Transitive Closure of Interval-valued Relations

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Abstract— *This paper introduces some concepts of interval-valued fuzzy relations and some of their properties: reflexivity, symmetry, T-transitivity, composition and locally reflexivity. It is also introduced the concept of T-transitive closure for an interval-valued fuzzy relation. An algorithm to compute the T-transitive closure of finite interval-valued relations, some properties and some examples are given.*

Keywords— Generalized t-norms, Interval-valued Fuzzy Relations, Interval-valued Fuzzy Sets, t-norms, t-representable t-norms, t-transitive closure.

1 Introduction

Fuzzy sets, $\mathcal{FS}$, were introduced by Zadeh in 1.965 [1]. Since then many generalizations of fuzzy sets have been proposed to model the uncertainty and the vagueness in linguistic variables replacing the unit interval by another structure such as posets or lattices [2, 3, 4]. One of these generalizations are type-2 fuzzy sets, $\mathcal{FS}2$, [5, 6, 7] were introduced by Zadeh. A Type-2 fuzzy set on a universe of discourse X, $\mathcal{FS}2$, is a fuzzy set whose membership function is another fuzzy set on [0,1]:

$$A = \{((x,u), \mu_A(x,u)) \mid \forall x \in X, \forall u \in [0,1]\}$$

Type-2 fuzzy sets have been widely studied and applied since in many cases the uncertainty can be better expressed by a fuzzy set than by a single numeric value. The problem with type2 fuzzy sets though is their computational complexity and the difficulty for an expert to select the adequate fuzzy subset as membership degree of an object to a linguistic label. This is why some simplifications have been proposed, such as the use of only some families of fuzzy sets such as triangular and trapezoidal ones. One of the most popular and useful simplification is the use of interval-valued fuzzy sets.

Interval-valued fuzzy sets ($\mathcal{IVFS}$) were introduced in the 60s by Grattan-Guinness [8], Jahn [9], Sambuc [10] and Zadeh [5]. They are extensions of classical fuzzy sets where the membership value between 0 and 1 is replaced an interval in [0,1]. They easily allow to model uncertainty and vagueness because sometimes it is easier for experts to give a "membership interval" than a membership degree to objects on a universe. $\mathcal{IVFS}$ are a special case of type-2 fuzzy sets that simplifies the calculations while preserving their richness as well. Intuitionistic fuzzy sets on X ($\mathcal{IFS}$) introduced by Atanassov [11]. In intuitionistic fuzzy sets each element has a membership degree, μ, and a non-membership degree, ν satisfying: $\mu + \nu \leq 1$.

$$A = \{(x, \mu(x), \nu(x)) \mid x \in X, \mu(x), \nu(x) \in [0,1]\}$$

The value $\pi = 1 - \mu - \nu$ is a measure of the uncertainty. This paper is organized as follows: In section three we introduce some important properties of interval-valued fuzzy relations such as reflexivity, symmetry, T-transitivity, composition and local reflexivity. Traditionally, the study of conjunctions between interval-valued fuzzy sets has been reduced to be modeled with t-representable t-norms. However, not all generalized t-norms are t-representable. Moreover, some of the non t-representable t-norms sometimes satisfy even more properties than t-representable t-norms [12]. Probably the most important property a fuzzy relation can fulfil is transitivity with respect a given t-norm. Since many times the data are given by a proximity relation P (i.e.: a reflexive and symmetric but not necessarily transitive fuzzy relation), there are some methods to obtain a transitive relation close to P to replace it when transitivity is required. The most popular way to do this is calculating its transitive closure. In section four we introduce the concept of T-transitive closure for an interval-valued fuzzy relation and its expression in a finite universe for any t-norm. A few methods to compute it and some examples are given.

2 Preliminaries

2.1 Interval-valued fuzzy sets

Definition 2.1 *[13] Let $\mathcal{L}^I = (L, \leq_L)$ be a lattice that satisfies:*

1. $L = \{[x_1, x_2] \in [0,1]^2 \text{ with } x_1 \leq x_2\}$.

2. $[x_1, x_2] \leq_L [y_1, y_2]$ *if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$*

   Trivially:

   $[x_1, x_2] <_L [y_1, y_2] \Leftrightarrow x_1 < y_1, x_2 \leq y_2$ or $x_1 \leq y_1, x_2 < y_2$
   $[x_1, x_2] =_L [y_1, y_2] \Leftrightarrow x_1 = y_1, x_2 = y_2$.

$0_L =_L [0,0]$ and $1_L =_L [1,1]$ are the smallest and the greatest elements in L respectively.

$\mathcal{L}^I$ is a complete lattice an the supremum and infimum are defined as follows.

