbb{H}(U, F)$$

is a weak equivalence.

Observation 2.10. Voevodsky calls a presheaf of spectra that satisfies descent *flasque*. Other authors call them *quasi-fibrant*.

Observation 2.11. When $\mathcal{C} = \mathbf{Sch}_k$, we will say that a presheaf of spectra satisfies Zariski or Nisnevich descent if it satisfies descent in the model category associated with the Nisnevich or Zariski topologies.

2.3 The Mayer-Vietoris property and Cech diagrams

Definition 2.12. We say that a commutative square of spectra

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "j"] & B \arrow[d, "f"] \\ C \arrow[r, "g"] & D \end{tikzcd}$$

is *homotopy cartesian* if the corresponding square in the homotopy category of spectra **HoSpt** induces a distinguished triangle

$$A \xrightarrow{(i,-j)} B \oplus C \xrightarrow{f+g} D \longrightarrow A[1]$$

Observation 2.13. This is equivalent to say that A is a homotopy pullback of the corresponding diagram or that D is a homotopy pushout.

Now we give an abstract version of the Mayer-Vietoris property. For the rest of this section the category $\mathcal{C}$ will be the category $\mathbf{Sch}_k$ together with the Nisnevich or cdh topology.

Definition 2.14. Let $F \in \mathbf{PreSh}(\mathcal{C}, \mathbf{Spt})$. We say that F has the *Mayer-Vietoris property* if it sends every distinguished square in $\mathcal{C}$ to a homotopy cartesian square of spectra.

A cartesian square in $\mathcal{C}$ which is sent to a homotopy cartesian square of spectra is said to be *F-acyclic*.

If we take a finite covering $\mathfrak{U} = \{U_i \to X\}_{i \in I}$ of more than two objects, we can generalise this in the Cech way. First some notation.

Definition 2.15. Let $\square_I^+$ the category associated to the partially ordered set of subsets of I. Let $\square_I$ full subcategory of $\square_I^+$ obtained by removing the object associated to the empty set.

That is, $\square_I^+$ has a cubical shape with the arrows pointing away from the object represented by the empty set.

Example 2.16.

$$\square_{\{1,2\}}^+ = \begin{tikzcd} \bullet & \bullet \arrow[l] \\ \bullet \arrow[u] & \bullet \arrow[l] \arrow[u] \end{tikzcd} \qquad \square_{\{1,2\}} = \begin{tikzcd} \bullet & \bullet \arrow[l] \\ \bullet \arrow[u] & \end{tikzcd}$$

We will use these categories as shapes for diagrams of spectra. A diagram of spectra of shape $\square_I^+$ will be a covariant functor $D\colon \square_I^+ \to \mathbf{Spt}$. Observe that a diagram of shape $\square_I^+$ can be thought as an augmented diagram of shape $\square_I$, that is, giving $\square_I^+$-diagram $D\colon \square_I^+ \to \mathbf{Spt}$ is the same as giving a $\square_I$-diagram $D\colon \square_I \to \mathbf{Spt}$ together with maps $D(\emptyset) \to D(\{i\})$ for every $i \in I$ making the obvious diagrams commutative.

So, given a presheaf of spectra and a covering $\mathfrak{U}$ we get a Cech cubical diagram of spectra $\check{C}_{\square^+}(\mathfrak{U}, F)\colon \square_I^+ \to \mathbf{Spt}$ such that for every $s \in \square_I^+$

$$\check{C}_{\square^+}(\mathfrak{U}, F)_s = F(\prod_{i \in s}{}_X U_i)$$

where the product must be understood as a fibered product over X. As $\check{C}_{\square^+}(\mathfrak{U}, F)_\emptyset = F(X)$, by the previous remark we can think this diagram as an augmented $\square_I$-diagram

$$F(X) \to \check{C}_\square(\mathfrak{U}, F)_\bullet.$$

Proposition 2.17. *A presheaf of spectra $F \in \mathbf{PreSh}(\mathcal{C}, \mathbf{Spt})$ has the Mayer-Vietoris propery if, and only if for every finite covering $\mathfrak{U} = \{U_i \to X\}$ the above augmentation induces a weak equivalence*

$$F(X) \to \operatorname*{holim}_{\square_I} \check{C}_\square(\mathfrak{U}, F)_\bullet$$

Proof. This is done by induction on the dimension of the cubes and apply the Fubini property of homotopy limits. □

Finally, we relate the Mayer-Vietoris property with the descent property.

