Strong labelings of linear forests

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Mathematics Subject Classifications: 05C78,05C05

Abstract

A (p, q)-graph G is called super edge-magic if there exists a bijective function $f : V(G) \cup E(G) \rightarrow \{1, 2, \ldots, p + q\}$ such that $f(u)+f(v)+f(uv)$ is a constant for each $uv \in E(G)$ and $f(V(G)) = \{1, 2, \ldots, p\}$.

In this paper, we introduce the concept of strong super edge-magic labeling as a particular class of super edge-magic labelings

[*]Supported by Slovak VEGA Grant 1/4005/07.

[†]Supported in part by Abdus Salam School of Mathematics, Lahore, Pakistan, where part of the research was conducted.

[‡]Supported by the Spanish Research Council under grant BFM2002-00412.

and we use such labelings in order to show that the number of super edge-magic labelings of an odd union of path-like trees, all of them of the same order m, grows at least exponentially with m.

1 Introduction.

Graphs considered in this paper are not necessarily simple, that is, they may contain loops. Also, for most of the graph theory terminology and notation utilized here, the authors refer the reader to Chartrand and Lesniak [12]; however, to make the paper reasonably self contained, we mention that for a graph G we denote the vertex set and edge set of G by $V(G)$ and $E(G)$, respectively.

For a graph or a digraph D, we will denote its adjacency matrix by $A(D)$, and for a digraph D we denote by $und(D)$ the underlying graph of D.

The seminal paper on edge-magic labelings was published in 1970 by Kotzig and Rosa [15], who called these labelings *"magic valuations"*. These were later rediscovered by Ringel and Lladó [19] who coined one of the now popular terms for them: edge-magic labelings. More recently, they have also been refereed to as edge-magic total labelings by Wallis [23]. For a (p, q)-graph G, a bijective function $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ is an *edge-magic labeling* of G if for each $uv \in E(G)$, $f(u) + f(v) + f(uv)$ is a constant k called the valence of f. If such a labeling exists, then G is said to be an edge-magic graph. In [10] Enomoto, Lladó, Nakamigawa and Ringel, defined an edge-magic labeling f of a graph G, to be a *super edge-magic labeling* of G if f has the additional property that $f(V(G)) = \{1, 2, \ldots, p\}$. Thus, a super edge-magic graph is a graph that admits a super edge-magic labeling. Super edge-magic graphs have been called strong edge-magic total graphs by Wallis [23].

In this paper, we will also consider a path P_n to be a particular case of a linear forest and for a linear forest G we introduce the concept of strong super edge-magic labeling as follows: Let G be a (p, q)-linear forest, and assume that $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ is a super edge-magic labeling of G with the extra property that if $uv \in E(G)$, $u'v' \notin E(G)$ and $d_G(u, u') = d_G(v, v') < \infty$, then we have that $f(u) + f(v) = f(u') + f(v')$. From now on, we will call this property *strong*. Then, we call f a *strong super edge-magic labeling* of G, and we call G a strong super edge-magic linear forest. For instance, for the path P_n as shown in Figure 1 the following labeling f described by Kotzig and Rosa in [15] is in fact a strong super edge-magic labeling of the path P_n.

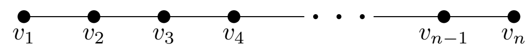

Figure 1: The path P_n.

$$f(v_i) = \begin{cases} \frac{i+1}{2} & \text{if } i \text{ is odd} \\ \lceil \frac{n}{2} \rceil + \frac{i}{2} & \text{if } i \text{ is even} \end{cases}$$

$$f(v_i v_{i+1}) = 2n - i;\ \ i \in \{1, 2, \ldots, n-1\}.$$

At this point, we define the complementary labeling $\bar{f}$ of a strong super-edge magic labeling f of a $(p, q)-$linear forest G as follows.

Let G be a (p, q)-graph, and let $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ be a strong super edge-magic labeling of G. The complementary labeling of f, denoted by $\bar{f}$, is the labeling defined by the rule

$$\begin{cases} \bar{f}(v) = p + 1 - f(v) & \forall v \in V(G) \\ \bar{f}(vw) = 2p + q + 1 - f(uv) & \forall vw \in E(G). \end{cases}$$

One can see that the following lemma is true.

Lemma 1.1 *Let G be a $(p, q)-$linear forest and let $f : V(G) \cup E(G) \to \{1, 2, \ldots, p+q\}$ be a strong super edge-magic labeling of G. Then the complementary labeling $\bar{f}$ is also a strong super edge-magic labeling of G.*

Next, consider the following result by Figueroa-Centeno *et al.* [13].