Definition 2.2 *[12] Let $\{[v_i, w_i]\}$ be a set of intervals on L. Then*

1. *Infimum:*
   $Inf_L\{[v_i, w_i]\} \equiv [infimun\{v_i\}, infimun\{w_i\}]$

2. *Supremum:*
   $Sup_L\{[v_i, w_i]\} \equiv [supremun\{v_i\}, supremun\{w_i\}]$

Definition 2.3 *[13] An interval-valued fuzzy set A on a universe X is a mapping:*

$$A = \{(a, [x_1, x_2]) \mid a \in X, [x_1, x_2] \in L\}$$

Definition 2.4 *[13] Let X be a universe and A and B two interval-valued fuzzy sets. The equality between A and B is defined as: $A =_L B$ if and only if $A(a) =_L B(a) \, \forall a \in X$.*

Definition 2.5 *[13] Let X be a universe and A and B two interval-valued fuzzy sets. The inclusion of A in to B is defined as: $A \subseteq_L B$ if and only if $A(a) \leq_L B(a) \, \forall a \in X$.*

Definition 2.6 *[13] A negation function for interval-valued fuzzy sets $\mathcal{N}$ is a decreasing function, $\mathcal{N} : L \to L$, that satisfies:*

1. *$\mathcal{N}(0_L) =_L 1_L$*

2. *$\mathcal{N}(1_L) =_L 0_L$*

If $\mathcal{N}(\mathcal{N}([x_1, x_2])) =_L [x_1, x_2]$ for all $[x_1, x_2]$ in L then $\mathcal{N}$ is called an involutive negation.

Definition 2.7 *A strong negation function for interval-valued fuzzy sets, $\mathcal{N}$, is a involutive function, $\mathcal{N} : L \to L$, that satisfies:*

1. *$\mathcal{N}(0_L) =_L 1_L$*

2. *$\mathcal{N}(1_L) =_L 0_L$*

T-norms are generalized to the lattice $\mathcal{L}^I$ in a straightforward way.

Definition 2.8 *A generalized t-norm function [13], $\mathcal{T}$, is a monotone increasing, symmetric and associative operator, $\mathcal{T} : L^2 \to L$, that satisfies: $\mathcal{T}(1_L, [x_1, x_2]) =_L [x_1, x_2]$ for all $[x_1, x_2]$ in L.*

Due to monotony, it is easy to show:

$$\mathcal{T}(Sup_L\{[v_i, w_i]\}, [y_1, y_2]) \geq_L Sup_L\{\mathcal{T}([v_i, w_i], [y_1, y_2])\}$$
$$\mathcal{T}(Inf_L\{[v_i, w_i]\}, [y_1, y_2]) \leq_L Inf_L\{\mathcal{T}([v_i, w_i], [y_1, y_2])\}$$

Due to the associativity of $\mathcal{T}$ the conjunction of three or more intervals can be defined inductively as:

$$\mathcal{T}(a, \mathcal{T}(b, c)) =_L \mathcal{T}(\mathcal{T}(a, b), c) =_L a \triangle b \triangle c \text{ where}$$
$$\triangle =_L \mathcal{T}.$$

where $a =_L [a_1, a_2]$, $b =_L [b_1, b_2]$ and $c =_L [c_1, c_2]$.

$\underline{T([x_1, x_2], [y_1, y_2])}$ and $\overline{T([x_1, x_2], [y_1, y_2])}$ will denote the lower and the higher values of $T([x_1, x_2], [y_1, y_2])$.

Definition 2.9 *[14] Let $\{x_i\}$ in $[0, 1]$. A t-norm T in $([0, 1], \leq)$ is left-continuous if it satisfies:*

$$T(Sup \, x_i, y) = Sup \, T(x_i, y)$$

Right-continuity can be defined in a similar way. This property is also called sup-preserving.

Definition 2.10 *[13] A generalized t-norm operator $\mathcal{T}$ is t-representable in $\mathcal{L}^I$ if there are two t-norms: T_1 and T_2 (T_1, T_2, in $([0,1], \leq)$) that satisfy:*

$$\mathcal{T}([x_1, x_2], [y_1, y_2]) =_L [T_1(x_1, y_1), T_2(x_2, y_2)]$$

where $T_1(v, w) \leq T_2(v, w) \, \forall v, w \in [0, 1]$.

Let $x =_L [x_1, x_2]$ and $y =_L [y_1, y_2]$ be two intervals on L:

Example 2.1
$Inf_L(\{[x_1, x_2], [y_1, y_2]\}) =_L [min(x_1, y_1), min(x_2, y_2)]$ *is t-representable by means of $T = min$ in $([0,1], \leq)$:*

Example 2.2 *The following product generalized t-norm $*_L$ is t-representable:*

$$*_L([x_1, x_2], [y_1, y_2]) =_L [x_1 * y_1, x_2 * y_2]$$

Example 2.3 *There are two generalizations of the Lukasiewicz t-norm [12]:*

- *$T_w([x_1, x_2], [y_1, y_2]) =_L$*
  $[max(0, x_1 + y_1 - 1), max(0, x_2 + y_2 - 1)]$

- *$T_W([x_1, x_2], [y_1, y_2]) =_L$*
  $[max(0, x_1 + y_1 - 1), max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)]$

Note that T_w is t-representable but T_W is not.