Theorem 2.18. *Let $\mathcal{C} = \mathbf{Sch}_k$ with the Nisnevich or cdh topology. A presheaf of spectra $F \in \mathbf{PreSh}(\mathcal{C}, \mathbf{Spt})$ satisfies descent if, and only if it has the Mayer-Vietoris property.*

Proof. It follows from [Voe00a] lemma 3.5 applied to the fibrant resolution morphism and the fact that fibrant presheaves have the Mayer-Vietoris property. □

2.4 Homotopy invariant presheaves of spectra

Definition 2.19. We say that a presheaf of spectra on $\mathbf{Sch}_k$ is *homotopy invariant* if for every scheme X, the projection $X \times \mathbb{A}^1 \to X$ induces a weak equivalence of spectra $F(X) \to F(X \times \mathbb{A}^1)$.

Proposition 2.20. *Let F be a presheaf of spectra on $\mathbf{Sch}_k$ satisfying Zariski descent. Then F is homotopy invariant if, and only if for every vector bundle $p\colon E \to X$ the induced map $F(X) \to F(E)$ is a weak equivalence.*

Proof. Let us assume F is homotopy invariant. Let $\mathfrak{U} = \{f_i\colon U_i \to X\}$ be a Zariski covering such that $E_i = p^*U_i \to U_i$ is a trivial vector bundle. The homotopy invariance says that $F(U_i) \to F(E_i)$ is a weak equivalence. Together with Zariski descent we have

$$\begin{aligned} F(X) &\simeq \operatorname{holim} \check{C}(\mathfrak{U}, F)_\bullet \\ &\simeq \operatorname{holim} \check{C}(p^*\mathfrak{U}, F)_\bullet \\ &\simeq F(E). \end{aligned}$$

□

3 Descent for blow-ups

In this section we prove that a presheaf of spectra on $\mathbf{Sm}_k$ which is homotopy invariant and satisfies Nisnevich descent automatically satisfies cdh descent. This is Theorem 3.3.1 in [Mor99].

The proof is done in two parts. The homotopy invariance condition is used to show that this result is true for the blow-up of the zero section embedding of a vector bundle. Next we use Nisnevich descent to reduce to this special case.

3.1 Special case: zero section of a vector bundle

Now we will study the blow-up of the zero section embedding of a vector bundle $p\colon E \to X$. Recall that the exceptional divisor of the blow-up is the projectivised of the normal bundle of the embedding $s\colon X \to E$, which in this case is $\mathbb{P}E$. So, we have a blow-up square

$$\begin{tikzcd} \mathbb{P}E \arrow[r] \arrow[d, "q"] & \tilde{E} \arrow[d, "p"] \\ X \arrow[r, "s"] & E \end{tikzcd}$$

Recall that on $\mathbb{P}E$ there is a tautological line bundle $\mathscr{O}_{\mathbb{P}E}(-1)$ whose total space is identified as the closed subscheme of q^*E whose fiber over $y \in \mathbb{P}E$ corresponds to the points in q^*E lying in the line y.

Proposition 3.1. *In the above situation, the blow-up square of the zero section embedding of the vector bundle $E \to X$ is isomorphic to the cartesian square*

$$\begin{tikzcd} \mathbb{P}E \arrow[r, "s'"] \arrow[d] & \mathscr{O}_{\mathbb{P}E}(-1) \arrow[d] \\ X \arrow[r, "s"] & E \end{tikzcd}$$

where s' is the zero section embedding of the line bundle $\mathscr{O}_{\mathbb{P}E}(-1) \to \mathbb{P}E$.