Lemma 1.2 *A (p, q)-graph G is super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$, such that the set*

$$S = \{f(u) + f(v) : uv \in E(G)\}$$

consists of q consecutive integers. In such a case, f can be extended to a super edge-magic labeling of G with valence $k = p + q + s$, where $s = min(S)$ and

$$S = \{k - (p+1), k - (p+2), \ldots, k - (p+q)\}.$$

A *graceful labeling* of a (p, q)-graph G is an injection $g : V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ such that, when each edge uv is assigned the label $|g(u) - g(v)|$, the resulting edge labels (or weights) are distinct. A graph that admits a graceful labeling is said to be *graceful*. When the graceful labeling g has the property that there exists an integer λ such that for each edge uv either $g(u) \leq \lambda < g(v)$ or $g(v) \leq \lambda < g(u)$, g is called an *α-labeling*. The number λ is called the *boundary value* of g. A graph with an α-labeling is necessarily bipartite and the boundary value must be the smallest of the two vertex labels that yield the edge label 1. A graph that admits an α-labeling is called an α-graph. Graceful labelings and α-labelings are probably the most popular kind among the several classes of the graph labelings. They were introduced by Rosa in [20]. The Ringel-Kotzig conjecture that all trees are graceful is a very popular open problem. Some methods for constructing the graceful labelings and α-labelings for certain families of trees can be found in [3, 9, 21, 22].

Let $0 \leq d < n - 1$ and let P_n be a path with $V(P_n) = \{v_i : 1 \leq i \leq n\}$ and $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\}$. Let f be an α-labeling of P_n. Then f will be called an α_d-labeling of P_n if $min\{f(v_1), f(v_n)\} = d$. It is known (see [16] or [7]) that if graph G of order n and size $n - 1$ admits an α-labeling, then G also admits a super edge-magic labeling.

Next lemma gives a relationship between α_d-labeling of P_n for n odd and $d = 0, 1, 2$, and super edge-magic labeling of cycles C_n.

Lemma 1.3 *Let P_n be a path on n vertices, $n \geq 3$ odd. If P_n admits an α_d-labeling for $d = 0, 1, 2$, then the cycle C_n admits a super edge-magic labeling.*

Proof Abrham and Kotzig [1] proved that if f is an α-labeling of the path P_n, $n = 2t+1$, and $min\{f(v_1), f(v_n)\} \leq t-1$, then $f(v_1)+f(v_n) = t = \lambda$. In this case the vertices with the values $> \lambda$ and the vertices with the values $\leq \lambda$ necessarily alternate.

Let f be an α_d-labeling of P_{2t+1}, for $d = 0, 1, 2$, satisfying $f(v_1) < f(v_n)$ and $f(v_1) \leq t - 1$. Consider the following labeling of the vertices of P_{2t+1}:

$$g(v_i) = \begin{cases} f(v_i) + 1 & \text{if} \quad i \quad \text{is even} \\ t + 1 - f(v_i) & \text{if} \quad i \quad \text{is odd.} \end{cases}$$

Since $f(v_1) \leq t-1 = \lambda-1$, then the labels assigned by g to the vertices v_i, i odd, are $1, 2, \ldots, \lambda, \lambda+1$, and those assigned to the vertices v_i, i even, are $\lambda + 2, \lambda + 3, \ldots, n$. Thus, g is an injection from $V(P_n)$ onto $\{1, 2, \ldots, n\}$. Furthermore $\{|f(v_i)-f(v_{i+1})| : i = 1, 2, \ldots, n-1\} = \{1, 2, \ldots, n-1\}$ since

f is an α-labeling and $\{g(v_i) + g(v_{i+1}) = t + 2 + |f(v_i) - f(v_{i+1})| : i = 1, 2, \ldots, n-1\} = \{t+3, t+4, \ldots, 3t+2\}$.

If $d = 0$ then $f(v_1) = 0$, $f(v_n) = t$ and $g(v_1) = t + 1$, $g(v_n) = 1$.

If $d = 1$ then $f(v_1) = 1$, $f(v_n) = t - 1$ and $g(v_1) = t$, $g(v_n) = 2$.

If $d = 2$ then $f(v_1) = 2$, $f(v_n) = t - 2$ and $g(v_1) = t - 1$, $g(v_n) = 3$.

For each previous case $g(v_1) + g(v_n) = t + 2$ and according to Lemma 1.2 the vertex labeling g can be extend to a super edge-magic labeling of cycle C_n with valence $5t + 4$. □

From now on, the symbol $N_d(n)$ will be used to denote the number of α_d-labeling of P_n. The next lemma gives an exponential lower bound for the number of super edge-magic labelings of the cycle C_n, n odd.

Lemma 1.4 *Let C_n be a cycle on n vertices, $n \geq 11$ odd. The number of super edge-magic labelings of the cycle C_n is at least $\frac{5}{4}2^{\lfloor \frac{n}{3} \rfloor} + 1$.*

Proof Abrham and Kotzig [1] proved that $N_0(n) = 1$ for every $n \geq 2$,$N_1(n) \geq \frac{1}{4}2^{\lfloor \frac{n}{3} \rfloor}$ for every $n \geq 6$, and $N_2(n) \geq 2^{\lfloor \frac{n}{3} \rfloor}$ for every $n \geq 10$. With respect to Lemma 1.3 and Abrham and Kotzig result, we have that for every $n \geq 11$ odd, the number of super edge-magic labelings of the cycle C_n is at least $N_0(n) + N_1(n) + N_2(n) \geq \frac{5}{4}2^{\lfloor \frac{n}{3} \rfloor} + 1$. □

At this point, we will state a version of Lemma 1.2 for strong super edge-magic labelings, which follows immediately from Lemma 1.2 and the definition of strong super edge-magic labeling.