Definition 2.11 *[13] A generalized t-conorm $\mathcal{S}$ is an increasing, commutative, and associative operator $\mathcal{S} : L^2 \to L$, that satisfies: $\mathcal{S}(0_L, [x_1, x_2]) =_L [x_1, x_2]$ and $\mathcal{S}(1_L, [x_1, x_2]) =_L 1_L$.*

Due to the associativity of $\mathcal{S}$ we can write:

$$\mathcal{S}(a, \mathcal{S}(b, c)) =_L \mathcal{S}(\mathcal{S}(a, b), c) =_L a \bigtriangledown b \bigtriangledown c \text{ where } \bigtriangledown =_L \mathcal{S}.$$

For example, $\mathcal{S} = Sup_L$ is a generalized t-conorm.
In this paper the next definition is proposed.

Definition 2.12 *Let $\{[v_i, w_i]\}$ be in L. A generalized t-norm operator $\mathcal{T}$ is left-continuous if and only if:*

$$\mathcal{T}(Sup_L\{[v_i, w_i]\}, [y_1, y_2]) =_L Sup_L\{\mathcal{T}([v_i, w_i], [y_1, y_2])\}$$

Right-continuity can be defined in a similar way. A t-norm, a negation and the dual t-conorm of $\mathcal{T}$ with respect to $\mathcal{N}$ is called a De Morgan triplet.

3 Interval-valued fuzzy relations

This section contains some definitions on interval-valued fuzzy relations that generalize the ones of fuzzy relations valued on [0,1]. In the next section, $\mathcal{T}$-transitive relations will be generated using the $Sup - \mathcal{T}$ product.

Definition 3.1 *[15] Let X_1 and X_2 be two universes of discourse. An interval-valued fuzzy relation $R : X_1 \times X_2 \to L$ (where $\mathcal{L}^I = (L, \leq_L)$ is a lattice) is a mapping:*

$$R = \{((a, b), [x, y]) \mid a \in X_1, b \in X_2, [x, y] \in L\}$$

where $x = \underline{R(a, b)}$ and $y = \overline{R(a, b)}$.
In the rest of the paper, it is assumed that $X_1 = X_2 = X$.

Definition 3.2 *An interval-valued relation $R : X^2 \to L$ is reflexive if:*

$$R(a, a) =_L 1_L \, \forall a \in X$$

Definition 3.3 *An interval-valued relation* $R : X^2 \to L$ *is* $[\alpha_1 - \alpha_2]$−*reflexive if:*

$$R(a,a) \geq_L [\alpha_1, \alpha_2] \; \forall a \in X$$

where $\alpha_1, \alpha_2 \in [0,1]$ *and* $\alpha_1 \leq \alpha_2$.

Definition 3.4 *An interval-valued relation* $R : X^2 \to L$ *is symmetric if:*

$$R(a,b) =_L R(b,a) \; \forall a,b \in X$$

Definition 3.5 *Let* $\mathcal{T}$ *be a generalized t-norm operator and let* R *interval-valued fuzzy relation on* X. R *is* $\mathcal{T}$-*transitive if:*

$$\mathcal{T}(R(a,b), R(b,c)) \leq_L R(a,c) \; \forall a,b,c \in X$$

Proposition 3.1 *If* $\mathcal{T}$ *is t-representable with* T_1 *and* T_2 *(*$\mathcal{T} = [T_1, T_2]$*) then an interval-valued relation* $R : X^2 \to L$ *is* $\mathcal{T}$-*transitive if and only if* $\underline{R}$ *is* T_1-*transitive and* $\overline{R}$ *is* T_2-*transitive:*

Definition 3.6 *Let* $\mathcal{T}$ *a generalized t-norm. Let* $\mathcal{S}$ *a generalized t-conorm from* $\mathcal{S}$ *and let be* $\triangledown$ *the n-ary generalized t-conorm defined by associativity. Let* R_1 *and* R_2 *be two interval-valued fuzzy relations on a finite set* $X = \{c_1, ... c_m\}$. *The* $\mathcal{S} - \mathcal{T}$-*composition of* R_1 *and* R_2 *is defined as follows:*

$$(R_1 \diamond_{(\mathcal{ST})} R_2)(c_q, c_r) =_L \triangledown_{c_k \in X} \mathcal{T}(R_1(c_q, c_k), R_2(c_k, c_r))$$

where $\triangledown_{c_k \in X} \mathcal{T}(R_1(c_q, c_k), R_2(c_k, c_r)) =_L$ $\mathcal{T}(R_1(c_q, c_1), R_2(c_1, c_r)) \triangledown ... \triangledown \mathcal{T}(R_1(c_q, c_m), R_2(c_m, c_r))$.

Definition 3.7 *The local equality relation of a fuzzy interval-valued fuzzy relation* R *on* X *is the fuzzy relation* E_R *is:*

$$E_R(a,b) = \begin{cases} Sup_L \, _{\forall c \in X} max(R(a,c), R(c,a)) & a = b; \\ [0,0], & a \neq b. \end{cases}$$

Definition 3.8 *An interval-valued relation* R *is locally reflexive if* $E_R \subseteq_L R$.