Proof. The property of being a blow-up square is (Zariski) local on X, so restricting to an open cover we can assume the vector bundle is trivial.

Assume for a moment that $X = \operatorname{Spec} k$. The zero section embedding is nothing more than a point in $\mathbb{A}^n$, and the blow-up $\tilde{E}$ can be described as

$$\tilde{E} = \left\{ (y, z) \in \mathbb{P}^{n-1} \times \mathbb{A}^n \mid z \in y \right\}.$$

This coincides with the total space of $\mathscr{O}_{\mathbb{P}E}(-1)$, so in that case the blow-up of the zero section embedding is given by the square

$$\begin{tikzcd} \mathbb{P}^{n-1} \arrow[r] \arrow[d] & \mathscr{O}_{\mathbb{P}^{n-1}}(-1) \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

Now taking the fibered product (over $\operatorname{Spec} k$) of the previous diagram with X, gives the cartesian square

$$\begin{tikzcd} X \times \mathbb{P}^{n-1} \arrow[r] \arrow[d] & X \times \mathscr{O}_{\mathbb{P}^{n-1}}(-1) \arrow[d] \\ X \arrow[r] & X \times \mathbb{A}^n \end{tikzcd}$$

which is also a blow-up square. But now we see that this is the blow-up of the zero section embedding of a trivial vector bundle over $E = X \times \mathbb{A}^n \to X$, and that the top right corner corresponds to the line bundle $\mathscr{O}_{\mathbb{P}E}(-1)$ over $\mathbb{P}E$, precisely what we wanted to show. □

Corollary 3.2. *Let $F\colon \mathbf{Sm}_k^{op} \to \mathbf{Spt}$ be a homotopy invariant presheaf of spectra satisfying Zariski descent. Then the square*

$$\begin{tikzcd} \mathbb{P}E \arrow[r] \arrow[d, "q"] & \tilde{E} \arrow[d, "p"] \\ X \arrow[r, "s"] & E \end{tikzcd}$$

obtained by blowing-up the zero section embedding of a vector bundle E is F-acyclic.

Proof. Observe that the two horizontal maps of the blow up square of the zero section embedding are zero sections of vector bundles. The lower one by hypothesis, and the upper one is the zero section of the line bundle $\mathscr{O}_{\mathbb{P}E}(-1)$ by the proposition. But now, homotopy invariance together with Zariski descent imply that F sends the horizontal maps to weak equivalences, and the resulting square of spectra is trivially homotopy cartesian. □

3.2 Reduction to the special case

First we recall a theorem in [EGA4] which says how a closed embedding of smooth schemes looks like locally in the étale topology.

Proposition 3.3. *Let $f\colon Y \to X$ be a closed embedding of smooth schemes. Suppose that $n = \dim X$ and $k = \dim Y$. Then there exist an open cover of X by affine open subschemes U_i together with étale maps $g_i\colon U_i \to \mathbb{A}^n$ such that they fit in pull-back squares*

$$\begin{tikzcd} Y \times_X U_i \arrow[r, "f"] \arrow[d] & U_i \arrow[d, "g_i"] \\ \mathbb{A}^k \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

where the morphism $\mathbb{A}^k \to \mathbb{A}^n$ is the inclusion in the first k coordinates.

Proof. See [EGA4] Corollaire 17.12.2.d. □

Observation 3.4. This proposition says that, locally in the étale topology, a closed embedding between smooth schemes looks like the embedding $\mathbb{A}^k \to \mathbb{A}^n$.

Let's give now a local description of a closed embeddings between smooth schemes for the Nisnevich topology. It will follow from the following result.