Lemma 1.5 *A $(p, q)-$linear forest G is strong super edge-magic if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$, such that:*

1. *the set $S = \{f(u) + f(v) \ : \ uv \in E(G)\}$ consists of q consecutive integers,*
2. *if $uv \in E(G)$ and $d_G(u, u') = d_G(v, v') < \infty$ for two vertices $u', v' \in V(G)$, and $u'v' \notin E(G)$, then $f(u) + f(v) = f(u') + f(v')$.*

In such a case, f can be extended to a strong super edge-magic labeling of G with valence $k = p + q + s$, where $s = min(S)$ and

$S = \{f(u) + f(v) : uv \in E(G)\} = \{k - (p + 1), k - (p + 2), \ldots, k - (p + q)\}$.

Thus, due to Lemma 1.5, it is sufficient to exhibit the vertex labels of a strong super edge-magic labeling.

In his Ph.D. thesis, Barrientos [8] introduced the concept of a path-like tree as follows:
We embed the path P_n as a subgraph of the 2-dimensional grid, this is to say the cartesian product $P_r \times P_t$ of a path on r vertices with a path on t vertices. Given such an embedding, we consider the ordered set of subpaths $L_1, L_2, \dots, L_h$ which are maximal straight segments in the embedding, where the end of L_i is the beginning of L_{i+1} for any $i = 1, 2, \dots, h-1$. Suppose that $L_i \cong P_2$ for some i, $1 < i < h$, $V(L_i) = \{u_0, v_0\}$, thus $u_0 \in V(L_{i-1}) \cap V(L_i)$ and $v_0 \in V(L_i) \cap V(L_{i+1})$. Let $u \in V(L_{i-1}) - \{u_o\}$ and $v \in V(L_{i+1}) - \{v_o\}$ such that their distance on the grid is 1. The replacement of the edge u_0v_0 by the new edge uv is called an *elementary transformation* of the path P_n. We say that a tree T of order n is a *path-like tree*, when it can be obtained after a sequence of elementary transformations on an embedding of P_n in the 2-dimensional grid.

The concept of path-like tree is very similar to the concept of T_p-tree. Although in [17] it was shown that the two concepts are different. For further information on T_p-trees, the interested reader may consult [2], [11] and [17].

The labeling properties of path-like trees have been studied by many authors lately, for instance Barrientos [8], Bača *et al.* [4], [5], [6] and Ngurah *et al.* [18].

In a recent work Bača *et al.* [5] studied the super edge-magic properties of an odd number of copies of path-like trees and they proved the following result.

Theorem 1.6 *Let T_j, $1 \leq j \leq m$, be a path-like tree of order n. If m is odd, $m \geq 3$ and $n \geq 4$, then a forest $F \cong \bigcup_{j=1}^{m} T_j$ admits a super edge-magic labeling.*

It is clear that if mP_n is strong super edge-magic for m odd, then the forest $F \cong \bigcup_{j=1}^{m} T_j$, where each T_j is a path-like tree of order n, is super edge-magic. The authors in [5] provided a strong super edge-magic labeling of mP_n for m odd and it was the technique used in order to prove Theorem 1.6. The main goal of this paper is to show that the number of non-isomorphic strong super edge-magic labelings of the graph mP_n, for m odd and any n, grows very fast with m. This allows us to generate an exponential number of non-isomorphic super edge-magic labelings of the forest $F \cong \bigcup_{j=1}^{m} T_j$, where each T_j is a path-like tree of order n and m is an odd integer. We do this, using a technique introduced in [14] that involves products of digraphs. In the next lines we will describe this digraph operation.

Let D be a digraph and let $\Gamma = \{F_1, F_2, \ldots, F_s\}$ be a family of digraphs that meet the following conditions:

1) $V(F_i) = V$ for every $i \in \{1, 2, \ldots, s\}$.

2) $out(v) = in(v) = 1$ for every $v \in V(F_i)$.

3) F_i may contain loops. In other words, each F_i of the same order is either a cycle or union of cycles (possibly loops) such that each component has been oriented cyclically.

4) Each vertex of F_i takes the name of a super edge-magic labeling of F_i.

Consider any function $h : E(D) \rightarrow \Gamma$. Then the product $D \bigotimes_h \Gamma$ is a digraph with vertex set $V(D \bigotimes_h \Gamma) = V(D) \times V$ and

$((a, b), (c, d)) \in E(D \bigotimes_h \Gamma) \Longleftrightarrow [(a, c) \in E(D) \wedge (b, d) \in E(h(a, c))]$.

Notice that the adjacency matrix of $D \bigotimes_h \Gamma$, denoted by $A(D \bigotimes_h \Gamma)$, is obtained by multiplying every 0 entry of $A(D)$, where $A(D)$ denotes the adjacency matrix of D, by the $|V| \times |V|$ null square matrix, and every 1 entry of $A(D)$ by $A(h(a, c))$, where $A(h(a, c))$ denotes the adjacency matrix of $h(a, c)$. Notice that when the function h is constant, we have the classical Kronecker matrix product.