It is easy to prove that if R is an reflexive interval-valued relation then R is a locally reflexive interval-valued relation.

4 $\mathcal{T}$-transitive closure for IVFSs

In this section it is proved the existence of the $\mathcal{T}$-transitive closure of an interval valued fuzzy relation and its calculation via $sup - \mathcal{T}$. The $sup - \mathcal{T}$ product is associative if $\mathcal{T}$ is left continuous of the universe is finite. Moreover, it is proved that the $sup - \mathcal{T}$ product is continuous and so is the calculation of the $\mathcal{T}$-transitive closure for finite universes. An algorithm to calculate the $\mathcal{T}$-transitive closure is provided.

Definition 4.1 *Let* $\mathcal{T}$ *be a generalized t-norm and let* $R : X^2 \to L$ *be an interval-valued relation on a finite universe* X. *The* $\mathcal{T}$-*transitive closure of* R *is the relation* $R^{\mathcal{T}} : X \times X \to L$ *that satisfies:*

1. $R \subseteq_L R^{\mathcal{T}}$.

2. *If* $R \subseteq_L R'$ *and* R' *is* $\mathcal{T}$-*transitive then* $R^{\mathcal{T}} \subseteq_L R'$.

Lemma 4.1 R *is* $\mathcal{T}$-*transitive if and only if* $R \diamond_{Sup_L T} R \subseteq_L R$.

Proof.

- R is $\mathcal{T}$-transitive $\Rightarrow R \diamond_{Sup_L T} R \subseteq_L R$:
  $(R \diamond_{Sup_L T} R)(a,b) =_L$
  $Sup_L \, _{c \in X} \{\mathcal{T}(R(a,c), R(c,b))\} \leq_L R(a,b)$
  due to $\mathcal{T}(R(a,c), R(c,b)) \leq_L R(a,b) \; \forall a,b,c \in X$

- $R \diamond_{Sup_L T} R \subseteq_L R \Rightarrow R$ is $\mathcal{T}$-transitive:
  $Sup_L \, _{d \in X} \{\mathcal{T}(R(a,d), R(d,b))\} \leq_L R(a,b) \; \forall a,b \in X$
  Then $\forall c \in X \; \mathcal{T}(R(a,c), R(c,b)) \leq_L R(a,b)\square$

Lemma 4.2 *Let* R *be an interval-valued relation. If* $R^{\mathcal{T}}$ *exists then it is unique.*

Proof. Let S_1 and S_2 be two relations. If S_1 and S_2 are transitive closures of R then according to definition 4.1: $S_1 \subseteq_L S_2$ and $S_2 \subseteq_L S_1$, consequently $S_1 =_L S_2\square$.

Lemma 4.3 *Let* R *be an interval-valued relation in a universe* X *and an let* $\mathcal{T}$ *be an arbitrary generalized t-norm. Then the transitive closure of* R *always exists.*

Proof. Consider the set Ω_R of $\mathcal{T}$-transitive fuzzy relations containing R. Let us define the fuzzy relation $S_R^*(a,b) =_L Inf_{L;S \in \Omega_R}\{S(a,b)\}$. We will prove that S_R^* is $\mathcal{T}$-transitive. Due to monotony of generalized $\mathcal{T}$-norms it is obtained:
$\mathcal{T}(S_R^*(a,b), S_R^*(b,c)) =_L$
$\mathcal{T}(Inf_L \, _{S_1 \in \Omega_R}\{S_1(a,b)\}, Inf_L \, _{S_2 \in \Omega_R}\{S_2(b,c)\})$
$\leq_L Inf_L \, _{S_1 \in \Omega_R} Inf_L \, _{S_2 \in \Omega_R} \{\mathcal{T}(S_1(a,b), S_2(b,c))\}$
$\leq_L Inf_L \, _{S \in \Omega_R} \{\mathcal{T}(S(a,b), S(b,c))\} =_L S_R^*(a,c)\square$

Proposition 4.1 *Let* X *be an arbitrary universe. Let* $\{S_1, ..., S_n\}$ *a set of interval-valued relations. If* $\mathcal{T}$ *is a left-continuous generalized* $\mathcal{T}$-*norm, then it satisfies:*

$$R \diamond_{Sup_L T} (Sup_L \, _{\forall i=1..n}\{S_i\}) =_L$$
$$Sup_L \, _{\forall i=1..n}\{R \diamond_{Sup_L T} S_i\}$$

Proof.