Proposition 3.5 ([Mor99] Lemme 3.2.9)**.** *Consider a pullback square*

$$\begin{tikzcd} Y \arrow[r, "i"] \arrow[d] & X \arrow[d] \\ \mathbb{A}^k \arrow[r] & \mathbb{A}^n \end{tikzcd}$$

where the right vertical map is étale and separated. Then there exist a smooth scheme Z fitting in a commutative diagram

$$\begin{tikzcd}
Y \arrow[r] & X \\
Y \arrow[u, "\mathrm{id}"] \arrow[r] \arrow[d, "\mathrm{id}"'] & Z \arrow[u] \arrow[d] \\
Y \arrow[r] & Y \times \mathbb{A}^{n-k}
\end{tikzcd}$$

where the two squares are cartesian and the right vertical maps are étale.

Proof. Consider the product of $Y \to \mathbb{A}^k$ and the identity $\mathrm{id}\colon \mathbb{A}^{n-k} \to \mathbb{A}^{n-k}$. This gives an étale map $Y \times \mathbb{A}^{n-k} \to \mathbb{A}^n$. Now define

$$Z' = X \times_{\mathbb{A}^n} (Y \times \mathbb{A}^{n-k}).$$

Note that the projections $\pi_1'\colon Z' \to X$ and $\pi_2'\colon Z' \to Y \times \mathbb{A}^{n-k}$ are étale maps.

We can form the following commutative cube

$$\begin{tikzcd}
& Y \arrow[rr] \arrow[dd] & & Y \times \mathbb{A}^{n-k} \arrow[dd] \\
Y \times_{\mathbb{A}^k} Y \arrow[ur, "\pi_2"] \arrow[rr] \arrow[dd, "\pi_1"'] & & Z' \arrow[ur] \arrow[dd] & \\
& \mathbb{A}^k \arrow[rr] & & \mathbb{A}^n \\
Y \arrow[ur] \arrow[rr] & & X \arrow[ur] &
\end{tikzcd}$$

where the upper horizontal map $Y \to Y \times \mathbb{A}^{n-k}$ on the back face is the zero section embedding.

Observe that the left, right, lower and back faces are pullbacks by definition, therefore the front and upper faces are also pullback squares. So we get two cartesian squares

$$\begin{tikzcd}
Y \times_{\mathbb{A}^k} Y \arrow[r] \arrow[d, "\pi_1"] & Z' \arrow[d, "\pi_1'"] \\
Y \arrow[r] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
Y \times_{\mathbb{A}^k} Y \arrow[r] \arrow[d, "\pi_2"] & Z' \arrow[d, "\pi_2'"] \\
Y \arrow[r] & Y \times \mathbb{A}^{n-k}
\end{tikzcd}$$

Now the diagonal embedding $Y \to Y \times_{\mathbb{A}^k} Y$ is open because $Y \to \mathbb{A}^k$ is unramified, and closed because it is separated. So we have a decomposition

$$Y \times_{\mathbb{A}^k} Y = Y \sqcup Y',$$

where Y and Y' are closed subschemes of $Y \times_{\mathbb{A}^k} Y$ and therefore both are closed subschemes of Z'. Finally, define

$$Z = Z' \setminus Y'$$

Restricting the upper map of the previous two pullback squares to $Y \to Z$ and sticking them together, we get

$$\begin{tikzcd} Y \arrow[r] & X \\ Y \arrow[u, "\mathrm{id}"] \arrow[r] \arrow[d, "\mathrm{id}"'] & Z \arrow[u] \arrow[d] \\ Y \arrow[r] & Y \times \mathbb{A}^{n-k} \end{tikzcd}$$

as we wanted. □

Observation 3.6. Note that that in the previous proposition, Z gives two Nisnevich distinguished squares

$$\begin{tikzcd} U \times_X Z \arrow[r] \arrow[d] & Z \arrow[d] \\ U \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} V \times_{Y \times \mathbb{A}^{n-k}} Z \arrow[r] \arrow[d] & Z \arrow[d] \\ V \arrow[r] & Y \times \mathbb{A}^{n-k} \end{tikzcd}$$

where $U = X \setminus Y$ and $V = Y \times \mathbb{A}^{n-k} \setminus Y$.

Propositions 3.3 and 3.5 together say that a closed embedding between smooth schemes, locally in the Nisnevich topology, looks like the zero section embedding of a vector bundle.