In [14] Figueroa-Centeno *et al.* call a digraph D super edge-magic if $und(D)$ is super edge-magic, and they proved the following two results:

Theorem 1.7 *Let D be a super edge-magic digraph for which each vertex takes the name of its label. Let $\Gamma = \{F_1, F_2, \ldots, F_s\}$ be a family of all super edge-magic $1-$regular labeled digraphs (each F_i, $1 \leq i \leq s$, is either a cycle or union of cycles such that each component has been oriented cyclically) of the same odd order each, where each vertex of F_i, $1 \leq i \leq s$, takes the name of its label. Consider any function $h : E(D) \longrightarrow \Gamma$. Then the digraph $D \bigotimes_h \Gamma$ is super edge-magic.*

Theorem 1.8 *Let $\vec{T}$ be any oriented tree. Let $\Gamma = \{F_1, F_2, \ldots, F_s\}$ be a family of $1-$regular digraphs of order m each. Consider any function $h : E(\vec{T}) \longrightarrow \Gamma$. Then $und(\vec{T} \bigotimes_h \Gamma) = mT$.*

As we already said before, in the paper we use this operation, in order to obtaining an exponential lower bound for the number of strong super edge-magic labelings of the union of paths, starting from the strong super edge-magic labeling of the path provided in [15].

2 Generating strong super edge-magic labelings for mP_n, m odd.

In this section, we will describe an algorithm, that will allow us to create strong super edge-magic labelings for the graph mP_n, $m = 2K + 1$, $K, n \in N$. Then, we will illustrate the algorithm with an specific example.

Input:

1. Oriented path $\vec{P_n}$ with:
   - Vertex set $V(\vec{P_n}) = \{v_i\}_{i=1}^{n}$ and $E(\vec{P_n}) = \{(v_i v_{i+1})\}_{i=1}^{n-1}$
   - Consider a function $f : V(\vec{P_n}) \longrightarrow \{1, 2, \ldots, n\}$ defined by the rule:
   $$f(v_i) = \begin{cases} \frac{i+1}{2}, & \text{if } i \text{ is odd} \\ \lceil \frac{n}{2} \rceil + \frac{i}{2}, & \text{if } i \text{ is even .} \end{cases}$$

   Observation: The labeling f, which is a strong super-edge magic labeling of $\vec{P_n}$, could be substituted by any strong super edge-magic labeling of the oriented path $\vec{P_n}$.
2. The set $\Gamma_m = \{F_1, F'_1, F_2, F'_2, \ldots, F_{\frac{s}{2}}, F'_{\frac{s}{2}}\}$ is the family of all 1−regular digraphs where each digraph of order $m = 2K+1$ is labeled in a super edge-magic way, and each vertex takes the name of its label. Each couple (F_j, F'_j) comes from the same underlying 2-regular graph, but it has been oriented in opposite way. That is to say, if a component is oriented clockwise in F_j, then the corresponding component is oriented counter clockwise in F'_j, and viceversa.
3. A function $h : E(\vec{P_n}) \longrightarrow \Gamma_m$ with
   $$h(v_{i-1}v_i) = \begin{cases} F_j, & \text{if } i \text{ is even} \\ F'_j, & \text{if } i \text{ is odd} \end{cases}$$
   for any fixed $j \in \{1, 2, \ldots, \frac{s}{2}\}$.

Observation: $$\begin{cases} f(v_{i-1}) & = x \\ f(v_i) & = x' \end{cases} \Longrightarrow h(v_{i-1}v_i) = \begin{cases} F_j, & \text{if } x + x' + \lceil \frac{n}{2} \rceil \text{ is even.} \\ F'_j, & \text{if } x + x' + \lceil \frac{n}{2} \rceil \text{ is odd.} \end{cases}$$

Algorithm:

1. Rename each vertex of $\vec{P}_n$ with the name of its label, creating a new graph $\vec{P}^l_n$.
2. Compute $\vec{P}^l_n \bigotimes_h \Gamma_m = \vec{Q}$.
3. Take $und(\vec{Q}) = Q$.
4. Let $(x_i, y_i) \in V(Q)$. Relabel the vertex (x_i, y_i) with z_i where z_i is computed using the formula

$$z_i = m(x_i - 1) + y_i$$

   creating the new graph Q^l.

Output:

$Q^l = (2K+1)P_n$ labeled in a strong super edge-magic way.

Proof By theorems 1.7 and 1.8, it is known that Q^l is super edge-magic and that $Q^l \cong (2K+1)P_n$. It only remains to be shown that the obtained labeling preserves the "strong property".

Let (x, y), (x', y') and (x'', y'') be the three vertices of Q such that

$$\left.\begin{array}{l} \{(x,y),(x',y')\} \\ \{(x',y'),(x'',y'')\} \end{array}\right\} \in E(Q).$$

If $x + x'$ is odd, then $x' + x''$ is even and viceversa. Since F_j and F'_j have opposite orientations, we obtain that $y = y''$.

Let (x_i, y_i), $1 \leq i \leq n$ be the labels of the n "consecutive" vertices of a component of Q. By the previous observation we have that $y_i = y_j$ if $|i-j|$ is even. Hence $y_i + y_{i+1} = y_{i-r} + y_{i+r+1}$ for every $i \in \{1, 2, \ldots, n-2\}$ $(r \leq min\{i-1, n-i-1, 1\})$.