$R \diamond_{Sup_L T} (Sup_L \, _{\forall i=1..n}\{S_i\})(a,c) =_L$
$Sup_L \, _{b \in X} \{\mathcal{T}(R(a,b), Sup_L \, _{\forall i=1..n}\{S_i(b,c)\}\}$
$=_L Sup_L \, _{\forall i=1..n} Sup_L \, _{b \in X} \{\mathcal{T}(R(a,b), S_i(b,c))\}$
$=_L (Sup_L \, _{\forall i=1..n}\{R \diamond_{Sup_L T} S_i\})(a,c)\square$

Definition 4.2 *Given a generalized t-norm* $\mathcal{T}$, *the* $\mathcal{T}$-*power* $R^{(n)_{\mathcal{T}}}$ *of a fuzzy relation* R *on* X *is recursively defined as follows:*

1. $R^{(1)_{\mathcal{T}}} \equiv R$

2. $R^{(n)_{\mathcal{T}}} \equiv R^{(n-1)_{\mathcal{T}}} \diamond_{Sup_L T} R$

Lemma 4.4 *If* $A \subseteq_L B$ *then* $A^{(k)} \subseteq_L B^{(k)}$

Proof.
Trivial due to monotony of $\mathcal{T}\square$.

Theorem 4.1 *Let* X *be an arbitrary universe and* $\mathcal{T}$ *a left-continuous t-norm. The transitive closure of* R *is:*

$$R^{\mathcal{T}} =_L Sup_L \, _{\forall k \in \mathbb{N}}\{R^{(k)_{\mathcal{T}}}\}$$

Proof.
Let $S = Sup_L \, _{\forall k \in \mathbb{N}}\{R^{(k)_{\mathcal{T}}}\}$ be. By proposition 4.1:

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$$S^{(2)\mathcal{T}} =_L$$
$$(Sup_L \,_{\forall k \in \mathbb{N}}\{R^{(k)\mathcal{T}}\}) \diamond_{Sup_L \mathcal{T}} (Sup_L \,_{\forall l \in \mathbb{N}}\{R^{(l)\mathcal{T}}\}) =_L$$
$$Sup_L \,_{\forall k,l \in \mathbb{N}}\{R^{(k+l)\mathcal{T}}\} =_L Sup_L \,_{\forall m \in \mathbb{N}\setminus\{0,1\}}\{R^{(m)\mathcal{T}}\} \subseteq S$$

By lemma 4.1, it follows that S is T-transitive.

Moreover, if $R \subseteq S'$ and S' is T-transitive, then again by lemma 4.1 and 4.4 it holds that:

$$R^{(2)\mathcal{T}} \subseteq_L S'^{(2)\mathcal{T}} \subseteq_L S', ..., R^{(k)\mathcal{T}} \subseteq_L S'^{(k)\mathcal{T}} \subseteq_L S'$$

hence $S \subseteq_L S'$ and $R^\mathcal{T} =_L S\square$.

Lemma 4.5 *Let $\mathcal{T}$ be any generalized t-norm, then:*

$$\mathcal{T}([x_1,x_2],[y_1,y_2]) \leq_L Inf_L([x_1,x_2],[y_1,y_2])$$
$$\forall [x_1,x_2],[y_1,y_2] \in L$$

Proof.
$\mathcal{T}([x_1,x_2],[y_1,y_2]) \leq_L \underline{\mathcal{T}([x_1,x_2],1_L) =_L [x_1,x_2]}$
$\Rightarrow [\underline{\mathcal{T}}([x_1,x_2],[y_1,y_2]), \overline{\mathcal{T}}([x_1,x_2],[y_1,y_2])] \leq_L [x_1,x_2]$
$\mathcal{T}([x_1,x_2],[y_1,y_2]) \quad =_L \quad \mathcal{T}([y_1,y_2],[x_1,x_2]) \quad \leq_L$
$\underline{\mathcal{T}([y_1,y_2],1_L) =_L [y_1,y_2]}$
$\Rightarrow [\underline{\mathcal{T}}([x_1,x_2],[y_1,y_2]), \overline{\mathcal{T}}([x_1,x_2],[y_1,y_2])] \leq_L [y_1,y_2]$
Therefore:
$\underline{\mathcal{T}([x_1,x_2],[y_1,y_2])} \leq minimum[x_1,y_1]$
$\overline{\mathcal{T}([x_1,x_2],[y_1,y_2])} \leq minimum[x_2,y_2] \Rightarrow$
$\mathcal{T}([x_1,x_2],[y_1,y_2]) \leq_L Inf_L([x_1,x_2],[y_1,y_2])\square$

Lemma 4.6 *Let $\mathcal{S}$ be any generalized t-conorm, then:*

$$\mathcal{S}([x_1,x_2],[y_1,y_2]) \geq_L Sup_L([x_1,x_2],[y_1,y_2])$$
$$\forall [x_1,x_2],[y_1,y_2] \in L$$

Proof.
$\mathcal{S}([x_1,x_2],[y_1,y_2]) \geq_L \underline{\mathcal{S}([x_1,x_2],1_L) =_L [x_1,x_2]}$
$\Rightarrow [\underline{\mathcal{S}}([x_1,x_2],[y_1,y_2]), \overline{\mathcal{S}}([x_1,x_2],[y_1,y_2])] \geq_L [x_1,x_2]$
$\mathcal{S}([x_1,x_2],[y_1,y_2]) \quad =_L \quad \mathcal{S}([y_1,y_2],[x_1,x_2]) \quad \geq_L$
$\underline{\mathcal{S}([y_1,y_2],1_L) =_L [y_1,y_2]}$
$\Rightarrow [\underline{\mathcal{S}}([x_1,x_2],[y_1,y_2]), \overline{\mathcal{S}}([x_1,x_2],[y_1,y_2])] \geq_L [y_1,y_2]$
Therefore:
$\underline{\mathcal{S}([x_1,x_2],[y_1,y_2])} \geq maximun[x_1,y_1]$
$\overline{\mathcal{S}([x_1,x_2],[y_1,y_2])} \geq maximun[x_2,y_2] \Rightarrow$
$\mathcal{S}([x_1,x_2],[y_1,y_2]) \geq_L Sup_L([x_1,x_2],[y_1,y_2])\square$