Now mimicking the suggested proof of theorem 3.3.1 in [Mor99], we have the following result.

Theorem 3.7. *Let* $F\colon \mathbf{Sm}/k^{\mathrm{op}} \to \mathbf{Spt}$ *be a homotopy invariant presheaf of spectra on the category of smooth schemes satisfying Nisnevich descent. Then* F *has the Mayer-Vietoris property for blow-up squares (i.e. sends blow-up squares to homotopy cartesian squares of spectra).*

Proof. Let

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & \tilde{X} \arrow[d] \\ Y \arrow[r] & X \end{tikzcd} \tag{1}$$

be a blow-up square of smooth schemes. By 3.3 X has a Zariski open cover $\mathfrak{U} = \{U_i \to X\}$ such that we have cartesian squares

$$\begin{tikzcd} Y_i \arrow[r] \arrow[d] & U_i \arrow[d] \\ \mathbb{A}^k \arrow[r] & \mathbb{A}^n \end{tikzcd} \tag{2}$$

with étale vertical maps. Here $Y_i = Y \times_X U_i$. By the proposition 3.5 we get a diagram

$$\begin{tikzcd} Y_i \arrow[r] & U_i \\ Y_i \arrow[u, "\mathrm{id}"] \arrow[r] \arrow[d, "\mathrm{id}"'] & Z_i \arrow[u] \arrow[d] \\ Y_i \arrow[r] & Y_i \times \mathbb{A}^{n-k} \end{tikzcd} \tag{3}$$

in which the two squares are cartesian and the two right vertical maps are étale. Let us call $E_i = Y_i \times \mathbb{A}^{n-k}$ and see it as a trivial vector bundle $E_i \to Y_i$. The lower map in (3) is the zero section embedding $Y_i \to E_i$. Now we blow-up U_i along Y_i and E_i along Y_i, obtaining two commutative squares attached to the diagram (3). Moreover, we can pull back both squares over the middle map $Y_i \to Z_i$ in (3) and both coincide with the blow up of Z_i along Y_i. So, blowing up the schemes on the right with center on the left gives a big commutative diagram

$$\begin{tikzcd}
& Y_i \arrow[rr] & & U_i \\
\tilde{Y}_i \arrow[ur] \arrow[rr] & & \tilde{U}_i \arrow[ur] & \\
& Y_i \arrow[uu] \arrow[rr] \arrow[dd] & & Z_i \arrow[uu] \arrow[dd] \\
\tilde{Y}_i \arrow[uu] \arrow[ur] \arrow[rr] \arrow[dd] & & \tilde{Z}_i \arrow[uu] \arrow[ur] \arrow[dd] & \\
& Y_i \arrow[rr] & & E_i \\
\tilde{Y}_i \arrow[ur] \arrow[rr] & & \tilde{E}_i \arrow[ur] &
\end{tikzcd} \tag{4}$$

We proceed in two steps. First of all we show that the upper face in the diagram is F-acyclic. The second step consists in gluing all the squares

$$\begin{tikzcd}
\tilde{Y}_i \arrow[r] \arrow[d] & \tilde{U}_i \arrow[d] \\
Y_i \arrow[r] & U_i
\end{tikzcd}$$

coming from the upper face in (4), which are shown to be F-acyclic in step 1, to prove that the commutative square (1) is also F-acyclic, as we wanted to prove.