Now, following the notation introduced in the algorithm, we denote by z_i the vertex of Q^l that corresponds to the vertex (x_i, y_i) in Q. We want to show that

$$z_i + z_{i+1} = z_{i-r} + z_{i+1+r}.$$

Notice that:

$$\left.\begin{array}{ll} z_i + z_{i+1} & = m[x_i + x_{i+1} - 2] + (y_i + y_{i+1}) \\ z_{i-r} + z_{i+1+r} & = m[x_{i-r} + x_{i+1+r} - 2] + (y_{i-r} + y_{i+1+r}) \end{array}\right\},$$

- $y_i + y_{i+1} = y_{i-r} + y_{i+1+r}$ by previous argument,
- $x_i + x_{i+1} = x_{i-r} + x_{i+1+r}$ since these are the same labels that we used in the oriented path $\vec{P}_n^l$, and the labeling of $\vec{P}_n^l$ is a strong super edge-magic labeling.

Therefore,

$$z_i + z_{i+1} = z_{i-r} + z_{i+1+r}.$$

□

In the following example we use the previous algorithm in order to obtain a strong super edge-magic labeling of $5P_6$. Let $\vec{P}_6$ be the following digraph, where each vertex of $\vec{P}_6$ takes the name of the strong super edge-magic labeling described in the algorithm.

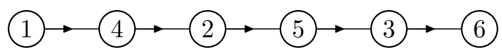

Figure 2: Vertex labeling of the digraph $\vec{P}_6$.

Let $\Gamma_5 = \{F_1, F_1', F_2, F_2', F_3, F_3'\}$ be the family of all super edge-magic 1− regular digraphs of order 5 such that each component is oriented cyclically and each vertex of each digraph takes the name of a super edge-magic labeling. Figures 3-14 illustrate the digraphs and their corresponding adjacency matrices.

Figure 15 depicts the adjacency matrix of the digraph $\vec{P}_6$.

Now, define the function $h : E(\vec{P}_6) \longrightarrow \{F_1, F_1'\}$ such that for every edge

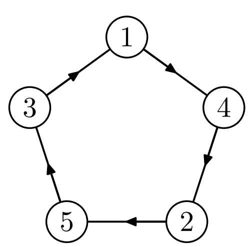

Figure 3: Digraph F_1.

$A(F_1) =$

	1	2	3	4	5
1	0	0	0	1	0
2	0	0	0	0	1
3	1	0	0	0	0
4	0	1	0	0	0
5	0	0	1	0	0

Figure 4: Adjacency matrix of F_1.

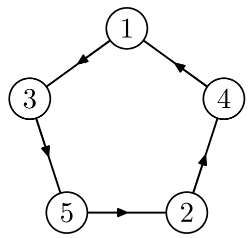

Figure 5: Digraph F_1'.

$A(F_1') =$

	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	1	0
3	0	0	0	0	1
4	1	0	0	0	0
5	0	1	0	0	0

Figure 6: Adjacency matrix of F_1'.

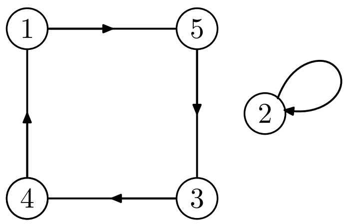

Figure 7: Digraph F_2.

$A(F_2) =$

	1	2	3	4	5
1	0	0	0	0	1
2	0	1	0	0	0
3	0	0	0	1	0
4	1	0	0	0	0
5	0	0	1	0	0

Figure 8: Adjacency matrix of F_2.

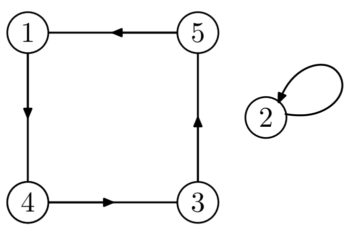

Figure 9: Digraph F_2'.

$A(F_2') =$

	1	2	3	4	5
1	0	0	0	1	0
2	0	1	0	0	0
3	0	0	0	0	1
4	0	0	1	0	0
5	1	0	0	0	0

Figure 10: Adjacency matrix of F_2'.

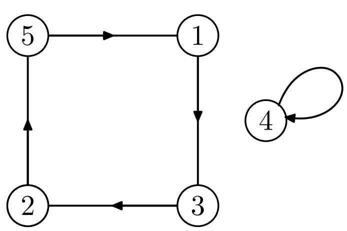

Figure 11: Digraph F_3.

$A(F_3) =$

	1	2	3	4	5
1	0	0	1	0	0
2	0	0	0	0	1
3	0	1	0	0	0
4	0	0	0	1	0
5	1	0	0	0	0

Figure 12: Adjacency matrix of F_3.

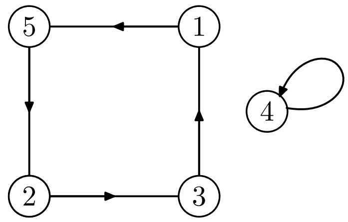

Figure 13: Digraph F'_3.