Due associativity of $\mathcal{T}$:

$$\mathcal{T}([x_1,x_2],\mathcal{T}([y_1,y_2],[z_1,z_2])) =_L$$
$$\mathcal{T}(\mathcal{T}([x_1,x_2],[y_1,y_2]),[z_1,z_2]) =_L [x_1,x_2] \triangle [y_1,y_2] \triangle$$
$$[z_1,z_2]$$
$$Inf_L([x_1,x_2],Inf_L([y_1,y_2],[z_1,z_2])) =_L$$
$$Inf_L(Inf_L([x_1,x_2],[y_1,y_2]),[z_1,z_2])) =_L [x_1,x_2] \triangle_{L-inf}$$
$$[y_1,y_2] \triangle_{L-inf} [z_1,z_2]$$

Lemma 4.7 *Let $\mathcal{T}$ be an arbitrary generalized t-norm. Let P be a path with a cycle:*

$$P \equiv R(a,a_1) \triangle R(a_1,a_2) \triangle ... \triangle R(a_{t-1},a_t) \triangle$$
$$R(a_t,c_1) \triangle R(c_1,c_2) \triangle ... \triangle R(c_q,a_t) \triangle R(a_t,a_{t+1}) \triangle$$
$$... \triangle R(a_k,b)$$

then

$$P \equiv R(a,a_1) \triangle R(a_1,a_2) \triangle ... \triangle R(a_{t-1},a_t) \triangle$$
$$R(a_t,c_1) \triangle R(c_1,c_2) \triangle ... \triangle R(c_q,a_t) \triangle R(a_t,a_{t+1}) \triangle$$
$$... \triangle R(a_k,b) \leq_L R(a,a_1) \triangle R(a_1,a_2) \triangle ... \triangle$$
$$R(a_{t-1},a_t) \triangle R(a_t,a_{t+1}) \triangle ... \triangle R(a_k,b)$$

Proof.
Trivial due to $\mathcal{T}([x_1,x_2],[y_1,y_2]) \leq_L [y_1,y_2]$ forall $[x_1,x_2],[y_1,y_2]$ in L and associativity of $\mathcal{T}\square$.

Theorem 4.2 *Let X be a finite universe with cardinality n. The transitive closure of R, $R^\mathcal{T}$, is:*

$$R^\mathcal{T} =_L Sup_L \,_{k=1..n}\{R^{(k)\mathcal{T}}\}$$

Proof.

$R^{(k)\mathcal{T}}(a,b) =_L$

$=_L Sup_L \,_{a_1,a_2,...,a_{k-1}} R(a,a_1) \triangle R(a_1,a_2) \triangle$
$... \triangle R(a_{k-1},b) \leq_L$
$=_L Sup_L \,_{a_1,a_2,...,a_{k-1}} R(a,a_1) \quad \triangle_{Inf_L}$
$R(a_1,a_2) \triangle_{Inf_L} ... \triangle_{Inf_L} R(a_{k-1},b)$ according to lemma 4.5.

Now suppose $k > n$. Then $R^{(k)} \subseteq_L R^{(n)}$ because it exists a cycle (lemma 4.7). Therefore, the cases $k > n$ need not be considered $\square$.

Corollary 4.1 *Let X be a finite universe with cardinality n. If R is a locally reflexive relation, the transitive closure of R is:*

$$R^\mathcal{T} =_L Sup_L \,_{k=1..n-1}\{R^{(k)\mathcal{T}}\}$$

Proof
If R is a locally reflexive relation then $R(a,b) \leq_L R(a,a)$. Consider a cycle of length n, then its strength is:
$R^{(n)\mathcal{T}}(a,a) =_L$

$=_L Sup_L \,_{a_1,a_2,...,a_{k-1}} R(a,a_1) \triangle R(a_1,a_2) \triangle$
$..., \triangle R(a_{k-1},a) \leq_L$
$Sup_L \,_{a_1,a_2,...,a_{k-1}} R(a,a_1) \quad \triangle_{Inf_L}$
$R(a_1,a_2) \triangle_{Inf_L} ... \triangle_{Inf_L} R(a_{k-1},a) \leq_L$
$R(a,a)\square$