Step 1: Looking at (4) we see the following

- It follows from 3.2 that the lower face is F-acyclic because it is the blow up of the zero section embedding of a vector bundle.
- Because the central horizontal square in (4) is obtained by pulling back either the upper or the lower horizontal squares, all the vertical maps on the left are identities. So both squares on the left face look like

$$\begin{tikzcd}
\tilde{Y}_i \arrow[r, "\mathrm{id}"] \arrow[d] & \tilde{Y}_i \arrow[d] \\
Y_i \arrow[r, "\mathrm{id}"] & Y_i
\end{tikzcd}$$

  and are obviously F-acyclic.
- The two squares on the right face are also F-acyclic. We show it for the upper one, but the same holds for the lower. Let $W_i = U_i \setminus Y_i$. Because the upper right square is a blow up, we can fit it in a commutative diagram

$$\begin{tikzcd}
W_i \times_X Z_i \arrow[r] \arrow[d] & \tilde{Z}_i \arrow[r] \arrow[d] & Z_i \arrow[d] \\
W_i \arrow[r] & \tilde{U}_i \arrow[r] & U_i
\end{tikzcd}$$

> The outer rectangle is a Nisnevich distinguished square as pointed out in the observation 3.6. For the same reason, the left square is also a Nisnevich distinguished square. As F satisfies Nisnevich descent, both, the outer rectangle and the left square are F-acyclic, so the right square is also F-acyclic.

Now we can conclude step 1. We have seen that the squares on the left, right and lower faces are F-acyclic. A small exercise on homotopy limits shows that the upper horizontal square is also F-acyclic, as we wanted.

Step 2: Let us denote the original blow up square (1) by $X_\bullet \to X$ considered as a $\square_2^{\mathrm{op}}$ augmented diagram, i.e. $X_\bullet$ represents the diagram

$$\begin{tikzcd} \tilde{Y} \arrow[r] \arrow[d] & \tilde{X} \\ Y & \end{tikzcd}$$

Recall that we obtained a Zariski covering $\mathfrak{U} = \{U_i \to X\}_{i \in I}$ of X such that the blow up square 1 restricted to each U_i is F-acyclic by step 1. So pulling back the covering $\mathfrak{U}$ to each X_t we get a covering $\mathfrak{U}_t$ and taking the Cech cubical diagrams associated to that coverings, we get a $\square_2 \times \square_I$-diagram of spectra that we denote by $\check{C}(\mathfrak{U}_\bullet, F)_\bullet$. The two dots indicate that this is a diagram of Cech diagrams, i.e. a $\square_2 \times \square_I$-diagram.

We want to check the Mayer-Vietoris propery of F on the square (1). This is equivalent to saying that the augmentation $F(X) \to F(X_\bullet)$ induces a weak equivalence

$$F(X) \to \operatorname*{holim}_{\square_2} F(X_\bullet).$$

Let's check this last condition holds,

$$\begin{aligned} F(X) &\simeq \operatorname*{holim}_{s \in \square_I} \check{C}(\mathfrak{U}, F)_s \\ &\simeq \operatorname*{holim}_{s \in \square_I} \operatorname*{holim}_{t \in \square_2} \check{C}(\mathfrak{U}_t, F)_s \\ &\simeq \operatorname*{holim}_{t \in \square_2} \operatorname*{holim}_{s \in \square_I} \check{C}(\mathfrak{U}_t, F)_s \\ &\simeq \operatorname*{holim}_{t \in \square_2} F(X_t). \end{aligned}$$

The first isomorphism comes from the generalised Mayer-Vietoris property of F for the Zariski covering $\mathfrak{U}$ of X. The second comes from the fact obtained in step 1 that the blow up square restricted to the open sets in $\mathfrak{U}$ are F-acyclic. The third isomorphism is the Fubini or interchange property of homotopy limits. Finally the fourth isomorphism comes again from the generalised Mayer-Vietoris property of F for the restricted Zariski covers $\mathfrak{U}_t$ on X_t. This concludes the proof. $\square$

Corollary 3.8. *If k admits resolution of singularities, for example if* $\operatorname{char} k = 0$*, then a presheaf of spectra satisfying the hypothesis of the previous theorem satisfies cdh descent.*

Proof. By 1.18 the blow-up squares of smooth schemes with smooth centers generate the topology of abstract blow-ups restricted on $\mathbf{Sm}_k$. Now the conclusion of theorem 3.7 together with theorem 2.18 say that F satisfies descent for the abstract blow-up topology. As F satisfies Nisnevich descent it follows that it satisfies cdh descent. $\square$

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