$A(F'_3) =$

	1	2	3	4	5
1	0	0	0	0	1
2	0	0	1	0	0
3	1	0	0	0	0
4	0	0	0	1	0
5	0	1	0	0	0

Figure 14: Adjacency matrix of F'_3.

$A(\overrightarrow{P_6}) =$

	1	2	3	4	5	6
1	0	0	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	0	1
4	0	1	0	0	0	0
5	0	0	1	0	0	0
6	0	0	0	0	0	0

Figure 15: Adjacency matrix of $\vec{P}_6$.

$xx' \in E(\vec{P}_6)$, we put

$$h(x,x') = \begin{cases} F_j, & \text{if } x + x' \equiv 1 \pmod 2 \\ F'_j, & \text{if } x + x' \equiv 0 \pmod 2 \end{cases}$$

The adjacency matrix of $\vec{P}_6 \bigotimes_h \{F_1, F'_1\}$ is obtained by multiplying every 0 entry of $A(\vec{P}_6)$ by the 5×5 null square matrix and every 1 entry of $A(\vec{P}_6)$ by $A(F_1)$ or $A(F'_1)$, see Figure 16.

The underlying graph of $\vec{P}_6 \bigotimes_h \{F_1, F'_1\}$ is isomorphic to $5P_6$. The adjacency matrix $A(\vec{P}_6 \bigotimes_h \{F_1, F'_1\})$ describes the corresponding vertex labeling of a strong super edge-magic labeling of $5P_6$, see Figure 17.

Let m be an odd positive integer, $m \geq 3$, and denote by $N(m)$ the number of non-isomorphic strong super edge-magic labelings of the graph mPn, $n \geq 4$. The next theorem gives an exponential lower bound for $N(m)$.

Theorem 2.1 *Let $m \geq 5$ be an odd integer. Then $N(m) \geq \frac{5}{2} 2^{\lfloor \frac{m}{3} \rfloor} + 1$.*

Proof In [14] (see Lemma 4.1) it was shown that if $h : E(D) \longrightarrow \Gamma$ and

$A(\overrightarrow{P_6} \otimes_h \{F_1, F_1'\}) =$

	1 … 5	6 … 10	11 … 15	16 … 20	21 … 25	26 … 30
1 ⋮ 5	0	0	0	$A(F_1)$	0	0
6 ⋮ 10	0	0	0	0	$A(F_1)$	0
11 ⋮ 15	0	0	0	0	0	$A(F_1)$
16 ⋮ 20	0	$A(F_1')$	0	0	0	0
21 ⋮ 25	0	0	$A(F_1')$	0	0	0
26 ⋮ 30	0	0	0	0	0	0

Figure 16: Adjacency matrix of $\vec{P}_6 \bigotimes_h \{F_1, F_1'\}$.

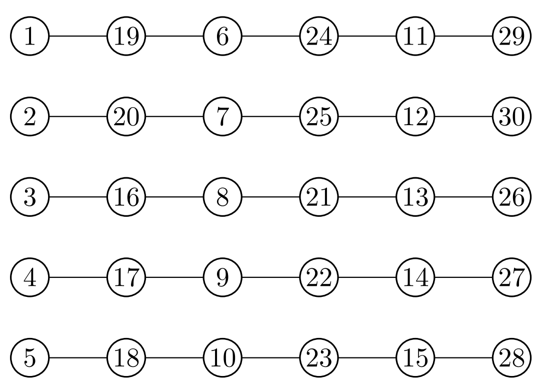

Figure 17: Vertex labeling of a strong super edge-magic labeling of $5P_6$.

$h' : E(D) \longrightarrow \Gamma$ are two different functions, then the labelings obtained applying the product are non-isomorphic.

Consider a strong super edge-magic labeling of P_n, $n \geq 4$, and $\Gamma_m = \{F_1, F_1', F_2, F_2', \ldots, F_{\frac{s}{2}}, F_{\frac{s}{2}}'\}$ as a family of super edge-magic 1-regular digraphs of order m. The family Γ_m consists of $\frac{s}{2}$ couples of the same super edge-magic labeled 1-regular digraphs (F_j, F_j'), $j = 1, 2, \ldots, \frac{s}{2}$, but with opposite orientations. With respect to the previous algorithm for generating strong super edge-magic labelings of the union of paths mP_n there are s different functions $h_j : E(D) \longrightarrow \{F_j, F_j'\}$, for $j = 1, 2, \ldots, \frac{s}{2}$ (each couple (F_j, F_j') has two possible orientations), and also s non-isomorphic strong super edge-magic labelings of mP_n.

It remains to investigate how many different couples (F_j, F_j') contains the family Γ_m.

Let us distinguish the four following cases, according to the order m.

Case 1. $m = 5$. We have three couples (F_1, F_1'), (F_2, F_2') and (F_3, F_3'), see Figures 3,5,7,9,11 and 13. With respect to two possible orientations of each couple we can see that in this case the lower bound is tight.