Theorem 4.3 *Let X be a finite universe with cardinality n. If R is a reflexive fuzzy relation on X then the transitive closure of R is:*

$$R^\mathcal{T} =_L R^{(n-1)\mathcal{T}}$$

Proof.
$R(a,b)^{(k)\mathcal{T}} =_L$

$=_L \mathcal{T}(R(a,a),R^{(k)\mathcal{T}}(a,b))$
$\leq_L Sup_L \,_{c \in X}\{T(R(a,c),R^{(k)\mathcal{T}}(c,b))\}$
$=_L R^{(k+1)\mathcal{T}}(a,b)$

Therefore $R(a,b)^{(k)\mathcal{T}} \leq_L R^{(k+1)\mathcal{T}}(a,b) \Leftrightarrow R^{(k)\mathcal{T}} \subseteq R^{(k+1)\mathcal{T}}\square$

Theorem 4.4 *Let $\mathcal{T}$ be a t-representable t-norm ($\mathcal{T} = [T_1,T_2]$) and let $R = [\underline{R},\overline{R}]$ be a interval-valued relation. Then $R^\mathcal{T} = [\underline{R}^{T_1},\overline{R}^{T_2}]$.*

Proof.

$R^{(k)\mathcal{T}}(a,b) =_L$

$=_L \quad Sup_{L \ a_1,a_2,...,a_{k-1}} R(a,a_1) \quad \triangle \quad R(a_1,a_2) \quad \triangle$

$..., \triangle \ (a_{k-1}, b)$

$=_L \quad Sup_{L \ a_1,a_2,...,a_{k-1}}[\underline{R(a,a_1)} \quad \triangle_1 \quad \underline{R(a_1,a_2)} \quad \triangle_1$

$..., \triangle_1 \quad \underline{R(a_{k-1},b)}, \overline{R(a,a_1)} \quad \triangle_2 \quad \overline{R(a_1,a_2)} \quad \triangle_2$

$..., \triangle_2 \ \overline{R(a_{k-1},b)}]$

$=_L \quad [max_{a_1,a_2,...,a_{k-1}} \underline{R(a,a_1)} \quad \triangle_1 \quad \underline{R(a_1,a_2)} \quad \triangle_1$

$..., \triangle_1 \ \underline{R(a_{k-1},b)}$

$, max_{a_1,a_2,...,a_{k-1}} \overline{R(a,a_1)} \ \triangle_2 \ \overline{R(a_1,a_2)} \ \triangle_2 ..., \triangle_2$

$\overline{R(a_{k-1},b)}]$

$=_[\quad \underline{R^{(k)_{T_1}}(a,b)}, \overline{R^{(k)_{T_2}}(a,b)}]$

where $\triangle = [\triangle_1, \triangle_2]$. Then $\mathcal{T} = [T_1, T_2] \Rightarrow R^{\mathcal{T}} = [\underline{R}^{T_1}, \overline{R}^{T_2}]\square$.

Definition 4.3 *The distance d of the supremum on L is defined for all $x_1, x_2, y_1, y_2 \in [0,1]$ by $d([x_1,x_2],[y_1,y_2]) = max(|x_1 - y_1|, |x_2 - y_2|)$.*

Definition 4.4 *Let R,S be two interval-valued fuzzy relatons on a set X. The distance d between R and S us defined by:*

$$d(R,S) = sup_{x,y \in X} d(R(x,y), S(x,y))$$

Lemma 4.8 *d is a distance on the set R_X of interval-valued fuzzy relations on X $\square$.*

Proof.

The supremum of distances is a distance.

Theorem 4.5 *Let T be a generalized continuous t-norm and R_X the set of interval-valued fuzzy relations on X. R_X with the sup-$\mathcal{T}$ product is an ordered topological semigroup.*

Proof.

- Associativity is a straightforward exercise.

- The interval-valued relation $E(x,y) = \begin{cases} [1,1], & x=y; \\ [0,0], & \text{otherwise.} \end{cases}$ is the identity element of R_X.

- Continuity: Since T is defined on a compact set, it is uniformly continuous. Therefore:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that}$$
$$\forall m,n,m',n',a,b,a',b' \in [0,1]$$
$$max(|m-a|,|m'-a'|,|n-b|,|n'-b'|) \Rightarrow$$
$$|\mathcal{T}([m,m'],[n,n']) - \mathcal{T}([a,a'],[b,b']) < \epsilon (*)$$

We want to prove that given two interval-valued fuzzy relations $A, B \in R_X$:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } \forall M, N \in R_X$$
$$max(d(M,A), d(N,B)) < \delta \Rightarrow d(M \circ N, A \circ B) < \epsilon$$

Given ϵ, we take $\delta > 0$ satisfying (*). Then:

$$d(M \circ N, A \circ B) =$$
$$sup_{x,y \in X} |sup_{z \in X} \mathcal{T}(M(x,z), N(z,y)) -$$
$$sup_{z \in X} \mathcal{T}(A(x,z), B(z,y))| \leq$$
$$sup_{x,y \in X} sup_{z \in X} |\mathcal{T}(M(x,z), N(z,y)) -$$
$$\mathcal{T}(A(x,z), B(z,y))| \leq \epsilon$$

- Monotonicity is an immediate consequence of the monotonicity of $\mathcal{T}\square$.