Case 2. $m = 7$. There are at least 14 couples. They are described in [14]. Let us rewrite them in Table 1.

graph	vertex labeling	number of possible orientations
$C_5 \cup C_1 \cup C_1$	$1-4-7-2-6 \cup 3 \cup 5$	2
$C_6 \cup C_1$	$1-6-3-2-4-7 \cup 5$	2
$C_6 \cup C_1$	$1-4-6-5-2-7 \cup 3$	2
$C_3 \cup C_3 \cup C_1$	$1-5-6 \cup 2-3-7 \cup 4$	4
C_7	$1-5-2-6-3-7-4$	2
C_7	$1-6-5-3-7-2-4$	2
C_7	$1-7-3-6-5-2-4$	2
C_7	$1-4-3-7-2-6-5$	2
C_7	$1-7-2-3-4-6-5$	2
C_7	$1-6-4-7-2-3-5$	2
C_7	$1-6-2-3-7-4-5$	2
C_7	$1-5-2-3-6-4-7$	2
C_7	$1-6-5-4-2-3-7$	2

Table 1: Super edge-magic $2-$regular graphs of order 7.

Case 3. $m = 9$. There are at least 39 couples. Table 2 shows these super edge-magic $2-$regular graphs of order 9, where each component has been oriented cyclically.

graph	vertex labeling	number of possible orientations
$C_3 \cup C_3 \cup C_3$	$1-5-9 \cup 2-6-7 \cup 3-4-8$	8
$C_3 \cup C_3 \cup C_3$	$1-6-8 \cup 2-4-9 \cup 3-5-7$	8
$C_5 \cup C_4$	$1-5-2-6-8 \cup 3-7-4-9$	4
$C_5 \cup C_4$	$2-9-5-8-4 \cup 1-6-3-7$	4
$C_5 \cup C_3 \cup C_1$	$2-9-5-4-8 \cup 1-6-7 \cup 3$	4
$C_5 \cup C_3 \cup C_1$	$1-7-6-4-8 \cup 2-5-9 \cup 3$	4
$C_5 \cup C_3 \cup C_1$	$1-5-6-2-8 \cup 3-4-9 \cup 7$	4
$C_5 \cup C_3 \cup C_1$	$2-6-4-3-9 \cup 1-5-8 \cup 7$	4
$C_6 \cup C_1 \cup C_1 \cup C_1$	$1-6-2-9-4-8 \cup 3 \cup 5 \cup 7$	2
$C_7 \cup C_1 \cup C_1$	$1-7-4-9-5-2-8 \cup 3 \cup 6$	2
$C_7 \cup C_1 \cup C_1$	$1-5-8-2-9-3-6 \cup 4 \cup 7$	2
$C_7 \cup C_1 \cup C_1$	$1-6-8-4-9-2-7 \cup 3 \cup 5$	2
$C_7 \cup C_1 \cup C_1$	$1-8-3-9-4-2-6 \cup 5 \cup 7$	2
$C_7 \cup C_1 \cup C_1$	$1-9-2-5-7-6-8 \cup 3 \cup 4$	2
$C_7 \cup C_1 \cup C_1$	$1-8-5-3-4-2-9 \cup 6 \cup 7$	2
$C_8 \cup C_1$	$1-9-4-7-5-2-6-8 \cup 3$	2
$C_8 \cup C_1$	$1-5-2-7-6-8-3-9 \cup 4$	2
$C_8 \cup C_1$	$1-7-2-4-9-3-8-6 \cup 5$	2
$C_8 \cup C_1$	$1-9-5-8-3-4-2-7 \cup 6$	2
$C_8 \cup C_1$	$1-6-3-5-8-4-2-9 \cup 7$	2
C_9	$1-6-2-7-3-8-4-9-5$	2
C_9	$1-5-7-2-6-8-3-4-9$	2
C_9	$1-5-9-2-6-7-3-4-8$	2
C_9	$1-5-7-3-4-9-2-6-8$	2
C_9	$1-6-7-3-5-9-2-4-8$	2
C_9	$1-6-7-2-4-8-3-5-9$	2
C_9	$1-6-8-3-5-7-2-4-9$	2

Table 2: Super edge-magic $2-$regular graphs of order 9.

Case 4. $m \geq 11$. If we consider only super edge-magic cyclically oriented cycles of order m, as elements of the family Γ_m, then from Lemma 1.4 it follows that, for $m \geq 11$, there exist at least $\frac{5}{4}2^{\lfloor \frac{m}{3} \rfloor}+1$ couples, where each couple comes from the same super edge-magic labeled cycle but with opposite orientations.

Since in the last three cases each couple of oriented cycles has two possible orientations, then there is at least $\frac{5}{2}2^{\lfloor \frac{m}{3} \rfloor}+1$ non-isomorphic strong super edge-magic labelings of the graph mP_n. □