Corollary 4.2 *For any positive integer n the map assigning to an interval-values fuzzy relation M on a finite set its nth power is non-decreasing and continuous if the corresponding generalized t-norm $\mathcal{T}$ is continuous.*

Corollary 4.3 *If a generalized t-norm $\mathcal{T}$ is continuous, the map that assigns the $\mathcal{T}$-transitive closure to the interval-valued fuzzy relations on a finite set is non-decreasing and continuous.*

Theorem 4.6 *Given a method to calculate the transitive closure of a fuzzy relation. Nevertheless, there are more effective methods and algorithms. For example it is possible to use an extension of the Floyd-Warshall algorithm [16]*

4.1 Algorithm to compute the transitive closure of an interval-valued fuzzy relation

Let R be an interval-valued relation on a finite universe X with cardinality n and let $\mathcal{T}$ be a generalized t-norm. It is possible to compute $R^{\mathcal{T}}$ using the next algorithm:

```
for k ← 1 until n do
    for i ← 1 until n do
        for j ← 1 until n do
            R(i,j)← Sup_L(R(i,j), T(R(i,k), R(k,j)))
        end for
    end for
end for
```

Example 4.1 *Let $\mathcal{T}$ be a generalized t-norm $T_w([x_1,x_2],[y_1,y_2]) =_L [max(0, x_1 + y_1 - 1), max(0, x_2 + y_2 - 1)]$ and let $R : X \times X \to L$ be the following interval-valued relation:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

The computation of the $\mathcal{T}$-transitive closure of R is the following:

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.8] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

Note that by theorem 4.4, $R^{\mathcal{T}} = [\underline{R^W}, \overline{R^W}]$

Example 4.2 *Let $\mathcal{T}$ be a generalized t-norm $T([x_1,x_2],[y_1,y_2]) =_L [W(x_1, y_1), Min(x_2, y_2)]$ and let $R : X \times X \to L$ be the following interval-valued relation:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

The computation of the $\mathcal{T}$-transitive closure of R is the following:

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.9] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

Note that by theorem 4.4, $R^{\mathcal{T}} = [\underline{R^W}, \overline{R^{Min}}]$

Example 4.3 *Let $\mathcal{T}$ be a generalized t-norm $T_W([x_1, x_2], [y_1, y_2]) =_L [max(0, x_1 + y_1 - 1), max(0, x_1 + y_2 - 1, x_2 + y_1 - 1)]$ and let $R : X \times X \to L$ be the following interval-valued relation:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0,0] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

The computation of the $\mathcal{T}$-transitive closure of R is the following:

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.8] & [0.6,0.9] & [0.2,0.5] \\ & [1,1] & [0.4,0.9] & [0.6,0.9] \\ & & [1,1] & [0.6,0.9] \\ & & & [1,1] \end{pmatrix}$$

Note that the generalized $\mathcal{T}$-norm in this example is not t-representable.

5 An Application

It is possible to model social relations using Interval-valued Fuzzy Sets. For example, it is possible to model "friendship". Let R be a relation which describes the degree of friendship between pairs of people. If A and B are friends and B and C are friends, then it will probably exist some degree of friendship between A and C. In this context, the $\mathcal{T}$-transitive closure can be used to learn friendship of individuals from a chain of friends, for example, from common friends.

Example 5.1 *Let R be an interval-valued relation which models the degree of friendship between four people $\{a, b, c, d\}$. It is possible to model the transitivity of friendship using the generalized t-norm $*_L([x_1, x_2], [y_1, y_2]) =_L [x_1 * y_1, x_2 * y_2]$:*

$$R = \begin{pmatrix} [1,1] & [0.6,0.9] & [0.6,0.7] & [0,0] \\ & [1,1] & [0,0] & [0.2,0.8] \\ & & [1,1] & [0,0] \\ & & & [1,1] \end{pmatrix}$$

Initially, d has only friendship with b. In addition, the uncertainty of degree of friendship between b and d is wide. It is interesting to see how friendship is propagated.

$$R^{\mathcal{T}} = \begin{pmatrix} [1,1] & [0.6,0.9] & [0.6,0.7] & [0.16,0.72] \\ & [1,1] & [0.48,0.63] & [0.2,0.8] \\ & & [1,1] & [0.096,0.5] \\ & & & [1,1] \end{pmatrix}$$

In $R^{\mathcal{T}}$ d has some degree of friendship with a, c learned by a common friend b. The uncertainty of degree of friendship of d with the rest of them has been propagated as well.

6 Conclusions

Traditionally, the study of interval-valued fuzzy sets has been reduced to t-representable t-norms. We have generalized the main properties of interval-valued fuzzy relations: reflexivity, symmetry, composition, local reflexivity and t-transitivity. We also have generalized the T-transitive closure for any t-norm and an algorithm to compute it is provided. It is analyzed the case of t-representable t-norms and proved that in this case the T-transitive closure of R can be obtained from the two classical T-transitive closures of the lower and higher relations (theorem 4.4). Some examples are provided.

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