According to the previous two cases, for $m = 7$ and 9, we can observe that

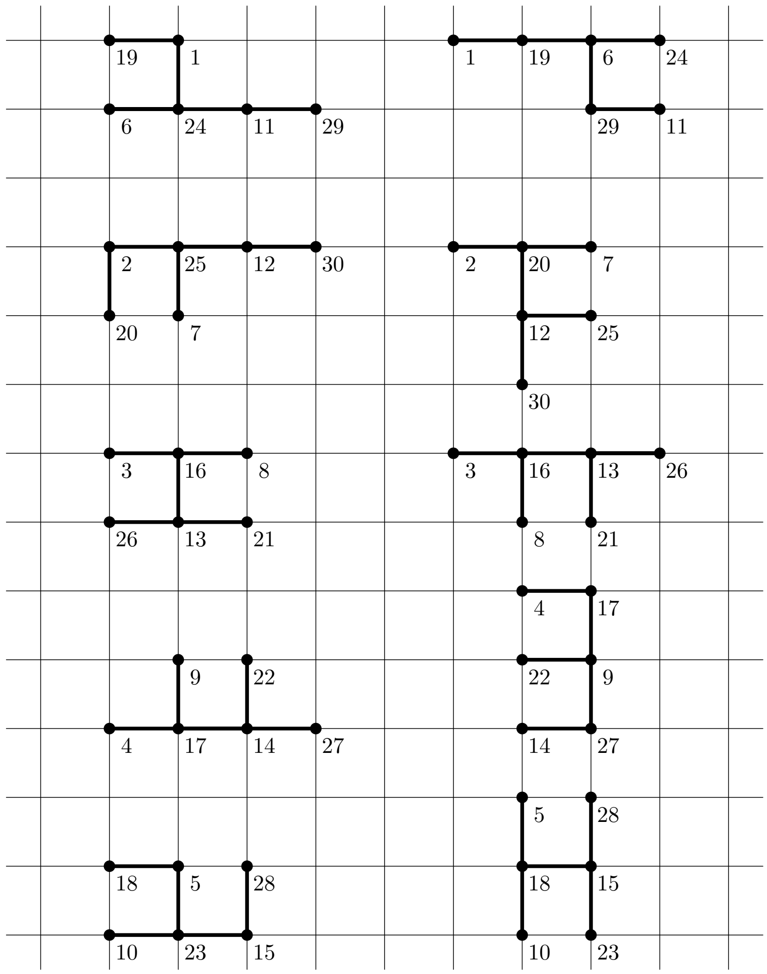

Figure 18: Vertex labeling of super edge-magic labelings of $\bigcup_{j=1}^{5} T_j$.

there exist also super edge-magic 1-regular disconnected graphs of order m. It means that really the lower bound of Theorem 2.1 is bigger.

Theorem 2.2 *Let P_n be strong super edge-magic for $n \geq 4$. If m is odd, $m \geq 5$, then a forest $F \cong \bigcup_{j=1}^{m} T_j$, where each T_j is a path-like tree of order n, admits at least $\frac{5}{2}2^{\lfloor \frac{m}{3} \rfloor} + 1$ non-isomorphic super edge-magic labelings.*

Proof From Theorem 2.1 it follows that if P_n dmits a strong super edge-magic labeling then there are at least $\frac{5}{2}2^{\lfloor \frac{m}{3} \rfloor} + 1$ non-isomorphic strong super edge-magic labelings of the disjoint union of paths $\bigcup_{j=1}^{m} P_j$ for m odd.

Consider an embedding of the disjoint union of paths $P_1, P_2, \ldots, P_m$ in the 2−dimensional grid. Let $P_j = T_j^0, T_j^1, T_j^2, \ldots, T_j^{s_j} = T_j$ be the series of trees obtained by successively applying the appropriate elementary transformations of P_j to obtain T_j, for $j = 1, 2, \ldots, m$, which keep the super edge-magic character of the path P_j.

There are different series of trees $T_j^0, T_j^1, T_j^2, \ldots, T_j^{s_j}$ for different s_j, i.e., the forest F is a disjoint union of different path-like trees $T_1, T_2, \ldots, T_m$, each of order n. For each strong super edge-magic labeling of $\bigcup_{j=1}^{m} P_j$, m odd, there exists a super edge-magic labeling of the forest $F \cong \bigcup_{j=1}^{m} T_j$.

Thus, the forest $F \cong \bigcup_{j=1}^{m} T_j$ admits at least $\frac{5}{2}2^{\lfloor \frac{m}{3} \rfloor} + 1$ non-isomorphic super edge-magic labelings. □

Figure 18 depicts two different super edge-magic labelings of the disjoint union of five path-like trees obtained by applying the appropriate elementary transformations on $5P_6$.

3 Conclusions

In this paper we introduced an algorithm that allows us to construct an exponential number of non-isomorphic strong super edge-magic labelings for $G \cong (2K + 1)P_n$. However at this point we do not know anything in general about the existence of strong super edge-magic labelings for the graph $G \cong (2K)P_n$, except for the fact that $(2K)P_2$ is not super edge-magic. To know an answer to this question is of great interest, since it would help us to understand the super edge-magic properties of an even union of path-like trees. Therefore we conclude this section with the following open problems.

Open Problem 1 *Let $G \cong (2K)P_n$; $n \neq 2$. Is G a strong super edge magic?*

Assuming that the answer to open problem 1 is yes, then it leads to open problem 2.

Open Problem 2 *Let $G \cong (2K)P_n$; $n \neq 2$. How many non-isomorphic strong super edge magic labelings does G admit?*

Open Problem 3 *Let $G \cong \bigcup_{j=1}^{2K} T_j$ be a union of an even number of path-like trees, all of them of the same order, and such that $T_j \neq P_2$ for $j = 1, 2, \ldots, 2K$. Is G a super edge-magic graph?*

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